On Learning Equilibria*

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Abstract

We investigate an inflationary overlapping generations model where households predict future inflation rates by running a least squares regression of inflation rates or prices on their past levels. We critically examine the results on learning equilibria obtained by Bullard (1994) and Schönhofer (1999) in this framework. They show that an increase in the money growth rate may lead to limit cycles and endogenous business cycles. We suggest an alternative estimation procedure, that starts from the same perceived law of motion, but is more sensible from an econometrician's point of view. We prove that for this estimation procedure there is global convergence on the monetary steady for a large set of savings functions. We also study, in an heterogeneous agents framework, evolutionary competition between the two estimation procedures, where the fraction of the population using a certain estimation procedure is determined by its past average quadratic forecast error. Interestingly, the more sensible estimation procedure is not always able to drive out the other estimation procedure, and endogenous business cycles may still be observed in this heterogeneous world.

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1 Introduction

There is a growing literature pointing out the limitations of the rational expectations hypothesis. In particular, it has been perceived as unsatisfactory that this hypothesis

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endows economic agents with precise information about the structure of the economy and the beliefs of other agents as well as unbounded reasoning abilities to deal with this information. A number of authors have suggested that the rational expectations hypothesis still is valid as a description of long run behavior, since economic agents learn over time and eventually arrive at a rational expectations steady state. The rational expectations hypothesis can therefore be supported by a learning story (see Lucas, 1976, Marcet and Sargent, 1989 and Evans and Honkapohja, 2001). In such a learning model *boundedly rational* agents are generally assumed to have no structural information about their economic environment other than time series observations on certain economic variables. They use these observations to make inferences about the economic environment. In his book on bounded rationality Sargent (1993, p.22) writes: "We can interpret the idea of bounded rationality broadly as a research program to build models populated by agents who behave like working economists or econometricians." Since the perceptions of agents influence their behavior, the learning feeds back into the actual realizations of economic variables. Hence, the learning procedure itself is one of the determinants of the evolution of the economic variables. With respect to this learning procedure Bullard (1994, p.468) states:

> *"A common research question, asked increasingly often in the recent literature, is how this learning takes place, and more importantly, if it makes any difference for inferences from dynamic general equilibrium models whether the learning is explicitly modeled."*

In his interesting paper Bullard shows, in an overlapping generations framework, that explicitly modelling agents as econometricians might create equilibrium paths different from the rational expectations steady state. Some of these *learning equilibria* can be characterized by endogenous fluctuations in inflation rates and agents beliefs. Moreover, Schönhofer (1999) shows that *chaotic* learning equilibria exist.

The objective of the present paper is twofold. First, we show that the non-convergence results of Bullard (1994) and Schönhofer (1999) depend heavily on the estimation procedure their agents use. The net effect of their procedure is that agents weight their past observations with an exponentially (over time) decreasing factor. In other words, their agents forget quickly. But then it is not surprising that the rational expectations steady state may not be learned. Moreover, the estimators generated by this procedure are not consistent and do not converge to the true parameter value as time goes to infinity. In fact, we will show that for an estimation procedure that is more sensible from a statistical point of view, the steady state of the learning model is globally stable for a large set of savings functions (including those studied by Bullard, 1994 and Schönhofer, 1999). An important observation is that the perceived law of motion is the same for both procedures, the only difference is in the way this perceived law is estimated. The main point is that it is more sound from an econometric point of view to run a regression on a stationary time series than on a nonstationary time series. Since in the presented model agents want to predict the inflation rates and since the time series of price levels is nonstationary, the estimation procedure should be in terms of inflation rates instead of price levels. The learning equilibria are therefore driven by the estimation procedure and not by the beliefs of the agents.

In this paper we identify a different source of complicated dynamics in this inflationary overlapping generations framework. We are interested whether all households will persistently use estimation procedures that do not lead to consisitent and converging estimates. Therefore, the second objective of our paper is to study an environment where both types of estimation procedures are available to the households, and where they choose one of them on the basis of their past performance (with past performance being measured in terms of forecasting accuracy). The main research question then is whether the estimation procedure suggested by Bullard (1994) and Schönhofer (1999) will still be viable when a more natural procedure is available, or whether the latter will drive out the former. We investigate this issue by employing the Brock and Hommes (1997) model of evolutionary competition between heterogeneous beliefs. Surprisingly enough, we find that both rules survive this evolutionary competition and that endogenous business cycles are still possible in this heterogenous beliefs framework, albeit for a significantly smaller set of overlapping generation economies. The intuition behind this result is that, far away from the steady state the "stable" estimation procedure will perform better (in terms of forecast errors) and eventually most households of future generations will use this rule, which will stabilize the inflation rates. However, for inflation rates close to the steady state, evolutionary pressure against the "unstable" estimation procedure will diminish and more households of future generations will start using it, which may destabilize the inflation dynamics. After this the whole story repeats.

The rest of the paper is organized as follows. Section 2 describes the overlapping generations model studied in Bullard (1994) and discusses the existence of learning equilibria. In Section 3 a learning procedure based upon inflation rates is introduced and the main stability results are given. Section 4 introduces evolutionary competition between the two different learning procedures and Section 5 concludes.

2 Learning equilibria

We consider a standard two period overlapping generations model, with only one commodity, where in each period a generation is born that lives for two periods. The generation born in period t solves

$$\max_{c_0,c_1} U(c_0,c_1) \quad \text{subject to} \quad p_t c_0 + p^e_{t+1} c_1 \leq p_t w_0 + p^e_{t+1} w_1,$$

where $U : \mathbb{R}^2_+ \to \mathbb{R}$ is a strictly monotone, strictly quasi concave utility function, c_0 and c_1 are consumption in the first and second period of the agent's life and w_0 and w_1 are his endowments of the commodity in these periods. Furthermore, p_t is the price of the commodity in period t and p^e_{t+1} is the price expected for period $t+1$.

The optimization problem gives the optimal level of consumption in the first period of the agent's life as a function of expected inflation $\frac{p^e_{t+1}}{p_t}$, i.e. $c_0 = c_0\left(\frac{p^e_{t+1}}{p_t}\right)$. Optimal saving of the young generation is then given by the savings function

$$S\left(\frac{p^e_{t+1}}{p_t}\right) = w_0 - c_0\left(\frac{p^e_{t+1}}{p_t}\right).$$

From now on we will assume that the savings function is twice differentiable and positive, i.e. $S(\xi) > 0$ for all ξ (this corresponds to the *Samuelson* case, where people save when young).[1] The demand for real balances in period t is given by

$$\frac{M_t}{p_t} = S\left(\frac{p^e_{t+1}}{p_t}\right). \tag{1}$$

The only means of saving is money. The money stock M_t is controlled by the government and grows over time to finance government expenditures. The monetary policy rule is

$$M_t = \vartheta M_{t-1}. \tag{2}$$

Combining the demand for real balances (1) with the monetary policy rule (2), we arrive at the following market clearing condition

$$S\left(\frac{p^e_{t+1}}{p_t}\right)p_t = \vartheta S\left(\frac{p^e_t}{p_{t-1}}\right)p_{t-1}.$$

In terms of gross inflation rates $\pi_t \equiv \frac{p_{t+1}}{p_t}$ this equilibrium condition becomes

$$\pi_{t-1}S\left(\pi^e_t\right) = \vartheta S\left(\pi^e_{t-1}\right). \tag{3}$$

At the monetary steady state, $\pi^* = \vartheta$, the inflation rate is equal to the money growth rate.

The model is closed by specifying the way in which agents form expectations about future inflation rates. Under rational expectations or perfect foresight we have $\pi^e_{t+1} = \pi_{t+1}$. It is well-known that for a downward sloping savings function the monetary steady state ϑ is unstable under perfect foresight. For non-monotonic savings functions more complicated perfect foresight dynamics, such as cycles and chaotic fluctuations, may occur (see e.g. Grandmont, 1985).

The assumption of perfect foresight requires that agents exactly know the market equilibrium equations as well as other agents' beliefs and are able to use this information to compute the market clearing prices for the future. An alternative approach is to assume that economic agents make inferences about their environment by means

[1] Since the savings function corresponds to an aggregate excess demand function, any continuous function corresponds to a savings function that is consistent with utility maximization, if we would extend the number of different agents per generation to at least 2 (see Sonnenschein, 1973).

of a learning procedure. Such a procedure uses time series observations to make forecasts about the future development of variables. Consider the following example of such a learning procedure.

Agents believe that the inflation rate is constant (which is indeed the case at the rational expectations steady state), that is, in terms of prices, they have the following *perceived law of motion*

$$p_t = \beta p_{t-1}. \tag{4}$$

The agents have no a priori knowledge about β, however. Bullard (1994) assumes that agents run a least squares regression on prices in order to estimate β and that they use this estimate to form predictions on the inflation rate. The least squares regression estimate for agents born in period t, using data available through time $t-1$, is

$$\beta_t = \frac{\sum_{s=1}^{t-1} p_{s-1} p_s}{\sum_{s=1}^{t-1} p_{s-1}^2}, \tag{5}$$

and hence their forecast of the inflation rate is $\pi_t^e = \beta_t$. Given this forecast, the *implied actual law of motion* for the price dynamics of the model becomes

$$p_t = \vartheta \frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)} p_{t-1}. \tag{6}$$

Equations (5) and (6) together form an *expectations feedback system*. Realized prices influence perceptions agents have about their economic environment and these perceptions feed back into the actual dynamics and determine which prices will be realized. The complete system (5)-(6) can be written as a recursive dynamic system by introducing the variable $g_t = p_{t-1}^2 \left[\sum_{s=1}^{t} p_{s-1}^2\right]^{-1}$. By furthermore defining the auxiliary variable $\gamma_t \equiv \beta_{t-1}$, we can write the learning model as a system of first-order difference equations

$$\begin{aligned} \beta_{t+1} &= \beta_t + g_t \left[\vartheta \frac{S\left(\gamma_t\right)}{S\left(\beta_t\right)} - \beta_t\right], \\ \gamma_{t+1} &= \beta_t, \\ g_{t+1} &= \left[g_t^{-1} \left(\vartheta \frac{S\left(\gamma_t\right)}{S\left(\beta_t\right)}\right)^{-2} + 1\right]^{-1}. \end{aligned} \tag{7}$$

Define the *inflation elasticity of savings* as $a\left(\pi\right) = -\pi \frac{S'(\pi)}{S(\pi)}$. The following proposition describes the main result on learning equilibria.

Proposition 1 ***(Bullard, 1994)*** *Assume $\vartheta > 1$ and $S\left(.\right)$ is twice differentiable and downward sloping. Then (7) generically undergoes a Hopf bifurcation at the monetary steady state at that value ϑ^* of ϑ, for which*

$$\left(1 - \vartheta^{-2}\right) a\left(\vartheta\right) = 1. \tag{8}$$

Moreover, if $\left(1-\vartheta^{-2}\right) a(\vartheta) < (>) 1$ *the monetary steady state is locally stable (unstable).*

It will be useful for us to investigate the details of this result a little further. The Jacobian matrix of (7) evaluated at the monetary steady state $(\beta^*, \gamma^*, g^*) = \left(\vartheta, \vartheta, 1-\vartheta^{-2}\right)$ is

$$\mathbf{J} = \begin{pmatrix} \vartheta^{-2} + \left(1-\vartheta^{-2}\right) a(\vartheta) & -\left(1-\vartheta^{-2}\right) a(\vartheta) & 0 \\ 1 & 0 & 0 \\ \frac{\partial g}{\partial \beta} & \frac{\partial g}{\partial \gamma} & \vartheta^{-2} \end{pmatrix}. \tag{9}$$

One of the eigenvalues is equal to ϑ^{-2} and hence lies inside the unit circle for $\vartheta > 1$. The other two eigenvalues are complex and lie on the unit circle when $\left(1-\vartheta^{-2}\right) a(\vartheta) = 1$. Moreover, the eigenvalues cross the unit circle with positive speed as ϑ changes. The Hopf bifurcation described in the proposition leads to an invariant closed curve around the steady state of the learning dynamics. This closed curve can be attracting or repelling, and motion on the closed curve can be periodic or quasi-periodic.[2] Bullard (1994) calls these cycles "learning equilibria" since they correspond to equilibria of the learning dynamics, which are not equilibria under rational expectations. Their existence can therefore be attributed to the learning process. If ϑ is increased further, the time series of the inflation rates can become even more complicated. Schönhofer (1999) gives, for a particular set of examples, numerical evidence for the existence of homoclinic orbits and chaos in the learning dynamics.

Let us now try to develop an intuition for the fact that the recursive least squares estimates do not converge to the monetary steady state. Ordinary least squares algorithms are so-called *decreasing gains* algorithms. Different observations receive the same weights in the regression which implies that, as time goes on and the number of observations increases, the impact or gain of individual new observations becomes smaller. In (7) this gain is represented by the variable $g_t = p_{t-1}^2 / \sum_{s=1}^{t} p_{s-1}^2$. If price levels are bounded g_t will converge to 0 which, if it does not result in convergence to the monetary steady state, at least leads to ever smaller changes in the estimate of β. In the present model, however, price levels are unbounded and in fact, at the steady state they grow at a constant rate $\vartheta > 1$. This implies that the equilibrium value of the gain g_t is strictly positive, $g^* = 1 - \vartheta^{-2} > 0$. Hence, even after many observations, one new observation on the price level may lead to a significant change in the beliefs of the agents, which therefore keep on fluctuating, implying endogenous and persisting fluctuations in the inflation rates. Notice that all observations still get the same weight in the regression, but that the gain does not approach 0 because the observations become larger themselves.

[2] If the savings function is nonmonotonic similar phenomena occur. In that case, the monetary steady state may also lose stability through a *period-doubling* bifurcation. This happens at that value ϑ^* of ϑ for which $\frac{\vartheta^2-1}{\vartheta^2+1} a(\vartheta) = -\frac{1}{2}$.

In fact, least squares learning on *price levels* is closely related to the *adaptive expectations rule* on *inflation rates*. Adaptive expectations (Nerlove, 1958) corresponds to updating the expectation in the direction of the last observation, i.e., $\pi^e_{t+1} = \pi^e_t + \alpha(\pi_{t-1} - \pi^e_t)$, with $0 < \alpha \le 1$. Notice that the weight α is constant, and adaptive expectations therefore correspond to a *constant gains* algorithm. Introducing adaptive expectations into (3) yields the following second order difference equation

$$\beta_{t+1} = \beta_t + \alpha \left(\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_t \right). \tag{10}$$

Notice that the only difference between (7) and (10) is that for the latter the weight α is constant whereas for the former it depends upon the realization of the prices. However, if the weight in (10) equals $\alpha = g^* = 1 - \vartheta^{-2}$, (10) has the same local stability properties as (7).[3] Hence, the learning scheme proposed by Bullard (1994) turns out to be closely related to adaptive expectations. Although the weight or gain g_t in (7) is not constant, it is certainly not (monotonically) decreasing over time.

The residuals or forecast errors from the regression (5) turn out to be

$$e_t = p_t - \beta_{t-1} p_{t-1} = \left(\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} \right) p_{t-1}. \tag{11}$$

With respect to these forecast errors we can make the following claim.

Lemma 2 *Consider the dynamical system (7). If the economy is not converging to the monetary steady state ϑ, when $\vartheta > 1$, and if the β_t remain bounded for all t, then the forecast errors (11) grow without bound.*

Proof. Note that the implied actual law (6) yields

$$p_{t+n} = \vartheta^n \frac{S(\beta_t)}{S(\beta_{t+n})} p_t. \tag{12}$$

Boundedness of the β_t implies that the sequence $\{\beta_t\}$ has a converging subsequence, and hence, by the continuity of S, that $S(\beta_t)/S(\beta_{t+n})$ is as close to 1 as we wish, for appropriately chosen $n = n_t$. Hence, since $\vartheta > 1$, the prices p_t grow exponentially, asymptotically with rate ϑ.

Therefore, if forecast errors are bounded, then

$$\lim_{t \to \infty} \vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} = 0,$$

[3] This follows from the fact that the upper 2×2 matrix of (9) (which is the relevant part) is equal to the Jacobian of (10) evaluated at the monetary steady state.

and, *a fortiori*, for the ergodic limit evolution,

$$\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} = 0,$$

for all t. But then the learning model (7) implies that $\beta_{t+1} = \beta_t$ for all t, and consequently, that $\beta_t = \vartheta$, contrary to the assumption that the dynamics is not at the monetary steady state. ■

This lemma provides us with another intuition for the nonconvergence of the recursive system (7): because the forecast errors grow indefinitely (in absolute value), the estimates keep changing significantly, despite the fact that the weight attached to each individual observation decreases as time goes by. Each new observation can upset the current estimate and lead to a radical change in the estimated perceived law of motion which, of course, is an unsatisfactory property of an estimation procedure. Given the exploding forecast errors, and the fluctuations in the beliefs, agents will be inclined to change their estimation procedure. Apart from that, it seems to be not too sensible to run a least squares regression on an exploding time series.

3 An alternative learning procedure

In the previous section it was argued that the nonstationary nature of the price time series may lead to endogenous and persisting fluctuations in inflation rates. According to the perceived law of motion (4) agents believe that the systematic part of the inflation rate is constant. We can rewrite (4) into the following perceived law of motion

$$\pi_t = \beta. \tag{13}$$

Notice that the economic agents' perceptions underlying both (4) and (13) is that the inflation rate is constant.

Now suppose agents try to learn the correct value of β in (13) by running a least squares regression of inflation rates on a constant, which corresponds to averaging over past inflation rates, that is,

$$\beta_{t+1} = \frac{1}{t}\sum_{s=0}^{t-1}\pi_s = \frac{1}{t}\left[\sum_{s=0}^{t-2}\pi_s + \pi_{t-1}\right] = \left(1 - \frac{1}{t}\right)\beta_t + \frac{1}{t}\pi_{t-1}.$$

The evolution of inflation rates and dynamics is then described by

$$\beta_{t+1} = \beta_t + \frac{1}{t}\left(\vartheta\frac{S\left(\beta_{t-1}\right)}{S\left(\beta_t\right)} - \beta_t\right). \tag{14}$$

This updating rule is closely related to (7) and (10), the main difference lying in the fact that the weight factor $1/t$ approaches 0 as t goes to infinity. Hence, the contribution of new observations will decrease over time and the stability properties of (14) turn out to be dramatically different from the stability properties of (7). In an earlier version of the current paper (Tuinstra and Wagener, 2000) we showed that the monetary steady state ϑ is globally stable under (14) for a large set of savings functions (including those studied in Bullard, 1994 and Schönhofer, 1999).

In the present version of this paper we consider a slightly different algorithm. Note that the total relative price increase between period 0 and period t can be written as

$$\frac{p_t}{p_0} = \frac{p_t}{p_{t-1}} \cdot \frac{p_{t-1}}{p_{t-2}} \cdot \ldots \cdot \frac{p_1}{p_0} = \prod_{s=0}^{t-1} \pi_s.$$

The average inflation rate over this period should therefore be computed as its *geometric* mean

$$\overline{\pi} = \left(\prod_{s=0}^{t-1} \pi_s \right)^{\frac{1}{t}},$$

and this is what our agents use as a predictor for the future inflation rate β_{t+1}. Notice that this is equivalent with

$$\ln \beta_{t+1} = \frac{1}{t} \sum_{s=0}^{t-1} \ln \pi_s = \ln \beta_t + \frac{1}{t} \left(\ln \pi_{t-1} - \ln \beta_t \right), \tag{15}$$

which corresponds to the *arithmetic* mean of the logarithm of the inflation rate

Now define $x_t = \ln \beta_t - \ln \vartheta$, and $\sigma(x_t) = \ln S(\vartheta \exp[x_t])$. Using (6), (15) can be written as the following nonautonomous second order difference equation

$$x_{t+1} = x_t + \frac{1}{t} \left(\sigma(x_{t-1}) - \sigma(x_t) - x_t \right). \tag{16}$$

The remainder of this section is devoted to analyzing the stability properties of (16).

Recall that a function $\sigma : I\!R \to I\!R$ is *locally Lipschitz continuous*, if it satisfies a Lipschitz condition on any bounded interval. In this section we show that the steady state $x = 0$ of the dynamic system (16) is locally stable for any locally Lipschitz continuous function σ, and that $x = 0$ is globally stable for any function satisfying a Lipschitz condition on $I\!R$.

The first property of our system is that, if a sufficiently long initial segment of the sequence $\{x_t\}$ is bounded then the whole sequence is bounded.

Proposition 3 *Let $K > 2$ be an arbitrary positive constant, and assume that $\sigma : [-K, K] \to I\!R$ is Lipschitz continuous with Lipschitz constant L. Let the sequence $\{x_t\}_{t=1}^{\infty}$ satisfy the recurrence relation (16). If for $1 \leq t \leq 10L + 1$ the condition*

$$|x_t| < \frac{3}{10} K \tag{17}$$

holds, then for all $t \geq 1$ we have $|x_t| < K$.

Proof. Introduce the increments $\delta_{t+1} = x_{t+1} - x_t$, and note that rewriting the recurrence equation (16) in terms of the increments gives

$$\delta_{t+1} = \frac{1}{t}\left(\sigma(x_t - \delta_t) - \sigma(x_t) - x_t\right). \tag{18}$$

The idea of the proof: if all x_t were, for instance, larger than zero, then using Lipschitz continuity, the increment δ_{t+1} can be bounded from above by

$$\delta_{t+1} \leq \frac{L}{t}|\delta_t|. \tag{19}$$

For large t, the factor L/t will be smaller than unity. Inequality (19) then implies that the sequence x_t is bounded from above. However, care is needed to treat the case that the x_t change sign, and to deal with the fact that (19) only gives an upper bound.

Set $T = 10L + 1$. Note that $|\delta_{t+1}| \leq |x_{t+1}| + |x_t| \leq \frac{3}{5}K$ for all $1 \leq t \leq T - 1$ by assumption. Moreover, as a consequence of (19) and the choice of T, the δ_{t+1} have the property that whenever $x_t > 0$ for $t \geq T$, then

$$\delta_{t+1} \leq \frac{1}{10}|\delta_t|.$$

We claim that this estimate implies $x_t < x_{T-1} + \frac{2}{3}K < K$ for all t.

Note generally that

$$x_t - x_{T-1} = \sum_{j=0}^{t-T} \delta_{T+j}.$$

1. First consider the case that $x_t > 0$ for all $t \geq T$. Let $T \leq s \leq t - 1$ be such that $\delta_s \leq 0$ and that $\delta_{s+j} > 0$, for $j = 1, \dots, n$. Since

$$\sum_{j=1}^{n} \delta_{s+j} \leq \sum_{j=1}^{n} 10^{-j}|\delta_s| < -\delta_s,$$

it follows that $\sum_{j=0}^{n} \delta_{s+j} < 0$. Hence, the sum $\sum_{j=0}^{t-T} \delta_{T+j}$ is bounded from above by the contribution of the first l terms, where l is such that $\delta_{T+j} \geq 0$ for $0 \leq j \leq l$ and $\delta_{T+l+1} < 0$. Since $|\delta_T| < \frac{3}{5}K$, this contribution is smaller than

$$\sum_{j=0}^{t-T} \delta_{T+j} < \frac{3}{5}K/\left(1 - \frac{1}{10}\right) = \frac{2}{3}K$$

and therefore

$$x_t = x_{T-1} + \sum_{j=0}^{t-T} \delta_{T+j} < \left(\frac{3}{10} + \frac{2}{3}\right)K < K.$$

2. In the case of a sign change of x_t, we argue by induction. It is assumed that $|x_j| < K$ for $1 \leq j \leq t$, and that $x_t < 0$ and $x_{t+j} \geq 0$ for some t and $j = 1, \ldots, n$. Then, by (18),

$$x_t < |\delta_{t+1}| \leq \frac{L}{t}|x_t - x_{t-1}| + \frac{|x_t|}{t} \leq \frac{L}{10L+1}2K + \frac{K}{10L+1} < \frac{3}{10}K.$$

Hence $x_{t+1} < \frac{3}{10}K$, and by the same argument as in 1. it follows that $\sum \delta_{t+j} < \frac{2}{3}K$ and consequently that $x_{t+j} < K$ for all $j = 1, \ldots, n$.

The proof that $x_t > -K$ for all $t \geq T$ is completely analogous. ∎

As a consequence we have

Corollary 4 *Let $\sigma : I\!R \to I\!R$ be (globally) Lipschitz continuous with Lipschitz constant L. If the sequence $\{x_t\}$ satisfies (16), then it is bounded.*

Proof. In view of the previous proposition, all we have to know is that

$$\max_{1 \leq t \leq 10L+1} |x_t| < \infty.$$

But this follows from the Lipschitz condition. ∎

In order to get some idea what types of savings functions are covered by Proposition 3 let $\xi = \vartheta \exp[x]$ and $\phi = \vartheta \exp[y]$. The condition that $\sigma(x)$ satisfies a global Lipschitz condition $|\sigma(x) - \sigma(y)| < L|x - y|$ is equivalent to

$$\left(\frac{\xi}{\phi}\right)^{-L} < \frac{S(\xi)}{S(\phi)} < \left(\frac{\xi}{\phi}\right)^{L}.$$

In particular, for $\phi = 1$ this condition reads as:

$$S(1)\xi^{-L} < S(\xi) < S(1)\xi^{L}.$$

The first inequality says that S should not decrease faster than polynomially in ξ^{-1} as $\xi \to \infty$.

That the condition is quite sharp can be seen by considering $\sigma(x) = -x - x^2$. The corresponding savings function reads as

$$S(\xi) = \left(\frac{\xi}{\vartheta}\right)^{-(\ln \xi + c)},$$

where $c = 1 - \ln \vartheta$. Note that here the exponent of ξ^{-1} grows beyond all bounds. But for this choice of σ, the unbounded series $x_t = t$ is *a* solution of (16). However, also for this savings function the trajectories under (16) remain bounded for a large set of initial conditions.

Bounded sequences have the following attractive property.

Proposition 5 *Let* $\sigma : I\!R \to I\!R$ *be continuous. Any bounded orbit* $\{x_t\}$ *of (16) converges to* 0.

Proof. Recall that 0 corresponds to the monetary steady state in the original variables. Introduce, as before, the increment $\delta_{t+1} = x_{t+1} - x_t$, and let $K > 0$ be such that $|x_t| < K$ for all $t \geq 1$. Note that it follows from (18) that

$$|\delta_t| < \frac{M}{t}$$

for some $M > 0$.

We shall show that for arbitrary $\varepsilon > 0$, any point will move to the interval $[-2\varepsilon, 2\varepsilon]$, and that the points in this interval cannot escape too far if t is sufficiently large.

Fix $\varepsilon > 0$. Note that σ is uniformly continuous on $[-K, K]$, and hence that there exists an $\delta > 0$ such that if $|\delta_t| < \delta$, then $|\sigma(x_t - \delta_t) - \sigma(x_t)| < \varepsilon$. Choose $T > 0$ larger than M/ε and M/δ. Then for all $t > T$, we have that $|\delta_t| < \varepsilon$ and, from (18), that

$$\delta_{t+1} \leq \frac{\varepsilon - x_t}{t}. \tag{20}$$

Let $t_0 > T$. We ask whether it is possible that $x_t \notin [-2\varepsilon, 2\varepsilon]$ for all $t \geq t_0$. Since $|\delta_{t+1}| < \varepsilon$, points cannot move from positive to negative without going through $[-2\varepsilon, 2\varepsilon]$, and we may restrict our attention to the case that the x_t are positive (the other case would be treated similarly).

From (20) and $x_t > 2\varepsilon$, the following estimate is obtained:

$$x_t = x_{t_0} + \sum_{s=t_0}^{t-1} \delta_{s+1} \leq x_{t_0} - \varepsilon \sum_{s=t_0}^{t-1} \frac{1}{s}.$$

Since the harmonic series $\sum s^{-1}$ diverges, the right hand side cannot be larger than 2ε for all t, and hence there must be a first moment in time t_1 such that $x_{t_1} \in [-2\varepsilon, 2\varepsilon]$. But note that then for $t > t_1$, the state x_t cannot escape the interval $[-3\varepsilon, 3\varepsilon]$ any more: if $|x_{t-1}| < 2\varepsilon < |x_t|$, then $|x_t| < |x_{t-1}| + |\delta_t| < 2\varepsilon + \varepsilon$, and the next iterates move the state back to the interval $[-2\varepsilon, 2\varepsilon]$, as can be seen by noting that from (20) we have $\delta_{t+1} < 0$ for $x_t > \varepsilon$. Similarly we have, from (18), $\delta_{t+1} > 0$ for $x_t < -\varepsilon$.

Finally, we remark that if σ is of the form $\ln S(\vartheta \exp[x])$, with S a positive savings function, then σ has the special property that

$$\lim_{x \to -\infty} \sigma(x) = \ln S(0).$$

From this we obtain that an orbit $\{x_t\}$ of (16) will be bounded from below. To see this, fix $\varepsilon > 0$, and find $K > 0$ such that $|\sigma(x) - \ln S(0)| < \varepsilon$ if $x < -K$. Then for $x_t < -K$,

$$x_{t+1} \geq x_t - \frac{x_t + 2\varepsilon}{t} \geq x_t + \frac{K - 2\varepsilon}{t} \geq x_t,$$

while for $|x_t| \leq K$ we have that

$$x_{t+1} \geq x_t - \frac{2C + K}{t} \geq -2C - 2K,$$

where $C = \max_{|x| \leq K} \sigma(x)$. Hence $x_t \geq \min(x_1, -2C - 2K)$, and these orbits cannot run off to infinity. ■

From these results we may conclude that our dynamic system converges to the monetary steady state ϑ for a large set of savings functions (obviously those included in Bullard, 1994 and Schönhofer, 1999). If the inflation dynamics do not converge to this equilibrium inflation rates diverge to infinity (which corresponds to autarky). This only happens if savings decline very fast as the (expected) inflation rate increases. Duffy (1994) studies an overlapping generations model where this autarkic steady state is also stable under learning.

4 Competition between learning procedures

As shown above, the procedure that agents use to estimate a perceived law of motion is pivotal for the stability properties of the full economic system. Moreover, the endogenous business cycles studied by Bullard (1994) and Schönhofer (1999) do not result from the perceptions of the agents per se, but from the way in which these perceptions are updated as new information becomes available. From Propositions 3 and 5 we know that these endogenous business cycles disappear when agents use inflation rates, instead of price levels, to update their perceptions. Furthermore, it can be argued that using inflation rates is more appropriate, because the price series is nonstationary and should therefore not be used in a least squares estimation procedure.

However, the fact that an estimation procedure does not seem to be sensible from an econometrician's point of view or the fact that it is destabilizing does not necessarily imply that economic agents will not use it. It is therefore important to investigate what happens when both procedures are available to the agents, a problem which we take up in this section. We assume that each newborn agent chooses an estimation procedure on the basis of its past performance. In fact, the lower the average quadratic forecast error of the predictions generated by a certain procedure, the higher the fraction of the newborn generation that will choose this procedure. Our objective is to study whether, in this *evolutionary competition* between estimation procedures, the "unstable" procedure will be driven out. A priori, this does not have to be the case, since the presence of this procedure may disrupt the inflation dynamics to such an extent that the other estimation procedure will perform even worse.[4]

[4] This would be similar to the theoretical finding that on financial markets so-called "noise traders" cannot be driven out by rational investors, see De Long, Shleifer, Summers and Waldmann (1990).

We follow the framework developed in Brock and Hommes (1997) for analyzing evolutionary competition between estimation procedures. There are two types of agents, those using prices to estimate future inflation rates (index 1), as discussed in Section 2, and those using past inflation rates (index 2), as discussed in Section 3. The fraction of agents of type 1 of the generation born in period t is denoted by n_t. Demand for real money balances follows as (cf. equation (1))

$$\frac{M_t}{p_t} = n_t S\left(\frac{p^e_{1,t+1}}{p_t}\right) + (1-n_t)\, S\left(\frac{p^e_{2,t+1}}{p_t}\right).$$

Using the money growth rule (2), the market clearing condition becomes

$$\pi_{t-1} = \vartheta \frac{n_{t-1} S\left(\beta_{1,t-1}\right) + (1-n_{t-1})\, S\left(\beta_{2,t-1}\right)}{n_t S\left(\beta_{1t}\right) + (1-n_t)\, S\left(\beta_{2t}\right)}, \tag{21}$$

where $\beta_{1t} = p^e_{1,t+1}/p_t$ and $\beta_{2t} = p^e_{2,t+1}/p_t$, are the expected inflation rates of type 1 and type 2 agents, respectively, which evolve according to (cf. equations (5) and (15))

$$\begin{aligned} \beta_{1,t+1} &= \beta_{1t} + g_t\left[\pi_{t-1} - \beta_{1t}\right], \\ g_{t+1} &= \left[g_t^{-1}\left(\pi_{t-1}\right)^{-2} + 1\right]^{-1} \text{ and} \\ \beta_{2,t+1} &= \beta_{2t}^{1-\frac{1}{t}} \pi_{t-1}^{\frac{1}{t}}. \end{aligned} \tag{22}$$

Before introducing evolutionary competition between the learning procedures let us consider the case where the fraction n_t is fixed and exogenously given.

Lemma 6 *Consider the model given by (21) and (22), where the distribution of agents over learning procedures is fixed, i.e. $n_t = \overline{n}$ for all t. Assume furthermore that the savings function is positive but monotonically decreasing in the inflation rate and that $a'(\vartheta) \neq 0$. The monetary steady state ϑ then is locally stable if and only if*

$$\left(1-\vartheta^{-2}\right)\overline{n} a(\vartheta) < 1.$$

Proof. As long as inflation rates are bounded, β_{2t} will converge to the monetary steady state ϑ (this follows, for example, from (12)). Our model can therefore be approximated by the autonomous three-dimensional dynamic system

$$\begin{aligned} \beta_{t+1} &= \beta_t + g_t\left[\vartheta\frac{\overline{n}S(\gamma_t) + (1-\overline{n})\, S(\vartheta)}{\overline{n}S(\beta_t) + (1-\overline{n})\, S(\vartheta)} - \beta_t\right], \\ \gamma_{t+1} &= \beta_t \\ g_{t+1} &= \left[g_t^{-1}\left(\vartheta\frac{\overline{n}S(\gamma_t) + (1-\overline{n})\, S(\vartheta)}{\overline{n}S(\beta_t) + (1-\overline{n})\, S(\vartheta)}\right)^{-2} + 1\right]^{-1}. \end{aligned}$$

It is easily checked that the eigenvalues of the associated Jacobian matrix are complex and cross the unit circle when $\left(1-\vartheta^{-2}\right)\overline{n}a\left(\vartheta\right)=1$. ■

Clearly, a decrease in the fraction $\overline{n}$ of type 1 agents generates local stability of the monetary steady state for a larger set of savings functions and money growth rates ϑ.

Now we will endogenize n_t by assuming that the households from a newborn generation base their choice for the estimation procedure upon the past performance of these procedures. The appropriate performance measure is the *average quadratic forecast error*, since this is what agents are actually trying to minimize with their least squares algorithms.[5] The (negative of the) average quadratic forecast error for learning procedure i is given as

$$w_{i,t+1}=-\frac{1}{t-1}\left(\beta_{i,t-1}-\pi_{t-1}\right)^2+\frac{t-2}{t-1}w_{it},\quad i=1,2. \tag{23}$$

The relationship between $w_{1,t+1}$, $w_{2,t+1}$ and the fraction n_{t+1} is modelled as follows. We assume that each household i of the newborn generation can be identified by a parameter ε_i, which is distributed according to a distribution F, with mean 0. This parameter ε_i measures household i's preference for (or bias towards) estimation procedure 2. This household then chooses rule 1 if and only if the performance of rule 1 is significantly higher than that of rule 2, that is, if

$$w_{1,t+1}-w_{2,t+1}\geq\frac{1}{\eta}\varepsilon_i,$$

where η is a measure of the dispersion of the bias, in the sense that as η increases the bias toward rule 2 is distributed more tightly around 0. Moreover, as η approaches $+\infty$, all households always choose the estimation procedure with the lowest average quadratic forecast error. Assuming that there are many households in each generation and using a law of large numbers argument the fraction of the newborn generation using rule 1 is then given by

$$n_{t+1}=\Pr\left\{w_{1,t+1}-w_{2,t+1}\geq\frac{1}{\eta}\varepsilon_i\right\}=1-F\left(\eta\left(w_{1,t+1}-w_{2,t+1}\right)\right).$$

In the rest of this paper we will take the logistic distribution for F which gives us (for a discussion see Brock and Hommes, 1997)

$$n_{t+1}=\frac{1}{1+\exp\left[\eta\left(w_{2,t+1}-w_{1,t+1}\right)\right]}, \tag{24}$$

but other choices give qualitatively the same results. The parameter η is sometimes called the *intensity of choice* and measures how sensitive agents' choice of learning

[5] Alternatively, we could let the choice of estimator depend upon the realized utility from the savings decision. This, however, has no qualitative influence upon our main results.

procedure is with respect to differences in forecasting accuracy. In fact, as η approaches infinity all households of the current generation will choose the estimation procedure with the best forecasting accuracy in the past. The parameter η therefore serves as a measure of the level of "rationality" in the choice of learning procedure. Furthermore, if learning procedures perform equally well, agents will be distributed evenly over the learning procedures, i.e. $n_t = \frac{1}{2}$. Note the timing of the process. When a new generation is born in period $t+1$ the last price it observes is p_t and the last inflation rate it observes is π_{t-1}. The fraction n_{t+1} is determined by quadratic forecast errors, and the last observed forecast error therefore corresponds to $\beta_{i,t-1} - \pi_{t-1}$.

Combining (23) and (24) gives

$$n_{t+1} = \frac{1}{1+\exp\left[\eta W_{t+1}\right]}, \tag{25}$$

where

$$\begin{aligned} W_{t+1} &= \alpha_{t-1}\left(\left(\beta_{1,t-1}^2 - \beta_{2,t-1}^2\right) + 2\pi_{t-1}\left(\beta_{2,t-1} - \beta_{1,t-1}\right)\right) + \left(1-\alpha_{t-1}\right)\frac{1}{\eta}\ln\frac{1-n_t}{n_t}, \\ \alpha_t &= \frac{1}{t}. \end{aligned}$$

The full evolutionary model is now given by equations (21), (22) and (25). The steady state of the full model is $\left(\beta_1^*, g^*, \beta_2^*, n^*\right) = \left(\vartheta, 1-\vartheta^{-2}, \vartheta, \frac{1}{2}\right)$. Note that this is a nonautonomous dynamic system. As a first approximation we consider the autonomous system with $\beta_{2t} = \vartheta$ for all t and $\alpha_t = \overline{\alpha} > 0$ for all t, where $\overline{\alpha}$ is taken to be sufficiently close to 0. A local stability analysis of the monetary steady state reveals the following.

Proposition 7 *Assume the savings function S is decreasing and strictly positive. Consider the evolutionary model given by equations (21), (22) and (25) with $\beta_{2t} = \vartheta$ for all t and $\alpha_t = \overline{\alpha}$ for all t. Let the savings function be positive and monotonically decreasing in the inflation rate. The monetary steady state is locally stable if and only if*

$$\left(1-\vartheta^{-2}\right) a\left(\vartheta\right) < 2. \tag{26}$$

Proof. Let $\beta_{2t} = \vartheta$ and $\alpha_t = \overline{\alpha}$ for all t. After introducing auxiliary variables $m_t = n_{t-1}$ and $\gamma_{1t} = \beta_{1,t-1}$ we have a five-dimensional system of first order autonomous difference equations. The Jacobian matrix of this dynamic system, evaluated at the steady state $\left(\beta_1^*, \gamma_1^*, g^*, n^*, m^*\right) = \left(\vartheta, \vartheta, 1-\vartheta^{-2}, \frac{1}{2}, \frac{1}{2}\right)$, is

$$\begin{pmatrix} \vartheta^{-2} + \frac{1}{2}\left(1-\vartheta^{-2}\right)a\left(\vartheta\right) & -\frac{1}{2}\left(1-\vartheta^{-2}\right)a\left(\vartheta\right) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \vartheta^{-3}\left(1-\vartheta^{-2}\right)a\left(\vartheta\right) & -\vartheta^{-3}\left(1-\vartheta^{-2}\right)a\left(\vartheta\right) & \vartheta^{-2} & 0 & 0 \\ 0 & 0 & 0 & 1-\overline{\alpha} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This Jacobian matrix has eigenvalues

$$\begin{aligned}\mu_{1,2} &= \frac{1}{4\vartheta^2}\left(\left(\vartheta^2-1\right)a(\vartheta)+2\pm\sqrt{\left(\vartheta^2-1\right)^2 a(\vartheta)^2-\left(8\vartheta^2-4\right)\left(\vartheta^2-1\right)a(\vartheta)+4}\right)\\ \mu_3 &= \vartheta^{-2}, \mu_4 = 1-\overline{\alpha} \text{ and } \mu_5 = 0.\end{aligned}$$

It is easily checked that the first two eigenvalues are complex and cross the unit circle when

$$\frac{1}{2}\left(1-\vartheta^{-2}\right)a(\vartheta) = 1.$$

■

Observe that the local stability condition (26) is independent of the value of $\overline{\alpha}$. Also note the relation with condition (8) from Proposition 1 which characterises local stability in the case that only the type 1 estimation procedure is available. The interpretation underlying Proposition 7 is the following: if inflation rates converge to the monetary steady state, both types of forecasts will be correct and the fraction n_t will converge to $\frac{1}{2}$. However, if the monetary steady state is unstable when n is fixed at $\frac{1}{2}$ (apply Lemma 6 for $\overline{n} = \frac{1}{2}$) the autonomous dynamic system will be locally unstable at the monetary steady state ϑ. This same mechanism applies to the (nonautonomous) full dynamic system, with $\beta_{2,t+1} = \left(\prod_{s=0}^{t-1}\pi_s\right)^{\frac{1}{t}}$ and $\alpha_t = \frac{1}{t}$. Moreover, in the long run the behavior of the nonautonomous system is very similar to the behavior of the autonomous system with α small. Also note that, although instability of the steady state is still possible under evolutionary competition between the learning procedures, this instability occurs only under a subset of savings functions and money growth rates for which instability arises in the original model. The introduction of type 2 agents therefore indeed tends to stabilize the inflation and learning dynamics.

Let us consider a typical numerical simulation to illustrate the global dynamics of the model. We use an example from Bullard (1994), where the savings function is derived from the well-known CES utility function $U(c_0, c_1) = (c_0^\rho + c_1^\rho)^{1/\rho}$ with endowments $w_0 = 1$ and $w_1 = 0$ and $\frac{1}{2} < \rho < 1$. Taking $\rho = \frac{3}{4}$, the savings function and inflation elasticity follow as

$$S(\pi) = \frac{1}{1+\pi^3} \quad \text{and} \quad a(\pi) = \frac{3}{1+\pi^{-3}}.$$

From Proposition 1 we immediately find that, if all agents are of type 1 (i.e. $n_t = 1$ for all t), the monetary steady state is locally stable for $\vartheta < \frac{1}{2} + \frac{1}{2}\sqrt{3} \approx 1.366$. On the other hand, if n_t is determined by past performance, as in (25), the steady state is locally stable for $\vartheta < 2$, as can be seen from applying Proposition 7. Simulation results for the case with $\vartheta = 2\frac{1}{2}$ and $\eta = 3$ are shown in Figures 1 and 2.

The attractor in Figure 1 shows the long run behavior of the inflation rates for $\vartheta = 2\frac{1}{2}$ and $\eta = 3$. Inflation rates move quasi-periodically over an invariant closed

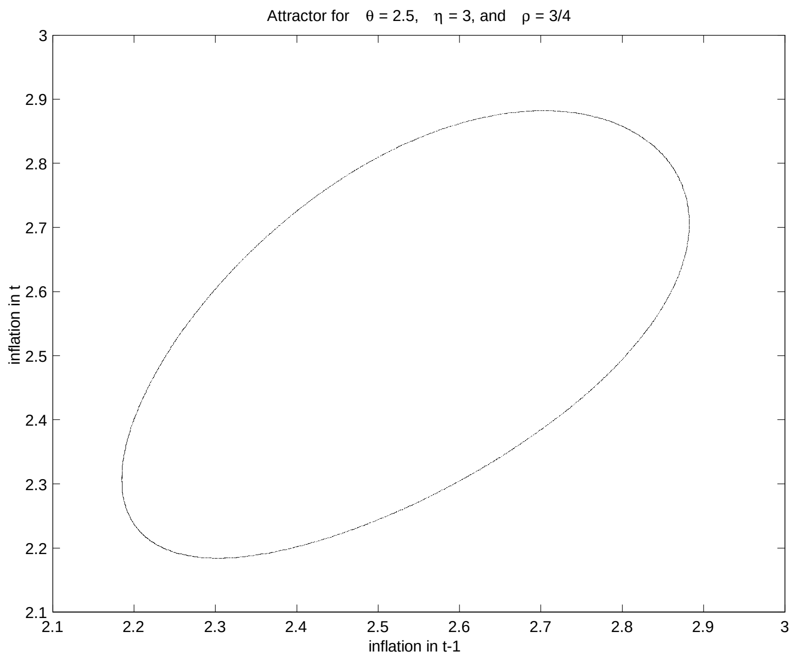

Figure 1: Attractor for the evolutionary model with $\vartheta = 2\frac{1}{2}$, $\eta = 3$ and CES utility function with $\rho = \frac{3}{4}$.

curve. Figure 2 shows the time series for the forecasts β_{1t} and β_{2t} and the fraction n_t. These time series provide a nice illustration of the mechanism underlying the global dynamics. When inflation rates are far away from the monetary steady state, as in the first 10 periods, predictor β_2 is much more accurate than β_1. This decreases n_t as can be seen in the lower panel of Figure 2. The decreased fraction of type 1 agents stabilizes the inflationary dynamics. However, as inflation rates converge to the monetary steady state, there is no evolutionary pressure against β_1 and the fraction of type 1 agents will increase again, until n_t has taken a value such that the inflation rates become unstable (it follows from Lemma 6 that for $\vartheta = 2\frac{1}{2}$ the critical value of n is given by $n^c = \frac{19}{45} \approx 0.4222$) and the story repeats. Eventually the fraction n_t converges to $\widehat{n} \approx 0.4248$. At this value of n the inflation rates *and* the forecasts of learning procedure 1 move over an invariant closed curve. Note that the fraction n_t converges because it is determined by the average forecast errors which converge as long as they are bounded.

Qualitatively similar results are obtained for simulations with different savings functions, parameter settings and\or initial conditions, although the transient behavior might be a little different.[6] Also, the behavior of the autonomous system

[6]Depending on the parameter values and the initial conditions, the transient behavior of the system might be quite long (this happens for example for high values of η).

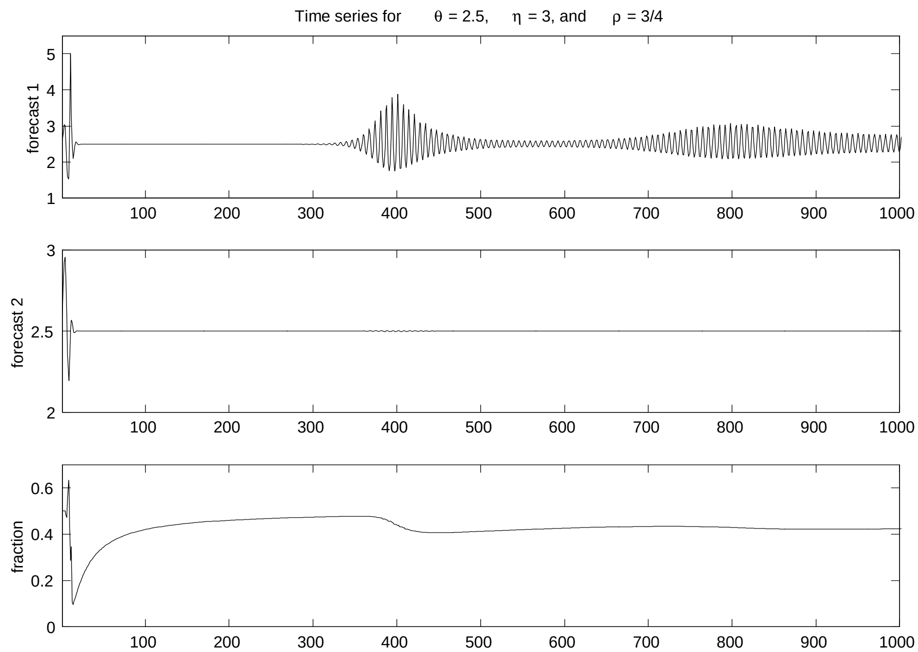

Figure 2: Timeseries for the evolutionary model with $\vartheta = 2\frac{1}{2}$, $\eta = 3$ and with CES utility function with $\rho = \frac{3}{4}$.

with $\beta_{2t} = \vartheta$ and $\alpha_t = \overline{\alpha}$ is similar to the behavior of the nonautonomous model in particular for small values of $\overline{\alpha}$. In all these examples n_t converges to a value below $\frac{1}{2}$.

5 Concluding remarks

Departing from the theory of rational expectations introduces infinitely many degrees of freedom in modelling agents' beliefs. This "wilderness of bounded rationality" can be restricted by considering agents that, if not unboundedly rational, at least are trying to be "sensible" in predicting the future development of economic variables. That is, they should have a perceived law of motion that is reasonable, in some sense, and they should use the proper econometric techniques to estimate this perceived law of motion. The learning equilibria obtained by Bullard (1994) and Schönhofer (1999), however, are partly obtained from a somewhat misguided application of econometric techniques, that is, a regression is applied on a nonstationary price time series. In this paper we have shown that a more reasonable estimation technique (estimating the perceived law of motion on the basis of the stationary time series of inflation rates) induces convergence to the monetary steady state. Recall that the perceived laws of motion and therefore agents' beliefs are the same for both models. Therefore, these learning equilibria are not the result of modelling agents as econometricians per se, but they are the result of modelling the agents as *naive* econometricians.

Another approach to discipline the "wilderness of bounded rationality" is to consider an evolutionary competition between the different types of beliefs, and then let this competition decide which estimation procedures will eventually be used. Applying such an evolutionary competition to the two learning procedures discussed in this paper shows that none of the learning rules will be driven out. This result can be explained by observing that at the steady state both learning procedures generate the same forecasts, and their performance will therefore be the same. At this steady state, therefore, there is no evolutionary pressure against the less sensible rule, but its presence may nevertheless destabilize the inflation rate dynamics and causes endogenous business cycles to emerge.

In this paper we have identified a new route to endogenous fluctuations in overlapping generations models. First we argued that in a homogeneous world, where all agents use the same estimation procedure, it does not seem to be reasonable to assume that they all use a procedure which uses nonstationary data, leads to exploding forecast errors and generates inconsistent and nonconverging estimators. Second, in a heterogeneous world, some agents might use such a procedure whereas others do not. As was established in the previous section, an evolutionary competition between the different rules might then, in a very natural way, lead to endogenous fluctuations.

The goal of this paper has not been to provide a counter-example to possible instability of learning models (Wenzelburger, 2002, for example, discusses for the same overlapping generations framework another learning process with nice stability properties). In fact, we saw in Section 4 that learning processes may lead to endogenous business cycles. However, the underlying mechanism driving these business cycles is different from that in the original model of Bullard (1994) and certainly more robust. Other examples of learning models that might lead to endogenous fluctuations are, for example, provided by Hommes and Sorger (1998) and Tuinstra (2003). In these models the perceived law of motion of the agents converges to some limit belief and given this limit belief prices keep fluctuating over some nontrivial attractor.

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