

# Financial Markets as Nonlinear Adaptive Evolutionary Systems.

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# 1 Introduction

The key difference between economics and the natural sciences is perhaps the fact that decisions of economic agents today depend upon their *expectations* or *beliefs* about the future. For example, after a couple of weeks of bad weather in the Netherlands and Western Europe in July 2000, the dreams and hopes of the Dutch about nice weather for summer holidays will *not* affect the weather in August. In contrast, the dreams and hopes of Dutch investors for excessive high returns on their investments in tulip bulbs in the seventeenth century may have contributed to or even caused what is nowadays known as the Dutch ‘tulip mania’, when the price of tulip bulbs exploded by a factor of more than 20 in the beginning of 1636 but ‘crashed’ back to its original level by the end of the year. Nowadays, in financial markets an overoptimistic estimate of future growth of ICT industries may contribute to an excessively rapid growth of stock prices and indices and might lead to over valuation of stock markets worldwide. Any dynamic economic system is in fact an *expectations feedback* system. A theory of expectation formation is therefore a crucial part of any economic model or theory.

Since its introduction in the sixties by Muth (1961) and its popularization in macroeconomics by Lucas (1971), the rational expectations hypothesis (REH) has become the dominating expectation formation paradigm in economic theory. According to the REH all agents are rational and take as their subjective expectation of future variables the objective prediction by economic theory. In a rational expectations model agents have perfect knowledge about the (linear) market equilibrium equations and use these to derive their expectations. Although many economists nowadays view rational expectations as something unrealistic, it is still viewed as an important benchmark. Despite a rapidly growing literature on *bounded rationality*, where agents use learning models for their expectations, it seems fair to say that at this point no generally accepted alternative theory of expectations is available.

In finance, the REH is intimately related to the Efficient Market Hypothesis (EMH). There are weak and strong forms of the EMH, but when economists speak of financial markets as being efficient, they usually mean that they view asset prices and returns as the outcome of a competitive market consisting of rational traders, who are trying to maximize their expected returns. The main reason why financial markets must be efficient is based upon an *arbitrage* argument (e.g. Fama (1970)). If markets were not efficient, then there would be unexploited profit opportunities, that could and would be exploited by rational traders. For example, rational traders would buy (sell) an underpriced (overpriced) asset, thus driving its price back to the correct, fundamental value. In an efficient market, there can be *no forecastable structure* in asset returns, since any such structure would be exploited by rational traders and therefore would be doomed to disappear. Rational agents thus process information quickly and this is reflected immediately in asset prices. The value of a risky asset is completely determined by its *fundamental price*, equal to the present discounted value of the expected stream of future dividends. In an efficient market, all traders are rational and changes in asset prices are completely random, solely driven by unexpected ‘news’ about changes in economic fundamentals.

In contrast, Keynes already questioned a completely rational valuation of assets, argu-

ing that investors sentiment and mass psychology (*'animal spirits'*) play a significant role in financial markets. Keynes used his famous *beauty contest* as a parable to financial markets. In order to predict the winner of a beauty contest, objective beauty is not that much important, but knowledge or prediction of others' perceptions of beauty is much more relevant. Keynes argued that the same may be true for the fundamental price of an asset: *'Investment based on genuine long-term expectation is so difficult as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes'* (Keynes, 1936, p.157). In Keynes view, stock prices are thus not governed by an objective view of 'fundamentals', but by 'what average opinion expects average opinion to be'.

New classical economists have viewed 'market psychology' and 'investors sentiment' as being *irrational* however, and therefore inconsistent with the REH. For example, Friedman (1953) argued that irrational speculative traders would be driven out of the market by rational traders, who would trade against them by taking long opposite positions, thus driving prices back to fundamentals. In an efficient market, 'irrational' speculators would simply lose money and therefore fail to survive evolutionary competition.

### **Financial markets as nonlinear evolutionary adaptive systems**

In a perfectly rational EMH world *all* traders are rational and it is *common knowledge* that all traders are rational. In real financial markets however, traders are different, especially with respect to their expectations about future prices and dividends. A quick glance at the financial pages of newspapers is sufficient to observe that *difference of opinions* among financial analysts is the rule rather than the exception. In the last decade, a rapidly increasing number of structural heterogeneous agent models have been introduced in the finance literature, see for example Arthur et al. (1997), Brock (1993, 1997), Brock and Hommes (1997, 1998), Brock and LeBaron (1996), Chiarella (1992), Chiarella and He (2000), Dacorogna et al. (1995), DeGrauwe et al. (1993), De Long et al. (1990), Farmer (2000), Frankel and Froot (1988), Gaunersdorfer (2000), Gaunersdorfer and Hommes (2000), Kirman (1991), Kurz (1997), LeBaron (2000), LeBaron et al. (1999), Lux (1995), Lux and Marchesi (1999a,b), Wang (1994) and Zeeman (1974), as well as many more references in these papers. Some authors even talk about a *Heterogeneous Market Hypothesis*, as a new alternative to the Efficient Market Hypothesis. In all these heterogeneous agent models different groups of traders, having different beliefs or expectations, co-exist. Two typical trader types can be distinguished. The first are rational, *'smart money'* traders or *fundamentalists*, believing that the price of an asset is determined completely by economic fundamentals. The second typical trader type are *'noise traders'*, sometimes called *chartists* or *technical analysts*, believing that asset prices are not determined by fundamentals, but that they can be predicted by simple technical trading rules based upon patterns in past prices, such as trends or cycles.

In a series of papers, Brock and Hommes (1997a,b, 1998, 1999), henceforth BH, propose to model economic and financial markets as *Adaptive Belief Systems (ABS)*. This paper reviews the main features of ABS and discusses a recent extension by Hommes and Gaunersdorfer (2000) as well as some recent experimental testing, jointly with my colleagues Joep Sonnemans, Jan Tuinstra and Henk van de Velden at CeNDEF. An ABS

is an evolutionary competition between trading strategies. Different groups of traders have different expectations about future prices and future dividends. For example, one group might be fundamentalists, believing that asset prices return to their fundamental equilibrium price, whereas another group might be chartists, extrapolating patterns in past prices. Traders choose their trading strategy according to an evolutionary ‘fitness measure’, such as accumulated past profits. Agents are boundedly rational, in the sense that most traders choose strategies with higher fitness. BH introduce the notion of *Adaptive Rational Equilibrium Dynamics (ARED)*, a coupling between market equilibrium dynamics and evolutionary updating of beliefs. Current beliefs determine new equilibrium prices, generating adapted beliefs which in turn lead to new equilibrium prices again, etc.. In an ARED, equilibrium prices and beliefs co-evolve over time.

Most of the heterogeneous agent literature is computationally oriented. An ABS may be seen as a tractable theoretical framework for the computationally oriented ‘artificial stock market’ literature, such as the Santa Fe artificial stock market of Arthur et al. (1997) and LeBaron et al. (1999). A convenient feature of an ABS is that the model can be formulated in terms of *deviations* from a benchmark fundamental. In fact, the perfectly rational EMH benchmark is nested within an ABS as a special case. An ABS may thus be used for experimental and empirical testing of possible significant deviations from any suitable RE benchmark.

The heterogeneity of expectations among traders introduces an important *nonlinearity* into the market. In an ABS there are also two important sources of *noise*: model approximation error and intrinsic uncertainty about economic fundamentals. Asset price fluctuations in an ABS are characterized by an irregular switching between phases of close-to-the-fundamental-price fluctuations, phases of optimism where most agents follow an upward price trend, and phases of pessimism with small or large market crashes. Temporary speculative bubbles (*rational animal spirits*) can occur, triggered by noise and reinforced by evolutionary forces. An ABS is able to generate some of the important stylized facts in many financial series, such as unpredictable returns, fat tails and volatility clustering.

In our discussion of ABS we will focus on the following questions central to the SFI workshop:

- Q1 Can technical analysts or habitual rule of thumb trading strategies *survive* evolutionary competition against rational or fundamental traders?
- Q2 Is an evolutionary adaptive financial market with competing heterogeneous agent *efficient*?
- Q3 Does heterogeneity in beliefs lead to *excess volatility*?

The paper is organized as follows. In section 2 we discuss the modeling philosophy emphasizing recent developments in nonlinear dynamics and their relevance to economics and finance. Section 3 presents ABS in a general mean-variance framework. In section 4 we present simple, but typical examples. Although the ABS are very simple, subsection 4.4 present an example where the autocorrelations of returns, squared returns and absolute returns closely resemble those of 40 years of S&P 500 data. Section 5 briefly

discusses some recent work concerning experimental testing of ABS. Finally, section 6 sketches a future perspective of the research program proposed here.

## 2 Philosophy of Nonlinear Dynamics

The past 25 years have witnessed an explosion of interest in nonlinear dynamical systems, in mathematics as well as in applied sciences. In particular, the fact that simple deterministic nonlinear systems exhibit bifurcation routes to chaos and strange attractors, with ‘random looking’ dynamic behavior, has received much attention. This section discusses some important features of nonlinear systems, emphasizing their relevance to economics and finance. Let us start by stating the main goal of our research program, namely to explain the most important ‘stylized facts’ in financial series, such as

- S1 Asset prices are *persistent* and have, or are close to having, a unit root.
- S2 Asset returns are fairly *unpredictable*, and typically have little or no autocorrelations.
- S3 Asset returns have *fat tails* and exhibit *volatility clustering* and *long memory*. Autocorrelations of squared returns and absolute returns are significantly positive, even at high order lags, and decay slowly possibly following a scaling law.
- S4 Trading volume is persistent and there is positive cross correlation between volatility and volume.

In this paper we will be mainly concerned with stylized facts S2 and S3<sup>1</sup>. The adaptive belief system introduced in the next section will be a nonlinear stochastic system of the form

$$X_{t+1} = F(X_t; n_{1t}, \dots, n_{Ht}; \lambda; \delta_t; \epsilon_t), \quad (1)$$

where  $F$  is a *nonlinear* mapping,  $X_t$  is a vector of prices (or lagged prices),  $n_{jt}$  is the fraction or weight of investors of type  $h$ ,  $1 \leq h \leq H$ ,  $\lambda$  is a vector of parameters and  $\delta_t$  and  $\epsilon_t$  are noise terms. In an ABS there are two types of noise terms which are relevant for financial markets. The noise term  $\epsilon_t$  is the *model approximation error* representing the fact that a model can only be an approximation of the real world. Approximation errors will also be present in a physical model, although the corresponding noise terms might be of smaller magnitude than in economics. In contrast to physical models however, in economic and financial models one almost always has to deal with *intrinsic uncertainty* represented here by the noise term  $\delta_t$ . In finance, for example one typically deals with investors’ uncertainty about economic fundamentals. In the ABS there will be uncertainty about future dividends and the noise term  $\delta_t$  represents unexpected random news about dividends. An important goal of our research program is to match

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<sup>1</sup>We will mainly focus on stationary systems here, but the example in subsection 4.4 will be close to having a unit root and matching the stylized fact S1. In future work, we also plan to investigate the relation between trading volume and volatility in stylized fact S4.

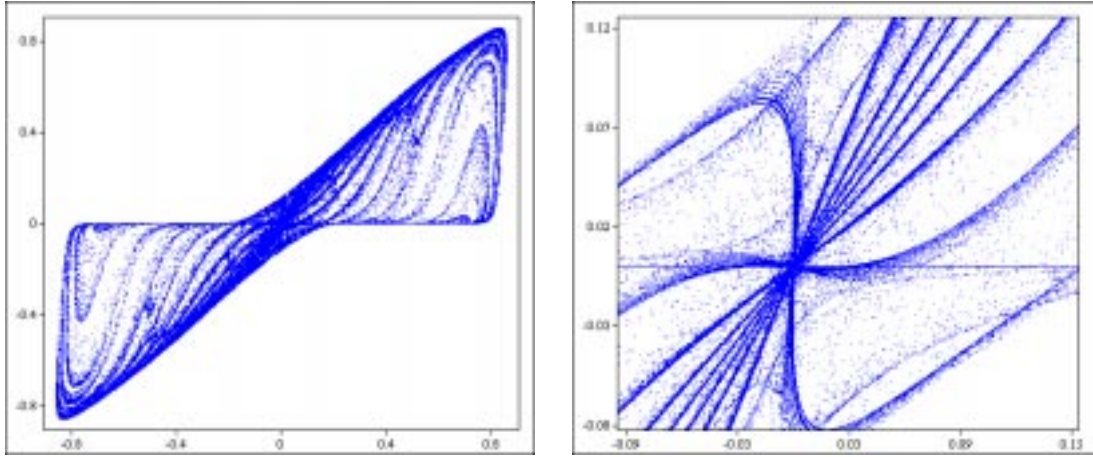


Figure 1: Strange attractor (left panel) of the adaptive belief system of subsection 4.3. and enlargement of its fractal structure (right panel).

the nonlinear stochastic model (1) to the statistical properties of the data, as closely as possible, and in particular to first match the most important stylized facts in the data.<sup>2</sup>

A special case of the nonlinear stochastic system (1) arises when all noise terms are set to zero. We will refer to this system as the (*deterministic*) *skeleton* denoted by<sup>3</sup>

$$X_{t+1} = F(X_t; n_{1t}, \dots, n_{Ht}; \lambda). \quad (2)$$

In order to understand the properties of the general stochastic model (1) it is important to understand the properties of the deterministic skeleton. In particular, one would like to impose as little structure on the noise process as possible, and relate the stylized facts of the general stochastic model (1) directly to generic properties of the underlying deterministic skeleton. There are three important, generic features of nonlinear deterministic systems which may play an important role in generating some of the stylized facts in finance and may in particular cause volatility clustering:

- F1 Chaos and strange attractors due to *homoclinic bifurcations*.
- F2 Simultaneous co-existence of different attractors.
- F3 Local bifurcations of steady states.

Figure 1 illustrates feature F1 and shows an example of a strange attractor in the ABS discussed in subsection 4.3. The motion on the strange attractor is highly unpredictable, with asset prices jumping irregularly over the complicated fractal set. Although research in nonlinear dynamics has been stimulated much by computer simulations in the past

<sup>2</sup>Other approaches to modeling the stylized facts, especially volatility clustering and fat tails, in finance include the (G)ARCH models initiated by Engle (1982), and more recently the multi-fractal modeling approach of Mandelbrot (1997, 1999) and the ‘scaling law approach’ in econophysics as surveyed in Mantegna and Stanley (2000). These approaches are statistically or time series oriented however and, although important and useful, lack structural economic modeling.

<sup>3</sup>This terminology is used e.g. by Tong (1990)

25 years, the ‘roots of chaos’ date back to the mathematician Henri Poincaré at the end of the previous century.<sup>4</sup> Poincaré introduced the notion of *homoclinic orbits*, in his investigations of the three body problem. Nowadays it is a mathematical fact that homoclinic orbits are a key feature of chaotic systems, and so-called homoclinic bifurcations leading to strange attractors seem to be the rule rather than the exception. Brock and Hommes (1997a, 1998) have shown that evolutionary adaptive systems with heterogeneous agents using competing trading strategies is a natural nonlinear world full of homoclinic bifurcations and strange attractors.

Nonlinear dynamic models can generate a wide variety of irregular patterns. In particular, nonlinear dynamic models can generate any given autocorrelation pattern.<sup>5</sup> A nonlinear, chaotic model, buffeted with dynamic noise, with almost no autocorrelations in returns but at the same time persistence in squared returns, with slowly decaying autocorrelations, may thus provide a structural explanation of the unpredictability of asset returns and volatility clustering in financial assets. In fact, the phenomenon of *intermittency*, as introduced by Pomeau and Manneville (1980), is well suited as a description of the phenomenon of volatility clustering. Intermittency means chaotic asset price fluctuations characterized by phases of almost periodic fluctuations irregularly interrupted by sudden bursts of erratic fluctuations. In an ABS intermittency occurs characterized by close to the RE fundamental steady state fluctuations suddenly interrupted by price deviations from the fundamental triggered by technical trading.

The second generic feature F2, *coexistence of attractors*, is also naturally suited to describe volatility clustering. In particular, the ABS exhibits coexistence of a stable steady state and a stable limit cycle. When buffeted with dynamic noise, irregular switching occurs between close to the fundamental steady state fluctuations, when the market is dominated by fundamentalists, and periodic fluctuations when the market is dominated by chartists. It is important to note that both, intermittency and coexistence of attractors, are *persistent* phenomena, which are by no means special for our ABS, but occur naturally in nonlinear dynamic models, and moreover are robust with respect to and sometimes even reinforced by dynamic noise.

The third generic feature F3, a local bifurcation of a steady state, that is, a change in the stability of the steady state, is related to feature F2. A local bifurcation occurs when the linearized system is at the border of stability, having at least one unit root. Close to the bifurcation point there can be regions in the parameter space where the nonlinear system has co-existing attractors. It turns out that an ABS with parameters close to a local bifurcation point of the steady state can generate some of the stylized facts in finance such as volatility clustering.

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<sup>4</sup>Gleick (1987) present a nice overview of the history of nonlinear dynamics. A good introduction to the mathematics of ‘chaos’ for non-specialists is Ruelle (1991). A mathematical treatment with recent advances in homoclinic bifurcation theory is e.g. Palis and Takens (1993).

<sup>5</sup>To see that a higher dimensional chaotic map can generate any desired autocorrelation structure, consider the nonlinear difference equation  $(x_t, y_t) = (a_1 x_{t-1} + \dots + a_L x_{t-L} + y_{t-1}, 1 - 2y_{t-1}^2)$ . As is well known, the second coordinate  $y_t$  follows a chaotic process with zero mean and zero autocorrelations at all lags. Since  $y_t$  is generated independently of past values of  $x_t$ , the series  $x_t$  and  $y_t$  are uncorrelated. The first coordinate  $x_t$  thus follows a linear AR( $L$ ) process driven by a chaotic series with zero autocorrelations at all lags, and thus has the desired autocorrelation structure.

In the last decade several attempts have been made to test for chaos in economic and financial data. Most empirical studies have rejected the hypothesis that economic or financial data are generated by low dimensional, purely deterministic chaos but strong evidence for nonlinearity is found (see e.g. the survey in Brock, Hsieh and LeBaron, 1991). It should be stressed though that the methods for detecting chaos are very sensitive to noise, and that nonlinear models with noise, as our proposed ABS may be consistent with the data. *A noisy nonlinear model may explain a significant part of observed fluctuations and stylized facts in economic and financial markets.*

### 3 Adaptive Belief Systems

This section reviews the notion of an *Adaptive Belief System (ABS)*, as introduced in Brock and Hommes (1997a,1997b,1998). An ABS is in fact a standard discounted value asset pricing model derived from mean-variance maximization, extended to the case of *heterogeneous beliefs*. Agents can either invest in a risk free asset or in a risky asset. The risk free asset is perfectly elastically supplied and pays a fixed rate of return  $r$ ; the risky asset, for example a large stock or a market index, pays an uncertain dividend. Let  $p_t$  be the price per share (ex-dividend) of the risky asset at time  $t$ , and let  $y_t$  be the stochastic dividend process of the risky asset. Wealth dynamics is given by

$$\mathbf{W}_{t+1} = (1 + r)W_t + (\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t)z_t, \quad (3)$$

where bold face variables denote random variables at date  $t + 1$  and  $z_t$  denotes the number of shares of the risky asset purchased at date  $t$ . Let  $E_t$  and  $V_t$  denote the conditional expectation and conditional variance based on a publically available information set such as past prices and past dividends. Let  $E_{ht}$  and  $V_{ht}$  denote the ‘beliefs’ or forecasts of trader type  $h$  about conditional expectation and conditional variance. Agents are assumed to be myopic mean-variance maximizers so that the demand  $z_{ht}$  of type  $h$  for the risky asset solves

$$Max_{z_t} \{E_{ht}[\mathbf{W}_{t+1}] - \frac{a}{2}V_{ht}[\mathbf{W}_{t+1}]\}, \quad (4)$$

where  $a$  is the risk aversion parameter. The demand  $z_{ht}$  for risky assets by trader type  $h$  is then

$$z_{ht} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t]}{aV_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t]} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t]}{a\sigma^2}, \quad (5)$$

where the conditional variance  $V_{ht} = \sigma^2$  is assumed to be equal for all types and constant.<sup>6</sup> Let  $z^s$  denote the supply of outside risky shares per investor, assumed to be constant, and let  $n_{ht}$  denote the fraction of type  $h$  at date  $t$ . Equilibrium of demand and supply yields

$$\sum_{h=1}^H n_{ht} \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t]}{a\sigma^2} = z^s, \quad (6)$$

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<sup>6</sup>Gaunersdorfer (2000) investigates the case with time varying beliefs about variances and shows that the results are quite similar to those for constant variance.



where  $H$  is the number of different trader types. Without loss of generality, we can focus on the case of zero supply of outside shares, i.e.  $z^s = 0$ .<sup>7</sup> The market equilibrium equation can then be rewritten as

$$(1+r)p_t = \sum_{h=1}^H n_{ht} E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}], \quad (7)$$

with dividends interpreted as risk adjusted dividends.<sup>8</sup>

### 3.1 The EMH benchmark with rational agents

Let us first discuss the EMH-benchmark with rational expectations. In a world where all traders are identical and expectations are *homogeneous* the arbitrage market equilibrium equation (7) reduces to

$$(1+r)p_t = E_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}], \quad (8)$$

where  $E_t$  denotes the common conditional expectation of all traders at the beginning of period  $t$ , based on a publically available information set  $I_t$  such as past prices and dividends, i.e.  $I_t = \{p_{t-1}, p_{t-2}, \dots; y_{t-1}, y_{t-2}, \dots\}$ . This arbitrage market equilibrium equation (8) states that today's price of the risky asset must be equal to the sum of tomorrow's expected price and expected dividend, discounted by the risk free interest rate. It is well known that, using the arbitrage equation (8) repeatedly and assuming that the *transversality condition*

$$\lim_{t \rightarrow \infty} \frac{E_t[\mathbf{p}_{t+k}]}{(1+r)^k} = 0 \quad (9)$$

holds, the price of the risky asset is uniquely determined by

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[\mathbf{y}_{t+k}]}{(1+r)^k}. \quad (10)$$

The price  $p_t^*$  in (10) is called the EMH fundamental rational expectations (RE) price, or the *fundamental price* for short. The fundamental price is completely determined by economic fundamentals and, given by the discounted sum of expected future dividends. In general, the properties of the fundamental price  $p_t^*$  depend upon the stochastic dividend process  $y_t$ . We will mainly focus on the case of an IID dividend process  $y_t$ , with constant mean  $E[y_t] = \bar{y}$ . We note however that any other random dividend process  $y_t$  may be substituted in what follows<sup>9</sup>. For an IID dividend process  $y_t$  with constant mean, the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \quad (11)$$

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<sup>7</sup>This can be seen by introducing a risk adjusted dividend  $y_{t+1}^\# = y_{t+1} - a\sigma^2 z^s$  to obtain the market equilibrium equation (7), as argued in Brock (1997).

<sup>8</sup>In the examples of ABS in section 4, we will add a noise term  $\epsilon_t$  to the RHS of the market equilibrium equation, representing a model approximation error.

<sup>9</sup>Brock and Hommes (1997b) for example discuss a non-stationary example, where the dividend process is a geometric random walk .

There are two crucial assumptions underlying the derivation of the RE fundamental price. The first is that expectations are *homogeneous*, all traders are *rational* and it is *common knowledge* that all traders are rational. Only in such an ideal, perfectly rational world the fundamental price can be derived from economic fundamentals. In contrast, in a world with heterogeneous traders having different beliefs or expectations about future prices and dividends, derivation of a RE fundamental price requires perfect knowledge about the beliefs of *all* other traders<sup>10</sup>. In a real market understanding the beliefs and strategies of all other, competing traders is virtually impossible, and therefore in a heterogeneous world derivation of the RE-fundamental price becomes impossible. The second crucial assumption underlying the derivation of the fundamental price is the transversality condition (9), requiring that the long run growth rate of prices (and risk adjusted dividends) is smaller than the risk free growth rate  $r$ . In fact, in addition to the fundamental solution (10) so-called *speculative bubble solutions* of the form

$$p_t = p_t^* + (1 + r)^t(p_0 - p_0^*) \quad (12)$$

also satisfy the arbitrage equation (8). It is important to note that along the speculative bubble solution (12), traders have rational expectations. Solutions of the form (12) are therefore called *rational bubbles*. These rational bubble solutions are explosive and do *not* satisfy the transversality condition. In a perfectly rational world, traders realize that speculative bubbles can not last forever and therefore they will never get started and the finite fundamental price  $p_t^*$  is uniquely determined. In a perfectly rational world, all traders thus believe that the value of a risky asset equals its fundamental price forever. Changes in asset prices are solely driven by unexpected changes in dividends and random ‘news’ about economic fundamentals. In a heterogeneous evolutionary world however, the situation will be quite different.

### 3.2 Heterogeneous beliefs

In the asset pricing model with heterogeneous beliefs, market equilibrium in (7) states that the price  $p_t$  of the risky asset equals the discounted value of tomorrow’s expected price plus tomorrow’s expected dividend, *averaged* over all different trader types. In such a *heterogeneous* world temporary upward or downward bubbles with prices deviating from the fundamental may arise, when the fractions of traders believing in those bubbles is large enough. Once a (temporary) bubble has started, evolutionary forces may reinforce deviations from the benchmark fundamental. We shall now be more precise about traders’ expectations (forecasts) about future prices and dividends. It will be convenient to work with

$$x_t = p_t - p_t^*, \quad (13)$$

the *deviation* from the fundamental price. We make the following assumptions about the beliefs of trader type  $h$ :

B1  $V_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t] = V_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1 + r)p_t] = \sigma^2$ , for all  $h, t$ .

B2  $E_{ht}[\mathbf{y}_{t+1}] = E_t[\mathbf{y}_{t+1}]$ , for all  $h, t$ .

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<sup>10</sup>see e.g. Arthur (1995) for a lucid account of this point.

B3 All beliefs  $E_{ht}[\mathbf{p}_{t+1}]$  are of the form

$$E_{ht}[\mathbf{p}_{t+1}] = E_t[\mathbf{p}_{t+1}^*] + f_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t. \quad (14)$$

According to assumption B1 beliefs about conditional variance are equal and constant for all types, as discussed above already. Assumption B2 states that expectations about future dividends  $\mathbf{y}_{t+1}$  are the same for all trader types and equal to the conditional expectation. All traders are thus able to derive the fundamental price  $p_t^*$  in (10) that would prevail in a perfectly rational world. According to assumption B3, traders nevertheless believe that in a heterogeneous world prices may *deviate* from their fundamental value  $p_t^*$  by some function  $f_h$  depending upon past deviations from the fundamental. Each forecasting rule  $f_h$  represents the *model of the market* according to which type  $h$  believes that prices will deviate from the commonly shared fundamental price. For example, a forecasting strategy  $f_h$  may correspond to a technical trading rule, based upon short run or long run moving averages, of the type used in real markets. We will use the short hand notation

$$f_{ht} = f_h(x_{t-1}, \dots, x_{t-L}) \quad (15)$$

for the forecasting strategy employed by trader type  $h$ . Brock and Hommes (1998) have investigated evolutionary competition between the simplest linear trading rules with only one lag, i.e.

$$f_{ht} = g_h x_{t-1} + b_h. \quad (16)$$

Simple forecasting rules are more likely to be relevant in real markets, because for a complicated forecasting rule it seems unlikely that enough traders will coordinate on that particular rule so that it affects market equilibrium prices. Although the linear forecasting rule (16) is extremely simple, it does in fact represent a number of important cases. For example, when both the trend parameter and the bias parameter  $g_h = b_h = 0$  the rule reduces to the forecast of *fundamentalists*, i.e.

$$f_{ht} \equiv 0, \quad (17)$$

believing that the market price will be equal to the fundamental price  $p^*$ , or equivalently that the deviation  $x$  from the fundamental will be 0. Other important cases covered by the linear forecasting rule (16) are the pure *trend followers*

$$f_{ht} = g_h x_{t-1}, \quad (18)$$

and the pure *biased belief*

$$f_{ht} = b_h. \quad (19)$$

Notice that the simple pure bias (19) rule represents any positively or negatively biased price forecast that traders might have. Instead of these extremely simple habitual rule of thumb forecasting rules, some economists might prefer the rational, *perfect foresight* forecasting rule

$$f_{ht} = x_{t+1}. \quad (20)$$

We emphasize however, that the perfect foresight forecasting rule (20) assumes perfect knowledge of the heterogeneous market equilibrium equation (7), and in particular

perfect knowledge about the beliefs of *all* other traders. Although the case with perfect foresight certainly has theoretical appeal, its practical relevance in a complex heterogeneous world should not be overstated since this underlying assumption seems highly unrealistic.<sup>11</sup>

An important and convenient consequence of the assumptions B1-B3 concerning traders' beliefs is that the heterogeneous agent market equilibrium equation (7) can be reformulated in deviations from the benchmark fundamental. In particular substituting the price forecast (14) in the market equilibrium equation (7) and using the facts that the fundamental price  $p_t^*$  satisfies  $(1+r)p_t^* = E_t[p_{t+1}^* + y_{t+1}]$  and the price  $p_t = x_t + p_t^*$  yields the equilibrium equation in deviations from the fundamental:

$$(1+r)x_t = \sum_{h=1}^H n_{ht} E_{ht}[\mathbf{x}_{t+1}] \equiv \sum_{h=1}^H n_{ht} f_{ht}, \quad (21)$$

with  $f_{ht} = f_h(x_{t-1}, \dots, x_{t-L})$ . An important reason for our model formulation in terms of deviations from a benchmark fundamental is that in this general setup, the benchmark rational expectations asset pricing model is *nested* as a special case, with all forecasting strategies  $f_h \equiv 0$ . In this way, the adaptive belief systems can be used in empirical and experimental testing whether asset prices deviate significantly from anyone's favorite benchmark fundamental.

### 3.3 Evolutionary dynamics

The evolutionary part of the model describes how beliefs are updated over time, that is, how the fractions  $n_{ht}$  of trader types in the market equilibrium equation (21) evolve over time. Fractions are updated according to an *evolutionary fitness* or performance measure. The fitness measures of all trading strategies are publically available, but subject to noise. Fitness is derived from a random utility model and given by

$$\tilde{U}_{ht} = U_{ht} + \epsilon_{ht}, \quad (22)$$

where  $U_{ht}$  is the *deterministic part* of the fitness measure and  $\epsilon_{ht}$  represents noise. Assuming that the noise  $\epsilon_{ht}$  is IID across  $h = 1, \dots, H$  drawn from a double exponential distribution, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy  $h$  is given by the well known *discrete choice model* or 'Gibbs' probabilities<sup>12</sup>

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}}, \quad Z_{t-1} = \sum_{h=1}^H \exp(\beta U_{h,t-1}), \quad (23)$$

where  $Z_{t-1}$  is a normalization factor in order for the fractions  $n_{ht}$  to add up to 1. The crucial feature of (23) is that the higher the fitness of trading strategy  $h$ , the more traders will select strategy  $h$ . The parameter  $\beta$  in (23) is called the *intensity of choice*,

<sup>11</sup>see also subsection 4.1 for a brief discussion of rational versus fundamentalists traders.

<sup>12</sup>see Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993) for extensive discussion of discrete choice models and their applications in economics.

measuring how sensitive the mass of traders is to selecting the optimal prediction strategy. The intensity of choice  $\beta$  is inversely related to the variance of the noise terms  $\epsilon_{ht}$ . The extreme case  $\beta = 0$  corresponds to the case of infinite variance noise, so that differences in fitness can not be observed and all fractions (23) will be fixed over time and equal to  $1/H$ . The other extreme case  $\beta = \infty$  corresponds to the case without noise, so that the deterministic part of the fitness can be observed perfectly and in each period, *all* traders choose the optimal forecast. An increase in the intensity of choice  $\beta$  represents an increase in the degree of rationality w.r.t. evolutionary selection of trading strategies. The timing of the coupling between the market equilibrium equation (7) or (21) and the evolutionary selection of strategies (23) is crucial. The market equilibrium price  $p_t$  in (7) depends upon the fractions  $n_{ht}$ . The notation in (23) stresses the fact that these fractions  $n_{ht}$  depend upon *past* fitnesses  $U_{h,t-1}$ , which in turn depend upon past prices  $p_{t-1}$  and dividends  $y_{t-1}$  in periods  $t-1$  and further in the past as will be seen below. After the equilibrium price  $p_t$  has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions  $n_{h,t+1}$ . These new fractions  $n_{h,t+1}$  will then determine a new equilibrium price  $p_{t+1}$ , etc.. In the ABS, market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

A natural candidate for evolutionary fitness is accumulated *realized profits*, as given by

$$U_{ht} = (p_t + y_t - Rp_{t-1}) \frac{E_{h,t-1}[\mathbf{p}_t + \mathbf{y}_t - Rp_{t-1}]}{a\sigma^2} - C_h + wU_{h,t-1} \quad (24)$$

where  $R = 1 + r$  is the gross risk free rate of return,  $C_h$  represents an average per period *cost* of obtaining forecasting strategy  $h$  and  $0 \leq w \leq 1$  is a *memory* parameter measuring how fast past realized fitness is discounted for strategy selection. The cost  $C_h$  for obtaining forecasting strategy  $h$  will be zero for simple, habitual rule of thumb forecasting rules, but may be positive for more sophisticated forecasting strategies. For example, costs for forecasting strategies based upon economic fundamentals may be positive representing investors' effort for information gathering, whereas costs for technical trading rules may be (close to) zero. The first term in (24) represents last period's realized profit of type  $h$  given by the realized excess return of the risky asset over the risk free asset times the demand for the risky asset by traders of type  $h$ . In the extreme case with no memory, i.e.  $w = 0$ , fitness  $U_{ht}$  equals net realized profit in the previous period, whereas in the other extreme case with infinite memory, i.e.  $w = 1$ , fitness  $U_{ht}$  equals total wealth as given by accumulated realized profits over the entire past. In the intermediate case, the weight given to past realized profits decreases exponentially with time. It will be useful to compute the realized excess return  $R_t$  in deviations from the fundamental to obtain

$$\begin{aligned} R_t &= p_t + y_t - Rp_{t-1} = x_t + p_t^* + y_t - Rx_{t-1} - Rp_{t-1}^* \\ &= x_t - Rx_{t-1} + p_t^* + y_t - E_{t-1}[p_t^* + y_t] + E_{t-1}[p_t^* + y_t] - Rp_{t-1}^* \\ &\equiv x_t - Rx_{t-1} + \delta_t, \end{aligned} \quad (25)$$

where we used that  $E_{t-1}[p_t^* + y_t] - Rp_{t-1}^* = 0$  since the fundamental  $p_t^*$  satisfies the market equilibrium equation (8), and  $\delta_t \equiv p_t^* + y_t - E_{t-1}[p_t^* + y_t]$  is a martingale difference sequence. The random term  $\delta_t$  enters because the dividend process is stochastic,

and thus represents intrinsic uncertainty about economic fundamentals<sup>13</sup>. According to the decomposition (25) excess return consists of a conventional EMH term  $\delta_t$  and an additional *speculative* term  $x_t - Rx_{t-1}$  of the ABS theory. Our ABS theory thus allows for the possibility of *excess volatility*. The extra term is zero if either  $x_t \equiv 0$ , that is prices equal their fundamental value, or if  $x_t = Rx_{t-1}$ , that is when prices follow a RE bubble solution. The ABS theory predicts excess volatility in periods when asset prices grow faster or slower than the risk free rate of return, or when prices switch between a temporary bubble solution and the fundamental.

Fitness can now be rewritten in deviations from the fundamental as

$$U_{ht} = (x_t - Rx_{t-1} + \delta_t) \left( \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2} \right) - C_h + wU_{h,t-1}. \quad (26)$$

### Risk adjusted profits as fitness measure.

Although realized net profits are a natural candidate for evolutionary fitness, this fitness measure does *not* take into account the *risk* taken at the moment of the investment decision. In fact, given that investors are risk averse mean-variance maximizers, maximizing their expected utility from wealth (4) another natural candidate for fitness are the *risk adjusted profits*. Using the notation  $R_t = p_t + y_t - Rp_{t-1}$  for realized excess return, the realized risk adjusted profit for strategy  $h$  in period  $t$  is given by

$$\pi_{ht} = R_t z_{h,t-1} - \frac{a}{2} \sigma^2 z_{h,t-1}^2, \quad (27)$$

where  $z_{h,t-1} = E_{h,t-1}[R_t]/(a\sigma^2)$  is the demand by trader type  $h$  as in (5). Notice that maximizing expected utility from wealth in (4) is equivalent to maximizing expected utility from profits in (27). A risk adjusted fitness measure based on (27) is thus consistent with the investors' demand function derived from mean-variance maximization of expected wealth. The fitness measure (24) based upon realized profits does not take into account the variance term in (27) capturing the investors' risk taken before obtaining that profit. On the other hand, in real markets realized net profits or accumulated wealth may be what investors care about most, and the non-risk adjusted fitness measure (24) may thus be practically important.

The expressions for risk adjusted profit fitness will simplify when subtracting off the realized risk adjusted profit by rational (perfect foresight) traders from (27). This can be done without loss of generality since it does not affect expected utility from profits by trader type  $h$ . The risk adjusted profit  $\pi_{Rt}$  by rational agents is given by

$$\pi_{Rt} = R_t \frac{R_t}{a\sigma^2} - \frac{a}{2} \sigma^2 \frac{R_t^2}{a^2 \sigma^4} = \frac{R_t^2}{2a\sigma^2}. \quad (28)$$

Using  $z_{h,t-1} = E_{h,t-1}[R_t]/(a\sigma^2)$  a simple computation shows that

$$\begin{aligned} \pi_{ht} - \pi_{Rt} &= -\frac{1}{2a\sigma^2} (R_t - E_{h,t-1}[R_t])^2 \\ &= -\frac{1}{2a\sigma^2} (p_t - E_{h,t-1}[p_t] + \delta_{y,t})^2 \\ &= -\frac{1}{2a\sigma^2} (x_t - E_{h,t-1}[x_t] + \delta_t)^2, \end{aligned} \quad (29)$$

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<sup>13</sup>In the special case of an IID dividend process  $y_t = \bar{y} + \epsilon_t$  we simply have  $\delta_t = \epsilon_t$ .

where  $\delta_{y,t} = y_t - E_{t-1}[y_t]$  and  $\delta_t = p_t^* + y_t - E_{t-1}[p_t^* + y_t]$  are both MDS sequences<sup>14</sup>. The fitness measure risk adjusted profits is thus, up to an MDS sequence, equivalent to minus squared forecasting errors. The risk adjusted fitness measure is now formally defined as

$$V_{ht} = -\frac{1}{2a\sigma^2}(p_t - E_{h,t-1}[\mathbf{p}_t] + \delta_{y,t})^2 - C_h + wV_{h,t-1}, \quad (30)$$

or in deviations from the fundamental

$$V_{ht} = -\frac{1}{2a\sigma^2}(x_t - E_{h,t-1}[\mathbf{x}_t] + \delta_t)^2 - C_h + wV_{h,t-1}. \quad (31)$$

The random term  $\delta_t$  or  $\delta_{y,t}$  enters because the dividend process is stochastic, and thus again represents intrinsic uncertainty about economic fundamentals

## 4 Some simple examples

This section presents four simple, but typical examples of ABS. The first three subsections discuss the most important features of Brock and Hommes (1997b,1998,1999), with realized profits as the fitness measure. The fourth example discussed a modified ABS by Gaunersdorfer and Hommes (2000), with evolutionary fitness given by risk adjusted profits conditioned upon deviations from the fundamental. Time series properties of the latter example will be compared to 40 years of S&P 500 data.

BH present a number of simple, but typical examples of the evolutionary dynamics in adaptive belief systems with two, three or four competing *linear* forecasting rules (16), where the parameter  $g_h$  represents a perceived *trend* in prices and the parameter  $b_h$  represents a perceived upward or downward *bias*. The ABS then becomes (in deviations from the fundamental):

$$(1+r)x_t = \sum_{h=1}^H n_{ht}(g_h x_{t-1} + b_h) + \epsilon_t \quad (32)$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})} \quad (33)$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2} + \delta_{t-1})\left(\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}\right) + wU_{h,t-2} - C_h, \quad (34)$$

where the noise term  $\epsilon_t$  represents the model approximation error and  $\delta_{t-1}$  represents uncertainty about economic fundamentals as before. In order to keep the analysis of the dynamical behavior tractable, BH have mainly focused on the case where the memory parameter  $w = 0$ , so that evolutionary fitness is given by last period's realized profit. In subsection 4.3 we will discuss the role of the memory parameter  $w$ . We briefly discuss three important cases. A common feature of all examples is that, as the intensity of choice to switch prediction or trading strategies increases, the fundamental steady state becomes locally unstable and non-fundamental steady states, cycles or even chaos arise.

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<sup>14</sup>In the special case of an IID dividend process  $y_t = \bar{y} + \epsilon_t$  and corresponding constant fundamental price  $p^* = \bar{y}/r$ , we have  $\delta_{y,t} = \delta_t = \epsilon_t$ .

## 4.1 Fundamentalists with positive information costs versus trend followers

The simplest example of an ABS only has *two* trader types, with forecasting rules

$$f_{1t} = 0 \qquad \text{fundamentalists} \qquad (35)$$

$$f_{2t} = gx_{t-1}, \quad g > 0, \quad \text{trend followers} \qquad (36)$$

that is, the first type are fundamentalists predicting that the price will equal its fundamental value (or equivalently that the deviation will be zero) and the second type are pure trend followers predicting that prices will rise (or fall) by a constant rate. In this example, the fundamentalists have to pay a fixed per period positive cost  $C_1$  for information gathering; in all other examples discussed below information costs will be set to zero for all trader types.

For small values of the trend parameter,  $0 \leq g < 1 + r$ , the fundamental steady state is always stable. Only for sufficiently high trend parameters,  $g > 1 + r$ , trend followers can *destabilize* the system. For trend parameter,  $1 + r < g < (1 + r)^2$  the dynamic behavior of the evolutionary system depends upon the intensity of choice to switch between the two trading strategies<sup>15</sup>. For low values of the intensity of choice, the fundamental steady state will be stable. As the intensity of choice increases, the fundamental steady state becomes unstable due to a *pitchfork bifurcation* in which two additional non-fundamental steady states  $-x^* < 0 < x^*$  are created. The evolutionary ABS may converge to the positive non-fundamental steady state, to the negative non-fundamental steady state, or, in the presence of noise, switch back and forth between the high and the low steady state. As the intensity of choice increases further, the two non-fundamental steady states also become unstable, and limit cycles or even strange attractors can arise around each of the (unstable) non-fundamental steady states. The evolutionary ABS may cycle around the positive non-fundamental steady state, cycle around the negative non-fundamental steady state or, driven by the noise, switch back and forth between cycles around the high and the low steady state.

This example shows that, in the presence of information costs and with zero memory, when the intensity of choice in evolutionary switching is high fundamentalists can *not* drive out pure trend followers and persistent deviations from the fundamental price may occur. Brock and Hommes (1999) show that this result also holds when the memory in the fitness measure increases. In fact, an increase in the memory of the evolutionary fitness leads to bifurcation routes very similar to bifurcation routes due to an increase in the intensity of choice.

It is sometimes argued that fundamentalists are *not* rational since they do *not* take into account the presence of other trader types. Let us therefore briefly discuss the case of perfect foresight versus trend followers, that is, the case when the fundamentalists forecasting rule (35) is replaced by a perfect foresight rule  $f_{1t} = x_{t+1}$ . Brock and Hommes (1998, p.1247, lemma 1) show that in this case the first bifurcation is the same, that

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<sup>15</sup>For  $g > (1 + r)^2$  the system may become *globally unstable* and prices may diverge to infinity. Imposing a stabilizing force, for example by assuming that trend followers condition their rule upon deviations from the fundamental as in subsection 4.4, leads to a bounded system again, possibly with cycles or even chaotic fluctuations.



is, as the intensity of choice increases two non-fundamental steady states are created due to a pitchfork bifurcation. Although examples with perfect foresight certainly have theoretical appeal, there are two fundamental reasons arguing against perfect foresight in a heterogeneous world. The first is a methodological reason, since with one type having perfect foresight a *temporary equilibrium* model with heterogeneous beliefs as in (32) becomes an *implicitly defined* dynamical system with  $x_t$  on the LHS and  $x_{t+1}$  and e.g.  $x_{t-1}$  on the RHS. Typically such implicitly defined evolutionary systems can not be solved explicitly and often they are not even well-defined.<sup>16</sup> The second and perhaps more important reason is that perfect foresight assumes perfect knowledge of the beliefs of all other types, which seems at odds with reality. It seems more reasonable and closer to financial practice to focus on examples with fundamentalists traders.

## 4.2 Fundamentalists versus opposite biased beliefs

The second example of an ABS is an example with *three* trader types, with forecasting rules

$$f_{1t} = 0 \quad \text{fundamentalists} \quad (37)$$

$$f_{2t} = b \quad b > 0, \quad \text{positive bias (optimists)} \quad (38)$$

$$f_{3t} = -b \quad -b < 0, \quad \text{negative bias (pessimists)}. \quad (39)$$

The first type are fundamentalists again, but this time there will be *no* information costs. The second and third types have a purely *biased* belief, expecting a constant price above respectively below the fundamental price.

For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases the fundamental steady becomes unstable due to a *Hopf* bifurcation and the dynamics of the ABS is characterized by cycles around the unstable steady state. This example shows that when memory is zero, even when there are *no* information costs for fundamentalists, they can not drive out other trader types with opposite biased beliefs. In the evolutionary ABS with high intensity of choice, fundamentalists and biased traders co-exist with fractions varying over time and prices cycling around the unstable fundamental steady state. Moreover, Brock and Hommes (1998, p.1259, lemma 9) show that as the intensity of choice tends to infinity the ABS converges to a (globally) stable cycle of period 4. Average profits along this 4-cycle are equal for all three trader types. Hence, if the initial wealth is equal for all three types, then in this evolutionary system in the long run accumulated wealth will be equal for all three types. This example suggests that the Friedman argument that smart-fundamental traders will automatically drive out simple habitual rule of speculative traders should be considered with care.

In this example with three trader types, cycles can occur but chaos does not arise.<sup>17</sup> Therefore, even in the presence of (small) noise, price fluctuations will be fairly regular

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<sup>16</sup>Brock and Hommes (1997a) consider an evolutionary cobweb model with rational expectations at positive information costs versus freely available naive expectations. This example can be solved explicitly and exhibits bifurcations routes to cycles and chaos similar to the ABS presented here.

<sup>17</sup>This may be seen from the plot of the largest Lyapunov exponent in Brock and Hommes (1998, p.1261, fig. 9b), which is always non-positive.

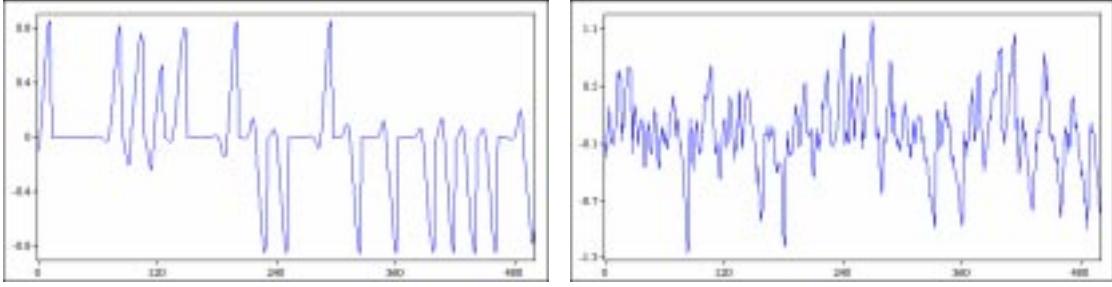


Figure 2: Chaotic (a) and noisy chaotic (b) time series of asset prices in adaptive belief system with four trader types.

and therefore returns will be predictable. This predictability will disappear however when we combine trend following with biased beliefs.

### 4.3 Fundamentalists versus opposite biased beliefs

The third example of an ABS is an example with *four* trader types, with linear forecasting rules (16) with parameters  $g_1 = 0$ ,  $b_1 = 0$ ;  $g_2 = 0.9$ ,  $b_2 = 0.2$ ;  $g_3 = 0.9$ ,  $b_3 = -0.2$  and  $g_4 = 1 + r = 1.01$ ,  $b_4 = 0$ . The first type are fundamentalists again, without information costs, and the other three types follow a simple linear forecasting rule with one lag. For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases, as in the previous three type example, the fundamental steady becomes unstable due to a *Hopf* bifurcation and a stable invariant circle around the unstable fundamental steady state arises, with periodic or quasi-periodic fluctuations. As the intensity of choice further increases, the invariant circle breaks into a strange attractor with chaotic fluctuations.

This example shows that when memory is zero, even when there are *no* information costs for fundamentalists, they can not drive out other simple trader types. As in the three type case, the opposite biases create cyclic behavior but apparently the additional trend parameters turn these cycles into chaotic fluctuations. In the evolutionary ABS fundamentalists and chartists co-exist with fractions varying over time and prices moving chaotically around the unstable fundamental steady state. The corresponding strange attractor, with the parameter values given by  $r = 0.01$ ,  $\beta = 90.5$ ,  $w = 0$  and  $C_h = 0$  for all  $1 \leq h \leq 4$ , was already shown in figure 1. Figure 2 shows a chaotic as well as a noisy chaotic time series. The (noisy) chaotic price fluctuations are characterized by an irregular switching between phases of close-to-the-EMH-fundamental-price fluctuations, phases of ‘optimism’ with prices following an upward trend, and phases of ‘pessimism’, with (small) sudden market crashes. Recall from subsection 3.1 that the asset pricing model with homogeneous beliefs, in addition to the benchmark fundamental price, has rational bubble solutions as in (12). One might say that in the ABS prices are characterized by an evolutionary switching between the fundamental value and these temporary speculative bubbles.

Brock and Hommes (1997a, 1998) show that for a high intensity of choice, the ABS

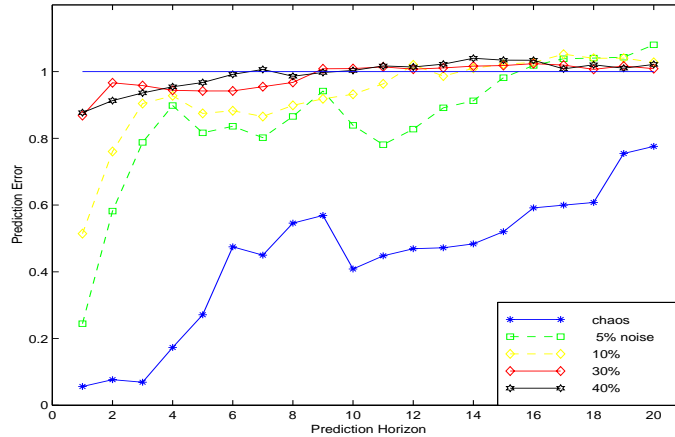


Figure 3: Forecasting errors for nearest neighbor method applied to chaotic returns series as well as noisy chaotic returns series, for different noise levels, in ABS with four trader types. All returns series have close to zero autocorrelations at all lags. The benchmark case of prediction by the mean 0 is represented by the horizontal line at the normalized prediction error 1. Nearest neighbor forecasting applied to the purely deterministic chaotic series leads to much smaller forecasting errors (lowest graph). A noise level of say 10% means that the ratio of the variance of the noise term  $\epsilon_t$  and the variance of the deterministic price series is 1/10. As the noise level slowly increases, the graphs are shifted upwards. Small dynamic noise thus quickly deteriorates forecasting performance, even at small lags.

system is close to having a *homoclinic orbit*, Poincaré’s classical notion and key feature of chaotic systems. In the purely deterministic chaotic case, the timing and the direction of the temporary bubbles seem hard to predict (see figure 2a). However, once a bubble has started, in the deterministic case, the length of the bubble seems to be predictable in most of the cases. In the presence of noise, as in figure 2b, the timing, the direction and the length of the bubble all seem hard to predict. In order to investigate this (un)predictability issue further, we employ a so called *nearest neighbor forecasting method* to predict the returns, at lags 1 to 20 for the purely chaotic as well as for several noisy chaotic time series, as illustrated in figure 3.<sup>18</sup> Nearest neighbor forecasting looks for past patterns close to the most recent pattern, and then yields as the prediction the average value following all nearby past patterns. It follows essentially from Takens’ embedding theorem that this method yields good forecasts for deterministic chaotic systems<sup>19</sup>. Figure 3 shows that as the noise level increases, the forecasting performance of the nearest neighbor method quickly deteriorates. Hence, in our simple nonlinear evolutionary ABS with noise it is hard to make good forecasts of future returns. Our simple nonlinear ABS with small noise thus captures the unpredictability of asset returns also present in real markets.

Finally, let us briefly discuss the issue of memory in the fitness measure. Figure 4 shows bifurcation diagrams of the deterministic ABS as well as the noisy ABS, with respect to the memory parameter  $w$ ,  $0 \leq w \leq 1$ . The parameters are as before, except for

<sup>18</sup>I would like to thank Sebastiano Manzan for providing this figure.

<sup>19</sup>See Takens (1981) and Packard et al. (1980). A recent treatment of nonlinear time series analysis and forecasting techniques is Kantz and Schreiber (1998).

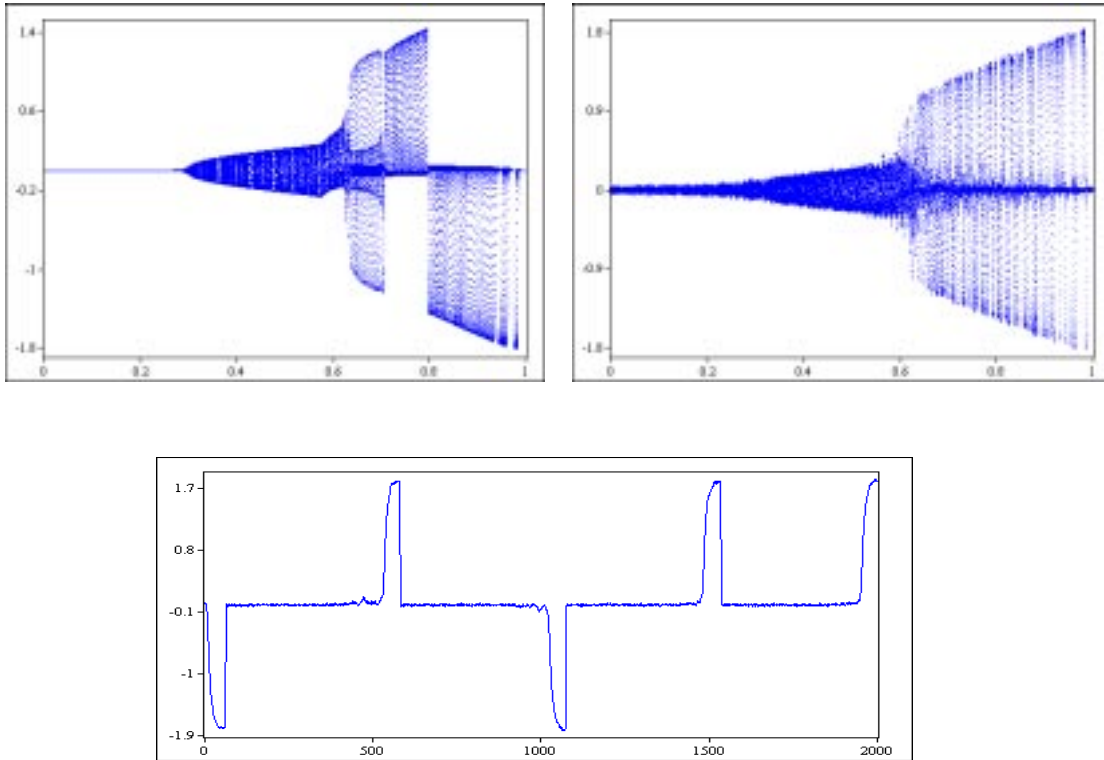


Figure 4: Bifurcation diagrams w.r.t. the memory parameter  $w$ , without noise (top left panel) and with small noise (top right panel). Time series for memory parameter  $w = 0.99$  in ABS with small noise (bottom panel).

the intensity of choice which has been chosen smaller,  $\beta = 40$ , so that in the case with zero memory ( $w = 0$ ) the fundamental steady state is *stable*. As the memory parameter increases, the fundamental steady becomes *unstable* for  $w \approx 0.3$ , due to a Hopf bifurcation, and as memory approaches 1, that is, the case where fitness is given by accumulated wealth, the dynamics becomes more complicated, even chaotic. The (long) time series in figure 4 shows that with memory close to 1 speculative bubbles still occur, although they are less frequently than for smaller memory. In fact, increasing memory yields a bifurcation route to instability, cycles and chaos similar to the bifurcation routes with respect to an increase in the intensity of choice. The intuition behind this result is as follows. Even when the intensity of choice to switch strategies is low, when memory in fitness is large, differences in accumulated profits can become sufficiently large to cause the majority of traders to switch to a trend following strategy and leading to a (temporary) bubble. More memory thus in general does *not* stabilize an evolutionary system, but may in fact be *destabilizing*.

#### 4.4 An example with volatility clustering

From a *qualitative* viewpoint, the price fluctuations in the simple examples of the nonlinear noisy ABS are similar to observed fluctuations in real markets. But do these (chaotic) fluctuations explain a significant part of stock price fluctuations? Brock and Hommes (1997b) calibrated the ABS of the previous subsection to ten years of monthly IBM prices and returns and found that the autocorrelations of the (noisy) chaotic time

returns and squared returns had no significant autocorrelations, similar as for the monthly IBM returns and squared returns. In this section we discuss a modified ABS of Gaunersdorfer and Hommes (2000) exhibiting volatility clustering and fat tails. Let there be two types of traders, with forecasting rules

$$p_{1,t+1}^e = f_{1t} = p^* + v(p_{t-1} - p^*), \quad 0 \leq v \leq 1, \quad \text{fundamentalists} \quad (40)$$

$$p_{2,t+1}^e = f_{2t} = p_{t-1} + g(p_{t-1} - p_{t-2}), \quad g \geq 0, \quad \text{trend extrapolators.} \quad (41)$$

Trader type 1 are fundamentalists, believing that tomorrow's price will move in the direction of the fundamental price  $p^*$  by a factor  $v$ . We will concentrate here on the special case  $v = 1$ , so that

$$f_{1t} = p_{t-1}, \quad \text{EMH-believer} \quad (42)$$

and we will refer to this type as an EMH-believer because the naive forecast is consistent with a random walk. Trader type 2 are simple trend extrapolators, extrapolating the latest observed price change. Market equilibrium (7) in a world with fundamentalists and chartists as in (40-41), with common expectations on IID dividends  $E_t[y_{t+1}] = \bar{y}$ , becomes

$$(1+r)p_t = n_{1t}(p^* + v(p_{t-1} - p^*)) + n_{2t}(p_{t-1} + g(p_{t-1} - p_{t-2})) + \bar{y} + \epsilon_t, \quad (43)$$

where  $n_{1t}$  and  $n_{2t}$  represent the fractions of fundamentalists and chartists respectively and  $\epsilon_t$  is an IID random variable representing model approximation errors.

Beliefs will be updated by *conditionally evolutionary* forces. The basic idea is that fractions are updated according to past fitness, conditioned upon the deviation of actual prices from the fundamental price. The evolutionary competitive part of the updating scheme follows the BH-framework with risk adjusted profits as the fitness measure; the additional conditioning upon deviations from the fundamental is similar to the approach taken in the Santa Fe artificial stock market in Arthur et al. (1997) and LeBaron et al. (1999). Using (30) with zero costs and zero memory, the evolutionary part of the updating of fractions yields the discrete choice probabilities

$$\tilde{n}_{ht} = \exp\left[-\frac{\beta}{2a\sigma^2}(p_{t-1} - f_{h,t-2} + \delta_t)^2\right]/Z_{t-1}, \quad h = 1, 2 \quad (44)$$

where  $Z_{t-1}$  is again the normalization factor such that the fractions add up to one. In the second step of updating of fractions, the conditioning on deviations from the fundamental by the technical traders is modeled as

$$n_{2t} = \tilde{n}_{2t} \exp[-(p_{t-1} - p^*)^2/\alpha], \quad \alpha > 0 \quad (45)$$

$$n_{1t} = 1 - n_{2t}. \quad (46)$$

The conditioning upon fundamentals part of the updating scheme may be seen as a stabilizing *transversality condition* in a heterogeneous world. Recall that in the perfectly rational world of subsection 3.1 the transversality condition excludes the rational expectations explosive bubble solutions (12). According to (45) the fraction of technical traders decreases more, the further prices deviate from their fundamental value  $p^*$ . As long as prices are close to the fundamental, updating of fractions will almost completely

be determined by evolutionary fitness (44), but when prices move far away from the fundamental, the correction term  $\exp[-(p_{t-1} - p^*)^2/\alpha]$  in (45) becomes small. The majority of technical analysts thus believes that temporary speculative bubbles may arise but that these bubbles can not last forever and that at some point a price correction towards the fundamental price will occur. The condition (45) may thus be seen as a weakening of the transversality condition in a perfectly rational world, allowing for temporary speculative bubbles.

The noisy conditional evolutionary ABS with fundamentalists versus chartists is given by (40–41), (43), (44) and (45–46). By substituting all equations into (43) a 4-th order nonlinear stochastic difference equation in prices  $p_t$  is obtained. It turns out that this nonlinear evolutionary system exhibits periodic as well as chaotic fluctuations of asset prices and returns; a detailed mathematical analysis of the bifurcation routes to strange attractors and coexisting attractors is given in Gaunersdorfer, Hommes and Wagener (2000). Here we focus on one simple, but typical example with EMH-believers, i.e.  $v = 1$ , versus trend followers<sup>20</sup>. Figure 5 compares time series properties of the noisy ABS to 40 years of S&P 500 data. The price series in the top panels are off course quite different, since S&P 500 is non-stationary and strongly increasing, whereas the ABS is a stationary model.<sup>21</sup> Prices in the ABS system are highly persistent however, and are in fact close to having a unit root. In the S&P 500 returns series the October 1987 crash and the two days thereafter have been excluded.<sup>22</sup> The ABS returns series ranges from  $-0.27$  to  $+0.29$ , which is larger than for the S&P 500, especially when the crash is excluded. The ABS returns series exhibits fat tails, with kurtosis coefficient  $k = 5.37$  versus  $k = 8.51$  for the S&P 500 returns, and strong volatility clustering. We estimated a GARCH(1,1) model of the form

$$R_t = c_1 + \delta_t, \quad \delta_t \sim N(0, \sigma_t^2) \quad (47)$$

$$\sigma_t^2 = c_2 + \rho_1 \delta_{t-1}^2 + \rho_2 \sigma_{t-1}^2 \quad (48)$$

on both returns series. The estimated parameters are  $\rho_1 = 0.068$  and  $\rho_2 = 0.929$ , with  $\rho_1 + \rho_2 = 0.997$ , for S&P 500 returns and  $\rho_1 = 0.071$  and  $\rho_2 = 0.914$ , with  $\rho_1 + \rho_2 = 0.985$ , for the ABS returns. The third panel in figure 5 shows that the autocorrelations of the returns, squared returns and the absolute returns of the ABS-model series are very similar to those of S&P 500, with (almost) no significant autocorrelations of returns and slowly decaying autocorrelations of squared and absolute returns. Although the ABS-system considered here is a nonlinear dynamic system with only 4 lags, it exhibits *long memory* with long range autocorrelations. The bottom panel shows a scaling law of the form

$$\rho_j = \frac{k}{j^\alpha} \quad (49)$$

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<sup>20</sup>Parameters are:  $r = 0.01$ ,  $\bar{y} = 3$ ,  $p^* = 300$ ,  $v = 1$ ,  $g = 1$ ,  $\beta = 5$ ,  $\alpha = 50$ ,  $\delta_t \equiv 0$  and  $\epsilon_t$  normally distributed with  $\sigma_\epsilon^2 = 11$ .

<sup>21</sup>By replacing our IID dividend process by a non-stationary dividend process, e.g. by a geometric random walk, prices in the ABS will also rapidly increase, similar to the S&P 500 series. We intend to study such non-stationary ABS in future work.

<sup>22</sup>The returns for these days were about  $-0.20$ ,  $+0.05$  and  $+0.09$ . In particular, the crash affects the autocorrelations of squared returns, which drop to small values of 0.03 or less for all lags  $k \geq 10$  when the crash is included.

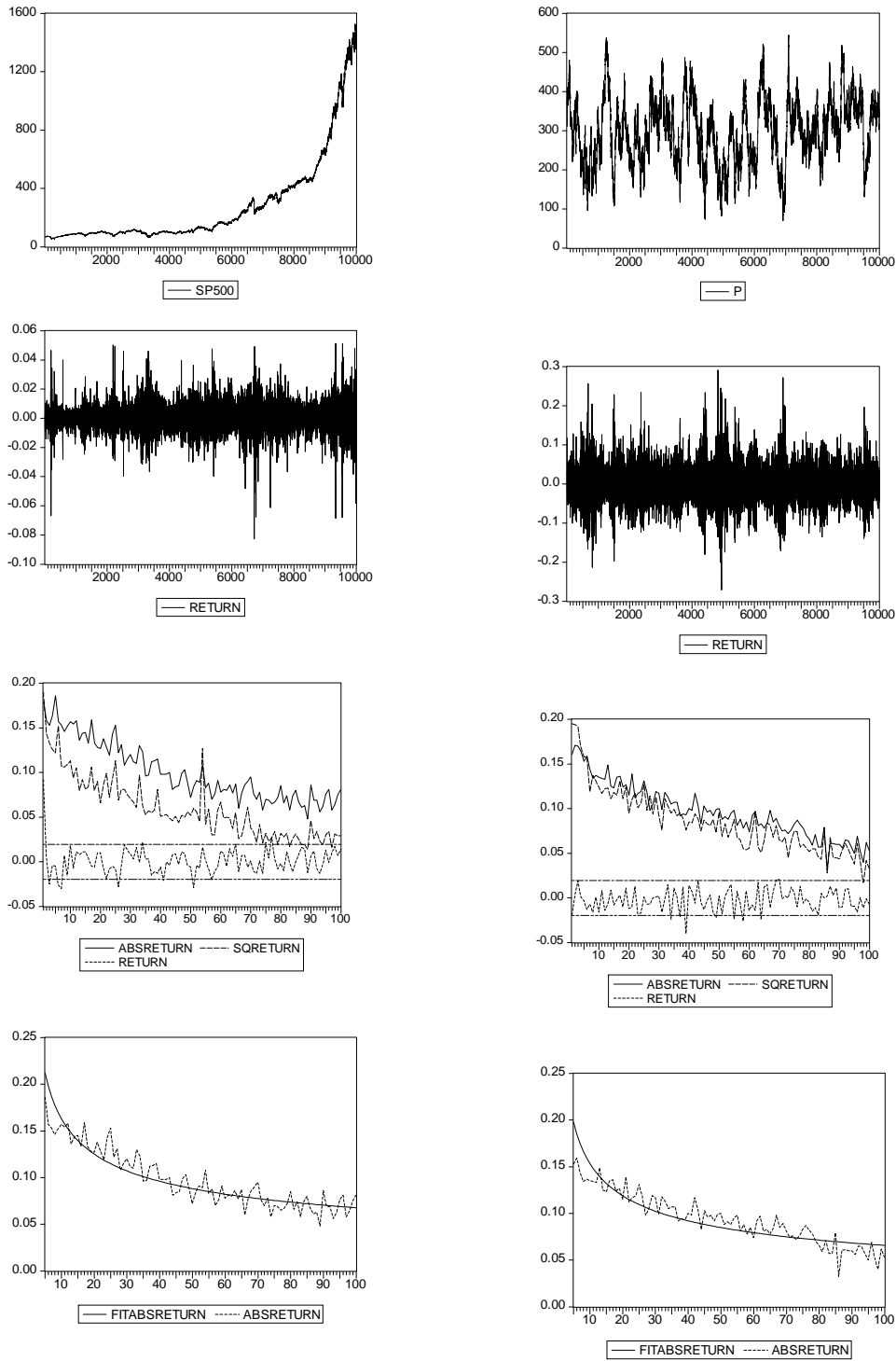


Figure 5: S&P 500 data (4 left panels) compared to ABS data (4 right panel). Prices (top panel), returns (second panel) defined as relative price changes, ACF's of returns, squared returns and absolute returns (third panels) and ACF's of absolute returns with fitted scaling laws (bottom panels).

fitted to the autocorrelations of absolute returns, for lags  $5 \leq j \leq 100$ . The estimated scaling exponents are 0.38 for S&P 500 and 0.37 for the ABS; see Mantegna and Stanley (2000) for a survey on the role of scaling laws in financial time series. Our simple ABS thus exhibits a number of important stylized facts of 40 years of S&P 500 returns data.

## 5 Asset pricing experiments

Much computational and theoretical work on expectation formation in heterogeneous agent systems has been done in the past decade, but it is hard to test the heterogeneous expectations hypothesis empirically and to infer the expectations hypothesis from economic or financial data. Survey data research, as for example in Shiller (1989) on stock market expectations, yields useful insight on expectation formation but also has its limitations, for example because of changing underlying economic fundamentals. Controlled laboratory experiments seem to be well suited to investigate expectation formation and forecasting behavior in particular situations. As noted e.g. by Sunder (1995), it is remarkable that, despite an explosion of interest in experimental economics, relatively few contributions have focused on expectation formation and learning in dynamic experimental markets with expectations feedback. An exception is the well known ‘bubble experiment’ by Smith, Suchanek and Williams (1988) in an experimental asset market. This study can not be viewed however as pure experimental testing of the expectations hypothesis, *everything* else being held constant, because dynamic market equilibrium is affected not only by expectations feedback but also by other types of human behavior, such as trading behavior.

In a series of three papers, Hommes et al. (1999a,b) and Sonnemans et al. (1999), jointly with my colleagues from CeNDEF, we have started to test what is perhaps the simplest dynamic expectations feedback system, the cobweb ‘hog cycle’ model, in the CREED experimental laboratory at the University of Amsterdam. A convenient feature of the cobweb model is that it has a *unique* rational expectations equilibrium, namely the steady state price at which demand and supply intersect. Market equilibrium equations were controlled and fixed during the experiment (although they were subject to unexpected exogenous shocks and/or noise). Subjects were asked to predict prices and their earnings were inversely related to their quadratic forecasting errors. Price realizations only depend upon subjects’ price expectations. Demand and supply were chosen such that under naive expectations the RE steady state is locally unstable and prices diverge away from the RE steady state and converge to a 2-cycle, with ‘systematic’ forecasting errors. An important motivation was whether individuals in the experiment would be able to learn from their forecasting errors and coordinate on the RE steady state. For the cobweb experiments, the main conclusions can be summarized as follows:

1. In most experimental treatments realized market prices converge to RE *in the mean*, that is, the null hypothesis that the sample mean of realized market prices equals the RE steady state price can *not* be rejected;
2. In all experimental treatments realized market prices exhibit significant *excess*



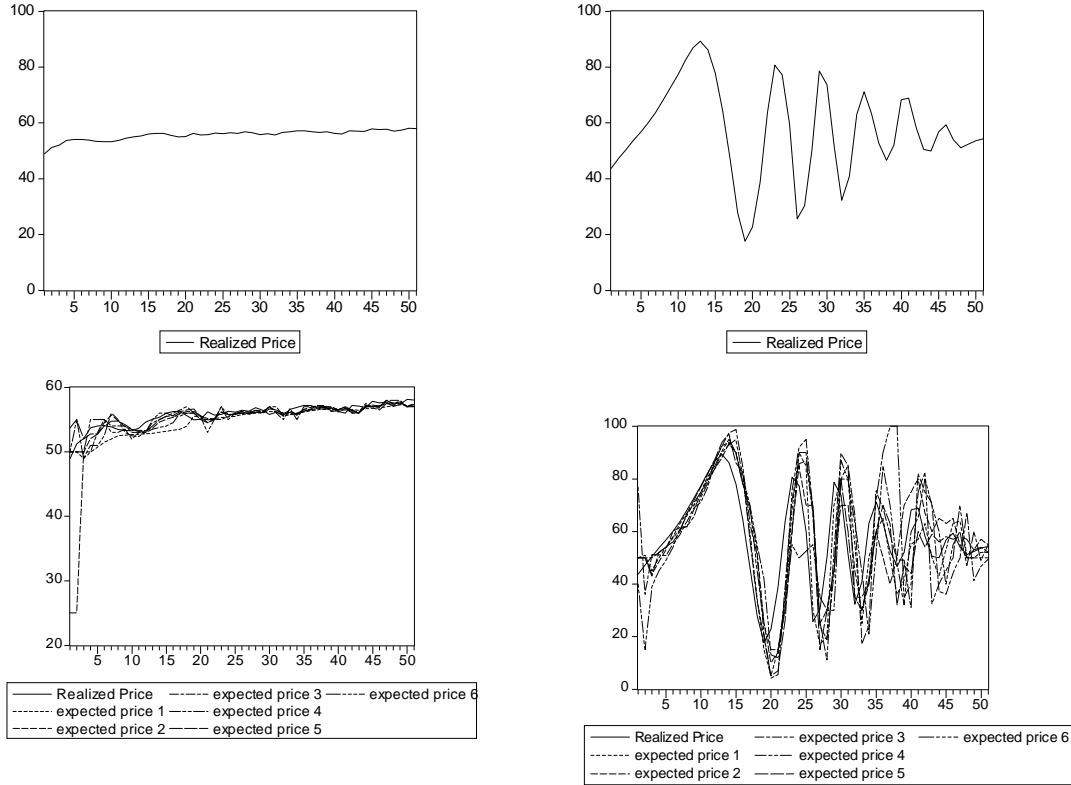


Figure 6: Asset pricing experiments. A stable market (top left panel) and a market with bubbles (top right panel). Individual expectations are included in lower panels.

*volatility*, that is, the null hypothesis that the sample variance of realized market prices is smaller than or equal to the RE-variance is strongly rejected;

3. Realized market prices are typically irregular and exhibit hardly any linear predictability, since sample autocorrelations are typically insignificant.

The (unstable) experimental cobweb economy thus seems to be consistent with heterogeneous boundedly rational agents, with diversity of boundedly rational forecasting rules creating excess price volatility around the RE mean.

Figure 6 shows the outcome of some first asset pricing experiments with expectations feedback, from Hommes, Sonnemans, Tuinstra and van de Velden (2000).<sup>23</sup> Participants were asked to predict the price  $p$ ,  $0 \leq p \leq 100$  of a risky asset and earnings were inversely related to forecasting accuracy, or equivalently to risk adjusted profits. Recall that the asset pricing model has RE ‘bubble solutions’ growing at the risk free rate of return  $1 + r$ . These bubble solutions are perfect foresight solutions, where forecasting errors are zero (or equal to the noise term). If participants in the experiments are motivated by minimizing forecasting errors, both these bubble solutions and the fundamental solution  $p^*$  are equally attractive. To avoid the existence of an unbounded bubble solution, we introduced a stabilizing fundamental type  $F$  robot trader in the experimental setup. The fundamental type  $F$  always expects the fundamental price to

<sup>23</sup>I would like to thank Henk van de Velden for providing these figures.

prevail, i.e.  $E_{Ft}(p_{t+1}) = p^*$ . In the asset pricing experiments the (unknown) market equilibrium is given by

$$(1+r)p_t = \sum_{h=1}^H n_{ht} E_{ht}(p_{t+1}) + n_{Ft} p^* + \bar{y} + \epsilon_t, \quad (50)$$

where  $n_{ht}$ ,  $1 \leq h \leq H$  are the weights (or fractions) of the participants upon realized market equilibrium prices and  $n_{Ft}$  is the weight (or fraction) of the fundamental robot traders in the market. In our experimental setup  $H = 6$  and the weights  $n_{ht}$ ,  $1 \leq h \leq H$  of the participants will decrease when prices move away from the fundamental price  $p^*$ . More precisely, these weight will be given by

$$n_{ht} = \frac{\exp(-\alpha|p_{t-1} - p^*|)}{H}, \quad 1 \leq h \leq H, \quad \alpha = 0.005 \quad (51)$$

and the weight of the fundamentalists by

$$n_{Ft} = 1 - \sum_{h=1}^H n_{ht}. \quad (52)$$

Hence, when prices move away from the fundamental  $p^*$ , the weights  $n_{ht}$ ,  $1 \leq h \leq H$ , of the participants upon realized market equilibrium price  $p_t$  decrease and the weight  $n_{Ft}$  of the fundamental type increases. The fundamental robot trader in our experimental setup thus stabilizes possible bubble solutions. Notice that all weights  $n_{ht}$  of the participants are equal, and close to  $1/H$  as long as realized market prices are close to the fundamental price. In the experiments, mean dividend  $\bar{y} = 3$  and the risk free rate of return  $r = 0.05$  were common knowledge. All participants thus were able to compute the fundamental price  $p^* = \bar{y}/r = 60$ . The noise  $\epsilon_t$  was normally distributed with standard deviation  $\sigma_\epsilon = 0.5$ , so that in a perfectly rational world prices would fluctuate between 59 and 61 most of the time.

In the first experiment (figure 6, top left) the market is stable and converges to a value close to the RE fundamental. The realized market price is however always *below* the RE fundamental, so that the market is in fact *undervalued*. In the second experiment (figure 6, top right) participants coordinate on a speculative bubble solution. The first speculative bubble lasts 13 periods, but reverses and becomes a stable oscillation due to the presence of the fundamental robot trader. The first maximum occurs after 13 time periods; thereafter a stable oscillation arises, with local maxima at periods 23, 29, 35 and 41. The bottom panels show the realized market price as well as the individual forecasts by all 6 participants in the experiments. In the stable case (bottom left) participants quickly coordinate on a price below but fairly close to the fundamental price  $p^* = 60$ . In the unstable case, all participants quickly coordinate on a speculative bubble. However, as time goes on, the coordination becomes weaker, and the bubble seems to stabilize.

It should be noted that the fundamental price in these experiments is constant and thus as simple as possible. Even in this simplest setting, (temporary) bubbles arise but they seem to stabilize toward the end of the experiment. It is not clear what would happen if the fundamental robot trader would not be present in the market. In future experiments we also hope to focus on other controlled fundamentals, for example a constant fundamental with larger noise levels or, what seems to be relevant to real stock markets, a time varying, growing fundamental.

## 6 Future Perspective

Is a significant part of changes in stock prices driven by ‘Keynesian animal spirits’? For many decades already, this question has led to heavy debates among economic academics as well as financial practitioners. In the evolutionary adaptive belief systems discussed here, price changes are explained by a *combination* of economic fundamentals and ‘market psychology’. Negative economic ‘news’ (e.g. on inflation or interest rates) may act as a trigger event for a decline in stock prices, which may become reinforced by investors sentiment and evolutionary forces. Price movements are driven by an interaction of fundamentalism and chartism, the two most important trading strategies in financial practice. The nonlinear evolutionary ABS generates a number of important stylized facts observed in many financial series, such as unpredictability of returns, fat tails and strong volatility clustering.

Let us now reflect upon the questions Q1-Q3 raised at the end of the introduction, which were central to the SFI workshop.

### **Can speculators survive evolutionary competition?**

Friedman (1953, p.175) has argued that speculators will not survive evolutionary competition: ‘*People who argue that speculation is generally destabilizing seldom realize that this is largely equivalent to saying that speculators lose money, since speculation can be destabilizing in general only if speculators on the average sell when the currency is low in price and buy when it is high*’ [emphasis added]. We have seen that in the evolutionary ABS technical trading rules do survive evolutionary competition and can in fact earn profits or attain wealth comparable to profits or wealth of fundamentalists, even when there are no information costs and memory in evolutionary fitness is high. The technical analysts start buying when prices are low, in the early stage of an upward price trend which may have been triggered by news about economic fundamentals, and sell as soon as their trading rule detects that the trend has reversed when the price is still high. The main reason why technical trading can survive evolutionary competition seems to be the fact that in markets for risky assets an optimistic or pessimistic mood leads to a self fulfilling speculative bubble when the optimism or pessimism is shared by a large enough group of investors.

### **Is a financial market with heterogeneous adaptive agents efficient?**

There seems to be disagreement about exactly what efficiency means. But there are at least two important factors. One is sometimes called *informationally efficiency*, meaning that a market should be difficult to forecast, since otherwise there would be obvious arbitrage opportunities. Our nonlinear evolutionary ABS is close to informationally efficiency, because asset returns are fairly unpredictable and have e.g. close to zero autocorrelations. Even employing methods such as nearest neighbor forecasting does not lead to very good prediction of returns, not even in the short run, due to the strong noise amplification in a nonlinear evolutionary systems. A second factor is sometimes referred to as *allocative efficiency* meaning that asset prices reflect the ‘true’ fundamental value of the underlying asset. In the evolutionary ABS large and persistent deviations from the fundamental can occur, possibly triggered by noise and reinforced by evolutionary forces. An evolutionary ABS thus allows for the possibility of allocative *inefficiency*. Knowledge about the true fundamentals is important in

this respect. If most traders agree about the ‘true’ fundamental and it is known with great precision, deviations from this commonly shared fundamental seem unlikely. But in an uncertain world, where nobody really knows what exactly the fundamental is, good news about economic fundamentals reinforced by evolutionary forces may lead to deviations from the fundamental and overvaluation of asset prices.

### **Does heterogeneity create excess volatility?**

The ABS theory implies a decomposition of returns into two terms, one martingale difference sequence part according to the conventional EMH theory, and an extra *speculative* term added by the evolutionary theory. The theory of evolutionary ABS thus can explain excess volatility. In an ABS the phenomenon of volatility clustering occurs due to the interaction of heterogeneous traders. In periods of low volatility the market is dominated by fundamentalists. High volatility may be triggered by news about fundamentals and may be amplified by technical trading.

Our evolutionary ABS may be seen as, what Sargent (1999) calls an *approximate rational expectations equilibrium*. Traders are boundedly rational and use relatively simple strategies. The class of trading rules is disciplined by evolutionary forces based upon realized profits or wealth. A convenient feature of our theoretical setup is that the benchmark rational expectations model is *nested* as a special case. This feature gives the model flexibility with respect to experimental and empirical testing. It is worthwhile noting that Chavas (1996) and Baak (1997) have run empirical tests for heterogeneity in expectations in agricultural data and indeed find evidence for the presence of boundedly rational traders in the hog and cattle markets. It may seem even more natural that heterogeneity and evolutionary switching between different trading strategies play an important role in financial markets. Understanding the role of market psychology seems to be a crucial part of understanding the huge changes in stock prices observed so frequently these days. But much more insight into ‘financial psychology’ is needed, before ‘market sentiment’ based policy advice can be given. Theoretical analysis of stylized evolutionary adaptive market systems, as discussed here, and its empirical and experimental testing may contribute in providing such insight.

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