Wealth Selection in a Financial Market with Heterogeneous Agents*

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Abstract

We study the co-evolution of asset prices and agents' wealth in a financial market populated by an arbitrary number of heterogeneous, boundedly rational investors. We model assets' demand to be proportional to agents' wealth, so that wealth dynamics can be used as a selection device. For a general class of investment behaviors, we are able to characterize the long run market outcome, i.e. the steady-state equilibrium values of asset return, and agents' survival. Our investigation illustrates that market forces pose certain limits on the outcome of agents' interactions even within the "wilderness of bounded rationality". As an application we show that our analysis provides a rigorous explanation for the results of the simulation model introduced in Levy, Levy, and Solomon (1994).

JEL codes: G12, D84, C62.

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1 Introduction

Consider a financial market where a group of heterogenous investors, each following a different strategy to gain superior returns, is active. The open question is to specify which agents will survive in the long run and how agents' interaction affects market returns. This paper seeks to make a contribution in this direction.

At this purpose we investigate the co-evolution of prices and agents' wealth in a stylized market for a long-lived financial asset, populated by an *arbitrary number* of heterogenous agents. Agents decide, period after period, which fraction of their wealth to invest in the financial asset. Their investment decisions are modeled in a *general* manner, as some smooth functions of past realizations of prices and dividends. An important feature of our model is that the dynamics of price and wealths are closely intertwined. In fact, agents impact market prices proportionally to their relative wealth, and, at the same time, market price realizations determine agents' wealths. In other words, the dynamics of price and wealths acts naturally as a selection device operating on the set of different investment strategies, while, at the same time, the successful strategies determine the price and wealth dynamics.

By focusing on the asset price dynamics in a market with heterogeneous agents, our paper clearly belongs to the growing field of Heterogeneous Agent Models (HAMs), see Hommes (2006) for a recent survey. We share the standard set-up of this literature and assume heterogeneous investors to decide whether to invest in a risk-free bond or in a risky financial asset.[1] In the spirit of Brock and Hommes (1997) and Grandmont (1998) we consider stochastic dynamical system and analyze the sequence of temporary equilibria of its deterministic skeleton. As opposed to the majority of HAMs, which consider only a few types of investors and concentrate on heterogeneity in expectations, here we develop a general framework which can be applied to a quite large set of investment strategies, so that heterogeneity with respect to risk attitude, expectations, memory and optimization task can be accommodated. In fact, we characterize long run behavior of asset prices and agents' wealth distribution for a general set of competing investment strategies.

An important feature of our model concerns the demand specification. Many HAMs (see e.g. Brock and Hommes (1998); Gaunersdorfer (2000); Brock, Hommes, and Wagener (2005)) employ the setting where agents' demand is not changing with wealth, i.e. exhibits constant absolute risk aversion (CARA). In contrast, we assume that demand increases linearly with agents' wealth, which corresponds to the so-called constant *relative* risk aversion (CRRA) property. An advantage of this approach is that wealth dynamics acts as a selection mechanism and thus determines long run survival. On the contrary, in CARA models a selection mechanism has to be introduced *ad hoc* time by time. Furthermore, experimental literature seems to lean in favor of CRRA rather than CARA (see e.g. Kroll, Levy, and Rapoport (1988) and Chapter 3 in Levy, Levy, and Solomon (2000)).

The analytical exploration of the CRRA framework with heterogeneous agents is difficult, because the wealth dynamics of every agent have to be taken into account. Despite this obstacle, the recent papers of Chiarella and He (2001, 2002), Anufriev, Bottazzi, and Pancotto (2006), Anufriev and Bottazzi (2006) and Anufriev (2008) make some progress. In particular, the last studies introduce a geometric tool, so-called Equilibrium Market Curve (EMC), which can be used to carry out the equilibrium analysis. All these studies, however, are based on the assumption that the price-dividend ratio is exogenous. This seems at odd with the

[1] Recently, also some models with heterogeneous agents operating in markets with multiple assets (Chiarella, Dieci, and Gardini, 2007) and with derivatives (Brock, Hommes, and Wagener, 2006) have been developed.

standard approach, where the dividend process is exogenously set, while the asset prices are endogenously determined. In our paper we overcome this problem, and use the EMC approach to analyze a market for a financial asset whose dividend process is exogenous, so that the price-dividend ratio is a dynamic variable.[2]

The CRRA setup with exogenous dividend process allows us to link our model with agent-based simulations, and in particular with one of the first agent-based model of a financial market introduced in Levy, Levy, and Solomon (1994) (LLS model, henceforth). Their work investigates whether some stylized empirical findings in finance, such as excess volatility or long periods of overvaluation of asset, can be explained by relaxing the assumption of a fully-informed, rational representative agent. Despite some success of the LLS model in reproducing the financial "stylized facts", all its results are based on simulations. Our general setup can be applied to the specific demand schedules used in the LLS model, and, thus, provides an analytical support to its simulations.

As we are looking at agents' survival in a financial market ecology, our work can be classified within the realm of *evolutionary finance*. A seminal work of Blume and Easley (1992), as well as more recent papers of Sandroni (2000), Hens and Schenk-Hoppé (2005), Blume and Easley (2006) and Evstigneev, Hens, and Schenk-Hoppé (2006), investigate how beliefs about the dividend process affect agents' dominance in the market. There are two main differences of our work with this literature. First, whereas their work focuses on portfolio selection, we focus on relating asset returns in terms of the risk-free rate. Indeed, in their model an investor can choose among a number of different risky assets, while in our model agents can invest either in a risky or in a riskless asset. Second, although these contributions assume that each investment strategy depends on the realization of exogenous variables, i.e. dividends, our investors can also condition on past values of endogenous variables such as past prices. As a consequence in our framework prices today influence prices tomorrow through their impact on the agents' demands. This is important especially when the stability of the surviving strategy is investigated. In fact, when the investment strategy is too responsive to price movements, fluctuations are typically amplified and unstable price dynamics are produced. Indeed, we show that local stability is related to how far agents look in the past.

This paper is organized as follows. Section 2 presents the model and leads to the definition of the stochastic dynamical system where prices and wealths co-evolve. Section 3 studies the equilibria of the deterministic version of the dynamical system and introduces the tools to investigate their stability. Section 4 applies the general model to the special case where agents are mean-variance optimizers and explains the simulation results of the LLS model. The analysis of the general model is completed in Section 5, while Section 6 summarizes our main results and concludes. Proofs are collected in Appendices at the end of the paper.

2 The model

Let us consider a group of N agents trading in discrete time in a market for a long-lived financial asset. Assume that the asset is in constant supply which, without loss of generality, can be normalized to 1. Agents can lend the share of wealth which is not invested in the financial asset in return of an exogenously given constant interest rate $r_f > 0$ per period, i.e. they can buy a risk-less asset. This asset serves as numéraire with price normalized to 1

[2]Recent model of Chiarella, Dieci, and Gardini (2006) is another example of the HAM built in the CRRA framework with exogenous dividend process. In contrast to our approach, the market in Chiarella et al. is populated by only two types of agents, fundamentalists and chartists, and it is cleared by a market-maker.

in every period. The financial (risky) asset pays a dividend D_t at every time t in units of the numéraire, while its price P_t is fixed through market clearing.

Let $W_{n,t}$ stand for the wealth of agent n at time t. It is convenient to express agent's demand for the risky asset in terms of the fraction $x_{n,t}$ of wealth invested in this asset, so that agent n invests an amount $x_{n,t}W_{n,t}$ in the risky asset at time t. If the dividend is paid before trade takes place, wealth of agent n evolves as

$$W_{n,t+1} = (1 - x_{n,t})\, W_{n,t}\, (1 + r_f) + x_{n,t}\, W_{n,t}\, \frac{P_{t+1} + D_{t+1}}{P_t}\,, \tag{2.1}$$

where the price at time $t+1$ is fixed through the market clearing condition

$$\sum\nolimits_{n=1}^{N} \frac{x_{n,t+1}\, W_{n,t+1}}{P_{t+1}} = 1\,. \tag{2.2}$$

In this system prices and wealth co-evolve because the price depends on the current agent's wealth via (2.2) and, at the same time, the wealth of every agent depends on the contemporaneous price via (2.1). Before deriving the solutions of these equations, we devote our attention to the individual investment decisions $x_{n,t}$.

2.1 Investment Functions

We intend to study the evolution of the asset price and agents' wealth under an investment strategies as general as possible. Therefore, we avoid any explicit formulation of the demand and suppose that the investment shares $x_{n,t}$ are general functions of past realizations of prices and dividends. Following Anufriev and Bottazzi (2006) we formalize this intuitive concept of investment strategy as an investment function.

Assumption 1. For each agent $n = 1, \ldots, N$ there exists an *investment function* f_n which maps the information set, containing realized dividends and prices, into an investment share:

$$x_{n,t} = f_n\big(D_t, D_{t-1}, D_{t-2}, \ldots; P_{t-1}, P_{t-2}, \ldots\big)\,. \tag{2.3}$$

Agents' investment decisions evolve following individual prescriptions. Notice that since the investment choices should be made before the trade starts, the information set contains all the past dividends up to D_t, and all the past prices up to P_{t-1}.

Assumption 1 leaves a high freedom in the demand specification. The only essential restriction is that the investment share does not depend on the contemporaneous trader's wealth. This implies that the demand for the risky asset is linearly increasing with the trader's wealth. In other words, *ceteris paribus* investors maintain a constant proportion of their invested wealth as their wealth level changes. Such behavior can be referred as a constant relative risk aversion (CRRA) framework.[3]

A number of standard demand specifications are consistent with Assumption 1. In Section 4.1, as an application, we consider agents who maximize mean-variance utility of their next period expected return. Other investment behavior can also be accommodated within our framework, for instance, one can consider agents behaving in accordance with the prospect theory of Kahneman and Tversky (1979).[4]

[3]The distinction between constant relative and constant absolute risk aversion (CARA) behavior was introduced in Arrow (1965) and Pratt (1964), who also relate these concepts with utility maximization. Under CARA framework agents maintain a constant demand for the risky asset as their wealth changes.

[4]This is shown, for instance, in Chapter 9 of Levy, Levy, and Solomon (2000).

The generality of the investment functions allows modeling the agents' *forecasting* practice with a big flexibility too. The formulation (2.3) includes as special cases both technical trading, when agents' decisions are driven by the observed price fluctuations, and more fundamental attitudes, e.g. when the decisions are made on the basis of the price-dividend ratio. It also includes the case of constant investment strategy, occurring when agents assume the stationarity of the *ex-ante* return distribution.

Despite high flexibility, our setup does not include a number of important behavioral rules of agents. Since the current wealth is not included as an argument in the investment function, all the demand functions of CARA type are not covered by our framework. Also the current price is not among the arguments of (2.3). Therefore, investors deriving their investment share conditional on the (hypothetical) current price cannot be reconciled with our setup.

2.2 Co-evolution of Wealth and Prices

Given the asset price and agents' wealths at time t together with the dividend and agents' investment strategies at $t+1$, prices and wealth at $t+1$ are simultaneously determined by the evolution of wealth (2.1) and by the market clearing condition (2.2). To obtain an explicit solution for P_{t+1}, and thereafter W_{t+1}, we can use (2.1) to rewrite the market clearing equation (2.2) as

$$P_{t+1} = \sum_{n=1}^{N} x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \Big(\frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} \Big) \right) .$$

The solution of this equation with respect to P_{t+1} gives

$$P_{t+1} = \frac{\sum_{n=1}^{N} x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \frac{D_{t+1}}{P_t} \right)}{1 - \frac{1}{P_t} \sum_{n=1}^{N} W_{n,t}\, x_{n,t+1}\, x_{n,t}} . \tag{2.4}$$

Wealth evolution (2.1) then determines individual wealth $W_{n,t+1}$ for every agent n. The resulting expressions can be conveniently written in terms of the price return and dividend yield defined, respectively, as

$$k_{t+1} = \frac{P_{t+1}}{P_t} - 1 \qquad \text{and} \qquad y_{t+1} = \frac{D_{t+1}}{P_t} ,$$

as well as in terms of agents' relative wealth

$$\varphi_{n,t} = \frac{W_{n,t}}{\sum_m W_{m,t}} .$$

Dividing both sides of (2.4) by P_t and using that $P_t = \sum x_{n,t} W_{n,t}$, one derives the evolution of price returns. Together with the resulting expression for the evolution of wealth shares, it gives the following system:

$$\begin{cases} k_{t+1} = r_f + \dfrac{\sum_n \big((1 + r_f)\,(x_{n,t+1} - x_{n,t}) + y_{t+1}\, x_{n,t}\, x_{n,t+1} \big) \varphi_{n,t}}{\sum_n x_{n,t}\,(1 - x_{n,t+1})\,\varphi_{n,t}} , \\ \varphi_{n,t+1} = \varphi_{n,t}\, \dfrac{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f)\, x_{n,t}}{(1 + r_f) + (k_{t+1} + y_{t+1} - r_f) \sum_m x_{m,t} \varphi_{m,t}} , \qquad \forall n \in \{1, \ldots, N\} . \end{cases} \tag{2.5}$$

According to the first equation, the return depends on the totality of agents' investment decisions for two consequent periods. High investment fractions for the current period tend to increase the current price and, hence, the return. This effect is due to an increase of current demand. Moreover, the effect of agents' decision on the price return is proportional to their relative wealth. The second equation in (2.5) shows that the relative wealth of every agent changes according to the agent's *relative* performance, where the return of each individual wealth should be taken as performance measure.

2.3 Dividend Process and Dynamical System

The last ingredient of the model is the dividend process. The previous analytical models built in the CRRA framework, such as Anufriev, Bottazzi, and Pancotto (2006), Anufriev and Bottazzi (2006) and Anufriev (2008), assume that the dividend yield is an i.i.d. process. This assumption implies that any change in the level of price causes an immediate change in the level of dividends. In reality, however, the dividend policy of firms is hardly so fast responsive to the performance of the firm's assets, especially when prices are driven up by speculative bubbles. In this paper we consider the alternative setting, where the dividend process is completely exogenous with respect to the financial market.

Assumption 2. The dividend realization follows a geometric random walk,

$$D_t = D_{t-1}\,(1 + g_t)\,, \tag{2.6}$$

where the growth rate, g_t, is an i.i.d. random variable.

Rewriting this assumption in terms of dividend yields and price returns we get

$$y_{t+1} = y_t\,\frac{1 + g_{t+1}}{1 + k_t}\,. \tag{2.7}$$

Equations (2.5) and (2.7) together with the investment functions (2.3), rewritten as functions of price returns and dividend yields, specify the evolution of the asset-pricing model with N heterogeneous agents.

3 Equilibrium Returns and Agents' Survival

The dynamics of our model is stochastic due to the fluctuations of the dividend process. Following the typical route in the literature (cf. Brock and Hommes (1997), Grandmont (1998)), we start from the analysis of the *deterministic skeleton* of this stochastic dynamics. The skeleton is obtained by fixing the growth rate of dividends in (2.6) at the constant level $g > -1$. Our simulations, discussed later in this paper, show that the local stability analysis of the deterministic skeleton gives considerable insight for the case when the growth rate of dividends is a random variable.[5]

Before we start, let us introduce the investment decision weighted with relative wealth as

$$\left\langle x_t \right\rangle_s = \sum_{n=1}^{N} x_{n,t}\varphi_{n,s}\,, \tag{3.1}$$

[5] It is important to keep in mind that the agents do take the risk due to randomness into account, when deriving their investment functions. Given agents' behavior, we, as the modelers, set the noise level to zero and analyze the resulting deterministic dynamics.

where the time of the decision, t, and the time of the weighting wealth distribution, s, can be different. The deterministic skeleton for N agents whose investment functions depend on the lagged price returns and dividend yields[6] can be then written as

$$\begin{cases} x_{n,t+1} = f_n\big(k_t, k_{t-1}, \ldots, k_{t-L+1}; y_{t+1}, y_t, \ldots, y_{t-L+1}\big)\,, & \forall n \in \{1,\ldots,N\} \\ \varphi_{n,t+1} = \varphi_{n,t}\,\dfrac{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\,x_{n,t}}{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\,\big\langle x_t \big\rangle}\,, & \forall n \in \{1,\ldots,N\} \\ k_{t+1} = r_f + \dfrac{(1+r_f)\big\langle x_{t+1} - x_t\big\rangle_t + y_{t+1}\,\big\langle x_t\, x_{t+1}\big\rangle_t}{\big\langle x_t\,(1 - x_{t+1})\big\rangle_t}\,, & \\ y_{t+1} = y_t\,\dfrac{1+g}{1+k_t}\,. & \end{cases} \tag{3.2}$$

Given the arbitrariness of the size of population N, of the memory span L, and the absence of any specification for the investment functions, the analysis of dynamics generated by (3.2) is highly non-trivial in its general formulation. However, as we show in Section 3.1, the constraints on the dynamics set by the dividend process, the market clearing equation and the wealth evolution are sufficient to (i) uniquely characterize the steady-state equilibrium level of price returns, (ii) describe the corresponding possible distributions of wealth among agents, and (iii) restrict the possible values of steady-state equilibria dividend yields to a well specified set. Furthermore, in Section 3.2 we derive general conditions under which convergence to these equilibria is guaranteed.

The primary issue is whether, when the variables are restricted to the set of economically relevant values, the dynamics is well specified. In particular, the requirements of positive prices and positive dividends imply, respectively, that the price returns should exceed -1 and the dividend yield should be larger than zero. The following result shows that under such restrictions and when, in addition, agents are not taking short positions in both assets, the dynamics in (3.2) can always be described by a well-defined dynamical system.

Proposition 3.1. *Assume that agents do not take short positions in both the risk-less and the risky assets. In other words, given investment functions f_n as defined in Ass. 1, assume that*

$$\text{the image of } f_n \text{ belongs to } (0,1) \text{ for every } n. \tag{3.3}$$

Then the system (3.2) *defines a $2N{+}2L$-dimensional dynamical system of first-order equations. The evolution operator associated with this system*

$$\mathcal{T}(x_1,\ldots,x_N;\varphi_1,\ldots,\varphi_N;k_1,\ldots,k_L;y_1,\ldots,y_L) \tag{3.4}$$

is well-defined on the set

$$\mathcal{D} = (0,1)^N \times \Delta_N \times (-1,\infty)^L \times (0,\infty)^L\,, \tag{3.5}$$

consisting respectively of investment shares x_n, wealth shares φ_n, price returns k_l and dividend yields y_l, where $n = 1,\ldots,N$ and $l = 1,\ldots,L$, and Δ_N denotes the unit simplex in N-dimensional space

$$\Delta_N = \Big\{(\varphi_1,\ldots,\varphi_N) : \textstyle\sum_{m=1}^{N} \varphi_m = 1,\ \varphi_m \geq 0 \quad \forall m\Big\}.$$

[6] In order to deal with a finite dimensional dynamical system, we restrict the memory span of each agent to a finite L. Notice however that L can be arbitrarily large.

Proof. We prove that the dynamics from $\mathcal{D}$ to $\mathcal{D}$ is well-defined. The explicit evolution operator $\mathcal{T}$, which is used in the stability analysis, is provided in Appendix A.

Let us start with period-t variables belonging to the domain $\mathcal{D}$ and apply the dynamics described by (3.2) to them. Since $k_t > -1$, the fourth equation is well define and y_{t+1} is positive. As a result, the first equation defines the new investment shares belonging to $(0, 1)$ in accordance with the assumption in (3.3). It, in turn, implies that in the right-hand side of the third equation all the variables are defined, and the denominator is positive. Thus, k_{t+1} can be computed. Moreover, the denominator does not exceed 1, as a convex combination of numbers non-exceeding 1. Then, a simple computation gives

$$k_{t+1} > r_f + \sum\nolimits_m \big((1+r_f)(-1)+0\big)\varphi_{m,t} = -1\,.$$

Finally, it is easy to see that both the numerator and the denominator of the second equation are positive and that $\sum \varphi_{m,t+1} = 1$. Therefore, the dynamics of the wealth shares is well-defined and takes place within the unit simplex Δ_N. $\square$

In addition to proving the existence of a well-defined map[7], this proposition also shows that when the short position are forbidden, the agents' wealth shares are bounded between 0 and 1. This makes sense because only short investment positions can give rise to a negative wealth for some agents. Throughout the remaining part of this paper we impose the no-short-selling condition (3.3).

3.1 Location of Steady-State Equilibria

In a steady-state, aggregate economic variables, such as price returns and dividend yields, are constant and will be denoted by k^* and y^*, respectively.[8] Every steady-state has also constant agents' investment shares $(x^*_1, \ldots, x^*_n)$, and wealth distribution $(\varphi^*_1, \ldots, \varphi^*_n)$. Concerning the latter we introduce the following definition.

Definition 3.1. In a steady-state equilibrium $(x^*_1, \ldots, x^*_N; \varphi^*_1, \ldots, \varphi^*_N; k^*; y^*)$ an agent n is said to *survive* if his wealth share is strictly positive, $\varphi^*_n > 0$.

In every state of the economy there exists at least one survivor. In a steady-state with M surviving agents ($1 \le M \le N$) we will always assume that the first M agents survive. The characterization of all possible steady-states of the dynamical system defined on the set $\mathcal{D}$ is given below.

Proposition 3.2. *Steady-state equilibria of the dynamical system* (3.2) *evolving on the set* $\mathcal{D}$ *exist only when the dividend growth rate g is larger than the interest rate r_f.*

Let $g > r_f$ and let $(x^*_1, \ldots, x^*_N; \varphi^*_1, \ldots, \varphi^*_N; k^*; y^*)$ be a steady-state of (3.2)*. Then:*

- *The steady-state price return is equal to the growth rate of dividends, $k^* = g$;*

- *All surviving agents have the same investment share $x^*_\diamond$, which together with the steady-state dividend yield y^* satisfy*

$$x^*_\diamond = \frac{g - r_f}{y^* + g - r_f}\,. \tag{3.6}$$

[7]With some abuse of language we usually refer to (3.2) and not to the first order map $\mathcal{T}$ in (3.4) as "the dynamical system". The explicit map $\mathcal{T}$ will be used only in the proofs.

[8]Notice that in the steady-state all L lagged values of the return and yield, which were introduced to define the evolution operator $\mathcal{T}$, are equal to their contemporaneous values, i.e. to k^* and y^*, respectively.

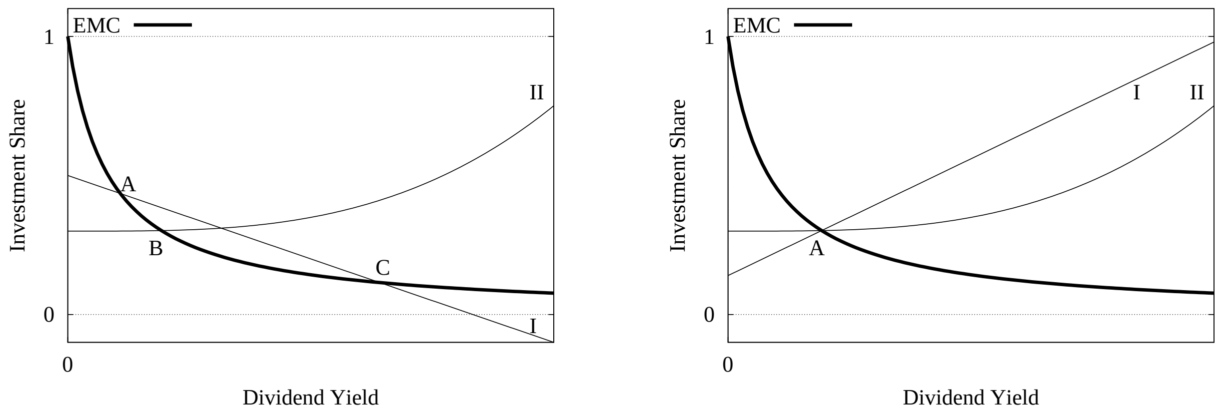

Figure 1: Location of equilibria for $g > r_f$. **Left panel:** The Equilibrium Market Curve is shown with two investment functions, I and II. In total there are three intersections with the EMC, corresponding to three different steady-states. In A and C agent I survives with $\varphi_I = 1$. In point B agent II survives with $\varphi_{II} = 1$. The abscissa of a point gives the corresponding dividend yield, while the ordinate gives the investment share of the survivor. **Right panel:** When the investment functions are intersecting at some point belonging to the EMC, there exists a set of steady-states. All of them are given by A and have the same dividend yield and agent's investment shares. However, the wealth shares are arbitrary non-negative numbers satisfying $\varphi_1 + \varphi_2 = 1$.

- *The steady-state wealth shares satisfy*

$$\begin{cases} \varphi^*_m \in (0,1] & \text{if } m \le M \\ \varphi^*_m = 0 & \text{if } m > M \end{cases} \qquad \text{and} \qquad \sum_{m=1}^{M} \varphi^*_m = 1\,. \tag{3.7}$$

Proof. A proof obtained with a simple algebra is provided in Appendix B. □

We have established that a steady-state can only exist when $g > r_f$. This result begs a question of what happens in the opposite case, when the dividend growth rate is smaller than r_f. The answer on this questions will be postponed to Section 5. For the moment we just assume that $g > r_f$. Then many situations are possible, including the cases with no steady-state, with multiple steady-states, and with different number of survivors in the same steady-state. Below we illustrate all these possibilities and discuss their economic implications.

Above all notice that in all the steady-state equilibria the price grows with the same rate as the dividend. In contrast, the steady-state level of dividend yield depends on agents' behaviors. Proposition 3.2 implies that the dividend yield, y^*, and the investment share of survivors, $x^*_\diamond$, are determined simultaneously by (3.6). To find out how, let us introduce the following definition.

Definition 3.2. The Equilibrium Market Curve (EMC) is the function

$$l(y) = \frac{g - r_f}{y + g - r_f} \qquad \text{defined for} \quad y > 0\,. \tag{3.8}$$

With this definition, since the investment share $x^*_\diamond$ is given by the value of the agent's investment function, (3.6) can be rewritten as the following system of M equations

$$l(y^*) = f_m\big(g, \ldots, g; y^*, \ldots, y^*\big) \qquad \forall\, 1 \le m \le M\,. \tag{3.9}$$

The solution y^* is the dividend yield, and $l(y^*)$ is the survivors' investment share in the steady-state. Condition (3.9) can be expressed graphically. Namely, all possible pairs $(y^*, x^*_\diamond)$ can be found as the intersections of the EMC with the *cross-section* of the investment function of each survivor by the set

$$\big\{k_t = k_{t-1} = \cdots = k_{t-L+1} = g; \quad y_{t+1} = y_t = \cdots = y_{t-L+1} = y\big\}\,. \tag{3.10}$$

We illustrate the use of the EMC in the left panel of Fig. 1 where a market with two agents is considered. Starting with arbitrary multi-dimensional investment functions, let us consider their cross-sections by the hyperplane (3.10). The resulting one-dimensional functions of the variable y are shown by the thin curves marked as I and II. Now draw the EMC (3.8) on the same diagram as a thick curve. According to (3.9), all possible steady-states of the dynamics are the *intersections* between the investment functions and the EMC. In this situation, obviously, there exist three steady-states. Point A corresponds to the case when (3.9) is satisfied for agent I. Therefore, in this steady-state he survives and, being alone, takes all available wealth, $\varphi^*_I = 1$. The equilibrium dividend yield y^* is the abscissa of point A, while the investment share of the survivor, x^*_I, is the ordinate of A. Finally, the equilibrium investment share x^*_{II} of the second agent can be found as a value of his investment function at y^*. Notice that in this case $x^*_I > x^*_{II}$, which is important for stability as will become clear later. In the other two steady-states the variables are determined in a similar way. In particular, agent I is the only survivor at C, while at B only the second agent survives.

In all the steady-states in the left panel of Fig. 1 only one agent survives. Proposition 3.2 implies that the survival of more than one agent requires existence of a common intersection between their investment functions and the EMC. Such situation is illustrated in the right panel of Fig. 1. Every steady-state must have yield y^* and average choice $x^*_\diamond$ as determined by the coordinates of the point A. However, any wealth distribution (i.e. all possible combinations of φ_I and φ_{II} satisfying to $\varphi_I + \varphi_{II} = 1$) defines some steady-state. Generally, the steady-states with multiple survivors, corresponding to the same point on the EMC, will form a manifold defined by condition (3.7). In all such steady-states the economic variables (return and yield) are the same, and the behavior of survivors must also be the same. Therefore, such equilibria may be indistinguishable when looking at the aggregate time series. However, the wealth shares of survivors can be different.

The EMC is an useful graphical tool which can be used to illustrate different long run possibilities arising under the dynamics of (3.2). It is, obviously, possible that the dynamics do not possess any steady-state (if no investment function intersects the EMC). It is also possible that the system possesses multiple steady-states with different levels of dividend yield (as in our first example), as well as multiple steady-states with the same level of yield (as in our second example). At the same time, the EMC shows that the often heard conjecture that, in the world of heterogeneous agents, "anything goes" is not necessarily valid. Even when the strategies of agents are generic, as in our framework, the market plays its role in shaping the aggregate outcome. The steady-states of system (3.2) can lie only on the EMC, which is a small subset of the original domain. The shape of the EMC is entirely determined by the exogenous parameters of the model, as g and r_f, and does not depend on agents' behavior.

To complete the discussion of the steady-states, notice from the EMC diagram that more "aggressive behavior" in the steady-state (i.e. investment of larger wealth fraction to the risky asset) will imply smaller equilibrium dividend yield. More aggressive behavior will push demand up, and lead to larger price level, which decreases the dividend yield. This negative effect, however, is exactly offset by the positive effect of taking the larger position in the risky asset. Indeed, the excess return in the steady-state according to (3.6) is given by

$$g + y^* - r_f = \frac{g - r_f}{x^*_\diamond} \,,$$

that is inverse proportional to the investment level $x^*_\diamond$. However, every survivor takes exactly $x^*_\diamond$ of this excess return. We formulate it as the following

Corollary 3.1. *At any steady-state equilibrium described in Proposition 3.2, the wealth return of each surviving agent is equal to g.*

Thus, for survivors all the points in the EMC are welfare-equivalent. We stress, however, that different equilibria generally have different survivors.

3.2 Stability of Equilibria

In order to characterize the long-run dynamics of our financial market, we now turn to derivation of the local stability conditions for the steady-states found in Proposition 3.2. For this purpose we assume that all the investment functions entering in the dynamics (3.2) are differentiable with respect to their arguments.

3.2.1 Evolutionary selection of agents

The positive excess return earned in the steady-states allows the market to play the role of a natural selecting force. In fact, it rewards some agents at the expense of others, shaping in this way the long-run wealth distribution. The first part of our stability analysis focuses on this "natural selection" which operates on agents' wealth shares. We show that some steady-states can be ruled out as unstable. The following general result holds.

Proposition 3.3. *Consider the steady-state equilibrium $\left(x^*_1, \dots, x^*_N; \varphi^*_1, \dots, \varphi^*_N; k^*; y^*\right)$ described in Proposition 3.2, where the first M agents survive and invest $x^*_\diamond$. It is (locally) stable if the following two conditions are met:*

1) *the investment shares of the non-surviving agents are such that*

$$1 - 2\,\frac{1+g}{g-r_f} < \frac{x^*_m}{x^*_\diamond} < 1 \qquad \forall m \in \{M+1, \dots, N\}\,. \tag{3.11}$$

2) *the steady-state $\left(x^*_1, \dots, x^*_M; \varphi^*_1, \dots, \varphi^*_M; k^*; y^*\right)$ of the reduced system, obtained by elimination of all the non-surviving agents from the economy, is locally stable.*

The system generically exhibits a fold *bifurcation when the rightmost inequality in* (3.11) *becomes an equality, and it exhibits a* flip *bifurcation if the leftmost inequality in* (3.11) *becomes an equality.*

Proof. To derive the stability conditions, the $(2N + 2L) \times (2N + 2L)$ Jacobian matrix of the system has to be computed and evaluated at the steady-state[9]. For the stability of the system, the eigenvalues of this Jacobian should be inside the unit circle. In Appendix C we show that condition ***1)*** is necessary and sufficient to guarantee that M eigenvalues of the Jacobian matrix lie inside the unit circle. Among the other eigenvalues there will be M zeros. Finally, all the remaining eigenvalues can be derived from the Jacobian associated with the "reduced" dynamical system, i.e. without non-surviving agents, evaluated in the steady-state. This implies condition ***2)***. □

[9]General references on the modern treatment of stability and bifurcation theory in discrete dynamical systems are Medio and Lines (2001) and Kuznetsov (2004).

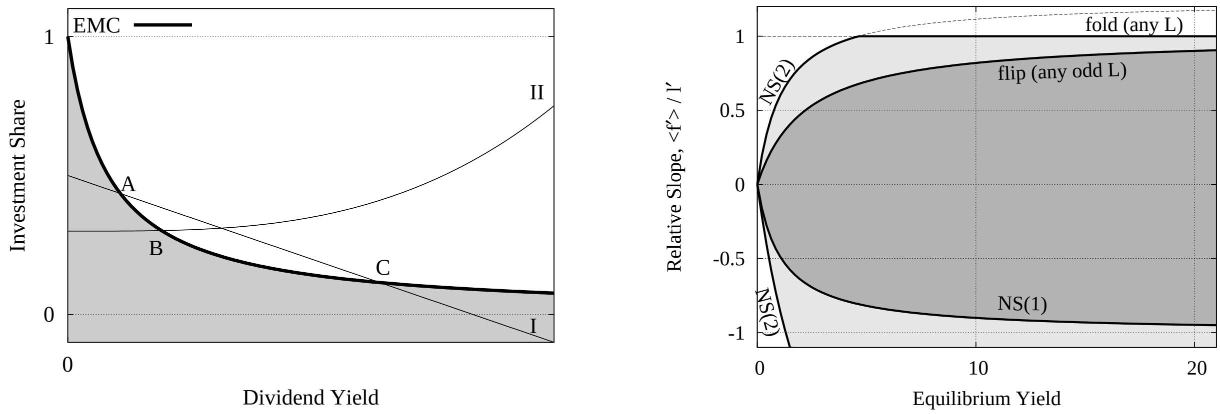

Figure 2: Stability conditions. **Left panel:** According to (3.11), the steady-state is stable (against the invasion by the non-survivors) if the non-survivors are less aggressive, i.e. their investment shares lie below the investment share of the survivors. **Right panel:** Stability of a steady-state when investment depends upon the average of past L total returns. In a stable steady-state for $L = 1$, the pair $\big(y^*, \langle f'(y^* + g)\rangle / l'(y^*)\big)$ belongs to the dark-grey area. When $L = 2$ the stability region expands and consists of the union of the dark- and light-grey areas. When $L \to \infty$ the stability region occupies all the space below "fold" line $\langle f'(y^* + g)\rangle / l'(y^*) = 1$. Crossing the border of the stability region causes the corresponding type of bifurcation, where NS stands for Neimark-Sacker.

This proposition gives an important necessary condition for stability. Namely, the investment shares of the non-surviving agents must satisfy (3.11). The leftmost inequality is always fulfilled for reasonable values of g and r_f. The rightmost inequality shows that the survivors should behave more "aggressively" in the stable steady-state, i.e. invest higher investment share than those who do not survive. This result is intuitively clear, because as long as the risky asset yields a higher average return than r_f, the most aggressive agent has also the highest total wealth return. For example, (3.11) implies the instability of equilibria B and C in the left panel of Fig. 1. We illustrate it more thoroughly in the left panel of Fig. 2. In the stable equilibrium the investment shares of the non-surviving agents should belong to the gray area, i.e. lie below the investment shares of survivors.

Proposition 3.3 can be also viewed as a result about survival of some investment strategies against an "invasion" by other investment strategies, i.e. *evolutionary stability.* At the steady-state the dynamics is not affected by the agents who are not surviving, so that the non-survivors can be discarded from the analysis. Nevertheless, if a new, more "aggressive" investment strategy is introduced with an infinitely small fraction of wealth, it destroys the old equilibrium.

3.2.2 Stability of equilibrium with survivors

According to condition **1)** in Proposition 3.3, when (3.11) is satisfied, the non-survivors can be eliminated from the market. The dynamics can then be described by the reduced system, that is, the same system (3.2) but with only M agents. When is the corresponding steady-state equilibrium stable? The general answer to this question is quite complicated, since the stability depends on the behavior of the survivors in a small neighborhood of the steady-state, i.e. on the slopes of their investment functions. More precisely, the stability is determined by the average values of partial derivatives of agents' investment functions weighted by the agents' equilibrium wealth shares.

Consider the steady-state $\left(x_1^*,\dots,x_M^*;\varphi_1^*,\dots,\varphi_M^*;k^*;y^*\right)$ and denote the vector of lagged returns and yields as $\boldsymbol{e}^* = (k^*,\dots,k^*;y^*,\dots,y^*)$. In the notation below index $m = 1,\dots,N$ and index $l = 0,\dots,L-1$. Denote the derivative of the investment function f_m with respect to the contemporaneous dividend yield as f_m^Y, the derivative with respect to the dividend yield of lag $l+1$ as f^{y_l}, and the derivative with respect to the price return of lag $l+1$ as f^{k_l}. Furthermore, consistently with our previous notation, we introduce

$$\langle f^Y \rangle = \sum_{m=1}^{M} \varphi_m^* f_m^Y(\boldsymbol{e}^*)\,, \quad \langle f^{y_l} \rangle = \sum_{m=1}^{M} \varphi_m^* f_m^{y_l}(\boldsymbol{e}^*)\,, \quad \langle f^{k_l} \rangle = \sum_{m=1}^{M} \varphi_m^* f_m^{k_l}(\boldsymbol{e}^*)\,,$$

which are the weighted derivatives of investment functions evaluated in the steady-state. Finally, we denote as $l'(y^*)$ the slope of the Equilibrium Market Curve $l(y)$ defined in (3.8) at the steady-state equilibrium y^*.

The next proposition reduces the stability problem to the exploration of the roots of a certain polynomial.

Proposition 3.4. *The steady-state $\left(x_1^*,\dots,x_M^*;\varphi_1^*,\dots,\varphi_M^*;k^*;y^*\right)$, described in Proposition 3.2, with M survivors is locally stable if all the roots of polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{1}{l'(y^*)}\Big(\langle f^Y\rangle\mu^L + \sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l} + (1-\mu)\frac{1+g}{y^*}\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big) \quad (3.12)$$

lie inside the unit circle. If, in addition, only one agent survives, then the steady-state is locally asymptotically stable.

The steady-state is unstable if at least one of the roots of polynomial $Q(\mu)$ is outside the unit circle.

When the investment functions are specified, this proposition provides a definite answer to the question about stability of a given steady-state. One has only to evaluate the polynomial (3.12) in this steady-state and compute (e.g. numerically) all its $L+1$ roots.

Even in its general formulation, Proposition 3.4 allows us to get some insight about the determinants of stability. For example, when the investment strategies of all survivors are non-responsive to a change in the yield (i.e. all the derivatives of the investment functions are 0 in the steady-state), the expression in parenthesis becomes zero and the stability condition is obviously satisfied. Using a continuity argument, this also implies that the steady-state is stable if the relative average slopes of the investment functions are small enough with respect to the slope of the EMC.

Recall that in the case of many survivors (as depicted in the right panel of Fig. 1), there exists a set of steady-states corresponding to different distributions of wealth among survivors. Since the stability conditions depend on the partial derivatives of the investment functions weighted with the equilibrium relative wealth shares, some of the steady-states on the same manifold (i.e. with the same dividend yield and investment share) can be stable, while other can be unstable.

3.2.3 An example with investment conditioned on the total return

Let us consider a special case which will also be important in the applications of Section 4. Suppose that agents' investment functions depend upon the average of past L *total* returns, given by the sum of price returns and dividend yields. For example, agents might care only about total returns and not track its two separate components. Formally, assume that individual investment shares are given by

$$x_{n,t} = f_n\left(\frac{1}{L}\sum_{\tau=1}^{L}\big(y_{t-\tau}+k_{t-\tau}\big)\right). \tag{3.13}$$

Simplifications in the polynomial (3.12) lead to

$$\tilde{Q}(\mu) = \mu^{L+1} - \frac{1+\mu+\cdots+\mu^{L-1}}{L}\Big(1+(1-\mu)\frac{1+g}{y^*}\Big)\frac{\sum_{m=1}^{M} f'_m(y^*+g)\varphi^*_m}{l'(y^*)}. \tag{3.14}$$

If all the roots of $\tilde{Q}(\mu)$, evaluated in the corresponding steady-state, lie inside the unit circle, the dynamics is locally stable. From the results of Section 3.1 it follows that the equilibrium yield is given as a solution of $l(y) = f(y+g)$. Thus, the last fraction in the polynomial (3.14) gives the relative slope of the "average" investment function of the survivors and the EMC at the steady-state.[10]

Proposition 3.3 and 3.4 give exhaustive characteristics of the stability conditions, but they are implicit. When $L = 1$ this requirement can be made explicit. Namely, the following result holds.

Corollary 3.2. *Consider a steady-state of the system* (3.2) *with investment functions* (3.13) *and lag $L = 1$, where all the non-survivors have been eliminated. The steady-state is locally stable if*

$$\frac{-y^*}{1+g+y^*} < \frac{\big\langle f'(y^*+g)\big\rangle}{l'(y^*)} < \frac{y^*}{y^*+2(1+g)}. \tag{3.15}$$

The steady-state generically exhibits flip *or* Neimark-Sacker *bifurcation if the right- or left-most inequality in* (3.15) *turns to equality, respectively.*

Proof. This follows from standard conditions for the roots of second-degree polynomial to be inside the unit circle. See appendix E for the details. □

Conditions (3.15) are illustrated in the right panel of Fig. 2 in the coordinates $(y^*, \langle f'\rangle/l')$. The steady-state is stable if the corresponding point belongs to the dark-grey area. From the diagram it is clear that the dynamics are stable for a low (in absolute value) relative slope $\langle f'\rangle/l'$ at the steady-state.

How does the stability depend on the memory span L? A mixture of analytic and numeric tools helps to reveal the behavior of the roots of polynomial (3.14) with higher L. The stability conditions for $L = 2$, derived in appendix E, can be confronted with the $L = 1$ case, see the right panel of Fig. 2. An increase of the memory span L brings stability to the system. With the next corollary we also prove that this is a general result.

Corollary 3.3. *Consider a steady-state of the system* (3.2) *with investment functions* (3.13). *Provided that*

$$\frac{\big\langle f'(y^*+g)\big\rangle}{l'(y^*)} < 1, \tag{3.16}$$

the corresponding steady-state is locally stable for high enough L.

[10] To be precise, recall that we use the one-dimensional cross-section, by the hyperplane (3.10), of a multi-dimensional investment function. It is the relative slope of this cross-section (averaged among survivors) and the EMC in the intersection that matters for the stability.

To summarize, if L is finite and low, the system can be stabilized by decreasing the average slope of survivors' investment functions with respect to the slope of the EMC. Furthermore, if inequality (3.16) holds, an increase of memory span always stabilizes the system.

4 An example with Mean-Variance Optimizers

In this section we analyze the co-evolution of price returns, dividend yields and wealth shares for a specific set of the agents' investment functions. We consider agents who are allocating wealth between the risky and the risk-less asset as to maximize their one period ahead mean-variance utility. We assume that agents' expectations of future variables are computed using averages of past observations. Agents are heterogeneous in their degree of risk aversion and in the length of memory they use to estimate future variables.

We proceed along two lines. First, we perform simulations of the system when the growth rate of dividends is stochastic and explain the simulations by the results of Section 3. Second, we apply these results to the analysis of the LLS model and resolve some puzzles put forward by Zschischang and Lux (2001) regarding the interplay between risk aversion and memory length in the simulations of Levy, Levy, and Solomon (1994) and Levy and Levy (1996).

4.1 Mean-Variance Optimizers

Throughout this section it is assumed that the growth rates of the dividends is given by

$$g_t = (1+g)\eta_t - 1\,,$$

where $\log(\eta_t)$ are i.i.d. normal random variables with mean 0. The variance of the dividend growth rate is denoted as σ_g^2. Thus Assumption 2 on the dividend process is satisfied, and in the deterministic skeleton the dividend grows with rate g. Let us further assume that $g > r_f$, so that results from Section 3 can be applied.

Agents maximize the mean-variance utility of total return

$$U = E_t[x_t(k_{t+1} + y_{t+1}) + (1 - x_t)r_f] - \frac{\gamma}{2} V_t[x_t(k_{t+1} + y_{t+1})]\,,$$

where E_t and V_t denote, respectively, the mean and the variance conditional on the information available at time t, and γ is the coefficient of risk aversion. Assuming constant expected variance $V_t = \sigma^2$, the investment fraction which maximizes U is

$$x_t = \frac{E_t[k_{t+1} + y_{t+1} - r_f]}{\gamma\sigma^2}\,. \tag{4.1}$$

Agents estimate the next period return as the average of L past realized returns. Following condition (3.3), the short positions are forbidden, and the investment shares are bounded in the interval $[0.01, 0.99]$. The investment functions are given as follows:

$$f_{\alpha,L} = \min\left\{0.99, \max\left\{0.01, \frac{1}{\alpha}\left(\frac{1}{L}\sum\nolimits_{\tau=1}^{L}(k_{t-\tau} + y_{t-\tau}) - r_f\right)\right\}\right\}\,, \tag{4.2}$$

where $\alpha = \gamma\sigma^2$ is the "normalized" risk aversion and L is the memory span. Notice that α and L can differ across agents.

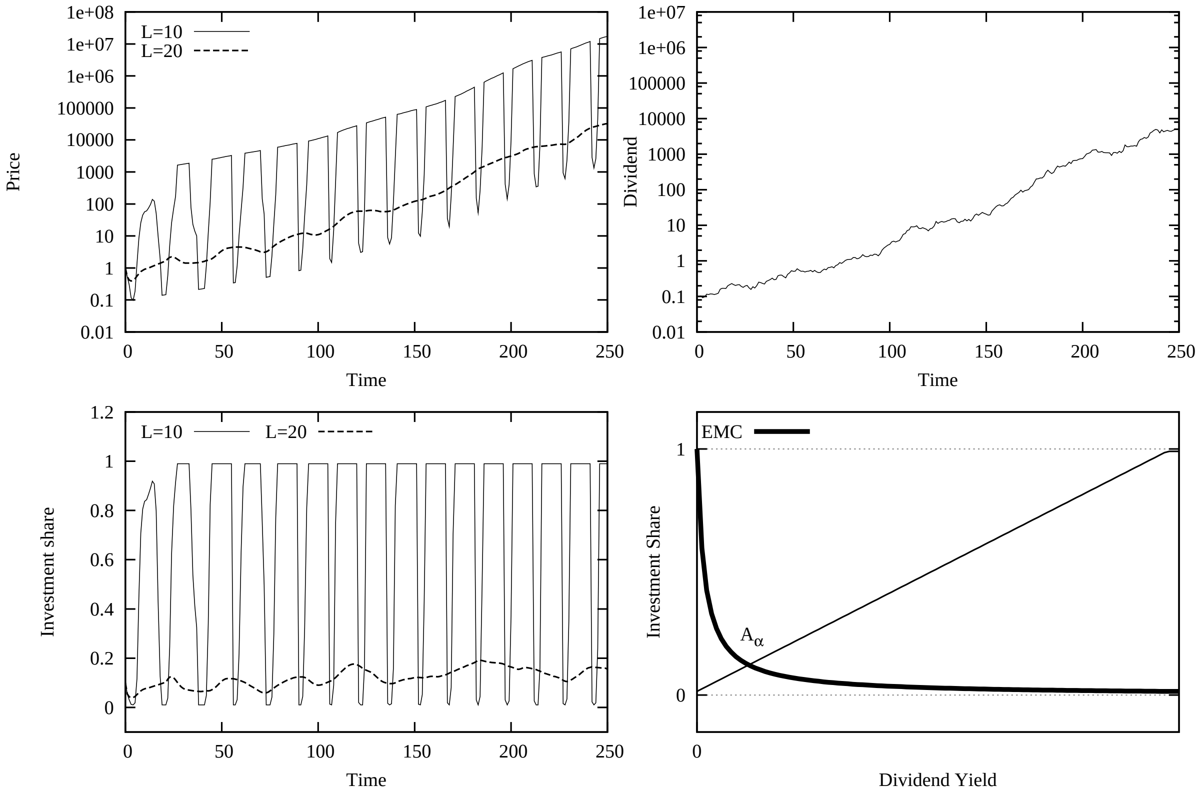

Figure 3: Dynamics with a single mean-variance maximizer in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span of the agent with lower risk aversion are compared. **Top-left panel:** log-price. **Bottom-left panel:** investment shares. **Top-right panel:** dividend process. **Bottom-right panel:** equilibrium on the EMC.

To derive all possible equilibria we turn to the deterministic skeleton, fixing $\sigma_g^2 = 0$, let the price return k^* be equal to g, and consider the cross-section of investment function (4.2) by the hyperplane (3.10). All the steady-state equilibria can be found as the intersections of the EMC with the cross-section function

$$\tilde{f}_\alpha(y) = \min\left\{0.99, \max\left\{0.01, \frac{y + g - r_f}{\alpha}\right\}\right\}, \tag{4.3}$$

which is shown by the thin curve on the bottom-right panel of Fig. 3. Notice that (4.3) does not depend on L, that is the memory span does not influence the location of the steady-states. Geometrically, all the multi-dimensional investment functions differed only in L collapse to the same one-dimensional curve. With a single investor the following result follows immediately from Proposition 3.2.

Corollary 4.1. *Consider the system* (3.2) *with* $g > r_f$ *and with a single agent investing according to* (4.2). *There exists a unique steady-state equilibrium* (x^*, k^*, y^*) *and it is characterized by* $k^* = g$ *and* $A_\alpha = (y^*, x^*)$ *with:*

$$y^* = \sqrt{\alpha\,(g - r_f)} - (g - r_f), \quad x^* = \sqrt{\frac{g - r_f}{\alpha}}. \tag{4.4}$$

The bottom-right panel of Fig. 3 illustrates this result. The market has a unique steady-state, A_α, whose abscissa, y^*, is the dividend yield, and whose ordinate, x^*, is the agent's investment share. The position of this steady-state depends on the (normalized) risk aversion coefficient α. It is immediate to see that when α increases, the line $x = (y + g - r_f)/\alpha$ rotates clockwise, so that the steady-state dividend yield increases, while the investment share decreases. Eq. (4.4) confirms it, as $\partial y^*/\partial\alpha > 0$ and $\partial x^*/\partial\alpha < 0$.

What are the determinants of stability of the steady-state equilibrium A_α? First of all, notice that we can apply the stability analysis of Section 3.2.3, because the investment function $f_{\alpha,L}$ is of the type specified in (3.13). The stability, therefore, is determined both by the memory span L and by the ratio of the slopes of the function $\tilde{f}_\alpha$ and the EMC in the point A_α. Straight-forward computations show that this ratio does not depend on α and it is always equal to -1. Corollary 3.3 then implies that for any given normalized risk aversion α the dynamics stabilizes with high enough memory span L.

To confirm that these results are applicable also for a stochastic system, we simulate the model with investment function $\tilde{f}_\alpha$ and stochastic dividend process. We plot the resulting dynamics in Fig. 3. The top-right panel shows the realization of dividend process. Given this process, simulations are performed for investment strategies with the same level of the risk aversion and two different memory spans, collapsed in the same curve as shown in the bottom-right panel. The left panels shows dynamics of prices (top) and investment shares (bottom). When the agents' memory span $L = 10$ (solid line), the steady-state is unstable and price fluctuates. These endogenous fluctuations are determined by the upper and lower bounds of the investment function and are much more pronounced than the fluctuations of the exogenous dividend process. When the memory span is increased to $L = 20$ (dotted line), the system converges to the stable steady-state equilibrium and observed fluctuations are only due to exogenous noise affecting the dividend growth rate.

We turn now to the analysis of a market with many agents, being particularly interested in assessing agents' survival in the long run. The bottom-right panel of Fig. 4 shows investment functions (4.3) for two different values of risk aversion, α and $\alpha' < \alpha$. According to Proposition 3.3, the survivor should have the highest investment share at the steady state. Since at y^*_α the agent with high risk aversion, α, invests less than the agents with low risk aversion, α', he cannot dominate the market. As a result the steady-state A_α is unstable. Whether the less risk averse agent can dominate the market depends on the stability of the second steady-state, $A_{\alpha'}$, that is on his memory span. If the memory span is high enough, the steady-state $A_{\alpha'}$ is stable and the less risk averse agent dominates the market.

Fig. 4 shows the market dynamics when one agent has risk aversion α and memory $L = 20$ (which produces a stable dynamics in a single agent case, cf. Fig. 3), and the other agent has risk aversion $\alpha' < \alpha$ and memory L'. Simulations for two different values of the memory span L' are compared. When the memory span of the less risk averse agent is low, $L' = 20$, the steady-state $A_{\alpha'}$ is unstable (see dotted lines). Wealth share of both agents keep fluctuating between zero and one. However, when the memory of the less risk averse agent increases to $L' = 30$, the new steady-state $A_{\alpha'}$ is stabilized and he ultimately dominates the market (solid lines). The steady-state return now converges, on average, to $g + y^*_{\alpha'} < g + y^*_\alpha$. Interestingly, in our framework low risk aversion leads to market dominance at the cost of lowering the market return. In fact, the agent with a lower risk aversion dominates the market, but produces lower equilibrium yield by investing a higher wealth share in the risky asset. However, as stated in Corollary 3.1, the total wealth return is g, independently from the survivor investment strategy.

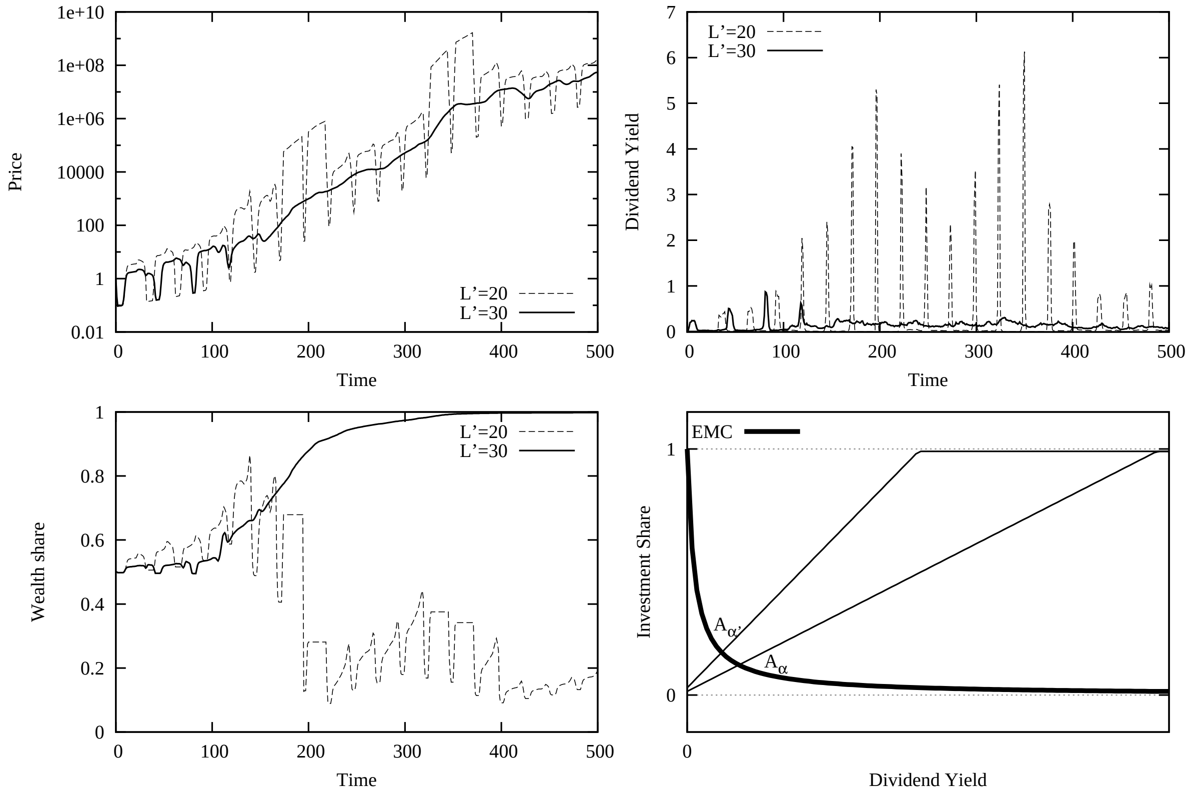

Figure 4: Dynamics with two mean-variance maximizers in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span of the agent with lower risk aversion are compared. **Top-left panel:** log-price. **Bottom-left panel:** wealth share of the agent with lower risk aversion. **Top-right panel:** dividend yield. **Bottom-right panel:** EMC and two investment functions. The agent with lower risk aversion α' produces the steady state $A_{\alpha'}$.

4.2 The LLS model revisited

The insights developed so far can be used to evaluate various simulations of the LLS model performed in Levy, Levy, and Solomon (1994); Levy and Levy (1996); Levy, Levy, and Solomon (2000); Zschischang and Lux (2001). In fact, as far as the co-evolution of prices and wealth is concerned, the LLS model is built on the same framework as considered in this paper.

In the LLS model, agents at period t maximize a power utility function $U(W_{t+1}, \gamma) = W_{t+1}^{1-\gamma}/(1-\gamma)$ with relative risk aversion $\gamma > 0$. Furthermore, to forecast the next period total return $z_{t+1} = k_{t+1} + y_{t+1}$, agents assume that any of the last L returns can occur with equal probability. Solutions of the maximization of a power utility are not available analytically but can be shown to give a wealth independent investment shares. This property holds for any perceived distribution, $g(z)$, of the next period total return, which is discrete uniform in this case. Let us denote the expected value of the total return as $\bar{z}$, and the corresponding investment function as $f^{EP}\big(\gamma, g(z)\big)$. As this investment function is unavailable in explicit form, the analysis of the LLS model relies on numeric solutions.

Since our results in Section 3 are valid for any functional form of the investment function, we are able to give an analytic support to the LLS model. In the example with mean-variance maximizers, we have seen that the risk aversion determines the capability of agents to invade the market, whereas the memory span influences the stability of the dynamics. These properties hold as long as the investment function on the "EMC plot" shifts upward with decrease of the risk aversion. As the following result shows (see Anufriev, 2008 for a proof), the function $f^{EP}\big(\gamma, g(z)\big)$ has this property.

Proposition 4.1. *Let f^{EP}_{γ} stand for the partial derivative of the investment function f^{EP} with respect to the risk aversion coefficient γ. Then the following result holds:*

$$\textit{If} \quad \bar{z} \gtreqless 0\,, \quad \textit{then} \quad f^{EP} \gtreqless 0 \quad \textit{and} \quad f^{EP}_{\gamma} \lesseqgtr 0 \quad .$$

In our setting, when a positive return is expected, agents with lower risk aversion invest higher share. Consequently, Propositions 3.3 and 3.4 provide rigorous analytic support to the simulation results of the LLS model.

In Levy and Levy (1996) the focus is on the role of the memory. The authors show that with a small memory span the log-price dynamics is characterized by crashes and booms. Our analysis shows that this result is due to the presence of an unstable steady-state and to the upper and lower bounds of the investment shares. Furthermore, this steady-state becomes stable if the memory is high enough. Simulations in Levy and Levy (1996) confirm this statement; when agents with higher memory are introduced, booms and crashes disappear and price fluctuations become erratic. In our view, these fluctuations around the steady-state of the deterministic system are simply due to the exogenous noise of the dividend process, and not due to the endogenous agents' interactions.

In Zschischang and Lux (2001) the focus is on the interplay between the length of the memory span and the risk aversion. Their simulations suggest that the risk aversion is more important than the memory span in the determination of the dominating agents, providing that the memory is not too short. The argument has not been put forward in a decisive way though, as the following quote from Zschischang and Lux (2001) (p. 568, 569) shows:

> "Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion. [...] It also appears that when adding different degrees of risk aversion, the differences of time horizons are not decisive any more, provided the time horizon is not too short."

Our analytic results make clear how and why this is the case. Agents with low risk aversion are indeed able to destabilize the market populated by agents with high risk aversion. However, this "invasion" leads to an ultimate domination only if the invading agents have sufficiently long memory. Otherwise, and this complements the conclusions of Zschischang and Lux (2001) and related works, agents with different risk aversion coefficients will coexist.

Another new result concerns the case of agents investing a constant fraction of wealth. In Zschischang and Lux (2001) the authors claim that such agents always dominate the market and add (p. 571):

> "Hence, the survival of such strategies in real-life markets remains a puzzle within the Levy, Levy and Solomon microscopic simulation framework as it does within the Efficient Market Theory."

Our analysis allows to make this statement more precise. The agents with constant investment fraction are characterized by the horizontal investment functions, for which Proposition 3.4 guarantees stability, independently of L. If these agents are able to invade the market successfully, they will ultimately dominate. However, their market invasion will fail, as soon as other agents are more aggressive in the steady-states created by invaders.

5 Non-selecting market dynamics

Our analysis so far dealt with a market where dividends are growing at a higher rate than the risk free asset. Indeed, the steady-state equilibria described in Proposition 3.2 can exist only for $g > r_f$. One may wonder what happens in the opposite case, when the dividends grow, on average, more slowly than r_f. For instance, some simulations of the LLS model were performed with positive risk-free rate and constant dividend (i.e. $g = 0$), and these simulations do converge. To understand why and where they converge in this Section we return to the setting with general investment functions and analyze the case of $g < r_f$. We show that prices are always growing at the rate r_f, no matter the initial set of investment strategies. Furthermore, the dividend yield converges to $y^* = 0$. As a result the wealth return is r_f for any investment strategy, and no selection on the set of investment strategies occurs.

The reason why we did not find any steady-state when $g < r_f$ despite a convergence of the LLS simulations is very simple. Since the domain $\mathcal{D}$ given in (3.5) is not a closed set, the dynamics can easily converge to a point which is outside of $\mathcal{D}$. One possibility is the point with zero dividend yield, another is where the price return $k^* = -1$. Both cases were not included in the definition of domain $\mathcal{D}$, because they lie outside the possibilities considered in our model. In the former case the dividend must be zero, while in the latter case the price sequence is not defined.

Formally, however, the dynamics (3.2) is well defined also on the set

$$\mathcal{D}' = (0,1)^N \times \Delta_N \times [-1,\infty)^L \times [0,\infty)^L \,. \tag{5.1}$$

Let us, therefore, extend our analysis on the dynamics defined on $\mathcal{D}'$, and in this way characterize possible behaviors of the system when it is *asymptotical* converge to a steady-state equilibrium with zero dividend yield. The next result applies.

Proposition 5.1. *Consider the dynamical system* (3.2) *evolving on the set* $\mathcal{D}'$ *introduced in* (5.1) *and assume that the no-short selling constraint* (3.3) *is satisfied. Apart from the steady-state equilibria described in Proposition 3.2, the system has other steady-states equilibria* $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$ *where:*

- *The price return is equal to the risk free rate,* $k^* = r_f$.
- *The dividend yield is zero,* $y^* = 0$.
- *The wealth shares satisfy*

$$\begin{cases} \varphi_m^* \in (0,1] & \text{if } \ m \le M \\ \varphi_m^* = 0 & \text{if } \ m > M \end{cases} \qquad \text{and} \qquad \sum_{m=1}^{N} \varphi_m^* = 1 \,. \tag{5.2}$$

Proof. See Appendix F. □

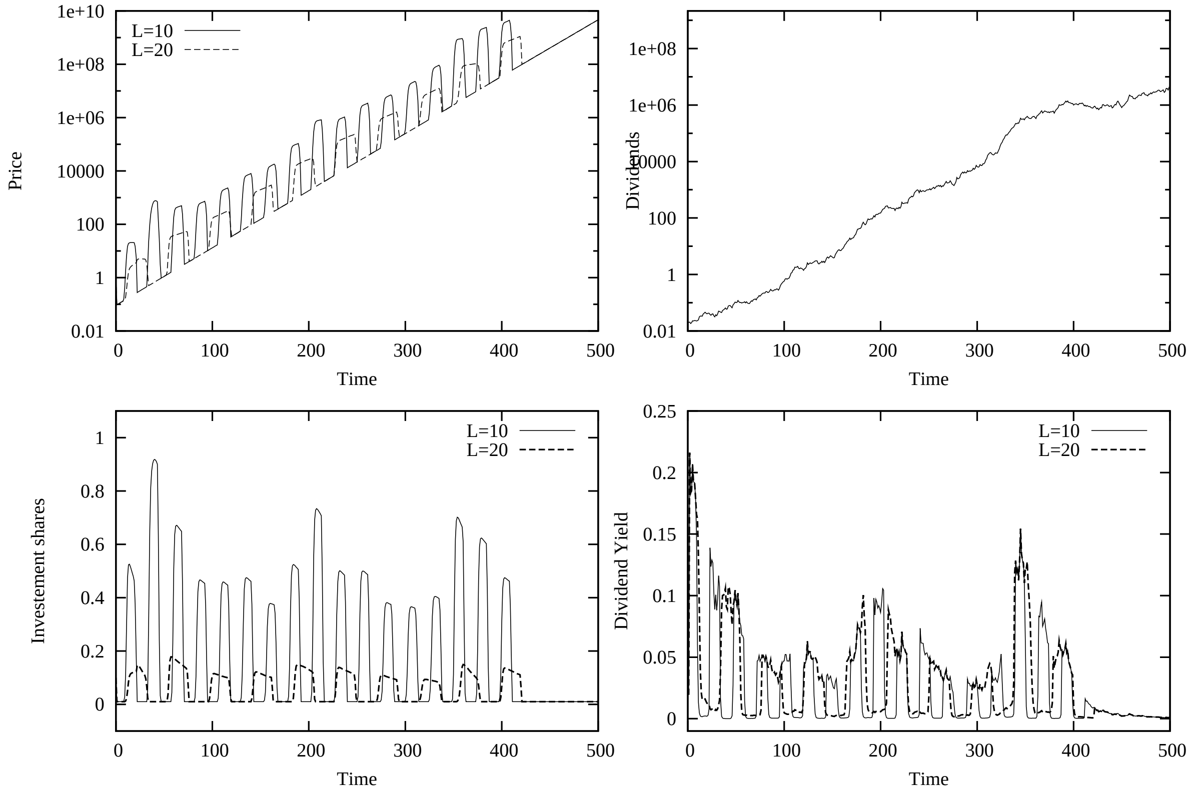

Figure 5: Dynamics with a single mean-variance maximizer in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. **Top-left panel:** log-price for two different values of memory span L. **Bottom-left panel:** investment shares. **Top-right panel:** dividend process. **Bottom-right panel:** dividend yield.

Contrary to the steady-state equilibria with positive dividend yield, in the steady-states derived in Proposition 5.1, the total return of the asset, $k^* + y^*$, coincides with r_f, the return of the risk-less asset. In these steady-states, therefore, there is no difference in the investment opportunities. The investment shares of agents are unambiguously determined through the investment functions, while the wealth shares are free of choice, so that any number of agents can survive. Survivors may behave differently, i.e. homogeneous behavior is not necessary, as opposed to the steady-states with positive yield. This is due to the fact that the total return is the same for all investment strategies.

Corollary 5.1. *At any steady-state equilibrium with $y^* = 0$ as found in Proposition 5.1, the wealth return of each agent is equal to r_f.*

The local stability of the steady-state equilibria with zero dividend yield can be analyzed along the same lines of Proposition 3.3.

Proposition 5.2. *Steady states of the dynamical system* (3.2) *evolving on the set $\mathcal{D}'$, in which the dividend yield is zero, can be stable only if the dividend growth rate g is less than the risk-free interest rate r_f.*

Let $g < r_f$ and let $\left(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_N^*; k^*; y^*\right)$ be a fixed point of (3.2) *with $y^* = 0$. This point is locally stable if all the roots of polynomial*

$$Q_0(\mu) = \mu^{L+1} + \frac{1 + r_f}{\left\langle x^*(1 - x^*)\right\rangle}(1 - \mu) \sum_{l=0}^{L-1} \left\langle f^{k_l}\right\rangle \mu^{L-1-l} \tag{5.3}$$

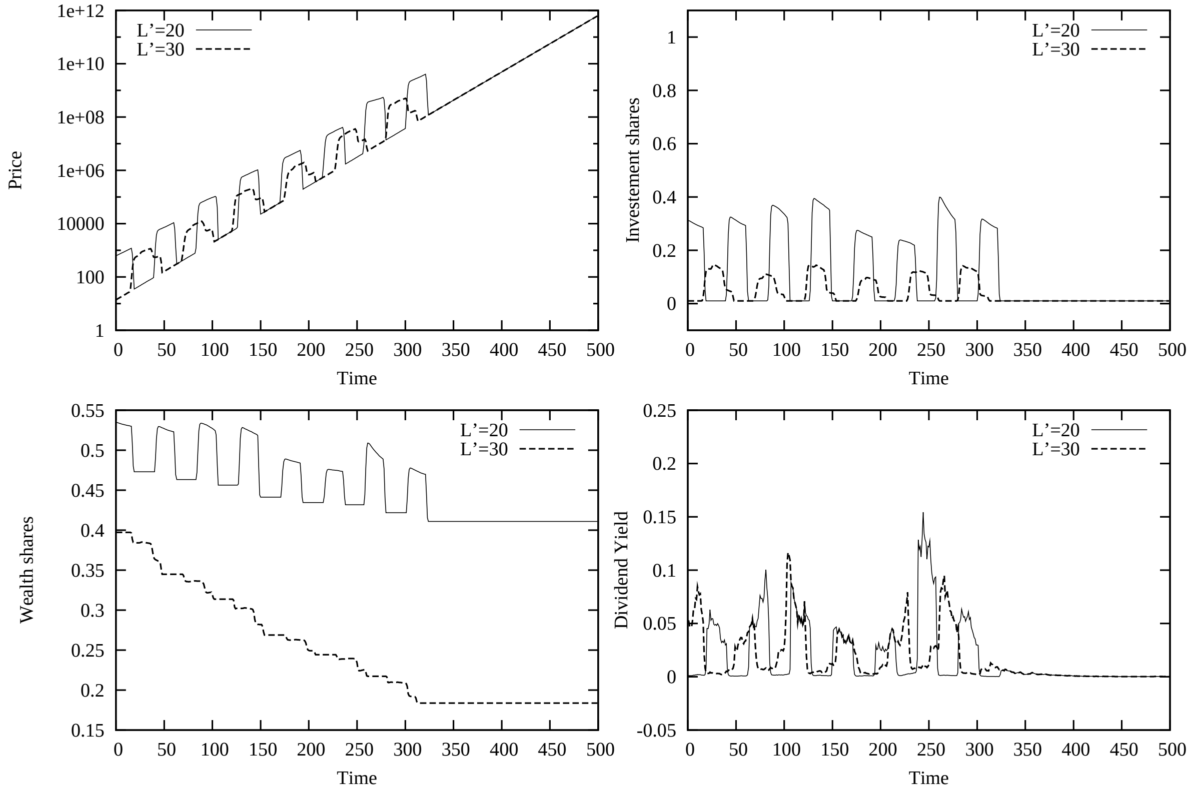

Figure 6: Dynamics with two mean-variance maximizers in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span of the agent with lower risk aversion are compared. **Top-left panel:** log-price dynamics. **Bottom-left panel:** wealth share of the agent with lower risk aversion α'. **Top-right panel:** investment shares. **Bottom-right panel:** dividend yield.

lie inside the unit circle.

The steady-state is unstable if at least one of the roots of polynomial $Q_0(\mu)$ is outside the unit circle.

Proof. See Appendix G. □

As a comparison with results from the previous section, in Figs. 5 and 6 we plot the results of market simulations when agents are mean-variance optimizers, with investment functions (4.2). Fig. 5 shows market dynamics for a single agent with memory span either $L = 10$ or $L = 20$. The only difference between simulations shown in Fig. 5 and those in Fig. 3 is the risk free rate, which is now equal to 0.05, so that $g < r_f$. Whereas with $g > r_f$ the market is stable with long memory and unstable with high memory, with $g < r_f$ the market dynamics stabilizes no matter the value of L.[11] Moreover, the price grows at the constant rate r_f (top-left panel), no matter the exogenous fluctuations of the dividend process (top-right panel). Since the price grows faster than the dividend, the dividend yield converges to 0 (bottom-right panel). Notice also that at the steady-state the agent is investing a constant fraction of wealth equal to the lower bound of (4.2), i.e. $x^* = 0.01$ in this case (bottom-left panel).

[11]By applying Proposition 5.2 to the investment function f in (4.2), and noticing that, due to the lower bound, the investment function is always flat at $y^* = 0$, it can be shown that the steady-state $y^* = 0$ is stable for any value of L.

Fig. 6 shows the market dynamics when two mean-variance optimizers, with different values of risk aversions, are active. It differs from Fig. 4 only for the value of risk free interest, now $g < r_f$. No matter the memory of the most aggressive agent, the price dynamics stabilizes (top-left panel). Prices grow at the constant rate r_f, despite the exogenous fluctuations of the dividend process. Since prices are growing faster than dividends, the dividend yield converges to 0 (bottom-right panel). At the steady-state both agents are investing $x^* = 0.01$ (bottom-left panel) and their wealth shares are both positive (top-right panel).

Summarizing, when $g < r_f$, in accordance with Proposition 5.1, market returns are equal to the risk free rate, independently from the noise of the dividend process and from the initial set of investment strategies. As a result, all the investor are gaining the same returns and the market is not selecting among them.

6 Conclusion

In his recent survey, LeBaron (2006) stresses that agent-based models do not require analytical tractability (as opposed to HAMs) and, therefore, are more flexible and realistic for what concerns their assumptions. In this paper we show that flexibility can be achieved in an analytically tractable framework as well. In fact, we perform an analytic investigation of a stylized model of a financial market with an arbitrary set of heterogenous agents. Under the assumption that impact of different agents on the market depends on the fraction of wealth which agents are able to accumulate, we derive existence and stability results for a general set of investment functions and an arbitrary number of agents. Moreover, we show that the steady-state equilibrium asset return and agents' survival can be characterized by looking at the intersection of each agent investment function with the equilibrium market curve (EMC). It turns out that our analysis of the corresponding deterministic skeleton is also helpful for the understanding of simulations with a stochastic dividend process.

As an application of our framework, we consider an example with mean variance maximizers agents, where each agent is characterized by two parameters, degree of risk aversion and length of memory used to estimate future variables. When the growth rate of dividends is bigger than the risk free rate, the agents with lower risk aversion dominate the market, provided that their memory spans are big enough. If it is so, the market dynamics converges to the stable steady state equilibrium, where prices are growing as fast as the dividends and the dividend yield increases with the risk aversion. In this case price fluctuations are due to the to the exogenous fluctuations of dividends. Otherwise, when the memory is not high enough, agents with different investment strategies coexist and the price fluctuations are endogenously determined. When, instead, the growth rate of dividends is smaller than the risk free rate, steady-state equilibrium asset returns are equal to risk free returns and the dividend yield converges to zero, no matter the ecology of agents. As a result wealths returns are equal for all investment functions and there is no selection.

Due to the generality of our approach, results from the analysis with mean-variance optimizing agents can be applied even when the functional form of the investment function is not known explicitly. Using this property, we have been able to relate our work to the simulation study of Levy, Levy, and Solomon (1994, 2000), and to resolve a number of issues concerning of the interplay between risk aversion and memory as reported in Zschischang and Lux (2001).

Appendix

A Dynamical System defined in Proposition 3.1

After Proposition 3.1 we have shown that the system of equations in (3.2) leads to the well-defined map from the domain $\mathcal{D}$, specified in (3.5), to itself. Here, we explicitly provide the evolution operator of the first-order dynamical system of $2N + 2L$ variables. We use the following notation for time t variables

$$x_{n,t}\,,\ \varphi_{n,t} \quad \forall n \in \{1, \ldots, N\} \qquad \text{and} \qquad k_{t,l}\,,\ y_{t,l} \quad \forall l \in \{0, \ldots, L-1\}\,, \tag{A.1}$$

where $k_{t,l}$ and $y_{t,l}$ denote the price return and the dividend yield at time $t - l$, respectively. We order the equations in the system in four separated blocks: $\mathcal{X}$, $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{Y}$. They define, respectively, N investment choices, N wealth shares, L price returns and L dividend yields. The last two blocks are needed to update the lagged variables. The map $\mathcal{T}$ referred in (3.4) is given by

$$\begin{aligned}
&\mathcal{X}: \left[\begin{array}{lcl}
x_{1,t+1} & = & f_1\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big) \\
\vdots & \vdots & \vdots \\
x_{N,t+1} & = & f_N\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big)
\end{array}\right. \\
&\mathcal{W}: \left[\begin{array}{lcl}
\varphi_{1,t+1} & = & \Phi_1\Big(x_{1,t}, \ldots, x_{N,t}; \varphi_{1,t}, \ldots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0}); \\
& & \quad K\Big[f_1\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big), \ldots, \\
& & \quad\quad f_N\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big); \\
& & \quad\quad x_{1,t}, \ldots, x_{N,t}; \varphi_{1,t}, \ldots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})\Big]\Big) \\
\vdots & \vdots & \vdots \\
\varphi_{N,t+1} & = & \Phi_N\Big(x_{1,t}, \ldots, x_{N,t}; \varphi_{1,t}, \ldots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0}); \\
& & \quad K\Big[f_1\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big), \ldots, \\
& & \quad\quad f_N\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big); \\
& & \quad\quad x_{1,t}, \ldots, x_{N,t}; \varphi_{1,t}, \ldots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})\Big]\Big)
\end{array}\right. \\
&\mathcal{K}: \left[\begin{array}{lcl}
k_{t+1,0} & = & K\Big[f_1\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big), \ldots, \\
& & \quad f_N\big(k_{t,0}, \ldots, k_{t,L-1}; Y(y_{t,0}, k_{t,0}), y_{t,0}, \ldots, y_{t,L-1}\big); \\
& & \quad x_{1,t}, \ldots, x_{N,t}; \varphi_{1,t}, \ldots, \varphi_{N,t}; Y(y_{t,0}, k_{t,0})\Big] \\
k_{t+1,1} & = & k_{t,0} \\
\vdots & \vdots & \vdots \\
k_{t+1,L-1} & = & k_{t,L-2}
\end{array}\right. \\
&\mathcal{Y}: \left[\begin{array}{lcl}
y_{t+1,0} & = & Y(y_{t,0}, k_{t,0}) \\
y_{t+1,1} & = & y_{t,0} \\
\vdots & \vdots & \vdots \\
y_{t+1,L-1} & = & y_{t,L-2}\,,
\end{array}\right.
\end{aligned} \tag{A.2}$$

where the following three functions Y, K, and Φ_n have been introduced. The function

$$Y(y, k) = y\,\frac{1 + g}{1 + k} \tag{A.3}$$

gives the dividend yield as a function of past realization of the yield and return, as in the right-hand side of the fourth equation in (3.2). The function

$$\begin{aligned}
&K\Big[z_1, \ldots, z_N; x_1, \ldots, x_N; \varphi_1, \ldots, \varphi_N; y\Big] = \\
&\qquad = r_f + \frac{(1 + r_f)\sum_{m=1}^{N}(z_m - x_m)\varphi_m + y\sum_{m=1}^{N} x_m z_m \varphi_m}{\sum_{m=1}^{N} x_m (1 - z_m)\varphi_m}\,,
\end{aligned} \tag{A.4}$$

gives the price return as a function of the investment choices, wealth shares and the dividend yield as in the right-hand side of the third equation in (3.2). Finally,

$$\begin{aligned}
&\Phi_n\big(x_1,\ldots,x_N;\varphi_1,\ldots,\varphi_N;y;k\big) = \\
&\qquad = \varphi_n \, \frac{1+r_f+(k+y-r_f)x_n}{1+r_f+(k+y-r_f)\sum_{m=1}^{N} x_m\varphi_m} \qquad \forall n\in\{1,\ldots,N\}\,,
\end{aligned} \tag{A.5}$$

specifies the wealth share of agent n as a function of the investment choices, wealth shares, the dividend yield and price return[12] as in the right-hand side of the second equation in (3.2). ☐

B Proof of Proposition 3.2

To solve for the equilibrium of the system (3.2), one can substitute time variables with equilibrium values and solve the resulting system for $(x_1^*,\ldots,x_N^*;\varphi_1^*,\ldots,\varphi_N^*;k^*;y^*)$. The system to be solved is as follows

$$\begin{cases}
x_n^* = f_n\left(k^*,\ldots,k^*;y^*,\ldots,y^*\right), \qquad \forall n\in\{1,\ldots,N\}\,,\\[4pt]
\varphi_n^* = \varphi_n^* \, \dfrac{(1+r_f)+(k^*+y^*-r_f)\,x_n^*}{(1+r_f)+(k^*+y^*-r_f)\big\langle x^*\big\rangle}\,, \qquad \forall n\in\{1,\ldots,N\}\,,\\[4pt]
k^* = r_f + \dfrac{y^*\big\langle x^{*2}\big\rangle}{\big\langle x^*\,(1-x^*)\big\rangle}\,,\\[4pt]
y^* = y^*\,\dfrac{1+g}{1+k^*}\,.
\end{cases} \tag{B.1}$$

Since y^* and investment shares are positive, from the third equation $k^* > r_f$, while the fourth equation fixes k^* to g. Thus, equilibria exist only when $g > r_f$. In particular, it means that $k^*+y^*-r_f > 0$. Then, the equations for the wealth shares imply that every surviving agent invests $x_n^* = \big\langle x^*\big\rangle$, which is independent of n. Therefore, all the survivors invest the same share, $x_\diamond^*$. Plugging this share into the third equation, one gets (3.6). ☐

C Proof of Proposition 3.3

A fixed point of the system (A.2) will be denoted as

$$\boldsymbol{x^*} = \big(x_1^*,\ldots,x_N^*;\varphi_1^*,\ldots,\varphi_N^*;\underbrace{k^*,\ldots,k^*}_{L};\underbrace{y^*,\ldots,y^*}_{L}\big)\,.$$

To derive the stability conditions in different equilibria, the Jacobian matrix has to be computed. The Jacobian depends on the derivatives of the functions Y, K and Φ_n introduced in Appendix A. We compute now the derivatives of these functions with respect to different arguments and perform the straightforward computation in the fixed point $\boldsymbol{x^*}$. For the function Y introduced in (A.3) the derivatives are given by

$$Y^y = \frac{\partial Y}{\partial y} = \frac{1+g}{1+k^*}\,, \qquad Y^k = \frac{\partial Y}{\partial k} = -y^*\,\frac{1+g}{(1+k^*)^2}\,. \tag{C.1}$$

[12] Notice that since the sum of the wealth shares is equal to 1 at any period, one of the equations in the system (e.g. the last equation of the block $\mathcal{W}$) is redundant and the dynamics can be fully described by the system of dimension $2N+2L-1$. However, the computations are more symmetric when the relation $\varphi_{N,t} = 1-\sum_{m=1}^{N-1}\varphi_{m,t}$ is not taken into account explicitly.

For the function K introduced in (A.4), for all $1 \le m \le N$, we have

$$\begin{aligned} K^{z_m} &= \frac{\partial K}{\partial z_m} = \varphi_m^* \,\frac{1 + r_f + (k^* + y^* - r_f) x_m^*}{\langle x^*(1-x^*)\rangle}\,, \\ K^{x_m} &= \frac{\partial K}{\partial x_m} = \varphi_m^* \,\frac{-1 - k^* + (k^* + y^* - r_f) x_m^*}{\langle x^*(1-x^*)\rangle}\,, \\ K^{\varphi_m} &= \frac{\partial K}{\partial \varphi_m} = x_m^* \,\frac{r_f - k^* + (k^* + y^* - r_f) x_m^*}{\langle x^*(1-x^*)\rangle}\,, \\ K^{y} &= \frac{\partial K}{\partial y} = \frac{\langle x^{*2}\rangle}{\langle x^*(1-x^*)\rangle}\,. \end{aligned} \tag{C.2}$$

Finally, for the function Φ_n introduced in (A.5) and for all $1 \le m \le N$, we have

$$\begin{aligned} \Phi_n^{x_m} &= \frac{\partial \Phi_n}{\partial x_m} = (k^* + y^* - r_f)\,\frac{(\delta_n^m - \varphi_m^*)\varphi_n^*}{1 + r_f + (k^* + y^* - r_f)\langle x^*\rangle}\,, \\ \Phi_n^{\varphi_m} &= \frac{\partial \Phi_n}{\partial \varphi_m} = \frac{\delta_n^m(1 + r_f) + (k^* + y^* - r_f)(\delta_n^m x_n^* - \varphi_n^* x_m^*)}{1 + r_f + (k^* + y^* - r_f)\langle x^*\rangle}\,, \\ \Phi_n^{y} &= \frac{\partial \Phi_n}{\partial y} = \varphi_n^* \,\frac{x_n^* - \langle x^*\rangle}{1 + r_f + (k^* + y^* - r_f)\langle x^*\rangle}\,, \\ \Phi_n^{k} &= \frac{\partial \Phi_n}{\partial k} = \varphi_n^* \,\frac{x_n^* - \langle x^*\rangle}{1 + r_f + (k^* + y^* - r_f)\langle x^*\rangle}\,, \end{aligned} \tag{C.3}$$

where δ_n^m is the Kronecker's delta. Using the block structure introduced in Appendix A, the Jacobian can be written in general form as:

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} & \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} & \frac{\partial \mathcal{W}}{\partial \mathcal{K}} & \frac{\partial \mathcal{W}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{K}}{\partial \mathcal{X}} & \frac{\partial \mathcal{K}}{\partial \mathcal{W}} & \frac{\partial \mathcal{K}}{\partial \mathcal{K}} & \frac{\partial \mathcal{K}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{W}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{K}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} \end{Vmatrix}\,. \tag{C.4}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. Since the investment choice of any agent does not explicitly depend on the investment choices in the previous period

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{X}}\right]_{n,m} = 0\,, \qquad 1 \le n, m \le N\,,$$

and this block is a zero matrix. The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. This is also a zero matrix and

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{W}}\right]_{n,m} = 0\,, \qquad 1 \le n \le N\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{X}/\partial\mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past price returns. Let us introduce a special notation for partial derivatives of the investment functions:

$$\frac{\partial f_n}{\partial k_{t-l}} = f_n^{k_l}\,, \qquad \frac{\partial f_n}{\partial y_{t+1}} = f_n^{Y}\,, \qquad \frac{\partial f_n}{\partial y_{t-l}} = f_n^{y_l}\,, \qquad 1 \le n \le N\,, \quad 0 \le l \le L-1\,.$$

Then

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{K}}\right]_{n,l} = \begin{cases} f_n^{k_0} + f_n^{Y} \cdot Y^k & \text{for } l = 0 \text{ (the first column)} \\ f_n^{k_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{X}/\partial\mathcal{Y}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past dividend yield. This block is given by

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{Y}}\right]_{n,l} = \begin{cases} f_n^{y_0} + f_n^Y \cdot Y^y & \text{for } l = 0 \text{ (the first column)} \\ f_n^{y_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is $N \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It holds

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \Phi_n^{x_m} + \Phi_n^k \cdot K^{x_m}\,, \qquad 1 \le n, m \le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It holds

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \Phi_n^{\varphi_m} + \Phi_n^k \cdot K^{\varphi_m}\,, \qquad 1 \le n, m \le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. For $1 \le n \le N$ and $0 \le l \le L-1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{K}}\right]_{n,l} = \begin{cases} \Phi_n^k \cdot \left(\sum_m K^{z_m}(f_m^{k_0} + f_m^Y Y^k) + K^y Y^k\right) + \Phi_n^y \cdot Y^k\,, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{z_m} f_m^{k_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{W}/\partial\mathcal{Y}$ is a $N \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged dividend yields. For $1 \le n \le N$ and $0 \le l \le L-1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{Y}}\right]_{n,l} = \begin{cases} \Phi_n^k \cdot \left(\sum_m K^{z_m}(f_m^{y_0} + f_m^Y Y^y) + K^y Y^y\right) + \Phi_n^y \cdot Y^y\,, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{z_m} f_m^{y_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{K}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple, since only the first line can contain non-zero elements. It reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{X}}\right]_{l,n} = \begin{cases} K^{x_n} & \text{for } l = 0 \text{ (the first column)} \\ 0 & \text{otherwise} \end{cases}\,, \qquad 0 \le l \le L-1\,, \quad 1 \le n \le N\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{W}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' wealth shares. It also has $L-1$ zero rows and reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{W}}\right]_{l,n} = \begin{cases} K^{\varphi_n} & \text{for } l = 0 \text{ (the first column)} \\ 0 & \text{otherwise} \end{cases}\,, \qquad 0 \le l \le L-1\,, \quad 1 \le n \le N\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{K}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves. It has a typical structure for such matrix with 1's under the main diagonal

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{K}}\right] = \begin{Vmatrix} \sum K^{z_m}(f_m^{k_0} + f_m^Y Y^k) + K^y Y^k & \sum K^{z_m} f_m^{k_1} & \dots & \sum K^{z_m} f_m^{k_{L-2}} & \sum K^{z_m} f_m^{k_{L-1}} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{Vmatrix}\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{Y}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to the lagged dividend yields. It is given by

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{Y}}\right] = \begin{Vmatrix} \sum K^{z_m}(f_m^{y_0} + f_m^Y Y^y) + K^y Y^y & \sum K^{z_m} f_m^{y_1} & \dots & \sum K^{z_m} f_m^{y_{L-2}} & \sum K^{z_m} f_m^{y_{L-1}} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{Vmatrix}\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{X}$ is a $L \times N$ matrix containing the partial derivatives of the lagged dividend yields with respect to the agents' investment choices. This block is a zero matrix

$$\left[\frac{\partial \mathcal{Y}}{\partial \mathcal{X}}\right]_{l,m} = 0\,, \qquad 0 \le l \le L-1\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{W}$ is a $L \times N$ matrix containing the partial derivatives of the lagged dividend yields with respect to the agents' wealth shares. This is also a zero matrix and

$$\left[\frac{\partial \mathcal{Y}}{\partial \mathcal{W}}\right]_{l,m} = 0\,, \qquad 0 \le l \le L-1\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{K}$ is a $L \times L$ matrix containing the partial derivatives of the lagged dividend yields with respect to the past price returns. The only non-zero element of this matrix is in the upper left corner, i.e.

$$\left[\frac{\partial \mathcal{Y}}{\partial \mathcal{K}}\right]_{l,j} = \begin{cases} Y^k & \text{for } l = j = 0 \text{ (the first row, the first column)} \\ 0 & \text{otherwise}\,. \end{cases}$$

Finally, the block $\partial\mathcal{Y}/\partial\mathcal{Y}$ is a $L \times L$ matrix containing the partial derivatives of the lagged dividend yields with respect to themselves. This matrix is given by

$$\left[\frac{\partial \mathcal{Y}}{\partial \mathcal{Y}}\right] = \left\| \begin{matrix} Y^y & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{matrix} \right\| .$$

With all these definitions, one obtains the following statement about the Jacobian in the equilibria of the system.

Lemma C.1. *Let $\boldsymbol{x}^*$ be an equilibrium of system* (3.2) *described in Prop. 3.2 and let first M agents survive in this equilibrium. The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure, where the actual values of non-zero elements, denoted by the symbols $\star$, are varying.*

$$\left\| \begin{array}{cccccc|cccccc|cccc|ccccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & &] & [& & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} & &] \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
[& \Phi^x &] & \vdots & \ddots & \vdots & [& \Phi^{\varphi}_S &] & [& \Phi^{\varphi}_{NS} &] & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{-y^*}{1+g} & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{array} \right\|$$

The solid lines divide the matrix in the same 16 *blocks as in* (C.4). *In addition, the first and the second column-blocks as well as the second row-block are split into the two parts of sizes M and $N-M$, corresponding to the survivors and the non-survivors, respectively.*

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial\mathcal{X}/\partial\mathcal{X}$ and $\partial\mathcal{X}/\partial\mathcal{W}$, are always zero. Two other blocks, $\partial\mathcal{X}/\partial\mathcal{K}$ and $\partial\mathcal{X}/\partial\mathcal{Y}$, in general contain non-zero elements and simplified, because in the equilibrium $k^* = g$ and therefore $Y^y = 1$ and $Y^k = -y^*/(1+g)$.

To simplify the second row-block, notice that $\Phi_n^k = \Phi_n^y = 0$ in this equilibrium. Indeed, the numerators of the corresponding general expressions in (C.3) are 0, because all the survivors invest the same share in the equilibrium (i.e. $x_n^* = \langle x^* \rangle = x_\diamond^*$ for all $n \le M$), while for the non-survivors $\varphi_n^* = 0$. This immediately implies that the two last blocks, $\partial\mathcal{W}/\partial\mathcal{K}$ and $\partial\mathcal{W}/\partial\mathcal{Y}$, contain only zero elements. Furthermore, from the Equilibrium Market Curve relation (3.6) we get in the equilibrium

$$1 + r_f + (g + y^* - r_f)x_\diamond^* = 1 + g\,. \tag{C.5}$$

Thus, in the equilibrium

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \Phi_n^{x_m} = \begin{cases} \varphi_n^*(\delta_n^m - \varphi_m^*)(y^* + g - r_f)/(1+g) & \text{for } n \le M \text{ (agent } n \text{ survives)} \\ 0 & \text{otherwise}\,, \end{cases}$$

and all the rows corresponding to the non-survivors in this block are zero rows. Moreover, all the columns corresponding to the non-survivors contain only zero elements as well, since then $\delta_n^m = \varphi_m^* = 0$. We denote as Φ^x the remaining (non-zero) part of the block $\partial\mathcal{W}/\partial\mathcal{X}$.

The simplifications in the next block lead to

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \Phi_n^{\varphi_m} = \begin{cases} \delta_n^m - \varphi_n^*(g - r_f)/(1+g) & \text{for } n, m \le M \\ -\varphi_n^* x_m^*(y^* + g - r_f)/(1+g) & \text{for } n \le M, m > M \\ \delta_n^m\big(1 + r_f + x_n^*(y^* + g - r_f)\big)/(1+g) & \text{for } n > M\,. \end{cases} \tag{C.6}$$

The block of the elements from the first line of the previous expression is denoted as Φ_S^φ; the block of the elements of the second line is denoted as Φ_{NS}^φ; while the block of the elements from the third line (i.e. when $n > M$) only the diagonal elements are non-zero.

It is obvious that in the next row-block with L rows the elements are zero in all the lines but the first. The only exception from this rule are the elements below the main diagonal in the block $\partial\mathcal{K}/\partial\mathcal{K}$ which are all equal to 1. To compute the elements in the first row consider the derivatives of function K derived in (C.2). For the first block, $\partial\mathcal{K}/\partial\mathcal{X}$, notice that for the non-surviving agents $K^{x_m} = 0$, while for the survivors, i.e. for $m \le M$

$$K^{x_m} = -\varphi_m^* \frac{1 + r_f}{(1 - x_\diamond^*)x_\diamond^*}\,. \tag{C.7}$$

Analogously, in the next block, $\partial\mathcal{K}/\partial\mathcal{W}$, for all the survivors $K^{\varphi_m} = 0$, while for all other agents $m > M$ the elements are given by

$$K^{\varphi_m} = x_m^* \frac{r_f - g + x_m^*(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*} = x_m^* \frac{(x_m^* - x_\diamond^*)(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*}\,, \tag{C.8}$$

where (C.5) was used to derive the last equality.

The simplifications in the blocks $\partial\mathcal{K}/\partial\mathcal{K}$ and $\partial\mathcal{K}/\partial\mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{z_m} for all the non-survivors are zeros, while for the survivors ($m \le M$) they are given by

$$K^{z_m} = \varphi_m^* \frac{1 + g}{(1 - x_\diamond^*)x_\diamond^*}\,. \tag{C.9}$$

Thus, all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. □

The rest of the proof of the Proposition is now clear. Consider the Jacobian matrix derived in Lemma C.1. The last $N - M$ columns of the left column-block contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N - M$. This eigenvalue does not affect stability. Moreover, these columns and the corresponding rows can be eliminated from the Jacobian. Analogously, in each of the last $N - M$ rows in the second row-block the only non-zero entries belong to the main diagonal. Consequently, $\Phi_n^{\varphi_n}$ for $n > M$ are the eigenvalues of the matrix, with multiplicity (at least) one, and the rows (together with the corresponding columns) can be eliminated from the Jacobian. Using the third line of (C.6) we get the following $N-M$ eigenvalues

$$\mu_n = \frac{1+r_f+x_n^*(y^*+g-r_f)}{1+g} = \frac{1+r_f+(g-r_f)(x_n^*/x_\diamond^*)}{1+g}$$

where the last equality follows from (C.5). Recall that the equilibria we consider, exist only when $g > r_f$. Then, with a bit of algebra, the stability conditions $-1 < \mu_n < 1$ can be simplified to conditions (3.11).

Finally, notice that the elimination of the rows and columns which we have performed reduce the Jacobian to the shape which correspond to the Jacobian of the same system in the same equilibrium but without non-surviving agents. □

D Proof of Proposition 3.4

Let us proceed with a reduced Jacobian obtained from the matrix in Lemma C.1 after eliminating the rows and columns corresponding to the survivors. We denote this Jacobian as $\boldsymbol{L}$, and an identity matrix of the same dimension $(2M+2L)\times(2M+2L)$ as $\boldsymbol{I}$. Then the characteristic polynomial whose roots are the eigenvalues of $\boldsymbol{L}$ is the determinant $\det(\boldsymbol{L}-\mu\boldsymbol{I})$. First, we analyze it and then we identify new eigenvalues.

Let us look at the second column block of the size M in this determinant. The only non-zero elements in this block lie in the rows of the second row block, in the part which was called $[\Phi_S^\varphi]$. The elements of this part have been computed in the first line in (C.6). Thus, this column block can be represented as $\left\Vert\, v\,\boldsymbol{b}+\boldsymbol{b}_1 \mid \ldots \mid v\,\boldsymbol{b}+\boldsymbol{b}_k \,\right\Vert$, where $v=(g-r_f)/(1+g)$ and the following column vectors have been introduced

$$\begin{array}{rcl}
\boldsymbol{b} & = & \left\Vert \; 0 \;\ldots\; 0 \;\middle|\; -\varphi_1^* \;\ldots\; -\varphi_M^* \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \; \right\Vert, \\
\boldsymbol{b}_1 & = & \left\Vert \; 0 \;\ldots\; 0 \;\middle|\; 1-\mu \;\ldots\; 0 \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \; \right\Vert, \\
 & \vdots & \\
\boldsymbol{b}_M & = & \left\Vert \; 0 \;\ldots\; 0 \;\middle|\; 0 \;\ldots\; 1-\mu \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \;\middle|\; 0 \;\; 0 \;\ldots\; 0 \; \right\Vert.
\end{array}$$

We consider each of the columns in the central block as a sum of two terms and, applying the multilinear property of the discriminant to the whole matrix $\boldsymbol{L}-\mu\boldsymbol{I}$, end up with a sum of 2^M determinants. Many of them are zeros, since they contain two or more columns proportional to the same vector $\boldsymbol{b}$. Actually, there are only $M+1$ non-zero elements in the expansion. The simplest has the structure $\left\Vert\, \boldsymbol{b}_1 \mid \ldots \mid \boldsymbol{b}_M \,\right\Vert$ in the second column block. Since the only non-zero elements in this block are the terms $1-\mu$ belonging to the main diagonal, the determinant of that part is equal to $(1-\mu)^M \det\boldsymbol{N}$, where the matrix $\boldsymbol{N}$ is defined as follows

$$\begin{array}{ccc|ccc|ccc}
-\mu & \ldots & 0 & f_1^{k_0}+c_1 f_1^Y & f_1^{k_1} & \ldots & f_1^{k_{L-1}} & f_1^{y_0}+f_1^Y & f_1^{y_1} & \ldots & f_1^{y_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -\mu & f_M^{k_0}+c_1 f_M^Y & f_M^{k_1} & \ldots & f_M^{k_{L-1}} & f_M^{y_0}+f_M^Y & f_M^{y_1} & \ldots & f_M^{y_{L-1}} \\
\hline
K^{x_1} & \ldots & K^{x_M} & c_2 \;\; f^{k_0} \;+c_1\;\; f^Y \;\; +c_1 K^y-\mu & c_2\;\; f^{k_1} & \ldots & c_2\;\; f^{k_{L-1}} & c_2\;\; f^{y_0} \;+\;\; f^Y \;\; +K^y & c_2\;\; f^{y_1} & \ldots & c_2\;\; f^{y_{L-1}} \\
0 & \ldots & 0 & 1 & -\mu & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & -\mu & 0 & 0 & \ldots & 0 \\
\hline
0 & \ldots & 0 & c_1 & 0 & \ldots & 0 & 1-\mu & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & -\mu & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -\mu
\end{array}.$$

The non-zero elements in this matrix have been computed during the proof of Lemma C.1. Namely, the constant $c_1 = Y^k = -y^*/(1+g)$, the values of K^{x_m} are given in (C.7), $c_2 = (1+g)/(x_\diamond^*(1-x_\diamond^*))$ comes from (C.9), and the derivative $K^y = x_\diamond^*/(1-x_\diamond^*)$ is computed from (C.2). Finally, by $\left\langle f^{k_l}\right\rangle$ and $\left\langle f^{y_l}\right\rangle$ for $l=1,\ldots,L$, as well as $\left\langle f^Y\right\rangle$ we mean the averages of the corresponding derivatives of the survivors' investment functions weighted by their equilibrium wealth shares.

Coming back to the computation of $\det(\boldsymbol{L}-\mu\boldsymbol{I})$, recall that there are other M non-zero blocks in the sum for this determinant. They are obtained when in the second column block all the vectors are $\boldsymbol{b}_i$ apart from the one column $v\,\boldsymbol{b}$. But all these determinants can be simplified since $M-1$ of their columns have only one non-zero element $1-\mu$ on the diagonal, and after eliminating the corresponding columns and rows the remaining column in the second block will contain the element $-v\,\varphi^*_\nu$ in the diagonal and zero elements in other positions. Therefore

$$\det(\boldsymbol{L}-\mu\boldsymbol{I}) = (1-\mu)^M \det\boldsymbol{N} - (1-\mu)^{M-1}\cdot\frac{g-r_f}{1+g}\sum_{\nu=1}^{M}\varphi^*_\nu \det\boldsymbol{N} = (1-\mu)^{M-1}\Big(1-\mu-\frac{g-r_f}{1+g}\Big)\det\boldsymbol{N}. \quad \text{(D.1)}$$

From this expression we obtain the eigenvalue equal to 1 of multiplicity $M-1$. Notice that when $M=1$ there are no such eigenvalues. That is why the system with one survivor is asymptotically stable (of course if all the roots of polynomial (3.14) are inside the unit circle.) When $M>1$ the eigenvalue 1 obviously corresponds to the movement of the system along the manifold of equilibria. Therefore, it is only the wealth distribution which is changing in the equilibria but not the other quantities.

Another eigenvalue obtained in the expansion (D.1) is $(1+r_f)/(1+g)$. It does not affect the stability, since $r_f<g$. All the remaining eigenvalues can be obtained from $(M+2L)\times(M+2L)$ matrix $\boldsymbol{N}$. We expand this matrix on the minors of the elements of the first row in the last block. Simplifying the resulting minors, we get

$$\det\boldsymbol{N} = (-1)^L c_1\mu^{L-1}\det\boldsymbol{N}_1(M) + (1-\mu)(-\mu)^{L-1}\det\boldsymbol{N}_2(M)\,, \quad \text{(D.2)}$$

where

$$\boldsymbol{N}_1(M)=\begin{array}{ccc|ccccc}
-\mu & \dots & 0 & f_1^{y_0}+f_1^Y & f_1^{y_1} & \dots & f_1^{y_{L-2}} & f_1^{y_{L-1}}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & -\mu & f_M^{y_0}+f_M^Y & f_M^{y_1} & \dots & f_M^{y_{L-2}} & f_M^{y_{L-1}}\\
\hline
K^{x_1} & \dots & K^{x_M} & c_2\ \ f^{y_0}\ +\ f^Y\ \ +K^y & c_2\ f^{y_1} & \dots & c_2\ f^{y_{L-2}} & c_2\ f^{y_{L-1}}\\
0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array},$$

and

$$\boldsymbol{N}_2(M)=\begin{array}{ccc|ccccc}
-\mu & \dots & 0 & f_1^{k_0}+c_1 f_1^Y & f_1^{k_1} & \dots & f_1^{k_{L-2}} & f_1^{k_{L-1}}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & -\mu & f_M^{k_0}+c_1 f_M^Y & f_M^{k_1} & \dots & f_M^{k_{L-2}} & f_M^{k_{L-1}}\\
\hline
K^{x_1} & \dots & K^{x_M} & c_2\ \ f^{k_0}\ +c_1\ f^Y\ \ +c_1K^y-\mu & c_2\ f^{k_1} & \dots & c_2\ f^{k_{L-2}} & c_2\ f^{k_{L-1}}\\
0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0\\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array},$$

The determinants of these two matrices of similar structure are computed in a recursive way. The following lemma is used.

Lemma D.1.

$$\begin{vmatrix}
x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n\\
1 & -\mu & 0 & \dots & 0 & 0\\
0 & 1 & -\mu & \dots & 0 & 0\\
0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \dots & -\mu & 0\\
0 & 0 & 0 & \dots & 1 & -\mu
\end{vmatrix} = (-1)^{n+1}\sum_{k=1}^{n} x_k\,\mu^{n-k} \quad , \quad \text{(D.3)}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k, whose corresponding sign is $(-1)^{k+1}$, is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-k}$ and the relation to be proved immediately follows. $\square$

Consider now the expansion of the matrix $\boldsymbol{N}_1(M)$ by the minors of the elements from the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{N}_1(M)$, which we denote as $\boldsymbol{N}_1(M-1)$. The minor associated with K^{x_1} has a left upper block with $M-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{M-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type (D.3). Applying Lemma D.1 one then has

$$\det \boldsymbol{N}_1(M) = (-\mu)\,\det \boldsymbol{N}_1(M-1) + (-1)^M K^{x_1}\mu^{M-1}(-1)^{L+1}\left(f_1^Y\mu^{L-1} + \sum_{l=0}^{L-1} f_1^{y_l}\mu^{L-1-l}\right).$$

Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end the lower right block of the original matrix remains, which is again a matrix similar to (D.3). Applying once more Lemma D.1 one has for $\boldsymbol{N}_1(M)$ the following

$$\begin{aligned}
\det \boldsymbol{N}_1(M) &= (-1)^{M+L+1}\,\mu^{M-1}\sum_{m=1}^{M} K^{x_m}\left(f_m^Y\mu^{L-1} + \sum_{l=0}^{L-1} f_m^{y_l}\mu^{L-1-l}\right) + \\
&\quad + (-1)^{M+L+1}\,\mu^M\left(\left(K^Y + c_2\langle f^Y\rangle\right)\mu^{L-1} + c_2\sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l}\right) = \\
&= (-1)^{M+L+1}\,\mu^{M-1}\left[\mu^L K^Y + \left(\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l}\right)\cdot\left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\right)\right].
\end{aligned}$$

The determinant of matrix $\boldsymbol{N}_2(M)$ can be computed analogously and we obtain

$$\begin{aligned}
\det \boldsymbol{N}_2(M) &= (-1)^{M+L+1}\,\mu^{M-1}\sum_{m=1}^{M} K^{x_m}\left(c_1 f_m^Y\mu^{L-1} + \sum_{l=0}^{L-1} f_m^{k_l}\mu^{L-1-l}\right) + \\
&\quad + (-1)^{M+L+1}\,\mu^M\left(-\mu^L + c_1\left(K^Y + c_2\langle f^Y\rangle\right)\mu^{L-1} + c_2\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\right) = \\
&= (-1)^{M+L+1}\,\mu^{M-1}\left[-\mu^{L+1} + c_1\mu^L K^Y + \left(c_1\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\right)\cdot\left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\right)\right].
\end{aligned}$$

Plugging the two last expressions in (D.2) we finally obtain

$$\begin{aligned}
\det \boldsymbol{N} &= (-1)^{M+1}\mu^{M+L-2}c_1\left[\mu^L K^Y + \left(\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l}\right)\cdot\left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\right)\right] + \\
&\quad + (1-\mu)(-1)^M\mu^{M+L-2}\left[-\mu^{L+1} + c_1\mu^L K^Y + \left(c_1\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\right)\cdot\left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\right)\right] = \\
&= (-1)^{M+1}\mu^{M+L-2}\left[(1-\mu)\mu^{L+1} + c_1\mu^L K^Y\left(1-(1-\mu)\right) + \right.\\
&\quad \left. + \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\right)\left(c_1\langle f^Y\rangle\mu^{L-1}\left(1-(1-\mu)\right) + c_1\sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l} - (1-\mu)\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\right)\right] = \\
&= (-1)^{M+1}\mu^{M+L-2}\left(1 + c_1K^y - \mu\right)\times \\
&\quad \times\left[\mu^{L+1} - c_2\left(c_1\langle f^Y\rangle\mu^L + c_1\sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l} - (1-\mu)\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\right)\right]
\end{aligned}$$

where in the last equality we used the relation $-c_2(1+c_1K^y) = -(1+r_f)/(x_\diamond^*(1-x_\diamond^*))$, which can be easily checked using the definitions of the constants c_2, c_1 and K^y.

Thus, we have found another zero eigenvalue of multiplicity $M+L-2$ and yet another eigenvalue $1+c_1K^y = (1+r_f)/(1+g)$ which lies inside the unit circle since $r_f < g$. The stability will depend only on the roots of the polynomial in the squared brackets. After some simplifications and using the relation $x_\diamond^*(1-x_\diamond^*) = -y^*l'(y^*)$, which can be directly checked from the definition of the Equilibrium Market Curve, we get the polynomial (3.14).

□

E Proof of Corollaries 3.2 and 3.3

When $L = 1$ the polynomial $\tilde{Q}(\mu)$ in (3.14) can be simplified and it is given by

$$\mu^2 + \mu\, C \frac{1+g}{y^*} - C\Big(1 + \frac{1+g}{y^*}\Big),$$

where $C = \sum_{m=1}^{M} f'_m(y^* + g)\varphi^*_m / l'(y^*)$. Let us introduce two quantities, trace and determinant, as follows $\mathrm{Tr} = -C(1+g)/y^*$ and $\mathrm{Det} = -C(1 + (1+g)/y^*)$. According to standard results for the second-degree polynomial (see e.g. Medio and Lines (2001)), we get the following conditions for stability, whose equality correspond to the bifurcation loci of fold, flip and Neimark-Sacker bifurcation respectively,

$$1 - \mathrm{Tr} + \mathrm{Det} > 0\,, \quad 1 + \mathrm{Tr} + \mathrm{Det} > 0\,, \quad \text{and} \quad \mathrm{Det} < 1\,.$$

Using our definitions of Tr and Det we get the conditions $C < 1$, $C < y^*/(y^*+2(1+g))$ and $C > -y^*/(1+g+y^*)$ respectively. The first condition is redundant, while the last two give (3.15).

For larger L the results on stability are limited. First, we can derive the loci of fold and flip bifurcations substituting, respectively, $\mu = 1$ and $\mu = -1$ into polynomial $\tilde{Q}(\mu)$ in (3.14). Straight-forward computations show that the line $C = 1$ is a locus of fold bifurcation for any L, while the curve $C = y^*/(y^* + 2(1+g))$ is a locus of flip bifurcation for any odd L (and there is no flip bifurcation, when L is even).

Second, plugging $\mu = e^{i\psi}$, where ψ is arbitrary angle and i is the imaginary unit, into equation $\tilde{Q}(\mu) = 0$ we can derive the locus of Neimark-Sacker bifurcation. In case of $L = 2$ the equation can be solved and, after tedious computations, one get the condition

$$C^2\left({y^*}^2 + 3(1+g)y^* + 2(1+g)^2\right) + 2\,C\,{y^*}^2 - 4\,{y^*}^2 = 0\,.$$

This second-order curve is depicted in the right panel of Fig. 2 in coordinates (y^*, C).

Finally, we analyze the case $L \to \infty$. Rewrite polynomial (3.14) as follows

$$\tilde{Q}(\mu) = \mu^{L-1}\left(\mu^2 - \frac{1}{L}\frac{1-(1/\mu)^L}{1-1/\mu}\Big(1 + (1-\mu)\frac{1+g}{y^*}\Big)\,C\right). \tag{E.1}$$

We want to proof that all the roots of this polynomial lie inside the unit circle of the complex plane for L high enough. Consider the region outside the unit circle (including the circle itself), fix $\mu = \mu_0$ and let $L \to \infty$. Since $|\mu_0| \geq 1$, the first term in (E.1) cannot be equal to zero. Therefore, μ_0 can be a root of the characteristic polynomial only if the expression in the parenthesis cancels out. First, assume that $|\mu_0| > 1$. Then when $L \to \infty$ the expression in the parenthesis leads to $\mu_0 = 0$ which contradicts our choice of μ_0. Second, let $|\mu_0| = 1$ but $\mu_0 \neq 1$. In this case the expression $|1 - (1/\mu)^L|$ is bounded (uniformly with L), and so, again taking the limit $L \to \infty$ we obtain $\mu_0 = 0$. Thus, the only remaining possibility is $\mu_0 = 1$, which wul imply $C = 1$, that is to the locus of fold bifurcation. Since we know that when the relative slope is $C = 0$, the steady-state is stable by continuity it follows that whenever $C < 1$ the steady state is stable too. This implies (3.16), and proofs Corollary 3.3. □

F Proof of Proposition 5.1

In Proposition 3.1 we have proved that the system is well defined on $\mathcal{D}$ given in (3.5). Along the same lines it is straightforward to show that $\mathcal{T}$ is also well defined in $\mathcal{D}'$. In particular, an extension for zero dividend yield does not create any problem. Since $\mathcal{D} \subset \mathcal{D}'$, the fixed points defined in Proposition 3.2 are also the fixed points in $\mathcal{D}'$. In those points, of course, $y^* \neq 0$.

In all other fixed points $y^* = 0$, while other quantities are obtained again from (B.1). From the third equation it immediately follows that $k^* = r_f$. Thus, the investment in the risky and the riskless asset yields the same return. Therefore, the wealths of all agents increase with the same rate, the second equations (B.1) are always satisfied, and no other restrictions on the agents' wealth shares are required. □

G Proof of Proposition 5.2

The procedure in this proof is analogous to the one we use for proving Propositions 3.3 and 3.4. In particular we use the derivatives and the general Jacobian structure which has been derived in Appendix C. The next Lemma, which is analogous to Lemma C.1 describes the Jacobian matrix for the steady-states with zero yield.

Lemma G.1. *Let $\boldsymbol{x}^*$ be a steady-state of dynamics (3.2) described in Proposition 5.1 and let the first M agents survive in this equilibrium. The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure, where the actual values of non-zero elements, denoted by the symbols $\star$, are varying.*

$$
\left\| \begin{array}{cccccc|cccccc|cccc|ccccc}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \star & \cdots & \star & \star & \star & \star & \cdots & \star & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & &] & [& & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} & &] \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \star & \cdots & \star & \star & \star & \star & \cdots & \star & \star \\
\hline
\star & \cdots & \star & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & \star & \cdots & \star & \star & \star & \star & \cdots & \star & \star \\
[& \Phi^x &] & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \Phi^k & &] & [& & \Phi^y & &] \\
\star & \cdots & \star & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & \star & \cdots & \star & \star & \star & \star & \cdots & \star & \star \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\star & \cdots & \star & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \star & \cdots & \star & \star & \star & \star & \cdots & \star & \star \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \frac{1+g}{1+r_f} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{array} \right\|
$$

The solid lines divide the matrix in the same 16 blocks as in (C.4). In addition, the first and second column-blocks and second row-block are split into the two parts of sizes M and $N - M$, corresponding to the survivors and the non-survivors, respectively.

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial\mathcal{X}/\partial\mathcal{X}$ and $\partial\mathcal{X}/\partial\mathcal{W}$, are always zero. Two other blocks, $\partial\mathcal{X}/\partial\mathcal{K}$ and $\partial\mathcal{X}/\partial\mathcal{Y}$, in general contain non-zero elements and simplified, because in the equilibrium $k^* = r_f$ and $y^* = 0$, and therefore $Y^y = (1+g)/(1+r_f)$ and $Y^k = 0$.

To simplify the remaining row-blocks, notice from (C.3) that $\Phi_n^{x_m} = 0$ and $\Phi_n^{\varphi_m} = \delta_n^m$, while $\Phi_n^k = \Phi_n^y = \varphi_n^*(x_n^* - \langle x \rangle)/(1+r_f)$ in this equilibrium. This follows immediately from the relation $k^* + y^* - r_f = 0$. At the same time from (C.2) we have $K^{\varphi_m} = 0$, $K^y = \langle {x^*}^2 \rangle / \langle x^*(1-x^*) \rangle$, and $K^{z_m} = -K^{x_m} = \varphi_m^*(1+r_f)/\langle x^*(1-x^*) \rangle$.

Thus, in the first block of the second row-block, $\partial\mathcal{W}/\partial\mathcal{X}$, the elements are equal to $\Phi_n^k \cdot K^{x_m}$ and they are zeros as soon as either n or m is larger than M. In the next block, $\partial\mathcal{W}/\partial\mathcal{W}$, all the elements are zeros, apart from the main diagonal elements. All the elements of the next block, $\partial\mathcal{W}/\partial\mathcal{K}$, contain the multiplying term Φ_n^k, so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^k]$. Similarly, in the block $\partial\mathcal{W}/\partial\mathcal{Y}$ all the elements are the sums containing either the term Φ_n^k or the term Φ_n^y, so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^y]$.

In the next row-block, with L rows, the elements are zeros in all the lines but the first. The only exception from thisrule are the elements below the main diagonal in the block $\partial\mathcal{K}/\partial\mathcal{K}$ which are all equal to 1. For the elements in the first row we use the derivatives of function K derived above. Consequently, in the first block, $\partial\mathcal{K}/\partial\mathcal{X}$, for the non-surviving agents we have $K^{x_m} = 0$. Analogously, in the next block, $\partial\mathcal{K}/\partial\mathcal{W}$, all the elements are zeros. The simplifications in the blocks $\partial\mathcal{K}/\partial\mathcal{K}$ and $\partial\mathcal{K}/\partial\mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{z_m} for all the non-survivors are zeros, therefore all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. □

In the remaining part of this proof we identify different multipliers of the matrix derived in the previous Lemma. From the first line in the fourth row-block we immediately obtain the eigenvalue $(1+g)/(1+r_f)$ and condition $g < r_f$ for stability. Elimination of this line together with the corresponding column creates a zero line in the same block. Proceeding recursively, we obtain the eigenvalue 0 with multiplicity $L-1$ and eliminate the fourth line- and column-block entirely.

From the second column-block we get the eigenvalue 1 with multiplicity N. These eigenvalues correspond to the directions of change in the wealth distribution between different agents. (Recall from Proposition 5.1 that the wealth shares are free of choice.) Consequently, there are no asymptotically stable equilibria. At the same time, it is clear that these eigenvalues, lying on the border of the unit circle, do not prevent the steady-state from stability.

From the last $N-M$ columns of the first column-block we obtain the eigenvalue 0 with multiplicity $N-M$. Eliminating corresponding columns and rows we get the following matrix

$$
\boldsymbol{N}_3(M) = \left(\begin{array}{ccc|ccccc}
-\mu & \dots & 0 & f_1^{k_0} & f_1^{k_1} & \dots & f_1^{k_{L-2}} & f_1^{k_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & f_M^{k_0} & f_M^{k_1} & \dots & f_M^{k_{L-2}} & f_M^{k_{L-1}} \\
\hline
K^{x_1} & \dots & K^{x_M} & \sum_m K^{z_m} f_m^{k_0} - \mu & \sum_m K^{z_m} f_m^{k_1} & \dots & \sum_m K^{z_m} f_m^{k_{L-2}} & \sum_m K^{z_m} f_m^{k_{L-1}} \\
0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array}\right),
$$

where, as we found in Lemma G.1, the derivatives are $K^{z_m} = -K^{x_m} = \varphi_m^*(1+r_f)/\langle x^*(1-x^*)\rangle$. This matrix has the same functional form as matrices $\boldsymbol{N}_1(M)$ and $\boldsymbol{N}_2(M)$ whose discriminant we computed in Appendix D. Proceeding in analogous way, we get

$$
\begin{aligned}
\det \boldsymbol{N}_3(M) = (-1)^{M+L+1}\,\mu^{M-1} \sum_{m=1}^{M} K^{x_m} \Big(\sum_{l=0}^{L-1} f_m^{k_l}\mu^{L-1-l}\Big) + \\
+ (-1)^{M+L+1}\,\mu^{M}\Big(-\mu^L + \sum_{l=0}^{L-1}\sum_m K^{z_m} f_m^{k_l}\mu^{L-1-l}\Big) = \\
= -(-1)^{M+L+1}\,\mu^{M-1}\,\frac{1+r_f}{\langle x^*(1-x^*)\rangle} \sum_{l=0}^{L-1} \langle f^{k_l}\rangle \mu^{L-1-l} + \\
+ (-1)^{M+L+1}\,\mu^{M}\Big(-\mu^L + \frac{1+r_f}{\langle x^*(1-x^*)\rangle} \sum_{l=0}^{L-1} \langle f^{k_l}\rangle \mu^{L-1-l}\Big) = \\
= (-1)^{M+L}\,\mu^{M-1}\Big(\mu^{L+1} + \frac{1+r_f}{\langle x^*(1-x^*)\rangle}(1-\mu) \sum_{l=0}^{L-1} \langle f^{k_l}\rangle \mu^{L-1-l}\Big).
\end{aligned}
$$

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