Heterogeneous beliefs and routes to complex dynamics in asset pricing models with price contingent contracts*

William A. Brock[1] and Cars H. Hommes[2]

[1] Department of economics, University of Wisconsin, 1180 Observatory Drive, Madison, WI 53706, USA, e-mail: wbrock@ssc.wisc.edu
[2] Center for Nonlinear Dynamics in Economics and Finance (CeNDEF), University of Amsterdam, Roetersstraat 11, NL-1018 WB Amsterdam, The Netherlands, e-mail: hommes@fee.uva.nl

Abstract. This paper discusses dynamic evolutionary multi-agent systems, as introduced in Brock and Hommes (1997). In particular, the heterogeneous agent dynamic asset pricing model of Brock and Hommes (1998) is extended by introducing derivative securities by means of price contingent contracts. Numerical simulations suggest that in a boundedly rational heterogeneous evolutionary world futures markets may be destabilizing.

Keywords: heterogeneous beliefs, bounded rationality, arrow securities, evolutionary dynamics.

1 Introduction

This paper extends a dynamic equilibrium asset pricing model with evolutionary selection of heterogeneous forecasting rules or strategies to include price contingent contracts. Our framework fits into equilibrium theory in the sense that there is market clearing. However, expectations are in 'disequilibrium' in the sense that agents may deviate from fully rational expectations or perfect foresight. One might describe our evolutionary adaptive belief systems (ABS) as an 'approximate rational expectations equilibrium'.

Before getting into details let us discuss the main issue we wish to deal with here. Brock and Hommes (1997, 1998), henceforth BH, attempt to develop a positive and normative equilibrium theory of endogenous dynamic belief formation in intertemporal markets, especially asset markets. Agents can choose from a finite set of competing forecasting rules or trading strategies. Agents are boundedly rational in the sense that they tend to use strategies

* We would like to thank an anonymous referee for detailed comments on an earlier draft of this paper. This research was supported by NSF under grant SES-9911251, the Vilas Trust, and by the Netherlands Organization for Scientific Research (NWO) under a NWO-MaG Pionier grant. None of the above are responsible for errors, opinions, and shortcomings in this paper.

that have performed well in the recent past. BH show that a *rational route to randomness*, that is a bifurcation route to local instability, limit cycles and chaos, may arise when evolutionary pressure for strategy selection increases. Stated differently, as the traders' sensitivity to differences in strategy performance increases, complicated and unpredictable asset price fluctuations may arise; see e.g. Brock (1997) or Hommes (2001) for recent reviews. However, BH (1998) did not consider the impact on belief heterogeneity of trading derivatives. The basic issue we wish to study here is the impact of introduction of a plentitude of "derivative" securities upon the dynamics of belief heterogeneity. *Does the introduction of derivatives in the asset market stabilize the rational routes to randomness?*

A common critique of any theory of diverse beliefs for economies with well developed asset markets is the following. The number of price contingent contracts (PCC's) (this terminology follows Kurz (1997)) such as derivative securities is rich enough in well developed markets, that enough agreement is forced by trading such objects that the quantitative effect remaining of initial belief diversity after several rounds of trading such securities is likely to be minor. Anyone who has visited an options exchange after opening rotation is completed is aware of how fast trading quiets down after the initial burst at the open (Brock and Kleidon (1992)). To put it another way, trading volume is typically higher at the open and at the close in comparison with volume during the rest of the day. This well known stylized fact is consistent with the view that several rounds of trading reduces belief heterogeneity, at least on time scales consistent with the securities being traded. The initial burst of trading at the open is consistent with the hypothesis that belief heterogeneity increased during the closed period before the opening. The rapid decrease in trading observed after the open is consistent with the hypothesis that trading decreases belief heterogeneity.

A corollary of this argument is that one should focus on researching relatively minor variations around some kind of natural focal baseline such as the Rational Expectations baseline (e.g. noisy rational expectations, adaptive learning centered at a rational expectations baseline, and so forth). Indeed, we have heard arguments that the presence of a plentitude of derivative securities in real markets should remove so much of the belief heterogeneity that the BH (1998) dynamics of belief heterogeneity would be "crushed" quite quickly into belief homogeneity. Magill and Shafer (1991) review General Equilibrium theory with Incomplete markets, hereafter called "GEI" theory. A basic idea of GEI theory is this. The more ways there are for agents at date t to move income across states at date t+1 via trading of different securities at date t, the less disagreement in beliefs across agents about date t+1 economic states. Intuitively, expected marginal rates of substitution across different agents across date $(t, t+1)$ event pairs are equated by trading of contracts at date t that pay one if event E_i occurs at date t+1 and zero otherwise. The more of these securities there are that enable the young to move income across states in old age by using available securities while young, the smaller the disagreement in beliefs about the future should be. It would seem that if there is a security for each state or if there is "spanning" (an equivalence), then disagreement in beliefs about future returns would vanish and that's the end of BH (1998) complex dynamical possibilities. Extreme versions of this type of argument suggest that the entire research program that studies the dynamics of belief heterogeneity in markets is essentially a waste of time. This paper argues that matters are rather more subtle and presents a numerical example suggesting that in a boundedly rational, heterogeneous evolutionary world price contingent contracts may actually *destabilize* the asset market.

The ultimate fate of dynamics of heterogeneous beliefs depends upon the dimension of the rule set relative to the number of derivatives or PCC's being traded among other things. Let us consider what Kurz (1997) has to say about belief heterogeneity even in contexts with a plentitude of PCC's. Despite the existence of such PCC's in real markets Kurz (1997) still gives a vigorous argument for quantitative importance of his alternative to Rational Expectations Theory (REE theory), which he calls Rational Beliefs Theory (RBE). He calls a belief, a "Rational Belief", provided that it can not be contradicted by intertemporal observed data. He works with a class of stochastic processes rich enough so that even though each Rational Belief is consistent with all possible limiting time averages generated by the economy, convergence to the true equilibrium stochastic process does not occur. Kurz works with an *exogenously fixed* set of RB's, but he imposes a strong consistency requirement with the data. Kurz (1997) stresses the importance of endogenous uncertainty created by asset markets not only for understanding the volatility patterns in returns data but also for policy purposes.

The objective of BH (1998) theory is similar to Kurz in that both theories wish to give a coherent and data disciplined approach to endogenous uncertainty and belief formation, but BH works with an *endogenous* set of beliefs that co-evolve over time. In the BH (1998) setting there is an additional evolutionary dynamics on the fraction of belief types as well as on prices. This additional dynamics adds extra complexity. BH theory pays the price that most of the beliefs will *not* be consistent with the data at all times, but the data are allowed to determine the set of "surviving" beliefs via the force of evolutionary selection based upon how well each belief type does in trading against the set of belief types in the economy as a whole. Structural rational expectations is always an equilibrium of the BH system if agents have free access to such structural information. If the information is expensive however, a dynamical tension develops which tends to create phases of "random" length where the economy runs close to rational expectations for a while then gradually drifts away from RE as some agents gradually realize they can get by with using naive predictors and not pay to get RE predictors. The size of departures away from RE and how long such excursions away from RE can be becomes a complicated function of underlying parameters of the economic system.

In BH financial modeling, the bond price (i.e. the return on bonds) was assumed to be exogenously fixed. We give a treatment of endogenous bond pricing in two period Overlapping Generations setups below. We restrict attention to a two periods OG setup, to avoid tractability problems that arise in T period settings of a heterogeneous agents economy when the horizon, T, is large. The BH-model with endogenous bond pricing is a true general equilibrium model whereas the original BH financial work is partial equilibrium. In a two period setup analytical problems are simplified for several reasons. First, the issue of wealth in a heterogeneous expectations BH setup and its book keeping problem to keep track of wealth accumulated by different traders that have switched strategies many times is simplified because the OG system is "re-initialized" each period as the young become old and pass from the scene. Brock (1990) generalizes a simple version of Lucas's Asset Pricing Model to OG setups. We can do the same here with the BH program. Second, "bubble" solutions are automatically crushed in OG models. "Up bubbles" are killed by the endowment of the young bounding the price of the asset from above. "Down bubbles" are killed off by limited liability. So, unlike BH mean variance theory, no exogenously specified "outside" force as used by Gaunersdorfer and Hommes (2000) is needed to crush bubbles. Bubbles are crushed by the economics of the models. Third, it is easy to insert Arrow securities (securities sold at date t a unit of which pays 1 if price plus dividend, $p_{t+1} + y_{t+1}$, next period takes a particular value and zero otherwise) into this OG model and explore their impact on forcing agreement of BH type heterogeneous expectors. We will propose arguments later that BH type learning should apply to the "p" part, i.e. the expected future price, of an Arrow security but perhaps algorithmic learning like Marcet/Sargent should apply to the "y" part, i.e. the expected future dividend part. See Evans and Honkapohja (2001) for an extensive recent treatment of algorithmic learning. The argument will revolve around Nature selecting and creating the y part, but Society co-creating the p part. We shall argue that it is not automatic that a full set of Arrow securities with learning in a BH framework gets rid of the dynamical complexity of BH learning. Hedging arguments against BH heterogeneity are not so easy because BH heterogeneity itself is unobservable and fluctuates unlike sunspots. So derivative security sunspot irrelevance type arguments do not automatically apply to BH heterogeneity. See Brock (1990) for discussion of Arrow securities in OG models. See Guesnerie's recent book (Guesnerie (2001))for a recent review of the sunspots literature.

Our contribution should be viewed as a sketch of ideas illustrated by some first, exploratory numerical simulations that will be developed more rigorously in future work. The paper is organized as follows. Section 2 presents a general two period OG-setup of the BH-model with heterogeneous beliefs and introduces PCC's in the model. Section 3 discusses an example with three belief types and two PCC's and presents numerical simulations of the dynamical behaviour. Finally, section 4 gives some concluding remarks and briefly discusses some future work.

2 An OG model with PCC's and heterogeneous beliefs

Consider an OG model with one risky asset, e.g. a stock, and one risk free bond with $h = 1, 2, ..., H$ different beliefs about the value of the risky asset. Assume agreement on the bond. Assume utility functions are the same across agents. Let u, v denote utility functions while young and old. Let p_t, y_t denote price and earnings of the risky asset at date t and let R_t denote gross return on the bond at date t. Following Brock (1990) let w_y, w_o denote endowments of young and old respectively. First let us assume homogeneous beliefs and solve for rational expectations equilibrium in order to establish a useful baseline. Assume the young solve (1) below at each date t,

$$\text{Maximize}\{u(w_y - b_t - p_t z_t) + E_t[v(w_o + (p_{t+1} + y_{t+1})z_t + R_t b_t)]\}, \qquad (1)$$

where b_t and z_t are the demands for the bond and the risky asset respectively. First order necessary conditions for maximum z_t, b_t are given by,

$$\begin{aligned} p_t u' &= E_t[(p_{t+1} + y_{t+1})v'] \\ u' &= R_t E_t(v'). \end{aligned} \qquad (2)$$

We will focus on the case where the dividends $\{y_t\}$ are independently identically distributed (IID).

Let the supply z_s of shares be fixed for each date t, normalized to one. In equilibrium, the supply of bonds is zero for society as a whole, thus, $b_t = 0$ in equilibrium. Thus, equilibrium $\{p_t\}$ must solve,

$$p_t u'(w_y - p_t) = E_t[(p_{t+1} + y_{t+1})v'(w_o + p_{t+1} + y_{t+1})]. \qquad (3)$$

As shown by Brock (1990), depending upon the shape of u and v, there can be many rational expectations solutions to (3). We concentrate on stationary ones here. Put

$$\begin{aligned} A(p) &= pu'(w_y - p) \\ B(p) &= E_t[(p + y_{t+1})v'(w_o + p + y_{t+1})]. \end{aligned} \qquad (4)$$

Since $\{y_t\}$ is IID, $B(p)$ does not depend upon time t. Assume $u' > 0, u'' < 0$. The $A(p)$ is increasing in p. Therefore if we restrict our search for solutions $\{p_t\}$ of (3) to deterministic sequences p_t, these must satisfy,

$$A(p_t) = B(p_{t+1}), \qquad \text{i.e. } p_t = F(p_{t+1}). \qquad (5)$$

The last follows because $A' > 0$ implies A is invertible. Notice that $B(p)$ can have many shapes, so cycles, sunspots, and many kinds of equilibria are possible for different B functions. Let $p^* = F(p^*)$ be a fixed point. Brock (1990) shows that there are usually at least two fixed points, one negative (even though $y > 0$) and one positive. He also shows that there can be many such fixed points. Assume the positive fixed point is unique. We call this the "fundamental price" p^*.

Notice also that if A cuts B from below at p^* it looks like "up bubbles" are possible in (5). Simply take an initial condition to the right of p* and solve (5) forward. However once $p_t > w_y$ it cannot be equilibrium because $p_t.z$ can not exceed the wealth in the economy, w_y. This simple argument shows that up bubbles can not be equilibria.

Notice that for a learning scheme where $p^e_{t+1} = G(p_{t-1}, ...)$ is an expectation of p_{t+1} that depends upon past observed prices, (5) generates a learning dynamics. Methods from the learning literature may be adapted and applied to this dynamics. We are interested in generalizing BH heterogeneous beliefs dynamics to this OG setup. Let b_{ht} and z_{ht} denote the demands of type h for the bond and the risky asset respectively. Expectational type h solves

$$\text{Maximize}\{u(w_y - b_{ht} - p_t z_{ht}) + E_{ht}[v(w_o + (p_{t+1} + y_{t+1})z_{ht} + R_t b_{ht})]\}. \quad (6)$$

Assume all h's agree on R_t. First order conditions for optimal choices of z_{ht}, b_{ht} are given by

$$p_t u'(w_y - b_{ht} - p_t z_{ht}) = E_{ht}[(p_{t+1} + y_{t+1})v'(w_o + (p_{t+1} + y_{t+1})z_{ht} + R_t b_{ht})]. \quad (7)$$

$$u'(w_y - b_{ht} - p_t z_{ht}) = R_t E_{ht}[v'(w_o + (p_{t+1} + y_{t+1})z_{ht} + R_t b_{ht})]. \quad (8)$$

We make an assumption about beliefs that parallels BH:
Assumption 1:

A1 $E_{ht}[(p_{t+1} + y_{t+1})v'(w_o + (p_{t+1} + y_{t+1})z_{ht} + R_t b_{ht})] =$

$$E_t[(p^* + f_{ht} + y_{t+1})v'(w_o + (p^* + f_{ht} + y_{t+1})z_{ht} + R_t b_{ht})],$$

A2 $R_t E_{ht}[v'(w_o + (p_{t+1} + y_{t+1})z_{ht} + R_t b_{ht})] =$

$$R_t E_t[v'(w_o + (p^* + f_{ht} + y_{t+1})z_{ht} + R_t b_{ht})],$$

where f_{ht} is a function of past prices $p_{t-1}, p_{t-2}, ...$, which have been observed at date t when h submits his or her demand function before the market has determined the equilibrium price p_t based upon aggregate demand and aggregate supply being equated at date t.

Notice how this assumption is the exact parallel of BH in a simpler mean variance model. It says that beliefs agree on the general functional form of the cumulative distribution function of $p_{t+1} + y_{t+1}$ but disagree on the form of the shift parameter in expectations about p_{t+1}. I.e. beliefs about $p_{t+1} + y_{t+1}$ are of the form

$$E_{ht}[p_{t+1} + y_{t+1}] = p^* + f_{ht} + E_t[y_{t+1}],$$

where there is agreement on the distribution of y_{t+1}, but disagreement on f_{ht} which is deterministic. Notice that this implies full agreement on the support of y_{t+1}. Below we will discuss an example where we allow disagreement on the support of y_{t+1} and we will show how the presence of PCC's forces a certain amount of agreement on the support of y_{t+1} in order for a PCC equilibrium to exist. The important point is that although existence of a PCC equilibrium may force agreement on supports, it does *not* force full agreement on probability masses over the support.

We are now ready to introduce PCC's. Put $q' = p' + y'$ for next period's values of p and y, $p + y$. Partition the non-negative real line as follows

$$0 < a_1 < a_2 < ... < a_n < \infty. \tag{9}$$

Define disjoint sets S_i as follows,

$$S_1 = [0, a_1), S_2 = [a_1, a_2), ..., S_i = [a_{i-1}, a_i), ..., S_n = [a_{n-1}, \infty). \tag{10}$$

The case $n = 1$ corresponds to $a_0 = 0$, $S_1 = S = [0, \infty)$ by convention, i.e. the case $n = 1$ corresponds to the case where only the risky asset itself is traded. We take $n > 1$ unless otherwise noted. Let p_{it} denote today's price of security i which pays $q'.1_i$ next period, where $1_i = 1[q' \in S_i]$ with $1[A]$ the indicator function for event A which is 1 if A occurs, zero otherwise; we will write p_{0t} for the price of the stock, i.e. the original risky asset, and refer to it as asset 0. Let z_{h0t} denote the demand for asset 0 and z_{hit} the demand for the i-th PCC. After introduction of n PCC's the type h young person's problem is to choose the demand vector $\boldsymbol{z} = (z_{h0t}, z_{h1t}, ..., z_{hnt}, b_{ht})$ to solve

$$\begin{aligned} \text{Maximize}\{u(w_y - b_{ht} - p_{0t}z_{h0t} - \textstyle\sum_{i=1}^{n} p_{it}z_{hit}) + \\ E_{ht}[v(w_o + q_{t+1}z_{h0t} + \textstyle\sum_{i=1}^{n} q_{i,t+1}z_{hit} + R_t b_{ht})]\}, \end{aligned} \tag{11}$$

where $q_{t+1} = p_{t+1} + y_{t+1}$ is a random variable and by definition,

$$q_{i,t+1} = q_{t+1}1[q_{t+1} \in S_i]. \tag{12}$$

The definition of $q_{i,t+1}$ implies the random variable q_{t+1} is a linear combination of $\{q_{i,t+1}\}$, i.e.

$$q_{t+1} = \sum_{i=1}^{n} q_{i,t+1}. \tag{13}$$

Therefore in order to prevent arbitrage portfolios (cf. Magill and Quinzii (1996), Chapter 2 and elsewhere) we must have that, at each date, the price of the stock, i.e. asset 0, equals the sum of the prices of the PCC's, i.e.

$$p_{0t} = \sum_{i=1}^{n} p_{it}. \tag{14}$$

This is easy to see. If (14) were not true, e.g. if p_{0t} were larger than the sum, the agent could borrow z shares of the asset 0, and buy z shares of each PCC i. By (13) this arbitrage results in a net income of zero at $t+1$ but generates positive income at t. It can be scaled up to infinity. Hence p_{0t} must be less than or equal to the sum. A similar argument forces p_{0t} equal to the sum. General arguments of this type and the restrictions they place on the prices of dependent random variables are discussed by Magill and Quinzii (1996). Our case is a very special case of this general discussion.

Let $\omega_{hit} = z_{h0t} + z_{hit}$ be the sum of the demands of asset 0 and the i-th PCC, and use (14) to rewrite (11) as follows,

$$\text{Maximize}\{u(w_y - b_{ht} - \sum_{i=1}^{n} p_{it}\omega_{hit}) + E_{ht}[v(w_o + \sum_{i=1}^{n} q_{i,t+1}\omega_{hit} + R_t b_{ht})]\}. \quad (15)$$

The treatment of Brock and Hommes (1998) corresponds to the special case, $u(.) = 0$ (i.e. there is no utility generated by consumption while young), and $v(c') = E(c') - (a/2)Var(c')$. The return, R, on the bond, b, was exogenously given in BH (1998). It is endogenous here. We are now ready to compare the dynamics of an Adaptive Belief System (ABS) with PCC's, to the dynamics of an ABS without PCC's. Let $\boldsymbol{\omega}_{ht}$ denote the demand vector $(\omega_{h1t}, ..., \omega_{hnt})$ and let $\mathbf{e}$ denote the vector of n ones. Let n_{ht} denote the fraction of agents using belief system h at date t. The heterogeneous agents market equilibrium conditions are

$$\sum_{h=1}^{H} n_{ht} b_{ht} = 0, \qquad \text{(bond market)} \qquad (16)$$

$$\sum_{h=1}^{H} n_{ht} \boldsymbol{\omega_{ht}} = z_s \boldsymbol{e}, \qquad \text{(n PCC markets).} \qquad (17)$$

Our dynamical system will be completely specified once we specify the fractions n_{ht} and the beliefs E_{ht} about q_{t+1} for each type h. As in BH (1998) the fractions n_{ht} will be updated by evolutionary selection through a discrete choice model, as will be discussed in the example in section 3. The beliefs we shall consider are specified by a cumulative probability function as follows

$$F_{ht}(x) = \text{Prob}\{E_{ht}[q_{t+1}] < x\} = \text{Prob}\{p^* + f_{ht} + y < x\} = F_y(x - p^* - f_{ht}), \quad (18)$$

where F_y is the cumulative probability function of the IID dividend process and

$$f_{ht} = f_h(x_{t-1}, x_{t-2}, ..., x_{t-L}). \qquad (19)$$

Recall that $x_s = p_s - p^*$. I.e. we are following BH (1998) in restricting the belief space to consist of common beliefs on the distribution of y and deterministic beliefs on p_{t+1} of the form $p^* + f_{ht}$ where f_{ht} is a function of L lags of past deviations from the “fundamental” p^*. This specification seems adequate for an initial exploration of the impact of PCC’s upon ABS dynamics. The special case of no PCC’s and exogenously determined R that was treated in BH (1998) is nested within our general framework.

Example

Up to now we have not seen a clear role in our models that PCC’s play in forcing agreement of beliefs, if any such force exists. We sketch here an example of two PCC’s that clearly exposes such a role. Let $S_1 = (-\infty, a_1)$ and let S_2 be the complement of S_1. At date t, let type $h = 1[h = 2]$ believe all support of $q_{t+1} = p_{t+1} + y_{t+1}$ is on $S_1[S_2]$. We consider subcases as follows. First, suppose the model is OG as in (1) above with budget sets for young and old as follows

$$\begin{aligned} w_y &= p_{0t} z_{h0t} + p_{1t} z_{h1t} + p_{2t} z_{h2t} + c_t \\ &= (p_{1t} + p_{2t}) z_{h0t} + p_{1t} z_{h1t} + p_{2t} z_{h2t} + c_t \\ &= p_{1t} \omega_{h1t} + p_{2t} \omega_{h2t} + c_t, \end{aligned} \tag{20}$$

$$c_{t+1} = w_o + q_{t+1} 1[q_{t+1} \in S_1] \omega_{h1t} + q_{t+1} 1[q_{t+1} \in S_2] \omega_{h2t}. \tag{21}$$

Notice that we do not have a riskless bond here. Clearly, since type 1 believes there is no support of q on S_2, if $p_2 > 0$ she will set $\omega_{h2t} < 0$ believing that she has nothing to repay when old. This way she can increase consumption while young to any desired level. This kind of move creates infinite supply of PCC #2 for type 2 and vice versa. The requirement that budget sets be bounded as well as existence of equilibrium will force agreement on supports of the random variable q_{t+1} for this case where utility is increasing in consumption while young.

Second, consider the same case but where there is zero utility for consumption while young. In this case type one might start to indulge in the same operation to increase ω_{h1t} to infinity by decreasing ω_{h2t} to negative infinity. However if $v(.)$ is risk averse (e.g. $v(.)$ is mean-variance) type one will not find it optimal to send ω_{h1t} to plus infinity. In this case we may locate sufficient conditions on $v(.)$ for finite demands even though the two types do not agree on the support of q_{t+1}.

Third, suppose we now add a risk free bond, but still assume there is zero utility from consumption while young. Consumption at t+1 when old is now given by

$$c_{t+1} = w_o + R_t b_{ht} + q_{t+1} 1[q_{t+1} \in S_1] \omega_{h1t} + q_{t+1} 1[q_{t+1} \in S_2] \omega_{h2t}. \tag{22}$$

Type one can now send ω_{h2t} to negative infinity at date t and purchase bonds long to receive $R_t b_{ht}$ while old. Type one believes that

$$q_{t+1} 1[q_{t+1} \in S_2] = 0, \tag{23}$$

so she believes it will cost her zero to pay back the "loan" $\omega_{h2t} < 0$ at date $t+1$. Obviously if $R_t > 0$ she can consume an infinite amount with no variance at date $t+1$ by scaling up this operation.

We have said enough at this point to convince the reader that trading of multiple securities quite easily forces agreement on supports of future price and earnings random variables. However, this alone does not automatically force agreement on the exact values of those random variables. These kind of examples are closely related to the classical conditions for boundedness of budget sets and existence of temporary equilibria in the literature reviewed by Grandmont (1982).

3 Mean variance setting with zero utility for consumption while young

In this section, we work out a special case with 3 belief types and 2 PCC's to allow direct comparison of the evolutionary dynamics with previous work without the presence of PCC's. Consider the case $u(.) = 0$ so that utility of consumption while young is zero. Let $V[c'] = E[c]' - (a/2)Var[c']$. As we said above we are now in the mean-variance setting of BH (1998). Neglecting terms that do not affect the optimal demand $\boldsymbol{\omega}$ and using the budget equation $w_y = b_{ht} + p_{0t}z_{h0t} + \sum_{i=1}^{n} p_{it}z_{hit}$ for the young the optimization problem for the young may now be written as follows,

$$E_{ht}[\sum_{i=1}^{n} A_{hit}\omega_{hit}] - (a/2)Var[\sum_{i=1}^{n} A_{hit}\omega_{hit})], \tag{24}$$

where $A_{hit} = q_{i,t+1} - R_t p_{it}$. Optimization produces the optimal demand

$$\boldsymbol{\omega_{ht}} = (1/a)V_{ht}^{-1}[\boldsymbol{m}_{ht} - R_t\boldsymbol{p_t}], \tag{25}$$

where $\boldsymbol{m}_{ht}$ is an n-vector with i'th element $E_{ht}[q_{i,t+1}]$, $\boldsymbol{p_t}$ is the n-vector of the prices p_{it} of the PCC's and V_{ht} is the covariance matrix whose elements are

$$s_{ij} = \mathrm{Cov}(q_{i,t+1}, q_{j,t+1}). \tag{26}$$

In general, this covariance matrix as well as the variance Var in (24) depends upon the belief type h. However, for analytical tractability, we will make the simplifying assumption that all belief types h use the same covariance matrix $V_{ht} = V$ of the fundamental belief types. This simplifying assumption is similar to the assumption made in BH 1998 that beliefs on the variance σ^2 of $q_{t+1} = p_{t+1} + y_{t+1}$ is the same for all belief types. Equilibrium in each PCC market requires that

$$\sum_{h=1}^{H} n_{ht}\boldsymbol{\omega_{ht}} = z_s\boldsymbol{e}. \tag{27}$$

Inserting the optimal demand (25), with $V_{ht} = V$, into the equilibrium equation (27) and multiplying by the risk aversion coefficient a yields

$$\sum_{h=1}^{H} n_{ht} V^{-1}[E_{ht}[\boldsymbol{q_{t+1}}] - R_t \boldsymbol{p_t}] = a z_s \boldsymbol{e}. \tag{28}$$

Multiplying (28) with the covariance matrix V and solving for the vector of PCC equilibrium prices with the gross rate of return of the risk free asset fixed at $R_t = R$ yields

$$R\boldsymbol{p_t} = \sum_{h=1}^{H} n_{ht} E_{ht}[\boldsymbol{q_{t+1}}] - a z_s V \boldsymbol{e}. \tag{29}$$

Once we specify how the fractions n_{ht} evolve over time, we have a well specified dynamical system for each array of PCC's. But let us first make an important observation how the dynamics of the n PCC prices in (29) are related to the dynamics of the equilibrium price of the underlying asset 0. We claim that summing up the n market equilibrium equations for PCC's in (29) exactly yields an equilibrium equation for the asset 0, given by

$$R p_{0t} = \sum_{h=1}^{H} n_{ht} E_{ht}[p_{t+1} + y_{t+1}] - a\sigma^2 z_s. \tag{30}$$

This follows immediately from the following three observations: (i) the price of the asset 0 is the sum of the prices of the n PCC's; (ii) by definition (13) it follows that $\sum_{i=1}^{n} E_{ht}[q_{i,t+1}] = E_{ht}[q_{t+1}] = E_{ht}[p_{t+1} + q_{t+1}]$; (iii) the sum of all elements of the covariance matrix V equals $\sigma^2 \equiv \sigma^2_{p_{t+1}+y_{t+1}}$. At first sight, it thus seems that summing up all equilibrium equations for the PCC's in (29) exactly leads to the BH-model (30) with one risky asset and no PCC's. But this is *not* true in general, since the fractions n_{ht} in the world with PCC's in (29) in general are *not* the same as the fractions in a world (30) without PCC's.

Let us now discuss evolution of belief types over time. BH 1997 have proposed an endogenous evolutionary updating of trading or forecasting strategies by a discrete choice model, where the fractions are given by

$$n_{ht} = \frac{exp(\beta U_{h,t-1})}{Z_{t-1}}, \qquad Z_{t-1} = \sum_{h=1}^{H} exp(\beta U_{h,t-1}), \tag{31}$$

where $U_{h,t-1}$ is the evolutionary fitness measure and Z_{t-1} is a normalization factor in order for the fractions n_{ht} to add up to 1. The crucial feature of (31) is that the higher the fitness of trading strategy h, the more traders will select strategy h. The parameter β in (31) is called the *intensity of choice*, measuring how sensitive the mass of traders is to selecting the optimal prediction strategy. Discrete choice models can be derived from a random utility model, where all agents observe the fitness measure with an error, applying a law of large numbers. The intensity of choice β is inversely related to the variance of the noise term. The extreme case $\beta = 0$ corresponds to the case of infinite variance noise, so that differences in fitness cannot be observed and all fractions (31) will be fixed over time and equal to $1/H$. The other extreme case $\beta = +\infty$ corresponds to the case without noise, so that the deterministic part of the fitness can be observed perfectly and in each period, *all* traders choose the optimal forecast. An increase in the intensity of choice β represents an increase in the degree of rationality w.r.t. evolutionary selection of trading strategies.

A natural candidate for evolutionary fitness is accumulated *realized profits*, which in the BH world without PCC's is given by

$$U_{ht} = (p_{0t} + y_t - Rp_{0,t-1}) \frac{E_{h,t-1}[p_{0t} + y_t - Rp_{0,t-1}]}{a\sigma^2} + wU_{h,t-1}, \qquad (32)$$

where $0 \leq w \leq 1$ is a *memory* parameter measuring how fast past realized fitness is discounted for strategy selection. In a world with n PCC's, realized profits for type h are given by

$$\pi_{ht} = \sum_{i=1}^{n} \omega_{hit}(q_{i,t} - Rp_{i,t-1}). \qquad (33)$$

A natural fitness measure in a heterogeneous world with n PCC's is therefore

$$U_{ht} = \pi_{ht} + wU_{h,t-1}. \qquad (34)$$

We are now ready to explore a specific example.

Example: three belief types, two PCC's

Let there be two PCC's with $S_1 = p_t + y_t < p^* + B$ and S_2 equal to the complement of S_1. We may vary B as a bifurcation parameter as well as other parameters. The covariance matrix is 2x2 with elements

$$s_{ij} = \mathrm{Cov}(q_{t+1}1[q_{t+1} \in S_i], q_{t+1}1[q_{t+1} \in S_j]). \qquad (35)$$

We assume all belief types at each date t are of the form

$$E_{ht}[q_{t+1}] = f_{ht} + E_t[y_{t+1}], \qquad f_{ht} = p^* + f_h(x_{t-1}, ..., x_{t-L}) \qquad (36)$$

where there is agreement on the cumulative distribution function of y_{t+1} but there may be disagreement on f_{ht} which is of the form

$$f_{ht} = p^* + f_h(x_{t-1}, ..., x_{t-L}). \qquad (37)$$

In order to get some understanding of the impact of introducing PCC's we study a simple case with three very simple belief types (cf. BH (1998, p. 1258, Example 4.2.1). Type 1 is fundamentalist with $f_{1t} \equiv 0$, type 2 is constant upward bias with $f_{2t} = b_2 > 0$, and type 3 is (opposite) downward bias with $f_{3t} = b_3$; we will focus on the symmetric case where $b_2 = -b_3 = b > 0$. The main questions that will be addressed here by numerical simulations are: *What is the impact on bifurcation values in the no PCC case when we add the two PCC's?*, and in particular *Do bifurcations towards instability occur "sooner" when PCC's are present or "later"?*

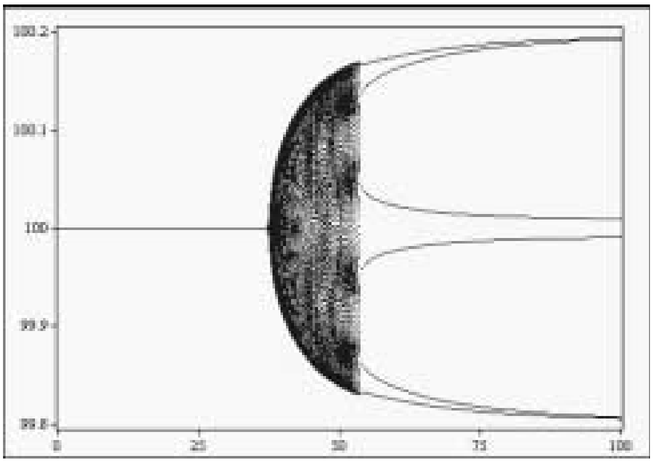

Fig. 1. BH-model with 3 types and no PCC's. Bifurcation diagram w.r.t. the intensity of choice β, $0 \leq \beta \leq 100$, with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$ and $w = 0$. A Hopf bifurcation of the fundamental steady state $p^* = 100$ occurs at $\beta = 37.5$.

Figure 1 shows a bifurcation diagram w.r.t. the intensity of choice parameter β for the three types world without PCC's. A Hopf bifurcation of the fundamental steady state $p^* = 100$ occurs at $\beta = 37.5$. For $\beta < 37.5$ the fundamental steady state is stable; for $\beta > 37.5$ the fundamental steady state is unstable and periodic and quasi-periodic asset price fluctuations arise, as illustrated in Figure 2. For $\beta > 55$ a stable 6-cycle arises. For the given belief parameters and fundamental parameters, fluctuations in asset prices p_t are relatively small. Fluctuations of fractions n_{1t} of fundamentalists are relatively small compared to fluctuations in fractions n_{2t} of optimists and fractions n_{3t} of pessimists.

Figure 3 shows the same bifurcation diagram w.r.t. the intensity of choice parameter β for the three types world with PCC's. These simulations suggests that the fundamental steady state is *unstable* for all β-values. For $0 < \beta < 25$ prices converge to a stable 2-cycle as illustrated in Figure 4; for $\beta > 25$ prices

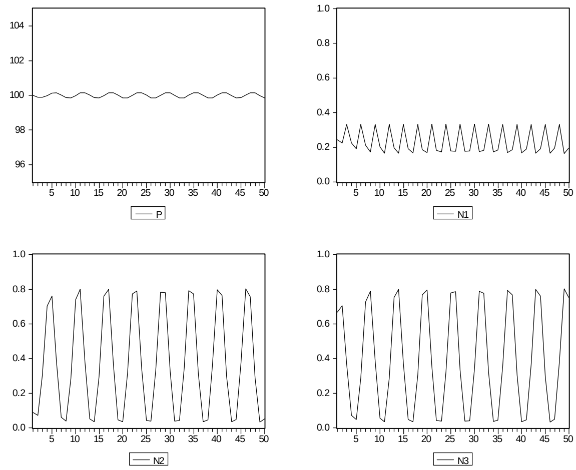

Fig. 2. BH-model with 3 types and no PCC's; (quasi-)periodic time series for $\beta = 50$, $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$ and $w = 0$. Fluctuations in asset prices p_t and fraction n_{1t} of fundamentalists are relatively small; fluctuations in fractions n_{2t} of optimists and fractions n_{3t} of pessimists are relatively large.

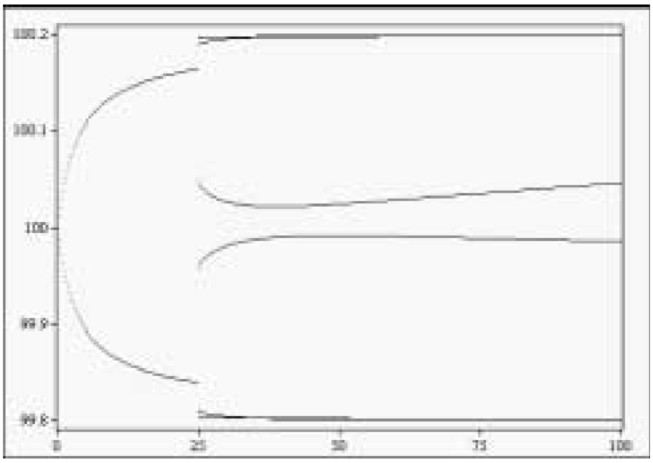

Fig. 3. BH-model with 3 types and 2 PCC's. Bifurcation diagram w.r.t. the intensity of choice β, $0 \leq \beta \leq 100$, with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$. For all positive β-values the fundamental steady state seems to be unstable.

converge to a stable 6-cycle as illustrated in Figure 5. This numerical example suggests that in this 3-type world, the introduction of PCC's is *destabilizing*. Another remarkable fact suggested by Figures 4 and 5 is that the fluctuations

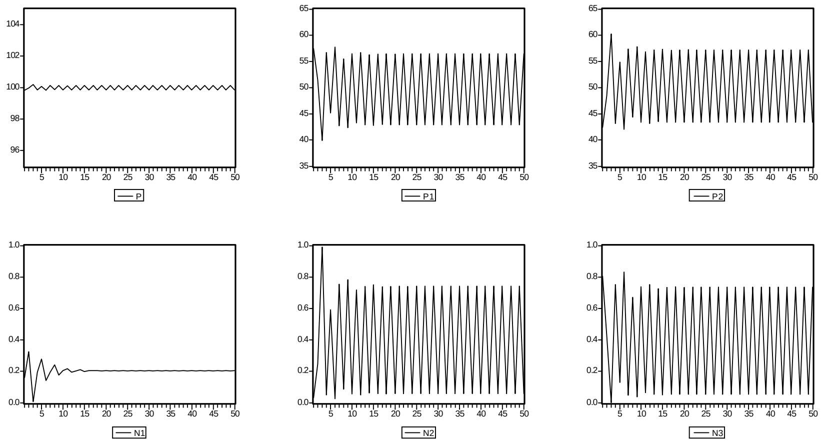

Fig. 4. BH-model with 3 types and 2 PCC's. Convergence to a 2-cycle for $\beta = 10$ with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$. Fluctuations in asset prices p_t are relatively small and of the same order of magnitude as in the case without PCC's. Fluctuations of prices p_{1t} and p_{2t} of the two PCC's are much larger than fluctuations in the asset price p_t. Fluctuations of the fraction n_{1t} of fundamentalists are relatively small; fluctuations in fractions n_{2t} of optimists and fractions n_{3t} of pessimists are relatively large.

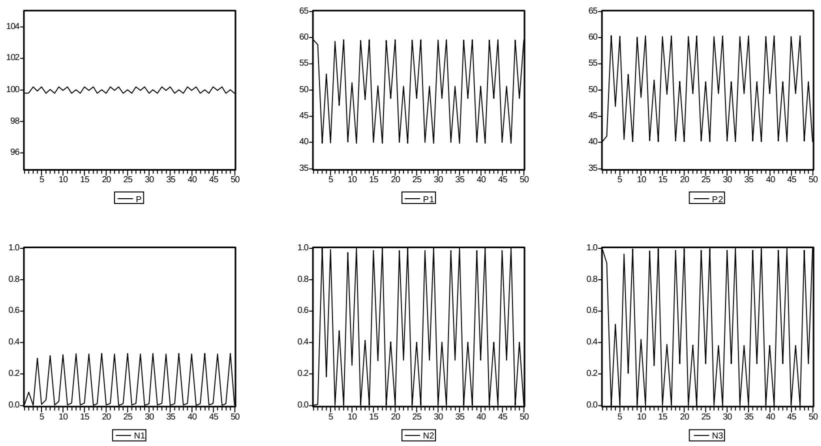

Fig. 5. BH-model with 3 types and 2 PCC's. Convergence to a 6-cycle for $\beta = 50$ with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$. Fluctuations in asset prices p_t are relatively small and of the same order of magnitude as in the case without PCC's. Fluctuations of prices p_{1t} and p_{2t} of the two PCC's are much larger than fluctuations in the asset price p_t. Fluctuations of the fraction n_{1t} of fundamentalists are relatively small; fluctuations in fractions n_{2t} of optimists and fractions n_{3t} of pessimists are relatively large.

in asset prices p_t are relatively small and of the same order of magnitude as in the case without PCC's, whereas fluctuations of prices p_{1t} and p_{2t} of the two PCC's are much larger than the fluctuations in the asset price p_t. This seems in accordance with real markets, where volatility of prices of derivatives is typically higher than volatility of the prices of the underlying asset.

Finally, we investigate the role of the memory parameter w in the evolutionary fitness measure. It is sometimes argued that more memory in the fitness measure should stabilize price fluctuations and force prices to the fundamental steady state. BH (1999) have presented a 2-type example, with costly fundamentalists versus trend followers, where this is *not* true and an increase in memory can actually destabilize price fluctuations. Figure 6 shows a bifurcation diagram w.r.t to the memory parameter w in the three type world with two PCC's. For $w = 0$, without memory, the fundamental steady state is unstable and the system has a stable 2-cycle as could be seen already for $\beta = 10$ in the bifurcation diagram of Figure 3. As the memory parameter w increases, price fluctuations become more complicated and chaotic price fluctuations arise for w close to 1. Figure 7 shows a strange attractor for $w = 0.99$ with corresponding chaotic time series shown in Figure 8.

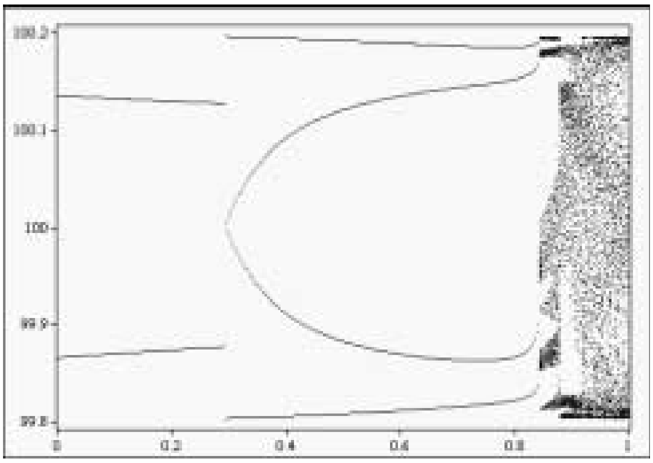

Fig. 6. BH-model with 3 types and 2 PCC's. Bifurcation diagram w.r.t. the weight factor w, $0 \leq w \leq 1$, of the evolutionary fitness measure, with the other parameters fixed at $\beta = 10$, $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$. An increase of memory leads to a 'rational route to randomness'.

Our numerical simulations thus suggest that the introduction of PCC's may *destabilize* asset price fluctuations. In future work we hope to get more analytical insight into the exact economic mechanism leading to this numerical observation. But based on our numerical simulations, the main intuition may be this. Consider a "correct bull", i.e. a type who was bullish at date $t - 1$

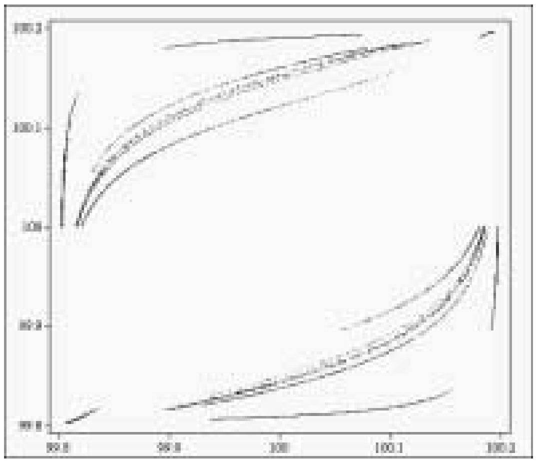

Fig. 7. BH-model with 3 types and 2 PCC's. A strange attractor for high memory $w = 0.99$, with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$.

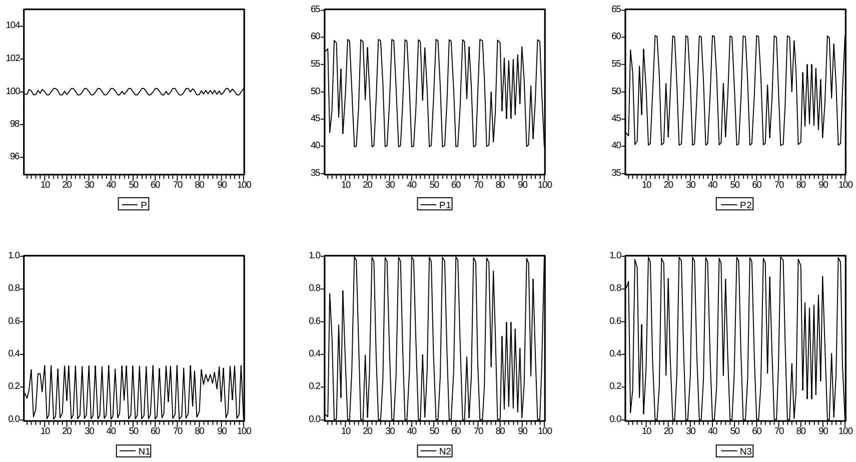

Fig. 8. BH-model with 3 types and two PCC's; chaotic time series corresponding to the strange attractor in Figure 7 for high memory parameter $w = 0.99$, with the other parameters fixed at $R = 1.01$, $\bar{y} = 1$, $z_s = 0$, $b_2 = 0.2$, $b_3 = -0.2$, $B = 0$ and $w = 0$. Fluctuations in asset prices p_t are relatively small, whereas the amplitude of fluctuations of prices p_{1t} and p_{2t} of the two PCC's is much larger. Fluctuations of the fraction n_{1t} of fundamentalists are relatively small compared to fluctuations in fractions n_{2t} of optimists and fractions n_{3t} of pessimists.

in using biased belief type $b_h = 0.2$ to forecast prices at t and turned out to be more right than the rest of the community in the sense that prices turned out to be above the fundamental. If there were no PCC's this correct bull would have been constrained to invest the same amount in PCC_1 and PCC_2 whereas the availability of PCC's allowed the correct bull to take a larger position in the upside PCC_2 based on his upside belief as well as possibly borrowing PCC_1 by going negative in it. It seems plausible that this extra freedom would tend to make profits turn out to be larger for correct bulls (and similar reasoning would apply to correct bears). When profit differences are larger, an evolutionary world where strategy selection is based upon past performance becomes more *unstable*.

If this admittedly loose speculation is right, at the minimum, the question of the contribution of the presence of PCC's and other derivatives to the stability or instability of the markets is shown to be a subtle one indeed. Of course one should investigate this same issue in the context of noisy rational expectations models where derivatives can play an additional "Hayekian/Grossmanian" role in transmitting information (Grossman (1989), de Fontnouvelle (2000)) before drawing any general conclusions. But we dare to speculate further that introduction of PCC type securities into a de Fontnouvelle dynamic noisy rational expectations framework may allow de Fontnouvelle's analog of "correct information bulls" discussed here to use PCC's to take larger positions and thus garner larger profits when correct. If indeed PCC's can be used to garner larger profits on the part of a belief type when it is on the "correct side" of the market, this will generate larger profits for that type, attracting copycats at a faster rate and, hence, possibly contribute to less stability of the markets rather than more stability of the markets.

4 General comments and discussion

This paper has sketched an approach to studying the impact of introducing additional securities into the ABS framework of BH (1998) upon the evolutionary dynamics studied by BH. This was done in an attempt to deal with one type of criticism of general evolutionary approaches with heterogeneous beliefs. Here are some other criticisms. First, the device of attaching higher probability of playing a strategy (e.g. trading on belief h at date t) based upon how well it performed in the past relative to the other available strategies runs the risk of reproducing the rather dumb type of "cobweb" behavior which was criticized by the original writers on rational expectations. Increasing β makes agents more responsive to past differences in relative performance which accentuates unstable "cobweb-like" behavior. To put it another way, the use of β as a tuning parameter that stands as a surrogate for a "dial of rationality" may be misplaced. A possible remedy might be to increase the strategy space to include condition-action beliefs of the form, "if h did well last period, believe h' this period, thinking most other traders will be believing and acting on h". One can easily add strategies of this form, attach performance indices to each of these new strategies based upon past profits that would have been garnered by such strategies, introduce a discrete choice model over this enlarged set of strategies and proceed as in BH (1998) but with a larger space. Of course there is no limit to the number of conditioning of this type that are placed before each action. But presumably the larger the number of such condition-action strategies that lie in the belief space the more plausible it might be to use β as a proxy for a "dial of rationality." One could also endogenize β, which may be a slow variable. Traders may become more sensitive to selecting the optimal prediction strategy when they are dissatisfied, that is at a low level of realized profits. This relates to the literature on bounded rationality that assumes that people economize on their cognitive activities when they are satisfied.

Second, we have seen a general tendency for a belief to do well if it puts the trader on the opposite side of the market from where the masses are and are moving towards. I.e. if most of the traders are bearish [bullish] today, a trader will do well to be on the bullish [bearish] side today, provided that the masses are not even more bearish [bullish] tomorrow. This is so because relative to the dividends that will be captured next period the price of the asset today is cheap [expensive today so it can be turned into a device to implement a cheap loan by borrowing it today, cashing it out today, investing the proceeds in some other asset, closing out the position tomorrow by paying out the relatively cheap dividends and repurchasing the asset tomorrow when, hopefully, it will be relatively cheap]. Any parameter change that magnifies a trader's move in the right direction (e.g. a decrease in risk aversion, a decrease in perceived variance) will tend to produce increased profits. This reasoning suggests that if PCC's can be used to reduce perceived variance, their presence will lead to more aggressive postures by traders. This force may cause the presence of PCC's to lead to more complex dynamics rather than less complex dynamics.

Finally, a third criticism revolves around what would happen if the time scale of the analysis was more appropriate to real world markets. I.e. might not patient traders with many time periods "smooth out" any dynamics so that in the end, in analogy with Levine and Zame (2000), much of the potential for endogenous complex dynamics vanishes?

Another major criticism of BH type theory is the lack of attention to wealth dynamics of traders who do a superior job of switching across the space of beliefs to garner profits over time. Wealthier traders should loom larger in asset price formation than "average" traders. Formidable analytical challenges must be faced because of the book keeping requirements that require tracking traders according to their switching histories. This makes the "state space" of the underlying dynamical system multiply up in dimension as time proceeds. An analytically tractable compromise is this. At each date t, one could attach a performance index W_{ht} to a belief h that gives the total wealth accumulated up to date t by trading on belief h from $s = 0$ to $s = t$. These wealth-based performance indices could be used in place of the utilities in the discrete choice systems of BH (1998). This would correspond to the performance records seen in actual mutual fund advertisements where an initial investment of size W_0 at $s = 0$ is tracked by the accumulated wealth at each date t. This record is typically displayed by a graph of W_t against t. The BH (1998) model treats investors as using a discrete choice model to choose at each date t, amongst $h = 1, 2, ..., H$ such "mutual funds" (e.g. "beliefs").

The fractions of belief types in BH (1998) possess dynamics which adds dimensions to the minimal characterization of the "state vector" of the economy at date t. This creates tractability problems if there are very many types, say four or more. However Large Type Limit theory developed by Brock, Hommes, and Wagener (2001) drastically reduces the dimension of the state space and removes many of the tractability problems. Hence, at this date two types of evolutionarily adaptive belief systems are analytically tractable: (i) Those with a very small number of types, (ii) Those with a very large number of types. Much more work within the themes *Equilibrium*, *disequilibrium* and *dynamics* needs to be done for a better understanding of adaptive evolutionary systems and their empirical and experimental relevance to economics and finance.

References

1. Brock, W.A. (1990), Overlapping generations models with money and transactions costs, In: Friedman, B., Hahn, F., (eds.), *Handbook of Monetary Economics, Volume I*, Amsterdam: North Holland, pp. 263-298.
2. Brock, W.A., (1997), Asset Price Behavior in Complex Environments, in: Arthur, W.B., Durlauf, S.N., and Lane, D.A., eds., *The Economy as an Evolving Complex System II*, Addison-Wesley, Reading, MA, 385–423.
3. Brock, W.A. and de Fontnouvelle, P., (2000), Expectational diversity in monetary economies, *Journal of Economic Dynamics and Control* 24, 725-759.
4. Brock, W.A., and Hommes, C.H., (1997) A rational route to randomness, *Econometrica* 65, 1059-1095.
5. Brock, W.A., and Hommes, C.H., (1998), Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *Journal of Economic Dynamics and Control* 22, 1235-1274.
6. Brock, W.A., Hommes, C.H. and Wagener, F.O.O. (2001), Evolutionary dynamics in financial markets with many trader types, *CeNDEF working paper* 01-01, University of Amsterdam.
7. Brock, W.A. and Kleidon, A., (1992) Periodic market closure and trading volume: a model of intraday bids and asks, *Journal of Economic Dynamics and Control* 16, 451-489.
8. Evans, G.W., and Honkapohja, S., (2001) *Learning in macroeconomics*, Princeton University Press.
9. de Fontnouvelle, P., (2000), Information dynamics in financial markets, *Macroeconomic Dynamics* 4, 139-169.

10. Gaunersdorfer, A. and Hommes, C.H., (2000), A nonlinear structural model for volatility clustering, *CeNDEF working paper* 00-02, University of Amsterdam.
11. Grandmont, J.-M. (1982), Temporary general equilibrium theory, In: Arrow, K.,J., and Intrilligator, M.D., (eds.), *Handbook of Mathematical Economics, Volume II*, Amsterdam: North Holland, Chapter 19, pp. 879-922.
12. Grossman, S.J., (1989), *The informational role of prices*, MIT Press, Cambridge.
13. Guesnerie, R. (2001), *Assessing rational expectations: sunspot multiplicity and economic fluctuations*, MIT Press: Cambridge, Mass.
14. Hommes, C.H., (2001), Financial markets as nonlinear adaptive evolutionary systems, *Quantitative Finance 1*, 149-167.
15. Kurz, M., (ed.), (1997), *Endogenous economic fluctuations: studies in the theory of rational beliefs*, Springer-Verlag: New York.
16. Levine, D. and Zame, W., (2000), Does market incompleteness matter?, Department of Economics, UCLA.
17. Magill, M. and Quinzi, M. (1996), *Theory of incomplete markets*, MIT Press: Cambridge, Mass.
18. Magill, M. and Shafer, W., 1991, Incomplete markets, In: Hildenbrand, W. and Sonnenschein, H., (eds.), *Handbook of Mathematical Economics, Volume IV*, Amsterdam: North Holland, Chapter 30.