Heterogeneity and Aggregation in a Financial Accelerator Model

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Abstract

In this paper we present a macroeconomic model in which changes in the variance (and higher moments of the distribution) of firm's financial conditions –i.e. "distributive shocks"– are bound to play a crucial role in the determination of output fluctuations. Firms differ by degree of financial robustness, which affect (optimal) investment in a bankruptcy risk context (à la Greenwald-Stiglitz). As to households, for the sake of simplicity, we assume that they are homogeneous in every respect so that we can adopt the representative agent hypothesis. We can explore the properties of the macro-dynamic model either via the study of the two-dimensional map defining the laws of motion of the average equity ratio and of the variance of the distribution or via simulations in a multiagent framework.

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1 Introduction

The representative agent assumption is still the cornerstone of most of contemporary macroeconomics but the awareness of its limitations[1] is spreading well beyond the circle of more or less dissenting economists. Also in mainstream macroeconomics, in fact, the representative agent is not as eagerly embraced as in the early years of the debate on microfoundations in the remote '70s and is still adopted mainly for lack of a workable alternative. In this paper we try to cast the Financial Accelerator story in a context of truly heterogeneous agents. True heterogeneity occurs when agents are different within the same group – in the present framework we will deal with heterogeneity of financial conditions across firms – so that we cannot rely upon the representative agent device even to describe the behaviour of a class of agents. True heterogeneity is obviously appealing but has a major disadvantage: we need an aggregation procedure to build the model from the bottom up. As it is well known from a large literature, aggregation is not an innocuous task in economics.

In order to take heterogeneity seriously in macroeconomic modelling, one should start with heterogeneous behavioural rules at the micro level and determine the aggregate (macroeconomic) quantity – such as GDP – by adding up the levels of a myriad of individual quantities. The increasing availability of computational power has allowed the implementation of this bottom-up procedure in multi-agent models. Not surprisingly, in the last ten years or so, a proliferation of agent-based models has paralleled the diffusion of research on issues concerning heterogeneity[2].

Multi-agent modelling is the most straightforward way of tackling the heterogeneity issue. In the profession at large, however, there is no agreement on the opportunity of following this methodology. While some colleagues, mainly in the unorthodox camp, eagerly embrace the new research strategy, some others, mainly in the mainstream, are skeptical or even dismissal. There are at least three reasons for this skepticism: (i) a basic distrust for the output of computer simulations, which is generally very sensitive to the choice of initial conditions and parameter values; (ii) a critique of adaptive micro-behavioural rules which are often considered ad hoc; (iii) the difficulty and sometimes the impossibility of thinking in macroeconomic terms, i.e. of using macro-variables in the theoretical framework.

The first type of skepticism is rapidly fading away. After all, also Real Business Cycle models are too complicated to be solved by pen and paper and must be simulated. In order to do so RBC theorists have developed procedures to calibrate their models which, with the passing of time and the spreading in the profession, have become standard tools – we can even call them protocols – of macroeconomic research.The same is true of agent-based models: Calibration and validation is ranking high in the agenda of multi–agent models' implementation.

[1] See Hartley (1997) for a detailed historical account of the development of the representative agent assumption and a thorough critique of its use and misuse.

[2] See Tesfatsion (2006) for a survey of agent based models.

As to the behavioural rules at the micro-level, it is true that some of the most enthusiastic believers in the heterogeneity mantra have seized the opportunity of agent based modelling to implement complex adaptive systems. Multi-agent models, in fact, allowe the comparison of the impact of different behavioural rules of thumb, which are often traced back to bounded rationality and adaptive behaviour. There is no reason, however, to assume that this is the only way of modelling individual choices. The multi-agent framework can also accomodate models of optimizing behaviour of heterogeneous agents. The model presented in this paper is a case in point as we will argue in a while.

Finally, the difficulty of thinking in macroeconomic terms can be circumvented by means of an appropriate aggregation procedure. In this paper we adopt a stochastic aggregation procedure – labelled the *Modified-Representative Agent* – which allows to resume macroeconomic thinking in a multi-agent framework[3].

The ambitious and apparently contradictory aim of the paper consists in building a macrodynamic model starting from the assumption, well corroborated by the existing evidence, that firms differ from one another according to their financial conditions. The diversity of firms' financial conditions is the only type of heterogeneity in the present framework. For the sake of analytical tractability, we keep the degree of heterogeneity at the lowest possible level, i.e. only one type of heterogeneity for only one class of agents. Therefore, we will stick to the old-fashioned representative agent assumption as far as households are concerned.

The paper is organized as follows. Section 2 describes the methodology. In section 3 we describe the behaviour of financially constrained firms. Section 4 analyzes the behaviour of households. In section 5 we describe and discuss the macroeconomic equilibrium. In section 6 we discuss the macroeconomic equilibrium under the assumption of zero growth. Finally in section 7 we derive and discuss the dynamics generated by the model. Technical details are shown in the appendix.

2 From micro to macro and return

We consider a closed economy populated by firms, households, financial intermediaries (banks) and the public sector (Government). Firms will be indexed by $i = 1, 2, .., z$ with z "large". Each firm is characterized by a certain degree of financial robustness, captured by the equity ratio that is the ratio of the equity base or net worth to the capital stock $a_{it} = \dfrac{A_{it}}{K_{it}}$. Starting from the distribution of the firms' equity ratio, we build a macrodynamic model in six steps.

Step 1. First of all we derive a behavioural rule at the microeconomic level for investment activity in an optimizing framework (section 3). We adopt

[3] The procedure has already been used. See Agliari et al. (2000). It is thoroughly discussed and compared with other aggregation procedures in Gallegati et al. (2006) where it is labelled the Variant-Representative-Agent methodology, with a somewhat paradoxical touch.

an optimizing perspective precisely to show that a multi-agent framework can accomodate optimizing behavior. Following Greenwald and Stiglitz (1993) each firm is assumed to maximize expected profit less expected bankruptcy costs. From the optimization we derive individual investment as a non-linear function of the individual equity ratio given the level of some macroeconomic variables – i.e. variables which are uniform across agents – such as the real wage and the real interest rate.

Step 2. Second, we apply the aggregation procedure mentioned above to the individual investment functions. Due to the non-linearity of the individual equations, we obtain average investment as a non-linear function of the moments of the distribution of the equity ratio. For the sake of simplicity, we cut short the procedure and consider only the first and second moments, that is the *mean and the variance of the equity ratio.* The distribution of the equity ratio is changing over time [4] and affects investment accordingly. In a sense, changes in the distribution of the equity ratio, proxied by changes in the first and second moments act as shocks of a distributive nature on investment.

Investment is a crucial part of any macroeconomic story. It is the most volatile component of aggregate demand and an engine of aggregate supply inasmuch as it expands capacity. Therefore, the changing distribution of financial conditions affects aggregate demand and supply through investment.

Step 3. The third step in developing the macroeconomic model consists in framing investment in a general equilibrium context. In order to do so, we have to analyze households' behaviour (section 4). We keep things as simple and as close to the mainstream conceptual framework as possible. The representative household chooses the optimal consumption plan and desired money balances maximizing utility over an infinite horizon subject to a sequence of budget constraints which incorporate money and bonds.

In equilibrium on the goods market, the consumption of the representative agent, together with investment and Government expenditure, yields a relationship between the interest rate and the output gap reminiscent of a IS curve. The moments of the distribution of the equity ratio are shift parameters of the IS curve.

In equilibrium on the money market, the demand for money of the representative agent, together with the (exogenous) money supply yields a relationship between the interest rate, the output gap and real money balances reminiscent of a LM curve.

Finally, imposing a zero-growth condition in order to focus on business fluctuations, we obtain a relationship between the interest rate and the moments of the distribution of the equity ratio. In the end we obtain a simple macroeconomic model, which can be solved for the equilibrium values of the interest rate, the output gap and the price level (section 5). All of the endogenous variables in equilibrium turn out to be functions of the moments of the distribution of the equity ratio.

[4] Greenwald and Stiglitz (1993) do consider the variance of agents'net worth as an important factor in the reaction of the macroeconomy to a shock but do not take up this issue in a dynamical context.

Steps one, two and three are the milestones of a route from the micro to the macro level. In a sense they provide the microfoundations of a macroeconomic model with heterogeneous agents. The difference between the traditional microfoundations based on the representative agent and the new ones is the explicit consideration in the latter of the moments of the distribution of the equity ratio[5]. Since moments are a concise measure of the shape of the distribution, by focusing on moments we resume macroeconomic thinking in its purest form, i.e. at a general, non microeconomic, level.

So far, we have treated the moments of the distribution as pre-determined variables. In order to endogenize the dynamics of the moments, we have to go back to the micro level and focus on the law of motion of the individual equity ratio which is a function, among other things, of the interest rate (section 7).

Step 4. The fourth step consists in plugging the equilibrium value of the interest rate – which is a function of the moments of the distribution – into the individual law of motion. As a consequence, the current equity ratio turns out to be a function not only of the individual lagged equity ratio but also of the lagged average equity ratio and variance. A mean field effect is at work: the average or macro state variable, in fact, affects the micro state variable. In a sense we incorporate a macrofoundation of the micro-dynamics.

Step 5. The fifth step consists in describing the dynamics of the moments. Two paths can be followed. The individual law of motion can be simulated in a multi-agent setting and macroeconomic aggregates can be determined by adding up individual quantities. The moments are computed directly from the empirical distribution obtained from simulated data. As an alternative, one can apply an aggregation procedure to the individual law of motion and determine a two dimensional non-linear dynamic system in discrete time which describes the evolution over time of the mean and the variance of the distribution itself.

Step 6. The sixth and final step consists in exploring the feedback from the dynamics of the moments to the aggregate variables. Both the steady state solution and transitional dynamics are interesting determinants of the aggregate.

3 Firms

Firms will be indexed by $i = 1, 2, .., z$ with z "large". They produce a homogeneous good by means of capital and labor and invest in order to expand capacity. The assumption of a large number of firms which produce an undifferentiated good implies that the market structure is competitive, i.e. firms are price takers.

Financial conditions. Firms are heterogeneous with respect to their financial robustness captured by the equity ratio a_{it}. In other words, the equity ratio of the i-th firm at time t a_{it} is a random variable with support $(0, 1)$, whose

[5] In our simple case, since we cut short the aggregation procedure and consider only the first and second moments, the only difference between old and new microfoundations is the variance of the distribution. In fact the first moment, i.e. the mean of the equity ratio, would be present also in the traditional microfoundations. The equity ratio of the representative agent coincides with the mean of the distribution of the equity ratio when the variance is zero.

distribution is characterized by expected value $E(a_{it}) = a_t$ and variance $E(a_{it} - a_t)^2 = V_t$. The expected value is the equity ratio of the average agent (average equity ratio for short). The variance measures the dispersion of the actual equity ratios around the average. The representative agent is a particular case of this framework: it coincides with the average agent when the variance is zero. In other words the representative agent is the zero-variance average agent.

Firms cannot raise external finance on the equity market (due to equity rationing: Myers and Majluf, 1984; Greenwald *et al.,* 1984) so that they have to rely on bank loans to finance investment. Therefore, they run the risk of bankruptcy. Banks extend credit to firms at an interest rate r which is uniform across firms and equal to the interest rate on bonds.

Technology and market structure. Each firm carries on production by means of a constant returns to scale technology that uses labor and capital as inputs. For simplicity we assume that technology is of the Leontief type and uniform across firms.The production function of the i-th firm therefore is $Y_i = \min(\lambda N_i, \nu K_i)$ where Y_i, N_i and K_i represent output, employment and capital (in the current period, i.e. at time t), ν and λ are positive parameters which measure the productivity of capital and labour respectively.

Assuming that labour is always abundant, we can write $Y_i = \nu K_i$ and $N_i = \frac{\nu}{\lambda} K_i = \frac{Y_i}{\lambda}$. ν is the reciprocal of the capital/output ratio. $\frac{\nu}{\lambda}$ is the reciprocal of the capital/labour ratio. Since these parameters are constant, by assumption output, capital and employment grow at the same rate. We will determine the rate of capital accumulation endogenously (see below) and will assume that output and employment grow at the same rate of the capital stock.

Profit. Profit of the i-th firm in real terms in the current period (π_i) is the difference between revenues ($u_i Y_i$) and total costs, which consist of production costs ($wN_i + rK_i$) and adjustment costs $\left(\frac{1}{2}\frac{I_i^2}{\bar{K}}\right)$:

$$\pi_i = u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} \tag{1}$$

u_i is the average revenue of the firm. For the sake of simplicity we assume that it is a random variable uniformly distributed over the interval $(0, 2)$ with $E(u_i) = 1$ where $E(u_i)$ is the expected value of the firms' average revenues, i.e. the expected average revenue; w is the real wage rate, r is the real interest rate, $I_i = K_i - K_{it-1}$ is investment[6] and $\bar{K} = \frac{K}{z}$ is the average capital stock, $K = \sum_{i=1}^{z} K_i$ being the aggregate capital stock. For the moment we assume that the real wage is given and constant.

The profit function is characterized by an idiosyncratic shock to revenues due, for instance, to a sudden change in preferences. Adjustment costs are quadratic in investment (as usual in investment theory) and decreasing in the average capital stock, i.e. we assume a *positive externality* in the accumulation

[6] For the sake of simplicity we assume that there is no depreciation.

of capital: the higher the economywide capital stock, the lower adjustment costs for the single firm. This is essentially a technical assumption, which allows to determine a relatively simple interior solution to the firm's optimization problem (see below).

Bankruptcy. The probability of bankruptcy for the i-th firm depends, among other things, on the equity ratio (see the appendix for a discussion). For the sake of analytical tractability we assume that firms adopt the following proxy of the probability of bankruptcy:

$$\Phi_i \approx \frac{\alpha}{a_{it-1}} \tag{2}$$

where $0 < \alpha < 1$. From (2) follows that the firm goes bankrupt with probability one if the equity ratio falls to α. Therefore the minimum of the equity ratio, i.e. the threshold the firm should not pass otherwise it goes bankrupt is α. On the other hand, since the maximum equity ratio is one, the minimum probability of bankruptcy is α. Hence both a_{it-1} and Φ_i are defined on the interval $(\alpha, 1)$. Also the definition (2) is essentially a technical assumption. More complicated formulations of the probability of bankruptcy would have made the model very difficult to manage without adding much to the results.

Bankruptcy is costly and the cost of bankruptcy is an increasing linear function of the capital that the firm owns, i.e. $CB_i = \beta K_i$ where β is a positive parameter.

The objective function of the firm V_i is the difference between expected profit $E(\pi_i)$ and bankruptcy (or borrower's) risk, i.e. bankruptcy cost in case bankruptcy occurs $CB_i\Phi_i$:

$$V_i = E(\pi_i) - CB_i\Phi_i = Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} - \beta K_i \frac{\alpha}{a_{it-1}} \tag{3}$$

In case there were no bankruptcy, i.e. $\beta = 0$, (3) would boil down to:

$$E(\pi_i) = Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} \tag{4}$$

i.e to expected profit. Comparing the objective functions (3) and (4) we see that (3) is smaller than (4) i.e. to expected profit.

Recall now that $Y_i = \nu K_i$ and $N_i = \frac{\nu}{\lambda}K_i$. As a consequence, the problem of the firm can be formulated as follows:

$$\max_{K_i} V_i = \left[\nu\left(1 - \frac{w}{\lambda}\right) - r\right]K_i - \frac{1}{2}\frac{(K_i - K_{it-1})^2}{\bar{K}} - \beta\alpha\frac{K_i}{a_{it-1}}$$

where the control variable is the individual capital stock[7]. Notice that, due to the Leontief technology, once the stock of capital has been optimally deter-

[7] In order to ensure that V_i is positive we impose the following restriction on parameters:

$$a_{it-1} > \beta\alpha \frac{K_i}{K_i\left[\nu\left(1 - \frac{w}{\lambda}\right) - r\right] - \frac{1}{2}\frac{I_i^2}{\bar{K}}}$$

mined solving the problem above, both output and employment follow being proportional to capital.

From the FOC we obtain:

$$\tau_i = \gamma - r - \frac{\beta\alpha}{a_{it-1}} \tag{5}$$

where $\tau_i \equiv \frac{I_i}{\bar{K}}$ is the investment ratio and $\gamma \equiv \nu\left(1 - \frac{w}{\lambda}\right)$.

In principle τ_i can be negative. $\tau_i < 0$ if the capital stock is shrinking, i.e. $I_i = K_i - K_{it-1} < 0$, a situation which we could not rule out – due for instance to a process of "creative distruction" which requires stripping obsolete machinery – but which should be relatively rare. The most common scenario in which capital is growing occurs if $\tau_i > 0$. According to (5) $\tau_i > 0$ iff $a_{it-1} > \frac{\beta\alpha}{\gamma - r}$.

According to (5), ignoring technological parameters, the individual investment ratio depends on two variables which are uniform across firms (the costs of primary inputs w, r) and on one individual variable, namely the degree of financial robustness a_{it-1}. In particular, as one could expect, the investment ratio is decreasing with the costs of the primary inputs: $\tau_{iw} = -\frac{\nu}{\lambda} < 0, \tau_{ir} = -1 < 0$ and increasing with the equity ratio. In fact

$$\frac{\partial\tau_i}{\partial a_{it-1}} = \frac{\beta\alpha}{a_{it-1}^2} > 0$$

In the absence of bankruptcy costs ($\beta = 0$) we obtain the first best:

$$\hat{\tau} = \gamma - r \tag{6}$$

According to (6), in the first best the investment ratio depends only on the costs of primary inputs w, r. Of course, financial robustness a_{it-1} has no role to play. Notice that, according to intuition, in the first best the investment ratio is always greater than in the presence of the risk of default: $\tau_i = \hat{\tau} - \frac{\beta\alpha}{a_{it-1}}$.

The average investment ratio τ is the average of individual investment ratios:

$$\tau = \frac{1}{z}\sum_{i=1}^{z}\tau_i = \hat{\tau} - \frac{\beta\alpha}{z}\left[\frac{1}{a_{1t-1}} + \frac{1}{a_{2t-1}} + ... + \frac{1}{a_{zt-1}}\right]$$

Hence, it depends on the costs of primary inputs w and r, which are uniform across firms and on the distribution of the firms'degree of financial robustness $(a_{1t-1}, a_{2t-1}, .., a_{zt-1})$.

In the following we will "summarize" the distribution with its first and second moments. In order to do so we approximate the individual investment ratio in the neighborhood of average financial robustness $(E(a_{it-1}) = a_{t-1})$ as follows:

$$\tau_i \approx \tau_R + \frac{\partial\tau_i}{\partial a_{it-1}}\mid_{a_{t-1}}(a_{it-1} - a_{t-1}) + \frac{1}{2}\frac{\partial^2\tau_i}{\partial a_{it-1}^2}\mid_{a_{t-1}}(a_{it-1} - a_{t-1})^2$$

where:

$$\tau_R = \hat{\tau} - \frac{\beta\alpha}{a_{t-1}}$$

$$\frac{\partial \tau_i}{\partial a_{it-1}} \mid_{a_{t-1}} = \frac{\beta\alpha}{a_{t-1}^2} > 0$$

$$\frac{\partial^2 \tau_i}{\partial a_{it-1}^2} \mid_{a_{t-1}} = -\frac{2\beta\alpha}{a_{t-1}^3} < 0$$

τ_R is the investment ratio of the Representative Agent[8].

We can compute the average investment ratio taking the expected value of the expression above:

$$\tau = E(\tau_i) \approx \tau_R + \frac{\beta\alpha}{a_{t-1}^2} E(a_{it-1} - a_{t-1}) - \frac{\beta\alpha}{a_{t-1}^3} E(a_{it-1} - a_{t-1})^2 \tag{7}$$

Notice that by definition of expected value $E(a_{it-1} - a_{t-1}) = 0$. Moreover $E(a_{it-1} - a_{t-1})^2 = V_{t-1}$ is the variance of the distribution of equity ratios. Therefore equation (7) boils down to:

$$\tau \approx \tau_R - \frac{\beta\alpha}{a_{t-1}^3} V_{t-1} = \hat{\tau} - \frac{\beta\alpha}{a_{t-1}} - \frac{\beta\alpha}{a_{t-1}^3} V_{t-1} = \gamma - r - \frac{\beta\alpha}{a_{t-1}} \left(1 + \frac{1}{a_{t-1}^2} V_{t-1}\right) \tag{8}$$

Ignoring technological coefficients, in the following we will refer to the average investment ratio with the expression:

$$\tau = \gamma - r - f(a_{t-1}, V_{t-1}) \tag{9}$$

where

$$f(a_{t-1}, V_{t-1}) = \frac{\beta\alpha}{a_{t-1}} \left(1 + \frac{1}{a_{t-1}^2} V_{t-1}\right)$$

is a function of the relevant moments of the distributions of the firms according to financial robustness. In the following, we will define a *positive distributive shock* as a change in one or more of the basic features of the distributions which boosts the average investment ratio. Therefore a positive distributive shock could be an increase of a_{t-1} or a decrease of V_{t-1}.

Notice that the average investment ratio in the presence of heterogeneity τ is smaller than the investment ratio of the representative agent τ_R which in turn is smaller than the investment ratio in the first best $\hat{\tau}$.

4 Households

As to households, for the sake of simplicity, we assume that they are homogeneous in every respect so that we can adopt the representative agent hypothesis.

[8] τ_R is greater than 0 iff $a_{t-1} > \frac{\beta\alpha}{\hat{\tau}}$.

Households demand consumption goods, financial assets (bonds) and money balances (deposits). Money balances are desirable because they provide "liquidity services" which are necessary if transactions require a means of payment, as we will assume. Assuming, for the sake of simplicity, that there is no currency, base money coincides with banks' reserves at the central bank and money coincides with deposits. We assume that deposits are not remunerated. The interest rate on bonds is the opportunity cost of holding money.

The representative household supplies inelastically one unit of labour. Since by assumption all the profits are retained within the firm, the only source of income for the representative agent is the wage rate if employed, the unemployment subsidy if unemployed. In symbols, income is w_k with $k = u, e$ where e stands for employed and u for unemployed: $w_e = w$ (i.e. the real wage rate), $w_u = \sigma$ (i.e. the unemployment subsidy).

The lifetime utility function of the representative household is:

$$U_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+\rho} \right)^s (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}} \right)^{1-\delta} \tag{10}$$

where c_t is consumption, m_t are (per capita) money balances, ρ is the rate of time preference, $0 < \delta < 1$.

The household's budget constraint is:

$$c_t + \frac{m_t + b_t}{P_t} = w_k + \frac{m_{t-1}}{P_{t-1}}\theta_{t-1} + (1+i)\frac{b_{t-1}}{P_{t-1}}\theta_{t-1} \tag{11}$$

where m_t, b_t are money and bonds (per capita) respectively. w_k is income, $\theta_{t-1} \equiv \dfrac{P_{t-1}}{P_t}$ is the real gross rate of return on money holdings. By definition, $\dfrac{1}{\theta_{t-1}} - 1$ is the inflation rate. The expression $(1+i)\,\theta_{t-1} = 1 + r_t$ is the real gross interest rate.

According to the budget constraint, the sum of consumption and the demand for money and bonds should be equal to income plus interest payments $(1+i)\,b_{t-1}$ and money balances m_{t-1} (carried over from the previous period).

The problem of the representative household, therefore, consists in maximizing (10) subject to (11) or:

$$\begin{aligned} &\max_{c_t, m_t/P_t, b_t/P_t} U_t = \textstyle\sum_{s=0}^{\infty} \left(\frac{1}{1+\rho} \right)^s (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}} \right)^{1-\delta} \\ &s.t. \quad c_t = w_k + \frac{m_{t-1}}{P_{t-1}}\theta_{t-1} + (1+i)\frac{b_{t-1}}{P_{t-1}}\theta_{t-1} - \frac{m_t}{P_t} - \frac{b_t}{P_t} \end{aligned} \tag{12}$$

From which we obtain the following Lagrangian:

$$
\begin{aligned}
L &= \sum_{s=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{s}(c_{t+s})^{\delta}\left(\frac{m_{t+s}}{P_{t+s}}\right)^{1-\delta}+ \\
&\quad+\left(\frac{1}{1+\rho}\right)^{s}\lambda_{t+s}\left[w_k+\frac{m_{t-1+s}}{P_{t-1+s}}\theta_{t-1+s}+\right. \\
&\quad\left.+(1+i)\frac{b_{t-1+s}}{P_{t-1+s}}\theta_{t-1+s}-\frac{m_{t+s}}{P_{t+s}}-\frac{b_{t+s}}{P_{t+s}}-c_{t+s}\right]
\end{aligned}
$$

The FOCs are:

$$
\frac{\partial L}{\partial c_t}=\delta\,(c_t)^{\delta-1}\left(\frac{m_t}{P_t}\right)^{1-\delta}=\lambda_t \tag{13}
$$

$$
\frac{\partial L}{\partial m_t/P_t}=(1-\delta)\,(c_t)^{\delta}\left(\frac{m_t}{P_t}\right)^{-\delta}-\lambda_t+\left(\frac{1}{1+\rho}\right)\lambda_{t+1}\theta_t=0 \tag{14}
$$

$$
\frac{\partial L}{\partial b_t/P_t}=-\lambda_t+\left(\frac{1}{1+\rho}\right)\lambda_{t+1}\,(1+i)\,\theta_t=0 \tag{15}
$$

Solving (13) (14) (15) for $c_t, m_t/P_t$ we obtain the following relation between optimal consumption and money demand:

$$
\frac{m_t}{P_t}=\frac{1-\delta}{\delta}\frac{1+i}{i}c_t \tag{16}
$$

We assume that changes in money supply are implemented by means of open market operations. Therefore:

$$
\frac{m_t}{P_t}-\frac{m_{t-1}}{P_{t-1}}\theta_{t-1}=-\left[\frac{b_t}{P_t}-(1+i)\frac{b_{t-1}}{P_{t-1}}\theta_{t-1}\right] \tag{17}
$$

Substituting (16) and (17) into (11) we obtain the optimal consumption function and money demand function for the representative household:

$$
\begin{aligned}
c_t &= w_k \\
\frac{m_t^d}{P_t} &= \frac{1-\delta}{\delta}\frac{1+i}{i}w_k
\end{aligned}
$$

Total consumption is the sum of consumption of the employed and consumption of the unemployed people. Each type of consumption, in turn, is the product of per-capita consumption times the number of agents in each group (employed and unemployed people respectively). Therefore:

$$
C_t=wN_t+\sigma(L-N_t) \tag{18}
$$

where L is total labour force.

Total demand for money is the sum of the demand for money of the employed and of the unemployed people. Each type of demand for money, in turn, is the product of per-capita demand times the number of agents in each group (employed and unemployed people respectively). Therefore:

$$\frac{M_t^d}{P_t} = \frac{1-\delta}{\delta}\frac{1+i}{i}\left[wN_t + \sigma(L-N_t)\right] \tag{19}$$

5 The macroeconomic equilibrium

In this economy there will be markets for labor, goods and financial assets. We will not impose an equilibrium condition on the labor market. In other words the labor market can be characterized by underemployment associated with equilibrium on the money and goods markets. Thanks to Walras' law, there will be also equilibrium on the market for financial assets.

The goods market is in equilibrium (planned expenditure is equal to actual expenditure) if $C_t + I_t + G_t = Y_t$ where C_t is aggregate consumption, I_t is aggregate investment, G_t is government expenditure. C_t is defined as in (18). Investment is $I_t = \tau K_t$ where $\tau = \gamma - r - f$ and $f = f\left(a_{t-1}, V_{t-1}\right)$(see equation (9)).

The Government carries on public expenditure, which can be thought of as investment (for instance in infrastructures). For the sake of simplicity, it does not raise taxes[9]. We assume that Government expenditure is proportional to the investment gap, i.e. the difference between the first best investment ratio and the current average investment ratio:

$$G_t = \varepsilon\left(\hat{\tau} - \tau\right)K_t = \varepsilon f K_t \tag{20}$$

with $0 < \varepsilon < 1$. Therefore, in equilibrium the following must hold true:

$$wN_t + \sigma(L-N_t) + \left(\gamma - r - f\right)K_t + \varepsilon f K_t = Y_t \tag{21}$$

Dividing by N_t, and recalling that $\frac{Y_t}{N_t} = \lambda$, $\frac{K_t}{N_t} = \frac{\lambda}{\nu}$ we can rewrite (21) as follows:

$$w + \sigma\left(\frac{1}{x_t} - 1\right) + \left(\gamma - r - f\right)\frac{\lambda}{\nu} + \varepsilon f\frac{\lambda}{\nu} = \lambda \tag{22}$$

where $x_t = \frac{N_t}{L}$ is the ratio of employment to population. [10]Notice that, thanks to the linearity of technology, x_t can be thought of also as the output gap[11]. Equation (22) can be solved for r, yielding:

$$r = \frac{\nu}{\lambda}\left(\frac{1}{x_t} - 1\right)\sigma - \left(1-\varepsilon\right)f \tag{23}$$

This relation between r and x represents the IS curve of our model.

[9] Therefore the budget deficit coincides with the sum of Government expenditure and unemployment subsidies.

[10] As a consequence, the unemployment rate is $u_t = 1 - x_t$

[11] In fact, $Y_i = \nu K_i = \lambda N_i$. Hence $N_t = Y_i/\lambda$ and $L = \hat{Y}/\lambda$ where $\hat{Y}$ is potential output.

We now turn to the *money market*. Total demand for money is represented by equation (19). In order to simplify the analysis we can assume that $\theta_t = 1$ (i.e. the price level is not growing over time) so that the nominal and real interest rate coincide . $i = r$. Moreover, imposing the equilibrium condition $M_t^d = \overline{M}_t^s$ where $\overline{M}_t^s$ is money supply and normalizing by L the equilibrium condition on the money market $\frac{\overline{M}_t^s}{P_t} = \frac{1-\delta}{\delta}\frac{1+r}{r}\left[wN_t + \sigma\left(L - N_t\right)\right]$ we get:

$$\frac{\bar{m}_t^s}{P_t} = \delta' \frac{1+r}{r}\left[\left(w - \sigma\right)x_t + \sigma\right] \tag{24}$$

where $\delta' = \frac{1-\delta}{\delta}$ and $\bar{m}_t^s = \frac{\overline{M}_t^s}{L}$ is per-capita money supply. This relation between r and x represents the LM curve of our model.

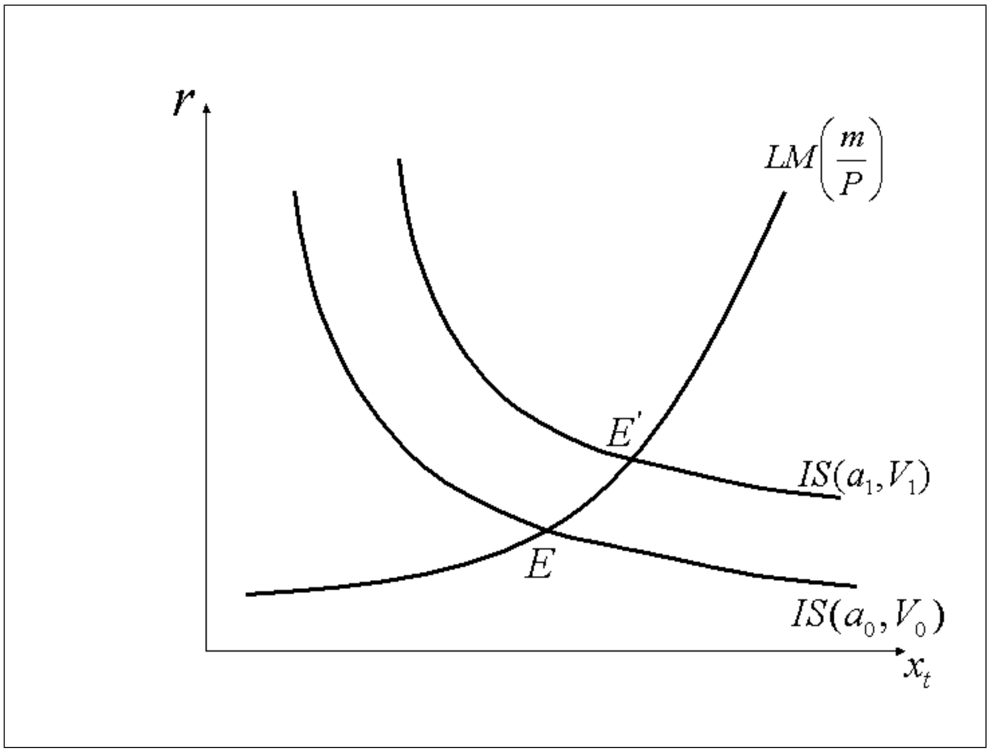

Figure 1: Macroeconomic Equilibrium

In figure 1 we represent the macroeconomic equilibrium. The moments of the distribution of the equity ratio are shift parameters of the IS curve. For each level of the real interest rate, the higher the mean and the lower the variance of the equity ratio, the higher will be employment and output. If for example we consider a certain value of the mean (a_0) and of the variance (V_0) of the distribution, given the LM curve, the economy converge to the equilibrium E. At this point we consider the case in which the mean of the distribution increases while the variance decreases, the IS curve shifts up and the new macroeconomic equilibrium is represented by the point E' with an higher level of both the outgap and the real interest rate. We are able to infer that changes in the moments of the distributions of the equity ratio have aggregate effects.

6 A convenient special case

Let's now turn to the supply side of the model. Due to linearity of the technology, aggregate output and employment grow at the same rate as the capital stock. The rate of growth of the capital stock – say g – in turn is an increasing non linear function of the investment ratio. It is easy to see, in fact, that $1+g=\dfrac{1}{1-\tau}$. In order to focus on business fluctuations, we impose a *zero-growth* condition $g=0$ which implies $\tau=0$. Imposing this condition into (9) we get:

$$r=\gamma-f \tag{25}$$

In this special case, in fact aggregate investments are equal to zero by definition and Government expenditure becomes $G_t=\varepsilon\hat{\tau}K_t$, i.e. it depends only on the first best investment ratio. Therefore, imposing the market clearing condition we get the IS curve under the zero growth assumption as follows:

$$r=\frac{\nu}{\lambda}\left(\frac{1}{x_t}-1\right)\frac{\sigma}{\varepsilon}-\nu\left(1-\frac{w}{\lambda}\right)\left(\frac{1}{\varepsilon}-1\right) \tag{26}$$

Notice that under the zero growth condition the IS curve does not depend on the function f, whose arguments are the moments of the distribution of the equity ratio.

The macroeconomic model in structural form with zero growth, therefore consists of equations (26) (**??**) and (25). From the third equation we obtain the equilibrium real interest rate:

$$r^*=\nu\left(1-\frac{w}{\lambda}\right)-f=r^*(a_{t-1},V_{t-1}) \tag{27}$$

Note that $r^*>0$ iff $f<\nu\left(1-\dfrac{w}{\lambda}\right)$. Substituting (27) into the (26) we are able to get the steady state value of the output gap x^*:

$$x^*=\frac{1}{\left(\lambda-w-\frac{\varepsilon}{\nu}f\right)\frac{1}{\sigma}+1} \tag{28}$$

Finally from the LM curve we get the steady state value of the real money balances:

$$\left(\frac{\overline{m}}{P}\right)^*=\delta'\frac{1+r^*}{r^*}\left[w+\sigma\left(\frac{1}{x^*}-1\right)\right] \tag{29}$$

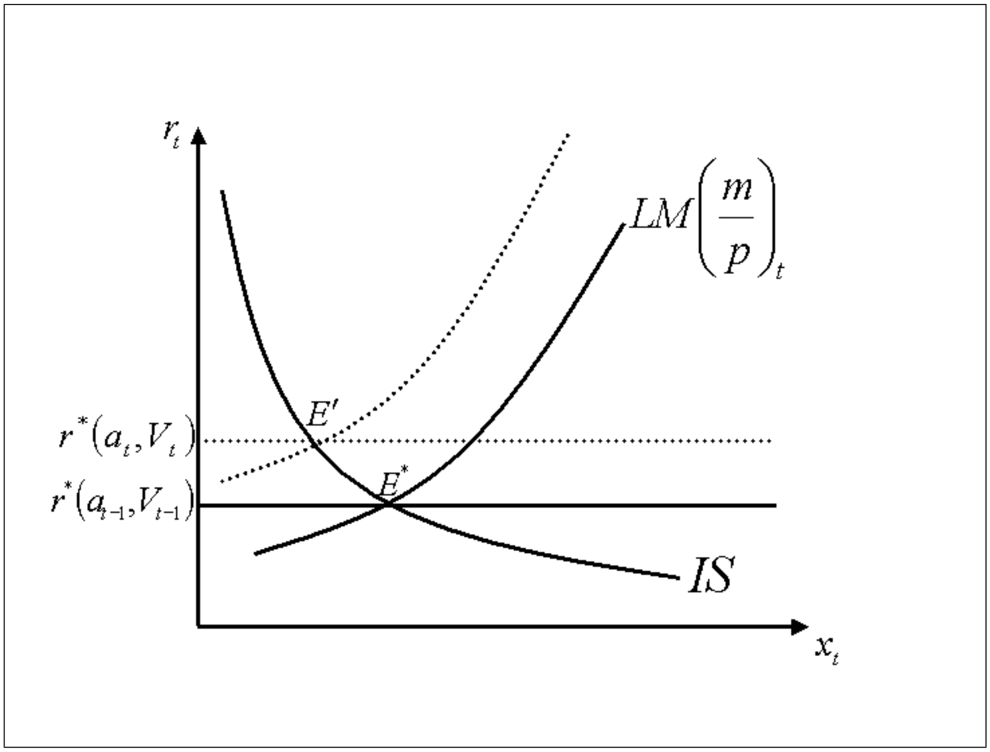

Figure 2: Effects of a change in the moments of the distribution of the equity ratio under zero growth assumption

Since the nominal money supply is exogenous, equation (29) implicitly determines the price level in equilibrium.

In figure 2 we represent the macroeconomic equilibrium under the assumption of zero growth. The equilibrium real interest rate $r^*(a_{t-1}, V_{t-1})$, as defined in equation (27) depends on the mean and the variance of the distribution. In particular the higher the mean and the lower the variance of the equity ratio, the higher will be the real interest rate. Given $r^*(a_{t-1}, V_{t-1})$ the LM curve crosses the IS curve in point E^* that represents the equilibrium of our economy. Consider know the case in which the mean of the distribution increases while the variance decreases, the real interest rate will increase ($r^*(a_t, V_t)$). Differently from the previous section, in this case the IS curve will not shift. Given the value of the real interest rate $r^*(a_t, V_t)$, the prices adjust so that the LM curve will move up and will intersect the IS curve in the new equilibrium point E', that is characterized by a lower output gap. Even if the IS curve is independent from the moments of the distribution of the equity ratio, changes in the mean and the variance of the distribution have aggregate effects.

7 Dynamics

In this section we explore the dynamics stemming from the macroeconomic model presented in the previous section. First of all we have to establish the law of motion of the individual net worth. Assuming that there are no dividends, the level of net worth in real terms for the i-th firm in t is $A_{it} = A_{it-1} + \pi_i$. Recalling (1) we get:

$$A_{it} = A_{it-1} + u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}}$$

Dividing by the capital stock we obtain the law of motion of the equity ratio:

$$a_{it} = a_{it-1}\frac{K_{it-1}}{K_i} + u_i\nu - w\frac{\nu}{\lambda} - r - \frac{1}{2}\frac{I_i^2}{K_i\bar{K}} \tag{30}$$

Recall that $\frac{K_{it-1}}{K_i} = 1 - \frac{I_i}{K_i} = 1 - \frac{\bar{K}}{K_i}\tau_i = 1 - \frac{\tau_i}{k_i}$ where $k_i \equiv \frac{K_i}{\bar{K}}$. Moreover $\frac{I_i^2}{K_i\bar{K}} = \frac{I_i^2}{K_i}\frac{\bar{K}}{\bar{K}^2} = \frac{\tau_i^2}{k_i}$. Plugging these expressions into (30) we obtain:

$$a_{it} = a_{it-1}\left(1 - \frac{\tau_i}{k_i}\right) + u_i\nu - w\frac{\nu}{\lambda} - r - \frac{1}{2}\frac{\tau_i^2}{k_i} \tag{31}$$

but, according to (5), $\tau_i = \nu\left(1 - \frac{w}{\lambda}\right) - r - \frac{\beta\alpha}{a_{it-1}}$ and according to (9) $r = \nu\left(1 - \frac{w}{\lambda}\right) - f$. Therefore:

$$\tau_i = f - \frac{\beta\alpha}{a_{it-1}} \tag{32}$$

Substituting (32) into (31) and assuming that $k_i \simeq 1$ – i.e. assuming that the firms differ in terms of financial conditions but have approximately the same size – after rearranging and simplifying we get:

$$a_{it} = a_{it-1}\left[1 - f\right] + \Gamma + f - \frac{1}{2}\left[f - \frac{\beta\alpha}{a_{it-1}}\right]^2 \tag{33}$$

where $\Gamma \equiv \beta\alpha - (1 - u_i)\,\nu$. The expression above represents the individual law of motion of the equity ratio.It is a non linear first order difference equation in the state variable a_{it}.

Equation (33) can be simulated. For instance, in figure 3 we have reported the first and second moment of the distribution of the equity ratio generated by (33). The two moments seem to be positively correlated. When on average the financial conditions of firms improve, also the dispersion of individual financial conditions around mean increases. In principle, the effects of a simultaneous increase of the mean and the variance on the aggregate macroeconomic performance are uncertain. In fact the model predicts that aggregate output goes up if either the mean increases or the variance decreases.

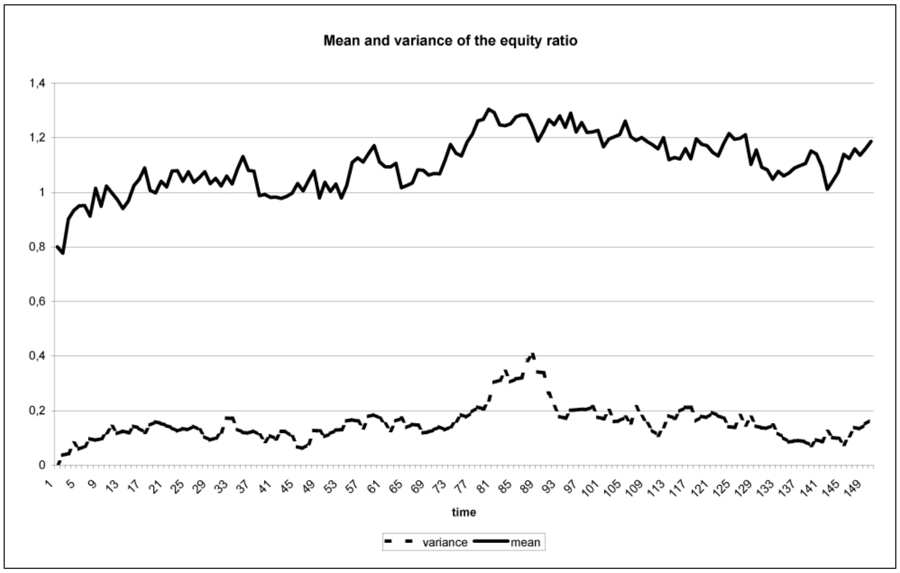

Figure 3: Mean and variance of the distribution

It is worth noting that refining the aggregation procedure described in section 3 and applied to the investment ratio, we could take into account also the impact of the evolution of higher moments – such as skewness – on aggregate variables. For the moment we stick to the simplest case.

In figure 4 we plot the real interest rate and the output gap. The two variables are clearly negatively correlated. This is due to the simplifying but restrictive zero growth assumption, which forces the economy to move along an IS curve that is independent from the moments of the distribution.

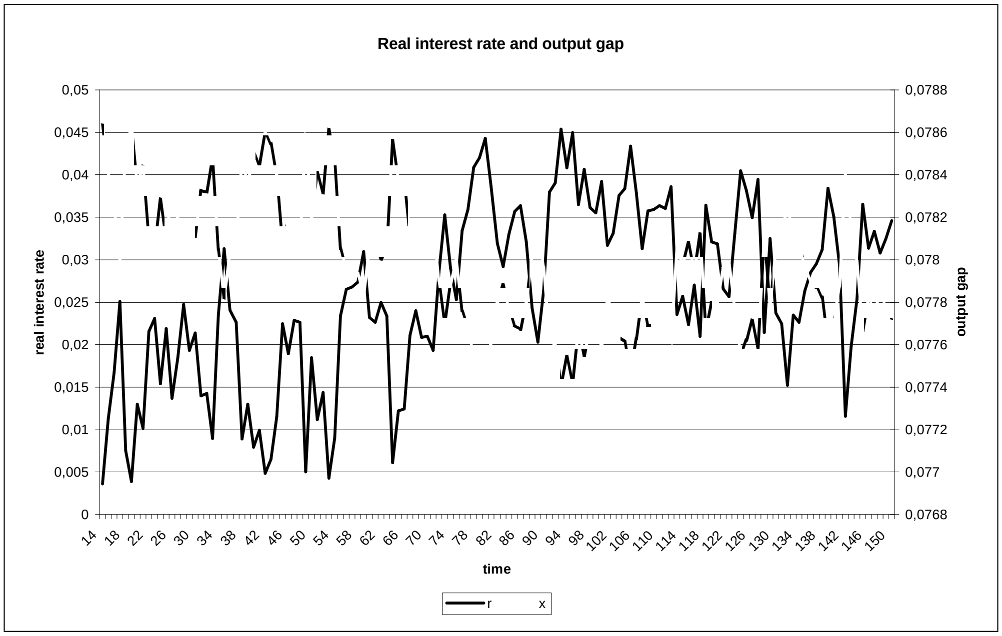

Figure 4: Real interest rate and output gap

A The probability of bankruptcy

The true probability of bankruptcy can be determined as follows.

Assuming that there are no dividends, the level of net worth in real terms for the i-th firm in t is $A_{it} = A_{it-1} + \pi_i$ where $\pi_i = u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}}$ represents the profit level. We define total cost as $TC_i = wN_i + rK_i + \frac{1}{2}\frac{I_i^2}{\bar{K}}$. Hence $A_{it} = A_{it-1} + u_i Y_i - TC_i$.

A firm goes bankrupt if $A_{it} < 0$, i.e. if:

$$u_i < AC_i - \frac{A_{it-1}}{Y_i} \equiv \bar{u}_i$$

where $AC_i = TC_i/Y_i$ is average cost. In words: the firm goes bankrupt if the realization of the random shock is smaller than a threshold $\bar{u}_i$ which in turn depends on equity, output, and the average cost. By assumption, the shock is a uniformly distributed random variable u_i with support $(0, 2)$, so that the probability of bankruptcy is:

$$\Pr(u_i < \bar{u}_i) = \frac{\bar{u}_i}{2} = \frac{1}{2}\left(AC_i - \frac{A_{it-1}}{Y_i}\right) \tag{34}$$

Let's assume, as in the text of the paper, that the cost of bankruptcy is $CB_i =$

βK_i. The objective function of the firm V_i is the difference between expected profit $E(\pi_i)$ and bankruptcy cost in case bankruptcy occurs $CB_i \Pr(u_i < \bar{u}_i)$:

$$V_i = E(\pi_i) - CB_i \Pr(u_i < \bar{u}_i) = Y_i - TC_i - \beta' Y_i \left(AC_i - \frac{A_{it-1}}{Y_i} \right)$$

with $\beta' = \beta/2\nu$. Rearranging one gets:

$$V_i = E(\pi_i) - CB_i \Pr(u_i < \bar{u}_i) = Y_i - \left(1 + \beta'\right) TC_i + \beta' A_{it-1} \tag{35}$$

The present formalization of the probability of bankruptcy makes clear that taking into account the expected bankruptcy cost in the objective function is tantamount to incurring an extra cost equal to $\beta' TC_i$ and gaining an extrarevenue equal to $\beta' A_{it-1}$.

The formalization, however, has a clear disadvantage in terms of tractability. In fact, plugging $Y_i = \nu K_i$ and $N_i = \dfrac{\nu}{\lambda} K_i$ into (34) and rearranging, the probability of bankruptcy turns out to be:

$$\Pr(u_i < \bar{u}_i) = \frac{\bar{u}_i}{2} = \frac{1}{2} \left\{ \frac{w}{\lambda} + \frac{r}{\nu} + \frac{1}{2} \frac{\left(K_i - K_{it-1}\right)^2}{\nu \bar{K} K_i} - \frac{a_{it-1}}{\nu K_i} K_{it-1} \right\}$$

The probability of bankruptcy is decreasing with the equity ratio but it depends on a large number of parameters and endogenous variables.

Moreover, maximizing (35) with respect to K_i yields:

$$\tau_i = \frac{\nu}{2 + (\beta/\nu)} - w\frac{\nu}{\lambda} - r$$

The interior solution to the maximization of V_i therefore is smaller than the first best $\hat{\tau} = \nu - w\dfrac{\nu}{\lambda} - r$ but is uniform across firms and independent of net worth. Therefore we would miss an important part of the financial fragility strory we want to tell. In order to keep net worth into the interior solution we can experiment with different bankruptcy cost functions, such as $CB_i = \beta K_i^2$. In this case however, the interior solution becomes rapidly very messy. With an acceptable loss of generality we adopt the approximation of the text.

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