Anything goes with heterogeneous, but not with homogeneous oligopoly

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December 10, 2007

Abstract

Corchón and Mas–Colell (1996) showed that in heterogeneous oligopoly (almost) everything is possible. Here it is shown that in order to obtain a similar result for homogeneous oligopoly, the reaction correspondences should fulfill a special condition.

Keywords: Cournot oligopoly, (in)stability, (limit) cycles, Morse theory.

JEL classification: D43, C62

1 Introduction

Half way the twentieth century economic theory was interested in the answers of the following three questions.

- *Does an equilibrium exist?*
- *If it exists, is it stable?*
- *If it exists, is it unique?*

There have always been economists believing the answer to all three questions *should* be *Yes!*

In General Equilibrium Theory (GET) rigorous proofs of existence were given by Arrow and Debreu [2] and McKenzie [21]. Their proofs depend on a number of, by now, standard conditions and one may find examples of non–existence of an equilibrium when these conditions are not fulfilled. In economic theory, *stable* means mostly *stable in the sense of Liapunov and attracting.* In [35] Scarf gave two examples of equilibria in an exchange economy. In the first example the equilibrium is stable in the sense of Liapunov but not attracting. All orbits different from the equilibrium are cycles. The equilibrium is a so

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called *center*[1] In Scarf's second example the equilibrium is unstable, in fact a *spiral source*, and there is an attracting limit cycle. Finally, Debreu [8] deals with economies with a finite number of equilibria. So, at least for GET, the answer to all three of the above questions is *No!*.

These answers raised the question: *Do Walras identity and continuity characterize the class of community excess demand functions?* (See Sonnenschein [39]). Or phrased different: what is possible for equilibria in GET? The answer was given by Debreu [9] and Mantel [19]: "everything is possible," sometimes formulated as: "anything goes."

Also with respect to oligopoly the answer to all three questions should be *No*. McManus [22] and Roberts and Sonnenschein [34] gave examples of oligopolies without an equilibrium. On this point see also [10, p. 67–69]. The first one to question the problem of stability was Theocharis who, in his seminal paper [40], showed that with a linear inverse demand function and a linear cost function, the (unique) Cournot equilibrium is stable when there are only two firms and unstable when there are four or more firms. When there are three firms, the equilibrium is stable in the sense of Liapunov, but not attracting. The instability was shown for a discrete Cournot tâtonnement process. In [11] and [3], as far as I know, for the first time in English, the possibility of multiple equilibria was mentioned.[2] In [11] it was shown that the questions about 'uniqueness' and 'stability' are related to each other. When there is more than one equilibrium, at least one of them is not stable.

In Furth [11] "stability and instability" for a homogeneous Cournot oligopoly model and a heterogeneous Bertrand oligopoly model was studied. In the homogeneous case there was a clear picture what was possible, but the heterogeneous model had many more possibilities. Stability of an equilibrium is always related to some dynamical (adjustment) process: the Cournot tâtonnement process. By studying dynamical systems, one always can choose between: (i) continuous dynamics; and: (ii) discrete dynamics. Here there has been chosen for the first, hence continuous time. In oligopoly theory one may choose between: (i) best reply dynamics and: (ii) gradient dynamics, see Corchón [5, p. 16].

An 'Anything goes'–result for heterogeneous Cournot and Bertrand oligopoly was derived by Corchón and Mas–Colell [6]. In section 3 it will be shown that such a result can not always be derived for a homogeneous Cournot oligopoly. This explains why in [11] I did not have a complete picture of the heterogeneous case. Normally one starts with an oligopoly, given by inverse demand– and cost functions and derives the reaction correspondences or the "marginal payoff" function from it. The reaction correspondence leads to best response dynamics, the marginal payoff function to gradient dynamics. Corchón and Mas–Colell start with arbitrary functions and show that one may find inverse demand– and cost functions such that the arbitrary functions are either the reaction functions

[1] See for the standard terminology Hirsch and Smale [16].

[2] Puu, in the introductory first chapter of [32], mentions a paper by Paleander ([29]) from 1939, in which Paleander showed the possibility of multiple equilibria. Unfortunately that paper is in Swedish. In later papers Puu also claims that Paleander was also the first to write about (in)stability in oligopoly.

or the marginal payoff functions of the oligopoly characterized by these inverse demand– and cost functions.

Corchón [5, proposition 1.1 page 11–12] shows that with an arbitrary set of strictly decreasing functions, one may find an inverse demand function and cost functions, such that the homogeneous Cournot oligopoly characterized by these functions have the functions one started with as best response functions.

The Corchón and Mas–Colell [6] result was derived for Bertrand oligopoly, but the derivation is similar for Cournot oligopoly. It will be shown that their result only holds for heterogeneous oligopoly, both Cournot and Bertrand. There is much literature on the comparison of the equilibria in (heterogeneous) Cournot and Bertrand oligopoly, see [15, 36, 4, 42, 28, 1]. As the Corchón and Mas–Colell [6] does not always hold for homogeneous oligopoly, in the present paper a comparison will be made between homogeneous and heterogeneous Cournot oligopoly. Tirole [41] considered the outcome of the homogeneous Bertrand oligopoly model as paradoxical. Reason enough for paying no attention to that model here. Those interested in a comparison of the equilibria in homogeneous Cournot and Bertrand oligopoly are referred to [7].

With an 'anything goes' result, one may try to construct oligopoly models with (i) multiple equilibria; (ii) unstable equilibria; and (ii) all kinds of dynamics, for instance chaotic dynamics or (limit) cycles. I have some examples of, what goes in heterogeneous Cournot oligopoly, does not always go for the homogeneous case.

A remark should be made here. The 'anything goes' results both in GET and oligopoly may lead to pathological cases. Although they may have little attraction to an economist, they can not be excluded on economic grounds. When one starts with 'strange' reaction functions or 'strange' marginal payoff functions, one may expect 'strange' results. But there is no reason why one should not study such strange, pathological cases. Forewarned is forearmed!

The first to show that *chaos* is possible in an duopoly model was Rand [33]. In one of Rand's examples, the reaction functions are *unimodular*. Rand assumes that, in a duopoly, such a reaction function can be obtained from the 'utility' functions of the two firms. As Rand considers the case of no costs, this utility is in fact the revenue of the firm. Revenue is equal to the quantity sold times a market clearing price. The market clearing price is, through the inverse demand a function of the total quantity sold. Hence Rand's result applies to *homogeneous* duopoly.

Following the Corchón and Mas–Colell procedure, indeed from a pair of unimodular functions one may find inverse demand functions, such that assuming these inverse demand functions and no costs will, for a duopoly, raise to the assumed functions as reaction functions. However the inverse demand functions give the prices of the commodities sold as function of both the quantities sold, not necessarily of the sum of the quantities sold. Hence the prices of the two brands may be different. The Corchón Mas–Colell result is one for *heterogeneous* oligopoly. In section 3 it will be shown that such a result can not be derived for a homogeneous Cournot oligopoly.

In section 2 first the oligopoly model is introduced. Section 3 proves the Corchón and Mas–Colell result for heterogeneous Cournot oligopoly. To derive the same result for homogeneous Cournot oligopoly, some extra conditions are needed. It will be shown that these conditions are necessary and sufficient for gradient dynamics, but only sufficient for best response dynamics. Many of the result of this paper heavily depend on the application of 'Morse Theory'. In an appendix therefor a 'rapid course' in this theory is given.

2 Cournot Oligopoly models

In this section it will be assumed that all functions are C^2, that is the functions itself and their (partial) derivatives of the first and second order are continuous.

As most functions in this paper are defined on closed sets, say S, derivatives at the boundary of S are not defined. It will be assumed that the functions are the restrictions to S of C^2–functions defined on an open neighborhood of S. The derivatives of the restrictions in boundary points of S are taken to be equal to the derivatives of the extensions in these points.

Let either $I = [0, M]$ with a certain $0 < M < \infty$, or $I = \mathbb{R}_+$, so always $0 \in I$. In the sequel the following notations will be used. $\mathbf{x} := (x_1, \ldots, x_n) \in I^n$ and for all $i \in N := \{1, \ldots, n\}$ the notations

$$\mathbf{x}_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in I^{n-1}$$

and

$$(x_i; \mathbf{x}_{-i}) := \mathbf{x}.$$

Similar notations will be used with p (prices) in stead of x (quantities) and for vector valued functions $\mathbf{f} = (f_1, \ldots, f_n)\colon U(\subset \mathbb{R}^n) \to \mathbb{R}^n$. Also define $X := \sum_{j=1}^{n} x_j$ and $X_{-i} := \sum_{j=1;j\neq i}^{n} x_j$.

There are $n(\geq 2)$ firms, each supplies a different good on an oligopolistic market. Let $x_i \in I$ be the quantity of the good supplied by firm $i \in N$. In heterogeneous Cournot oligopoly each firm $i \in N$ has a function

$$f_i\colon I^n \to \mathbb{R}_+.$$

For $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $p_i = f_i(x_1, \ldots, x_n)$ is called the *market clearing price* and the function f_i the *market clearing price function.* When the market clearing price functions $\mathbf{f} = (f_1, \ldots, f_n)\colon I^n \to \mathbf{f}(I^n)(\subset \mathbb{R}^n)$ do have an inverse, these inverse functions will be the demand functions $D_i(p_1, \ldots, p_n)$. Each D_i gives the demand in market i as a function of all (market) prices. In that case the f_i are the inverses of the demand functions. However, the possibility that the inverse functions of f_i do not exist is not excluded. Even when the functions f_i do not have an inverse, by abuse of language, they will be called the *inverse demand functions.*

In homogeneous oligopoly there is just one market clearing price and so there is only one inverse demand function:

$$f\colon J \to \mathbb{R}_+, \text{ with}$$
$$J = \sum_1^n I = nI = \left\{ \sum_{i=1}^n x_i \mid x_i \in I \text{ for all } i \in N \right\}.$$

The market clearing price $p = f(X)$ is a function of total supply. In the sequel also the notation $J_{-1} := (n-1)I = \left\{ \sum_{j=1;j\neq i}^n x_j \mid x_j \in I \right\}$ will be used.

In heterogeneous as well in homogeneous oligopoly, let $C_i\colon I \to \mathbb{R}_+$ be the cost function of firm i, who's costs are $C_i = C_i(x_i)$. Without too much loss of generality, it will mostly be assumed that $C_i(0) = 0$.

Oligopoly is fully characterized by the inverse demand function(s) and the cost functions. With these functions, in heterogeneous oligopoly profits of firm i are given by

$$\pi_i(\mathbf{x}) = x_i f_i(\mathbf{x}) - C_i(x_i),$$

while in homogeneous oligopoly, these profits are

$$\pi_i(\mathbf{x}) = x_i f(X) - C_i(x_i).$$

Firms will choose their output x_i, given the output of their rival $\mathbf{x}_{-i}$, such as to maximize profits. The *reaction correspondence* of firm i: $R_i\colon I^{n-1} \to \mathcal{P}(I)$ is defined by

$$R_i(\mathbf{x}_{-i}) := \arg\max_{x\in I} \pi_i(x; \mathbf{x}_{-i}).$$

In this $\mathcal{P}(I)$ is the *power set* of I, i.e. the set of all subsets of I. It should be clear that for homogeneous oligopoly $R_i(\mathbf{x}_{-i})$ is in fact $R_i(X_{-i})$, with $R_i\colon J_{-1} \to \mathcal{P}(I)$.

The reaction correspondences define a correspondence $\mathbf{R}\colon I^n \to \mathcal{P}(I^n)$ that, for heterogeneous oligopoly, maps

$$\mathbf{x} \mapsto \mathbf{R}(\mathbf{x}) := R_1(\mathbf{x}_{-1}) \times \cdots \times R_n(\mathbf{x}_{-n})$$

and for homogeneous oligopoly

$$\mathbf{x} \mapsto \mathbf{R}(\mathbf{x}) := R_1(X_{-1}) \times \cdots \times R_n(X_{-n}).$$

Definition 2.1 *A Cournot–Nash equilibrium* $\mathbf{x}^* \in I^n$ *of an oligopoly is a fixed point of the map* $\mathbf{R}$*. That is* $\mathbf{x}^* \in \mathbf{R}(\mathbf{x}^*)$*.*

$\mathbf{x}^*$ is a fixed point of $\mathbf{R}$ means for heterogeneous oligopoly, that $x_i^* \in R_i(\mathbf{x}_{-i}^*)$ for all i, and for homogeneous oligopoly that $x_i^* \in R_i(X_{-i}^*)$ for all i.

Define, for both heterogeneous and homogeneous oligopoly, the following *marginal payoff* functions.

$$\varphi_i(x_i; \mathbf{x}_{-i}) := \frac{\partial \pi_i(x_i; \mathbf{x}_{-i})}{\partial x_i}.$$

Consider the vectorfield[3] $\boldsymbol{\varphi}\colon I^n \to \mathbb{R}^n$, with

$$\boldsymbol{\varphi}(\mathbf{x}) = (\varphi_1(x_1; \mathbf{x}_{-1}), \ldots, \varphi_n(x_n; \mathbf{x}_{-n})).$$

Firms will choose their output, given the output of their rival, such as (locally) to maximize profits. The first order condition (from now on FOC) of a profit maximizing firm i reads

$$\varphi_i(x_i; \mathbf{x}_{-i}) = 0.$$

A solution $\mathbf{x}^*$ of the equation $\boldsymbol{\varphi}(\mathbf{x}) = 0$ is called a *zero* or a *singularity* of the vectorfield $\boldsymbol{\varphi}$.

Definition 2.2 *A singularity $\mathbf{x}^*$ of the vectorfield $\boldsymbol{\varphi}$ is called* (linearly) stable*, when all eigenvalues of the Jacobi matrix*

$$J(\boldsymbol{\varphi}(\mathbf{x}^*)) := \frac{\partial(\varphi_1(\mathbf{x}^*), \ldots, \varphi_n(\mathbf{x}^*))}{\partial(x_1, \ldots, x_n)}$$

have negative real parts.

The above definition of stability is convienient for the present paper. A mathematically 'correct' definition of (asymptotically) stability can be found for instance in Hirsch and Smale [16]. The property of the eigenvalues follows from the 'correct' definition.

The first order conditions (from now on the FOC's) of a profit maximizing firm i read

$$\varphi_i(x_i; \mathbf{x}_{-i}) = 0.$$

So a Cournot–Nash equilibrium $\mathbf{x}^*$ is a singularity of the vectorfield $\boldsymbol{\varphi}$. A Cournot–Nash equilibrium is an equilibrium that not only fulfills the FOC's, but also the Second Order Conditions (from now on SOC's). The SOC's require that for all i

$$\frac{\partial \varphi_i}{\partial o_i} < 0,$$

that is the diagonal elements of the Jacobi matrix $J(\boldsymbol{\varphi}(\mathbf{x}^*))$ are negative. So in a Cournot–Nash equilibrium, each firm is locally maximizing its profits. The set of Cournot–Nash equilibria is a subset of the set of singularities.

In the introduction it was mentioned that stability of an equilibrium is always related to some dynamical (adjustment) process. By studying dynamical systems, one always can choose between: (i) continuous dynamics; and: (ii) discrete dynamics. In this paper there has been chosen for the first, hence continuous time. In oligopoly theory one may choose between:

(a) *best reply dynamics*, which is defined by the following set of differential equations:[4] [5]

$$\dot{x}_i = R_i(\mathbf{x}_{-i}) - x_i \text{ for all } i\text{; and:}$$

[3] See the appendix on some mathematical notions.

[4] A dot above a variable denotes the derivative with respect to time t.

[5] In defining the best reply dynamics, it has been assumed that the reaction correspondences are singleton valued, hence in fact functions.

(b) *gradient dynamics* defined by:

$$\dot{x}_i = \lambda_i \varphi_i(x_i; \mathbf{x}_{-i}) \text{ for all } i,$$

where $\lambda_i > 0$ for each i. Different λ_i's represent a different adjustment speed. However in the sequel (without loss of generality) it will be assumed that all $\lambda_i = 1$.

The RHS (right hand sides) of the differential equations of the gradient dynamics, represents the *vectorfield* $\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_n)\colon I^n \to \mathbb{R}^n$. This vectorfield maps $\mathbf{x} \in I^n$ onto $(\varphi_1(x_1; \mathbf{x}_{-1}), \ldots, \varphi_n(x_n; \mathbf{x}_{-n}))$.

Let $\mathbf{x}^*$ be a singularity of the vectorfield $\boldsymbol{\varphi}$. That is one has, for all $i \in N$, that $x_i^* = R_i(\mathbf{x}_{-i}^*)$ and $\varphi_i(R_i(\mathbf{x}_{-i}^*); \mathbf{x}_{-i}^*) = 0$. With best reply dynamics firm i will increase its output x_i whenever $x_i < R_i(\mathbf{x}_{-i}^*)$ and with gradient dynamics whenever $\varphi_i(x_i; \mathbf{x}_{-i}^*) > 0$.

Lemma 2.1 *When $\frac{\partial}{\partial x_i}\varphi_i(x_i^*; \mathbf{x}_{-i}^*) < 0$ and x_i is in (a sufficient small) neighborhood of x_i^*, then $x_i < R_i(\mathbf{x}_{-i}^*)$ if and only if $\varphi_i(x_i; \mathbf{x}_{-i}^*) > 0$.*

Proof:
From the 'mean value theorem' one has

$$\varphi_i(x_i; \mathbf{x}_{-i}^*) = \varphi_i(R_i(\mathbf{x}_{-i}^*); \mathbf{x}_{-i}^*) + \frac{\partial}{\partial x_i}\varphi(\xi_i; \mathbf{x}_{-i}^*)\left(x_i - R_i(\mathbf{x}_{-i}^*)\right),$$

where $\xi_i = \lambda x_i + (1-\lambda)R_i(\mathbf{x}_{-i}^*)$ for a certain $0 \le \lambda \le 1$.

As $\frac{\partial \varphi_i}{\partial x_i}$ is continuous, there is an $\varepsilon > 0$ such that $\frac{\partial}{\partial x_i}\varphi_i(x_i; \mathbf{x}_{-i}^*) < 0$ whenever $|x_i - x_i^*| < \varepsilon$. As when $|x_i - x_i^*| < \varepsilon$, also $|\xi_i - x_i^*| = \lambda|x_i - x_i^*| < \varepsilon$. From this the conclusion of the lemma follows. □

The conclusion is when $x_i > R_i(\mathbf{x}_{-i}^*)$ and $\varphi_i(x_i; \mathbf{x}_{-i}^*) > 0$ hold simultaneously, the dynamical system is not the gradient and/or best response dynamics of some oligopoly model.

3 Anything goes in heterogeneous, but not in homogeneous oligopoly.

First the Corchón and Mas–Colell result is derived.

Proposition 3.1 *Let for $i = 1, \ldots, n$ be given the C^2–functions*

$$\varphi_i\colon I^n \to \mathbb{R}.$$

Then there exist inverse demand functions f_i and cost functions C_i, such that the functions φ_i are the marginal payoff functions of the oligopoly defined by f_i and C_i.

Proof: Choose $C_i(x_i) \equiv 0$. One should have

$$f_i + x_i \frac{\partial f_i}{\partial x_i} = \frac{\partial (x_i f_i)}{\partial x_i} = \varphi_i(x_i; \mathbf{x}_{-i}).$$

Define

$$\Phi_i(x_i; \mathbf{x}_{-i}) := \int_0^{x_i} \varphi_i(x; \mathbf{x}_{-i}) dx.$$

Choose

$$f_i(x_i; \mathbf{x}_{-i}) = \begin{cases} \frac{1}{x_i} \Phi_i(x_i; \mathbf{x}_{-i}), & \text{if } x_i \neq 0; \\ \varphi_i(0; \mathbf{x}_{-i}), & \text{if } x_i = 0. \end{cases} \qquad \square$$

Corollary 3.2 *Let for $i = 1, \ldots, n$ be given the C^2–functions*

$$R_i \colon I^{n-1} \to I.$$

Then there exist inverse demand functions f_i and cost functions C_i, such that the functions R_i are the reaction functions of the oligopoly defined by f_i and C_i.

Proof: Choose

$$\varphi_i(x_i; \mathbf{x}_{-i}) = R_i(\mathbf{x}_{-i}) - x_i$$

and apply Proposition 3.1. $\square$

The reason to give the proof of Proposition 3.1 is twofold. First: the proof by Corchón and Mas–Colell is for heterogeneous Bertrand oligopoly, while the (almost identical) proof given here is for heterogeneous Cournot oligopoly. And second: the functions Φ introduced in the proof are needed again in the proof of Proposition 3.4.

The proof of the following theorem, that deals with homogeneous Cournot oligopoly, can be found in Corchón [5, Proposition 1.1].

Proposition 3.3 *Let for $i = 1, \ldots, n$ be given the C^2–functions*

$$R_i \colon J_{-1} \to I.$$

When the functions R_i are strictly decreasing ($R_i'(X_{-i}) < 0$), then there exist an inverse demand function f and cost functions C_i, such that the functions R_i are the reaction functions of the oligopoly defined by f and C_i. $\square$

In the introduction it was mentioned that Rand's ([33]) example used unimodular reaction functions. These are defined as follows. Assume that for $i = 1, 2$ the reaction functions R_i are defined on a closed interval $I = [0, M]$. Such a reaction function R is called *unimodular* when $R(0) = R(M) = 0$, $R(x) \geq 0$ and there is a unique $\xi \in < 0, M >$ such that $R'(\xi) = 0$. To indicate the dependence of ξ of the reaction function R, it will be noted by $\xi(R)$.

Example 3.1 *With $M = 1$ the logistic maps $R_i(x_j) = \mu x_j(1 - x_j)$, for i and $j = 1, 2$ but $j \neq i$, are unimodular reaction functions with $\xi(R_i) = 1/2$. When $1 \leq \mu \leq 4$ $R_i \colon I \to I$.*

Applying Corollary 3.2, one may find inverse demand functions

$$f_i(x_i; x_j) = \mu x_j(1 - x_j) - \frac{1}{2} x_i. \qquad \diamond$$

It will be shown that, in order to find an inverse demand function for a homogeneous oligopoly, the marginal payoff functions and the reaction functions should fulfill a special condition. The unimodular reaction functions of Example 3.1 do not fulfill this condition.

For homogeneous Cournot oligopoly one has:

$$\varphi_i(x_i; \mathbf{x}_{-i}) = f(X) + x_i f'(X) - C_i'(x_i).$$

As the inverse demand function f is a function of X, define $g \colon I^n \to \mathbb{R}_+$ by $g(\mathbf{x}) = f(X)$. One has for all i:

$$f'(X) = \frac{\partial g(\mathbf{x})}{\partial x_i}.$$

In homogeneous oligopoly the partial derivatives of g with respect to all x_i, are necessarily the same.

Next we derive the special conditions on φ_i such that we may find an inverse demand function f. As before define the functions Φ_i in the following way.

$$\Phi_i(x_i; \mathbf{x}_{-i}) := \int_0^{x_i} \varphi_i(x; \mathbf{x}_{-i}) dx.$$

As in the heterogeneous case one may derive

$$f(x_i + X_{-i}) = \begin{cases} \frac{1}{x_i} \left[C_i(x_i) + \Phi_i(x; \mathbf{x}_{-i}) \right], & \text{if } x_i \neq 0; \\ C_i'(0) + \varphi_i(0; \mathbf{x}_{-i}), & \text{if } x_i = 0. \end{cases} \tag{3.1}$$

Let $AC_i(x_i)$ be the average costs of firm i. Then

$$\begin{aligned} f'(x_i + X_{-i}) &= AC_i'(x_i) + \frac{1}{x_i} \varphi_i(x_i; \mathbf{x}_{-i}) - \frac{1}{x_i^2} \Phi_i(x_i; \mathbf{x}_{-i}) \\ &= \frac{1}{x_i} \frac{\partial \Phi_i(x_i; \mathbf{x}_{-i})}{\partial x_j}. \end{aligned} \tag{3.2}$$

The first line of equation (3.2) holds for all $i = 1, \ldots, n$. The second line holds for all $j \neq i$. Applying equation (3.2) for different i and j, one gets the *integrability conditions*

$$\begin{aligned} AC_i'(x_i) &+ \frac{1}{x_i} \varphi_i(x_i; \mathbf{x}_{-i}) - \frac{1}{x_i^2} \Phi_i(x_i; \mathbf{x}_{-i}) \\ &= AC_j'(x_j) + \frac{1}{x_j} \varphi_j(x_j; \mathbf{x}_{-j}) - \frac{1}{x_j^2} \Phi_j(x_j; \mathbf{x}_{-j}); \end{aligned}$$

and

$$\frac{1}{x_i}\frac{\partial \Phi_i(x_i;\mathbf{x}_{-i})}{\partial x_j} = \frac{1}{x_j}\frac{\partial \Phi_j(x_j;\mathbf{x}_{-j})}{\partial x_i}.$$

The last of these equations is the special condition on the marginal payoff functions. This condition is necessary, the next proposition shows it is also sufficient.

Proposition 3.4 *Let for $i = 1, \ldots, n$ be given the C^2–functions*

$$\varphi_i \colon I^n \to \mathbb{R}.$$

Let Φ_i be defined as in the proof of Proposition 3.1.
When, for $i \neq j$,

$$\frac{1}{x_i}\frac{\partial \Phi_i(x_i;\mathbf{x}_{-i})}{\partial x_j} = \frac{1}{x_j}\frac{\partial \Phi_j(x_j;\mathbf{x}_{-j})}{\partial x_i}$$

holds, there exist an inverse demand functions f and cost functions C_i, such that the homogeneous oligopoly defined by these functions has the functions φ_i as its marginal payoff functions.

Proof: From equation (3.2) it follows that C_i should fulfill the following differential equation.

$$C_i'(x_i) = \frac{1}{x_i}C_i(x_i) + \Psi_i(x_i;\mathbf{x}_{-i});$$

where

$$\Psi_i(x_i;\mathbf{x}_{-i}) := \frac{1}{x_i}\Phi_i(x_i;\mathbf{x}_{-i}) - \varphi_i(x_i;\mathbf{x}_{-i}) + \frac{\partial \Phi_i(x_i;\mathbf{x}_{-i})}{\partial x_j}.$$

The solution of this differential equation is

$$C_i(x_i) = x_i\left(\int_0^{x_i}\frac{\Psi_i(x;\mathbf{x}_{-i})}{x}dx\right)$$

and so

$$f(X) = \int_0^{x_i}\frac{\Psi_i(x;\mathbf{x}_{-i})}{x}dx + \frac{1}{x_i}\Phi_i(x_i;\mathbf{x}_{-i}). \qquad \square$$

In fact in Proposition 3.4 and in its proof, everywhere $\mathbf{x}_{-i}$ may be replaced by X_{-i}. This is as all functions in n variables, such as φ_i, Φ_i and Ψ_i, are in fact functions in 2 variables x_i and X_{-i}. This remark is already taken care of in the following corollary.

Corollary 3.5 *Let for $i = 1, \ldots, n$ be given the C^2–functions*

$$R_i \colon J_{-1} \to I.$$

When for $i \neq j$

$$R_i'(X_{-i}) = R_j'(X_{-j})$$

holds, there exist an inverse demand functions f and cost functions C_i, such that the homogeneous oligopoly defined by these functions has the functions R_i as the reaction functions.

Proof: As in Corollary 3.2, let

$$\varphi_i(x_i; X_{-i}) = R_i(X_{-i}) - x_i.$$

These φ_i fulfill the conditions of Proposition 3.4. □

Example 3.2 *This example is a continuation of Example 3.1. As in that example a duopoly model is considered where the reaction functions are the unimodular functions $R_i(x_j) = \mu x_j(1 - x_j)$. One has*

$$R_i'(x_j) = \mu(1 - 2x_j) \neq \mu(1 - 2x_i) = R_j'(x_i).$$

The conditions of Corollary 3.5 are not fulfilled. ⋄

The conditions on the marginal payoff functions are necessary and sufficient. The conditions on the reaction functions are only sufficient. This is demonstrated in the following example.

Example 3.3 *(Puu [31]) Also this example is an example of a duopoly. Let $I = [0, M]$ where $M = \frac{1}{c} - \beta > 0$ and define $R_i \colon I \to I$ by*

$$R_i(x_j) = \sqrt{\frac{\beta + x_j}{c}} - \beta - x_j.$$

One easily shows that $R_i'(x_j) \neq R_j'(x_i)$ and the reaction functions do not fulfill the conditions of Corollary 3.5.

However Puu derives these reaction functions from a homogeneous oligopoly model with

$$f(x_1 + x_2) = \frac{1}{\beta + x_1 + x_2},$$

and $C_i(x_i) = cx_i$.

For this duopoly model

$$\varphi_i(x_i; x_j) = \frac{\beta + x_j}{(\beta + x_1 + x_2)^2} - c.$$

Hence

$$\frac{1}{x_1} \frac{\partial \Phi_1(x_1, x_2)}{\partial x_2} = \frac{1}{x_2} \frac{\partial \Phi_2(x_1, x_2)}{\partial x_1}.$$

That is the marginal payoff functions do fulfill the conditions of Proposition 3.4. ⋄

Example 3.4 *Also this example is an example of a duopoly. Choose* $I = [0, 1]$. *Define*

$$\varphi_i(x_i; x_j) = \mu x_i(1 - x_i) - x_j.$$

Then

$$\frac{1}{x_i}\frac{\partial \Phi_i(x_i; x_j)}{\partial x_j} = \frac{1}{x_j}\frac{\partial \Phi_j(x_j; x_i)}{\partial x_i} = -1.$$

The conditions of Proposition 3.4 are fulfilled and there are an inverse demand function f *and cost functions* C_i. *One finds*

$$C_i(x_i) = \gamma x_i - \frac{1}{2}(2 + \mu)x_i^2 + \frac{1}{3}x_i^3;$$

and

$$f(x_1 + x_2) = \gamma - x_1 - x_2.$$

In this γ *is an integration constant that should be taken large enough in order to make the market clearing price positive for a sufficient broad range of* x_1 *and* x_2. ◇

Notice that in Example 3.4 when $0 \leq x_j < \mu/4$, firm i may choose two values for x_i. In both points (x_1, x_2) the FOC (for firm i) are fulfilled, however only in one of the two points also the SOC hold. When $2 < \mu \leq 4$ there may be three equilibria, with x_1 and $x_2 > 0$. Only one of these three equilibria, the one for which $x_1 = x_2 = 1 - \frac{1}{\mu}$ is a Cournot–Nash equilibrium. When $0 < \mu < 2$, there is only one equilibrium, the one for which $x_1 = x_2 = 1 - \frac{1}{\mu}$, but this equilibrium is not a Cournot–Nash equilibrium.

4 Stability and (limit) cycles

4.1 Stability

Consider a heterogeneous Cournot oligopoly defined by inverse demand functions f_i and cost functions C_i. A homogeneous Cournot oligopoly will be considered as a heterogeneous one in which for all i one has $f_i(x_i; \mathbf{x}_{-i}) = f(X)$. The vectorfield $\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_n) \colon I^n \to \mathbb{R}^n$ maps $\mathbf{x} \in I^n$ onto

$$(\varphi_1(x_1; \mathbf{x}_{-1}), \ldots, \varphi_n(x_n; \mathbf{x}_{-n})).$$

In this section there has been chosen for $I = [0, M]$. Boundary points of I^n are points $(x_1, \ldots, x_n)$ for which there exists at least one $i = 1, \ldots, n$ such that either $x_i = 0$ or $x_i = M$. The vectorfield $\boldsymbol{\varphi}$ is said to be *inwards pointing* at the boundary of I^n, when

$$\varphi_i(x_i; \mathbf{x}_{-i}) \begin{cases} > 0, & \text{if } x_i = 0; \\ < 0, & \text{if } x_i = M. \end{cases}$$

Let

$$J(\boldsymbol{\varphi}) := \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}$$

be the obvious Jacobi matrix.

Assume the vectorfield $\boldsymbol{\varphi}$

(i) is a Morse–Smale vectorfield (see the appendix for the definition);

(ii) points inwards at the boundary of I^n;

(iii) has no cycles as integral curves.

Let c_j be the number of critical points of $\boldsymbol{\varphi}$ with Morse coindex j. So c_0 is the number of stable critical points, c_n the number of totally unstable ones. Now the following Morse inequalities hold.

$$\begin{aligned} c_0 &\geq 1 \\ c_1 - c_0 &\geq -1 \\ &\vdots \\ c_n - c_{n-1} + c_{n-2} + \cdots + (-1)^n c_0 &\geq (-1)^n. \end{aligned}$$

The last inequality is in fact an equality.

For each $(c_0, c_1, \ldots, c_n) \in \mathbb{Z}_+^{n+1}$ fulfilling the Morse inequalities, one may find a Morse–Smale vectorfield $\boldsymbol{\varphi}$ with $\sum_{\ell=0}^n c_\ell$ non–degenerate equilibrium points, such that there are precisely c_j points with Morse coindex j. From Proposition 3.1 it follows that there are inverse demand functions f_i and cost functions C_i such that the functions φ_i are the marginal payoff functions of the heterogeneous oligopoly defined by these inverse demand functions and cost functions. In [11] for instance, there was an example of a heterogeneous Bertrand duopoly for which $c_0 = 4$, $c_1 = 4$ and $c_2 = 1$. So (almost) anything goes for heterogeneous oligopoly. What about homogeneous oligopoly?

In homogeneous Cournot oligopoly, the Jacobi matrix $J(\boldsymbol{\varphi})$ has a special form. As

$$\frac{\partial \varphi_i(x_i; \mathbf{x}_{-i})}{\partial x_j} = \begin{cases} f'(X) + x_i f''(X), & \text{when } j \neq i; \\ 2f'(X) + x_i f''(X) - C_i''(x_i), & \text{when } j = i \end{cases}$$

the Jacobi matrix can be written as the sum of a diagonal matrix, with $f' - C_i''$ on the diagonal and a matrix of rank 1, in which the i^{th} row is given by $f' + x_i f''$. In Furth and Sierksma [12] such matrices are called M_1–matrices. The eigenvalues of such matrices are all real. Moreover one may say something on the sign of these eigenvalues. Assume that at an equilibrium for all i it is true that $f' - C_i'' < 0$, which is a generally made assumption.

Proposition 4.1 *Let a homogeneous Cournot oligopoly be given by the inverse demand function f and for $i = 1, \ldots, n$ cost functions C_i. When $f' - C_i'' < 0$ then the following holds. When at an equilibrium $\mathbf{x}^*$ for all i*

(i)

$$\frac{-1}{n-1} < \frac{f' + x_i^* f''}{2f' + x_i^* f'' - C_i''}$$

then the Morse coindex of the equilibrium is 0 *and the equilibrium is stable;*

(ii)

$$\frac{f' + x_i^* f''}{2f' + x_i^* f'' - C_i''} < \frac{-1}{n-1}$$

then the Morse coindex of the equilibrium is 1 *and the equilibrium is non stable.*

The proof of this proposition can be found in [11]. In case (i) of Proposition 4.1, all eigenvalues are negative, while in case (ii) all but one are negative, the remaining eigenvalue is positive. The case that

$$\frac{f' + x_i^* f''}{2f' + x_i^* f'' - C_i''} = \frac{-1}{n-1}$$

has been excluded as in this case there is an eigenvalue zero and the equilibrium is degenerate.

Proposition 4.1 shows that for a homogeneous oligopoly $c_2 = \cdots = c_n = 0$. From the Morse inequalities follows that $c_0 \geq 1$, hence there is at least one Cournot–Nash equilibrium, and that $c_1 \geq 0$. As the last Morse inequality is in fact an equality, it follows from this equality that $c_0 = c_1 + 1$. That is the total number of equilibria $c_0 + c_1 = 2c_1 + 1$ is odd.

From this one may conclude the following. For heterogeneous oligopoly, almost everything is possible: stable, unstable and totally unstable equilibria, as long as the Morse inequalities hold. For heterogeneous oligopoly the possibilities are restricted. Only stable and unstable equilibria are possible, the unstable equilibria have an *unstable manifold* of dimension 1. The number of stable equilibria is one more than the number of unstable equilibria.

Example 4.1 *Also this example is an example of a duopoly. Choose $I = [0, M]$. Let*

$$\varphi_i(x_i; x_j) = \gamma - 2\alpha x_i - \beta x_j,$$

then

$$\Phi_i(x_i; x_j) = (\gamma - \alpha x_i - \beta x_j)x_i.$$

It is assumed that α, β, γ and M are positive numbers.

These φ_i are the marginal payoffs of a heterogeneous duopoly with $f_i(x_i; x_j) = \gamma - \alpha x_i - \beta x_j$ as inverse demand function and $C_i(x_i) \equiv 0$ as cost function. But as

$$\frac{1}{x_i}\frac{\partial \Phi_i}{\partial x_j} = -\beta = \frac{1}{x_j}\frac{\partial \Phi_j}{\partial x_i},$$

there must be an inverse demand function f and cost functions C_i, such that the φ_i are the marginal payoff functions of a homogeneous duopoly given by these functions f and C_i. Clearly

$$f(x_1 + x_2) = \gamma - \beta(x_1 + x_2); \text{ and}$$
$$C_i(x_i) = (\alpha - \beta)x_i.$$

The equilibrium of this duopoly is given by

$$x_1^* = x_2^* = \frac{\gamma}{2\alpha + \beta}.$$

The matrix of Jacobi

$$J(\boldsymbol{\varphi}) = \begin{pmatrix} -2\alpha & -\beta \\ -\beta & -2\alpha \end{pmatrix}$$

has two (real) eigenvalues $\lambda_1 = -2\alpha - \beta$ and $\lambda_2 = -2\alpha + \beta$.

Both eigenvalues are negative when $\beta < 2\alpha$. The equilibrium is stable in that case. As the eigenvalues are not equal this is an example of a so called 'node'.

When $\beta > 2\alpha$, the eigenvalue λ_1 is still negative, but λ_2 is positive. The equilibrium is unstable in this case and an example of a so called 'saddle'. The stable manifold, that corresponds to λ_1, is the line with equation $y = x$. The unstable one, corresponding to the positive eigenvalue, is the line with equation $x_1 + x_2 = 2\gamma/(2\alpha + \beta)$.

In the unstable example, there is just one unstable equilibrium. In terms of the Morse indices $c_1 = 1$. As the number of stable equilibria is one more than the number of unstable equilibria, hence $c_0 = 2$. What has gone wrong? The answer is simple. the vectorfield $\boldsymbol{\varphi}$ is not pointing inwards at the boundary of $[0, M]^2$. This is so as the derivative of φ_i with respect to x_i is -2α always negative. It should be positive when $x_i = 0$. The two missing stable equilibria are boundary equilibria: $x_i^* = \gamma/2\alpha$ and $x_j^* = 0$. In these boundary equilibria, one of the two firms is a monopolist, while the other firm is not active. ◇

4.2 (Limit) Cycles

Example 4.2 *This example is a variant of Example 4.1. Also this is an example of a duopoly. Choose $I = [0, M]$. Let*

$$\varphi_1(x_1, x_2) = \gamma - 2\alpha x_1 - \beta x_2; \text{ and}$$
$$\varphi_2(x_1, x_2) = \beta x_1 + 2\alpha x_2 - \gamma,$$

then

$$\Phi_1(x_1, x_2) = (\gamma - \alpha x_1 - \beta x_2)x_1; \text{ and}$$
$$\Phi_2(x_1, x_2) = (\beta x_1 + \alpha x_2 - \gamma)x_2.$$

Notice that

$$\frac{1}{x_1}\frac{\partial \Phi_1}{\partial x_2} = -\beta \neq \beta = \frac{1}{x_2}\frac{\partial \Phi_2}{\partial x_1},$$

Hence one can not find inverse demand functions f and cost functions C_i, such that these φ's are the marginal payoff functions of the homogeneous oligopoly defined by these f and C_i's.

However these φ_i are the marginal payoff functions of a heterogeneous duopoly with

$$f_1(x_1, x_2) = \gamma - \alpha x_1 - \beta x_2; \text{ and}$$
$$f_2(x_1, x_2) = \beta x_1 + \alpha x_2 - \gamma.$$

as inverse demand functions and $C_i(x_i) \equiv 0$ as cost functions.

Also the equilibrium of this duopoly is given by

$$x_1^* = x_2^* = \frac{\gamma}{2\alpha + \beta}.$$

The matrix of Jacobi

$$J(\boldsymbol{\varphi}) = \begin{pmatrix} -2\alpha & -\beta \\ \beta & 2\alpha \end{pmatrix}$$

has two eigenvalues[6]

$$\lambda_{12} = \begin{cases} \pm\sqrt{4\alpha^2 - \beta^2}, & \text{if } 2\alpha > \beta; \\ \pm i\sqrt{\beta^2 - 4\alpha^2}, & \text{if } 2\alpha < \beta. \end{cases}$$

When $\beta < 2\alpha$ the eigenvalues are of opposite sign and the equilibrium is an unstable saddle. When $\beta > 2\alpha$ the eigenvalues are pure imaginary. The equilibrium is a so called 'center'.

For this example it is not too difficult to find the equation of the integral curves. One find these by solving the differential equation

$$\frac{dx_2}{dx_1} = -\frac{\beta x_1 + 2\alpha x_2 - \gamma}{2\alpha x_1 + \beta x_2 - \gamma}.$$

the solution is given by the following equation.

$$(2\alpha + \beta)\left(x_1 + x_2 - \frac{2\gamma}{2\alpha + \beta}\right)^2 - (2\alpha - \beta)(x_1 - x_2)^2 = \text{const.} \tag{4.1}$$

In this const *is an integration constant, the value of which depends on the initial conditions.*

When $\beta < 2\alpha$ the integral curves are hyperbolas, which of course they should be when the equilibrium is a saddle. When $\beta > 2\alpha$ the integral curves are ellipses, hence cycles. ◇

In the introduction I forewarned that when one starts with strange marginal payoff functions, one may get a strange oligopoly. In Example 4.2, in equilibrium

[6] In the following equation i is the complex number $\sqrt{-1}$ and not the name of a firm.

the market clearing price of firm 1 is positive, but that of firm 2 is negative. The reason of this to happen is that firm 2 will increase its output whenever $\varphi_2(x_1, x_2) > 0$, but in that case $x_2 > R_2(x_1)$. Lemma 2.1 does not hold for this example! Sometimes the cases are too pathological, even for me. The reason to consider this example is that it shows that with complex eigenvalues you may have cycles and/or oscillating behavior.

Assume that for a Cournot duopoly the marginal payoff functions are linear. Then there is a unique Cournot–Nash equilibrium $\mathbf{x}^*$. The possible Cournot–Nash equilibria can be classified by the properties of the eigenvalues of the Jacobi matrix.

A the eigenvalues of the Jacobi matrix are real,

- **(i)** both eigenvalues are negative, then the equilibrium is stable;
- **(ii)** one eigenvalue is negative, the other is positive, the equilibrium is a saddle;
- **(iii)** both eigenvalues are positive, the equilibrium is (totally) unstable;

B the eigenvalues are complex conjugated,

- **(iv)** the real part of the eigenvalues is negative, then the equilibrium is stable;
- **(v)** the real part is zero;
- **(vi)** the real part is positive, then the equilibrium is (totally) unstable.

In the case the eigenvalues are real and have the same sign, the equilibrium is called a *node* when the eigenvalues are different and a *focus* when the eigenvalues are equal. The integral curves of the dynamical system approach the equilibrium for $t \to \infty$, when the equilibrium is stable and for $t \to -\infty$ when the equilibrium is unstable.

When the eigenvalues are complex conjugated and the real part is not zero, the integral curves are spiraling around the equilibrium. When the integral curves approach the equilibrium for $t \to \infty$, the equilibrium is stable and called a *spiraling sink*. When the integral curves approach the equilibrium for $t \to -\infty$, the equilibrium is unstable and called a *spiraling source*.

Finally, when the eigenvalues are pure imaginary, that is complex conjugated but with a real part zero, the equilibrium is stable, but not attracting. All integral curves, except for the equilibrium, are closed cycles around the equilibrium. The equilibrium is called a *center* in this case.

All the above is possible for heterogeneous Cournot duopoly, but not for homogeneous Cournot duopoly, as in homogeneous Cournot duopoly all eigenvalues are real (see the remark just above Proposition 4.1).

Consider a duopoly, let R_i be the reaction function of firm i and let $\mathbf{x}^*$ be a (local) Cournot–Nash equilibrium. The best reply dynamics is given by $\dot{x}_i = R_i(x_j) - x_i$, for $i, j = 1, 2$ but $j \neq i$. A theorem due to Hartman [14] says that a vectorfield is locally equivalent to its linear part, as given by its Jacobi matrix, at a hyperbolic singularity. The Jacobi matrix, at the equilibrium, is equal to

$$\begin{pmatrix} -1 & R_1'(x_2^*) \\ R_2'(x_1^*) & -1 \end{pmatrix}$$

and its eigenvalues are $\lambda_{12} = -1 \pm \sqrt{R_1'(x_2^*)R_2'(x_1^*)}$. Eigenvalues are real whenever $R_1'(x_2^*)R_2'(x_1^*) > 0$. In this case both eigenvalues are negative whenever $R_1'(x_2^*)R_2'(x_1^*) < 1$, compare Example 4.1. One eigenvalue is negative and the other is positive whenever $R_1'(x_2^*)R_2'(x_1^*) > 1$, compare Example 4.2. Eigenvalues are complex conjugated whenever $R_1'(x_2^*)R_2'(x_1^*) < 0$, compare also Example 4.2.

When both eigenvalues are negative, the Cournot–Nash equilibrium is a sink and stable. According to Proposition 3.3, when we have two strictly decreasing functions, there is a homogeneous Cournot Duopoly with these functions as reaction functions. When one eigenvalue is positive and the other is negative, the equilibrium is a saddle and unstable. When both eigenvalues are positive, the equilibrium is a source and totally unstable When the eigenvalues are complex conjugated, the real part of the eigenvalues is -1, which is negative, hence the equilibrium is stable. In fact the integral curves are spiraling towards the equilibrium, compare Example 4.3 below. In this case one reaction function is (locally) decreasing, while the other is (locally) increasing.

Example 4.3 *As Example 4.2 also this example is a variant of Example 4.1. Choose $I = [0, M]$. Let*

$$\begin{aligned} \varphi_1(x_1, x_2) &= \gamma - 2\alpha x_1 - \beta x_2; \text{ and} \\ \varphi_2(x_1, x_2) &= \gamma + \beta x_1 - 2\alpha x_2. \end{aligned}$$

then

$$\begin{aligned} \Phi_1(x_1, x_2) &= (\gamma - \alpha x_1 - \beta x_2)x_1; \text{ and} \\ \Phi_2(x_1, x_2) &= (\gamma + \beta x_1 - \alpha x_2)x_2. \end{aligned}$$

Notice that

$$\frac{1}{x_1}\frac{\partial \Phi_1}{\partial x_2} = -\beta \neq \beta = \frac{1}{x_2}\frac{\partial \Phi_2}{\partial x_1},$$

Hence one can not find inverse demand functions f and cost functions C_i, such that these φ's are the marginal payoff functions of the homogeneous oligopoly defined by these f and C_i's.

However these φ_i are the marginal payoff functions of a heterogeneous duopoly with

$$\begin{aligned} f_1(x_1, x_2) &= \gamma - \alpha x_1 - \beta x_2; \text{ and} \\ f_2(x_1, x_2) &= \gamma + \beta x_1 - \alpha x_2. \end{aligned}$$

as inverse demand functions and $C_i(x_i) \equiv 0$ as cost functions.

The (unique) equilibrium of this duopoly is given by

$$x_1^* = \frac{2\alpha - \beta}{4\alpha^2 + \beta^2}\gamma \text{ and } x_2^* = \frac{\gamma}{2\alpha + \beta}.$$

The matrix of Jacobi

$$J(\boldsymbol{\varphi}) = \begin{pmatrix} -2\alpha & -\beta \\ \beta & -2\alpha \end{pmatrix}$$

has two eigenvalues

$$\lambda_{12} = -2\alpha \pm i\beta,$$

where as before i is the complex number $\sqrt{-1}$.

The eigenvalues are complex conjugated. As the real part of these eigenvalues (2α) is positive, the equilibrium is stable. In fact it is a so called spiral sink.

As in Example 4.2 it is not too difficult to solve the differential equations. One finds

$$x_1(t) = x_1^* + ke^{-2\alpha t}\cos(\theta_0 + \beta t); \text{ and}$$
$$x_2(t) = x_2^* + ke^{-2\alpha t}\sin(\theta_0 + \beta t).$$

In this k and θ_0 are integration constants, determined by the initial conditions. It can (easily) be shown that $k = \sqrt{(x_1(0) - x_1^*)^2 + (x_2(0) - x_2^*)^2}$. When $t \to \infty$, $x_i(t) \to x_i^*$ for $i = 1, 2$. Hence the integral curves are spiraling around the equilibrium and they are approaching it. As there is just one stable equilibrium $c_0 = 1$ and $c_1 = c_2 = 0$, hence the Morse inequalities hold. ◇

For completeness also consider a duopoly with the features of Example 4.2. The best reply dynamics is given by

$$\dot{x}_1 = R_1(x_2) - x_1; \text{ and}$$
$$\dot{x}_2 = x_2 - R_2(x_1),$$

where R_i is the reaction function of firm i. Let $\mathbf{x}^*$ be a (local) Cournot–Nash equilibrium. The Jacobi matrix, at the equilibrium, is equal to

$$\begin{pmatrix} -1 & R_1'(x_2^*) \\ -R_2'(x_1^*) & 1 \end{pmatrix}$$

and its eigenvalues are $\lambda_{12} = \pm\sqrt{1 - R_1'(x_2^*)R_2'(x_1^*)}$. Eigenvalues are real whenever $R_1'(x_2^*)R_2'(x_1^*) < 1$. In this case $\lambda_1 = -\lambda_2$ and the equilibrium is a saddle. Eigenvalues are complex conjugated whenever $R_1'(x_2^*)R_2'(x_1^*) > 1$ and the equilibrium is a center.

Let (x_1, x_2) be a point not on one of the two reaction curves. Both firms want to move towards their reaction curve. When the equilibrium is a sink, the vectorfield is pointing (more or less) in the direction towards the equilibrium.

When the equilibrium is a saddle, in some points the vectorfield is pointing towards the equilibrium, but there are also points where the vectorfield is pointing away from the equilibrium.

Oscillating behavior for a duopoly is only possible either when one reaction curve is (locally) increasing, while the other is (locally) decreasing, or when one firm moves towards its reaction curve while the other moves away from it. In this two cases there are complex conjugated eigenvalues with a non zero imaginary part. As for homogeneous duopoly, the eigenvalues are always real, oscillating behavior is not possible for homogeneous duopoly, but it may be possible for heterogeneous duopoly.

Example 4.4 *Consider The case that $n = 2$. For simplicity, write x for x_1, y for x_2, $\varphi(x, y)$ for $\varphi_1(x_1, x_2)$ and $\psi(x, y)$ for $\varphi_2(x_1, x_2)$. Consider the following dynamical system.*

$$\begin{aligned}\varphi(x, y) &= y - 2 + (x - 2)[1 - (x - 2)^2 - (y - 2)^2];\\ \psi(x, y) &= 2 - x + (y - 2)[1 - (x - 2)^2 - (y - 2)^2].\end{aligned}$$

The Jacobi matrix is

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} = \begin{pmatrix} 1 - 3(x - 2)^2 - (y - 2)^2 & 1 - 2(x - 2)(y - 2) \\ -1 - 2(x - 2)(y - 2) & 1 - (x - 2)^2 - 3(y - 2)^2 \end{pmatrix}$$

Let T be the trace of this matrix, D its determinant. By defining $\rho^2 := (x - 2)^2 + (y - 2)^2$ one may calculate that

$$\begin{aligned}T &:= 2(1 - 2\rho^2);\\ D &:= (3\rho^2 - 1)(\rho^2 - 1) + 1; \text{ and}\\ (1/4)T^2 - D &:= \rho^4 - 1.\end{aligned}$$

The eigenvalues of the Jacobi matrix are $\lambda_{12} = (1/2)T \pm \sqrt{(1/4)T^2 - D}$.

First notice that $D \geq 0$ for all ρ. When $0 \leq \rho^2 < 1/2$ T is positive $T^2 - 4D$ is negative. The eigenvalues are complex conjugated with positive real part. When $\rho^2 = 1/2$ $T = 0$ and $T^2 - 4D$ is negative. The eigenvalues are purely imaginary, that is complex conjugated with real part zero. When $1/2 < \rho^2 < 1$ T and $T^2 - 4D$ are negative. The eigenvalues are complex conjugated with negative real part. When $\rho^2 \geq 1$ the eigenvalues are real and negative.

The dynamical system is solved by going over on polar coordinates:

$$\begin{aligned}x &= 2 + r\cos(\theta)\\ y &= 2 + r\sin(\theta).\end{aligned}$$

The solution is

$$\begin{aligned}x &= 2 + \frac{\cos t}{\sqrt{1 + ke^{-2t}}};\\ y &= 2 + \frac{\sin t}{\sqrt{1 + ke^{-2t}}}.\end{aligned}$$

In this k is some integration constant,determined by the initial conditions. One derives

$$\rho^2 = (x-2)^2 + (y-2)^2 = \frac{1}{1+ke^{-2t}}.$$

It follows: (i) that $\rho^2 \to 1$ when $t \to \infty$, $(x-2)^2 + (y-2)^2 = 1$ is the equation of an attracting limit cycle; and: (ii) that $\rho^2 \to 0$ when $t \to -\infty$, $(x, y) = (2, 2)$ is a totally unstable (both eigenvalues have positive real part) equilibrium.

Is this dynamical system the gradient system of a Cournot duopoly? It follows directly form Proposition 3.1 that there is a heterogeneous Cournot duopoly that has (φ, ψ) as gradient system. On easily derives that

$$\Phi(x, y) = (1/2)x^2 + xy - 4x - (1/4)[x^4 - 8x^3 + 24x^2 - 16x] - (1/2)(x^2 - 4x)(y-2)^2;$$
$$\Psi(x, y) = (1/2)y^2 - xy - (1/2)(x-2)^2(y^2 - 4y) - (1/4)[y^4 - 8y^3 + 24y^2 - 16y].$$

From this it follows that

$$\frac{1}{x}\frac{\partial \Phi}{\partial y} = 1 - (x-4)(y-2) \neq -1 - (x-2)(y-4) = \frac{1}{y}\frac{\partial \Psi}{\partial x}$$

and, according to Proposition 3.4, (φ, ψ) is not the gradient system of a homogeneous Cournot duopoly. ◇

This example shows that a heterogeneous Cournot duopoly with a limit cycle is possible. However it does not show that a homogeneous Cournot duopoly with a limit cycle is impossible. In the example it was shown that when $0 \le \rho^2 < 1/2$ T is positive, when $\rho^2 = 1/2$ $T = 0$ and when $1/2 < \rho^2 < 1$ T is negative. The trace of the Jacobi matrix changing sign is a necessary condition for the existence of a cycle. This follows from the Bendixson Theorem, see [18, Theorem 2.4]

Proposition 4.2 *(Bendixson) Let (φ_1, φ_2) be a (smooth) vectorfield defined on a regular region $S \subset \mathbb{R}^2$. When the trace of the Jacobi matrix*

$$\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2}$$

has the same sign throughout S, than the dynamical system defined by this vectorfield has no cycle lying entirely in S as a solution. □

Apply Proposition 4.1 to a duopoly. Then the eigenvalues of the Jacobi matrix are either both negative, or one is negative and the other is positive. Assume the following.

(i) $f' - C_i'' < 0$ for $i = 1, 2$;

(ii) $f' + x_1 f'' \le f' + x_2 f''$,

Notice that the SOC implies that

$$2f' + x_i f'' - C_i'' < 0,$$

that is although $f' + x_i f''$ may be positive it should be smaller than $-(f' - C_i'')$. It can be shown that the eigenvalues of the Jacobi matrix are

$$\begin{aligned} \lambda_1 &= f' + x_1 f'' + (f' - C_1'') + (f' - C_2'') \\ \lambda_2 &= f' + x_2 f''. \end{aligned}$$

In [12] it was shown that

$$\lambda_1 < f' + x_1 f'' \leq \lambda_2.$$

When $f' + x_2 f'' < 0$ both eigenvalues are negative. As the trace of the Jacobi matrix is equal to the sum of the eigenvalues, also this trace is negative. When one eigenvalue is positive, this should be $\lambda_2 = f' + x_2 f''$. The sum of the eigenvalues is

$$\begin{gathered} f' + x_1 f'' + (f' - C_1'') + (f' - C_2'') + f' + x_2 f'' < \\ f' + x_1 f'' + (f' - C_1'') + (f' - C_2'') - (f' - C_2'') = \\ f' + x_1 f'' + (f' - C_1'') < 0. \end{gathered}$$

The inequalities, both in the first and the last line, follows from the SOC's. In all cases the sum of the eigenvalues, and therefor also the trace of the Jacobi matrix, is negative. It now follows from the Bendixson Theorem that gradient dynamics does not have a cycle as a solution.

So cycles are possible for heterogeneous Cournot duopoly, but not for homogeneous Cournot duopoly. Does this conclusion also hold when there are more than two firms? With a *Hopf bifurcation* a stable equilibrium turns into an attracting (limit)cycle when a pair of complex conjugated eigenvalues cross the imaginary ax. So a Hopf bifurcation is not possible in a homogeneous oligopoly. Said differently, in homogeneous Cournot oligopoly the existence of a (limit) cycle can not be due to a Hopf bifurcation. But does this exclude cycles? I do not know.

5 Conclusions

I would have liked to add a section on 'chaos' in general and on 'strange attractors' in particular. Starting with the differential equations for the Lorentz attractor, one may find a heterogeneous triopoly that has these differential equations as its gradient dynamics. One also can show that there is no homogeneous triopoly, having the same gradient dynamics.

The exercise would not much differ from Subsection 4.2. In Section 4 it was shown that all kind of instabilities are possible in heterogeneous oligopoly, but in an equilibrium of a homogeneous oligopoly, the dimension of the instable manifold is at most one. So heterogeneous oligopoly has properties that are not shared by homogeneous oligopoly. In Section 4.2 it was shown that oscillating behavior and/or (limit) cycles are possible in heterogeneous duopoly, but not in homogeneous duopoly. I would have liked to prove a similar result when there are more than two firms, but I have not been able to derive such a result. Due to the fact that I have not chosen for dynamics in discrete time, but for continuous time, in order to have examples of 'strange attractors' one needs at least three firms and the case of more than two firms turned out to be rather complicated. So I did not investigate it, as my point:

> Everything that is possible with heterogeneous oligopoly, is not always possible with homogeneous oligopoly,

has been made already.

A A rapid course in Morse theory

Good introductions to Morse theory are [23, 20]. It should be noted that all results in this appendix are generic. That is the cases where the results do not hold are rare. To give the intuition behind Morse theory, first a 'proof' of the following 'Theorem' will be given.

Theorem A.1 *Let be given an island. Let M be the number of mountains on the island, V the number of valleys and P the number of passes. The following equality holds:*

$$M + V = P + 1.$$

'Proof':

The proof is by induction on the number of mountains. When there is just one mountain $M = 1$ and $V = P = 0$, such that the equation holds.

Suppose the equation holds for $M = n$. Add a mountain. There are two possibilities.

(i) The new mountain 'touches' precisely one other mountain. In that case not only the number of mountains increases by one, but also the number of passes. The 'new' pass is between the two 'touching' mountains. The result is that the equality still holds.

(ii) The new mountain 'touches' two other mountains. Again there are two possibilities.

- **(a)** The new mountain closes a chain of mountains, turning it into a cycle of mountains. In this case the number of passes increases by two. But the cycle now encloses a new valley. So the number of mountains and the number of valleys increases by one, so the equality still holds.
- **(b)** The new mountain divides the valley between the two other mountains. Again the number of passes increases by two. The number of valleys increases by one, as an existing valley now splits into two new valleys. Again the equality holds. □

Corollary A.2 *An island does have at least one mountain.* □

With respect to this 'Theorem', there will follow now some remarks.
Remarks

1. To turn the 'Theorem' into a mathematical result. Let $C \subset \mathbb{R}^2$ be a compact subset of the two dimensional Euclidian space. Let $f\colon C \to \mathbb{R}$ be a continuous map. For the 'Theorem' C is the island and f is the 'height' function. That is $f(x, y)$ is the height (above sea level) at location $(x, y) \in C$ on the island. M corresponds to the number of maxima of f, V to the number of minima and P to the number of saddles.

2. Consider the *gradient* Df of the function f in the previous remark, so

$$Df := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

   The gradient $Df\colon C \to \mathbb{R}^2$ defines a vector in each point $(x, y) \in C$. The vector points into the direction of fastest increase. A maximum is characterized as a point from where you go down in all directions. A minimum is a point where you go up in all directions. Finally with a saddle, in some directions you go up, while in other directions you go down.

3. You may think that the induction in the proof of the theorem, could have been started with one valley. However when you have an island with just one valley (and no mountains or passes), once high tide and your island will be gone. This of course is not a mathematical argument. When you approach an island from sea, you will always go upwards, That is the gradient field points inwards at the boundary (the beach) of the islands. When there are no valleys and no passes, it follows from the equality that there is precisely one mountain. This proves the corollary. It proves more general: "When the gradient Df of $f\colon C \to \mathbb{R}$ points inwards at the boundary of C, f has at least one maximum."

4. What about a volcano? Is that a mountain or a valley? Difficult question, when we assume that the points of the edge of the crater are all at the same height. But fortunately, volcanos are mostly in regions where there are frequently earthquakes. After such an earthquake, the edge of the crater will be slightly tilted. As a consequence, one point of the edge will be the highest point and another the lowest. The heigh point corresponds to a maximum, because from it in each direction you go down. The low point is a saddle, because along the edge of the crater you go up, but in a traverse direction you go down. Finally the crater itself corresponds to a minimum. So consider the volcano as a valley, but increase the number of passes and mountains by one extra.

Let us try to formulate the above more general.

First one needs an n–dimensional equivalent of the 2–dimensional island. This will be a simply connected region V, defined in the following way. Let there be k functions $\kappa_\ell \colon \mathbb{R}^n \to \mathbb{R}$ with $\ell \in K := \{1, \dots, k\}$. Define for $\ell \in K$

$$V_\ell := \{\mathbf{x} = (x_1, \dots, n) \in \mathbb{R}^n \mid \kappa_\ell(\mathbf{x}) \geq 0\}$$

and let $V := \cap_{\ell \in K} V_\ell$. When the functions κ_ℓ are quasi–concave, the sets V_ℓ and V are convex and closed.[7]

Let U be an open neighborhood of V. Let

$$\boldsymbol{\psi} = (\psi_1, \dots, \psi_n) \colon U \to \mathbb{R}^n.$$

So $\boldsymbol{\psi}$ assigns to each point $\mathbf{x} \in U$ a vector $(\psi_1(\mathbf{x}), \dots, \psi_n(\mathbf{x})) \in \mathbb{R}^n$. It will be assumed that this is done in a smooth way. That means, it is assumed that the functions ψ_i have continuous (partial) derivatives of the first and the second order. $\boldsymbol{\psi} \colon U \to \mathbb{R}^n$ will be called *a vectorfield* on U.

For each point $\mathbf{x} \in V$ define the set

$$\mathcal{K}(\mathbf{x}) := \{k \in K \mid \kappa_k(\mathbf{x}) = 0\}.$$

When $k \in \mathcal{K}(\mathbf{x})$, the gradient of κ_k in $\mathbf{x}$ is defined as

$$D\kappa_k(\mathbf{x}) := (\frac{\partial \kappa_k(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial \kappa_k(\mathbf{x})}{\partial x_n}).$$

The set V is called a *regular region* whenever the vectors

$$\{D\kappa_\ell(\mathbf{x}) \mid \ell \in \mathcal{K}(\mathbf{x})\}$$

are linearly independent.

Notice that for a regular region there can be at most n elements in $\mathcal{K}(\mathbf{x})$. one says that a point $\mathbf{x}$ in a regular region V is

- an interior point when $\mathcal{K}(\mathbf{x}) = \emptyset$;
- a boundary point when $\mathcal{K}(\mathbf{x}) \neq \emptyset$;
- a corner when the number of elements in $\mathcal{K}(\mathbf{x})$ is precisely n.

∂V is the set of all boundary points of V.

As the vectors

$$\{D\kappa_\ell(\mathbf{x}) \mid \ell \in \mathcal{K}(\mathbf{x})\}$$

are linearly independent, there exist $\lambda_\ell \in \mathbb{R}$ such that

$$\boldsymbol{\psi}(\mathbf{x}) = \sum_{\ell \in \mathcal{K}(\mathbf{x})} \lambda_\ell D\kappa_\ell(\mathbf{x}). \tag{A.1}$$

A vectorfield $\boldsymbol{\psi}$ defined on $U(\supset V)$ is said to *point inwards* (resp. *point outwards*) at $\mathbf{x} \in \partial V$, when in equation (A.1) all λ's are positive (resp. negative).

[7] In the 'Theorem' $n = 2$, $V = C$ and C could be defined by just one function, for instance $\kappa(x, y) = r^2 - x^2 - y^2$.

A point $\mathbf{x} \in \partial V$ is called a (+)–Kuhn–Tucker point (resp. a (−)–Kuhn–Tucker point) when in equation (A.1) all λ's are positive (resp. negative).

Consider the Jacobi matrix:[8]

$$J(\boldsymbol{\psi})(\mathbf{x}) := \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(x_1, \ldots, x_n)}.$$

The set N in the following definition is the set $\{1, \ldots, n\}$.

Definition A.1 *1.* $\mathbf{x}^* = (x_1^*, \ldots, x_n^*) \in U$ *is called a* singular point *of the vectorfield* $\boldsymbol{\psi}$ *when for all* $i \in N$ *it is true that*

$$\psi_i(\mathbf{x}^*) = 0.$$

2. A singular point $\mathbf{x}^*$ *of the vectorfield* $\boldsymbol{\psi}$ *is called* non–degenerate *when the Jacobi matrix* $J(\boldsymbol{\psi})(\mathbf{x}^*)$ *is non–singular.*

3. A non-degenerate singular point is stable *when all eigenvalues of the Jacobi matrix* $J(\boldsymbol{\psi})(\mathbf{x}^*)$ *have negative real part.*

4. A non-degenerate singular point is unstable *when it is not stable.*

5. A non-degenerate singular point is totally unstable *when all eigenvalues of the Jacobi matrix* $J(\boldsymbol{\psi})(\mathbf{x}^*)$ *have positive real part.*

6. The Morse coindex of a non–degenerate singular point $\mathbf{x}^*$ *is the number of eigenvalues of the Jacobi matrix* $J(\boldsymbol{\psi})(\mathbf{x}^*)$ *with a positive real part.*

When $\mathbf{x}^* \in U$ is

(i) a stable singular point, hence the Morse coindex is 0, it will be called a *sink*;

(ii) an unstable singular point, with Morse coindex $1 \leq \ell \leq n-1$, it will be called a *saddle*;

(iii) a totally unstable singular point, hence the Morse coindex is n, it will be called a *source*.

When the vectorfield is in fact the gradient field of some function defined on V, sinks of the gradient field are maxima of that function, sources are minima and saddles are of course saddles.

In Definition A.1 stability and instability is mentioned. Stability of a singular point is always related to some dynamical (adjustment) process. By studying dynamical systems, one always can choose between: (i) continuous dynamics (i.e. given through a set of differential equations); and: (ii) discrete dynamics (i.e. given through a set of difference equations). Here there has been chosen for continuous time. The dynamical process with the vectorfield $\boldsymbol{\psi}$ on U is given through the differential equations[9]

$$\dot{x}_i = \psi_i(\mathbf{x}) \text{ for all } i.$$

[8] Some authors call this matrix the Jacobian. Here the name Jacobian will be only used for the determinant of the Jacobi matrix, of course only when this determinant exists.

[9] A dot above a symbol, such as x means that this x has been differentiated with respect to time t.

The solution of these differential equations is given as follows. For each $\mathbf{y} = (y_1, \ldots, y_n)$ in the interior of U let be defined a maximal interval $\mathcal{J}(\mathbf{y}) \subseteq \mathbb{R}$ with $0 \in \mathcal{J}(\mathbf{y})$. When the differential equations have a solution, this solution can be given as a map $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \colon \mathcal{J}(\mathbf{y}) \to U$, such that for all $t \in \mathcal{J}(\mathbf{y})$ and all $i \in N$ one has

$$\dot{\theta}_i(t) = \varphi_i(\boldsymbol{\theta}(t))$$

and $\boldsymbol{\theta}(0) = \mathbf{y}$.

$\boldsymbol{\theta}(\mathcal{J}(\mathbf{y})) \subset U$ is called an *integral curve* through $\mathbf{y}$. A *cycle* is a closed integral curve. That is for a cycle $\gamma \subset U$ it holds that for all $\mathbf{y} \in \gamma$ one has that

(i) $\mathcal{J}(\mathbf{y}) = \mathbb{R}$;

(ii) there is a $\tau \in \mathbb{R}$ such that $\boldsymbol{\theta}(k\tau) = \mathbf{y}$ for all $k \in \mathbb{Z}$.

The smallest τ for which (ii) holds is called the *period* of γ.

The following definition is a technical one, in which not all notions are defined. For the full definition see Smale [37, 38] and /or Palis and De Melo [30]. The vectorfield $\boldsymbol{\psi}$ defined on U is a Morse Smale vectorfield when

(i) it has a finite number of critical elements, singular points and cycles, all hyperbolic;

(ii) there are no saddle connections, integral curves that start and end in a saddle.

A singular point $\mathbf{x}^*$ of the vectorfield $\boldsymbol{\psi}$ is *hyperbolic*, when the Jacobi matrix $J(\boldsymbol{\psi})(\mathbf{x}^*)$ does not have purely imaginary eigenvalues.

A *nice* vectorfield on $V(\subset U)$ is a Morse–Smale vectorfield $\boldsymbol{\psi} \colon U \to \mathbb{R}^n$, that points inwards at the boundary of V, and has no cycles as integral curves. For a nice vectorfield $\boldsymbol{\psi}$ one may proof the following (see Milnor [23] or Morse and Cairns [25]).

Let c_j be the number of (non degenerate) singular points of $\boldsymbol{\psi}$ with Morse coindex j. So c_0 is the number of stable singular points, c_n the number of totally unstable ones. For a nice vectorfield we let $m_i = c_i$. The following Morse inequalities hold.

$$\begin{aligned} m_0 &\geq 1 \\ m_1 - m_0 &\geq -1 \\ &\vdots \\ m_n - m_{n-1} + m_{n-2} + \cdots + (-1)^n m_0 &\geq (-1)^n. \end{aligned}$$

The last inequality is in fact an equality.[10] Therfore one may write it as

$$m_0 - m_1 + m_2 - \cdots + (-1)^n m_n = 1,$$

[10] You should recognize, for $n = 2$, the equality of the 'Theorem' in it.

from which one derives $m_0 + m_2 + \cdots = 1 + m_1 + m_3 + \cdots$ and the total number of singular points is odd: $\sum_{\ell=1}^{n} m_\ell = 1 + 2(m_1 + m_3 + \cdots)$.

For each $(m_0, m_1, \ldots, m_n) \in \mathbb{Z}_+^{n+1}$ fulfilling the Morse inequalities, one may find a nice vectorfield $\boldsymbol{\psi}$ with $\sum_{\ell=0}^{n} m_\ell$ non–degenerate singularities, such that there are precisely m_j points with Morse coindex j.

The above treatment of Morse theory differs in two respect with the usual treatment.

Marston Morse's book [24], in which he introduced Morse theory, is called: "The calculus of variations in the large." In the beginning of the theory, the main field of applications was physics. Now physicists are mainly interested in minima and rarely in maxima. But economists are more interested in maxima and rarely in minima. Classical Morse theory deals with the critical points of a smooth real valued function f on some manifold M, or with the critical points of a vectorfield $\boldsymbol{\psi}$. So in order to get results for maxima, one has to apply classical Morse theory either to the critical points of the function $-f$ or of the vectorfield $-\boldsymbol{\psi}$. So the Morse coindex of a non–degenerate singular point $\mathbf{x}^*$ of the vectorfield $\boldsymbol{\psi}$ is equal to the Morse index of $\mathbf{x}^*$ of the vectorfield $-\boldsymbol{\psi}$. When the vectorfield $\boldsymbol{\psi}$ points inwards at $\mathbf{x} \in \partial V$, the vectorfield $-\boldsymbol{\psi}$ points outwards at $\mathbf{x} \in \partial V$.

The second difference is that Morse theory deals with functions or vectorfields defined on a general manifold, possibly with boundary, edges and corners, see [17] or [11, Appendix B] for a definition of a manifold with boundary, edges and corners and [13] for Morse theory for functions or vectorfields on such manifolds. For general manifolds the right hand sides of the Morse inequalities may be different. For instance, for the earth as a whole, the number of mountains plus the number of valleys is two more than the total number of passes. It should be understood that these numbers should include the number of mountains, valleys and passes at the bottom of the sea. When there is a planet that has the shape of a torus, the number of mountains plus the number of valleys is equal to the number of passes.

There still remain two questions to be answered.

(i) What when the vectorfield points outwards at some points of ∂V?

(ii) What when the vectorfield does have cycles?

In the 'Theorem' it was seen that on an island the first does not hold, but in general it may. Let $\mathbf{x}^* \in \partial V$ be a $(-)$–Kuhn–Tucker point. For each point $\mathbf{x} \in V$ the set

$$\mathcal{K}(\mathbf{x}) := \{k \in K \mid \kappa_k(\mathbf{x}) = 0\}$$

was defined. Define the set

$$\Sigma(\mathbf{x}^*) := \{\mathbf{x} \in V \mid \kappa_k(\mathbf{x}) = 0 \text{ for all } k \in \mathcal{K}(\mathbf{x}^*)\}.$$

$\Sigma(\mathbf{x}^*)$ is called the *stratum* to which $\mathbf{x}^*$ belongs. V is sometimes called a *stratified set*. For each $\mathbf{x} \in \Sigma(\mathbf{x}^*)$ let $\tilde{\boldsymbol{\psi}}(\mathbf{x})$ be the (orthogonal) projection of $\boldsymbol{\psi}(\mathbf{x})$ on $T_{\mathbf{x}}\Sigma(\mathbf{x}^*)$, the tangent space to the stratum $\Sigma(\mathbf{x}^*)$. The Morse index of $\mathbf{x}^*$ is the number of eigenvalues with negative real part of the Jacobi matrix $J(\tilde{\boldsymbol{\psi}})(\mathbf{x}^*)$. The Morse coindex (of the $(-)$–Kuhn–Tucker point $\mathbf{x}^* \in \partial V$) is defined as $n-$the Morse index. Notice, that in general the dimension of the stratum is smaller than n. When the dimension of the stratum is $m(< n)$, the Jacobi matrix $J(\tilde{\boldsymbol{\psi}})(\mathbf{x}^*)$ is an $m \times m$ matrix, not an $n \times n$ matrix. Let c_j be the number of (interior) points with Morse coindex equal to j, let a_j be the number of $(-)$–Kuhn–Tucker points with Morse coindex equal to j and let $m_j = c_j + a_j$, then the Morse inequalities still hold, see [25].

In the 'Theorem' it was seen how to solve in the presence of a volcano (cycle). In general, let γ be a cycle of the vectorfield $\boldsymbol{\psi}$. A *local cross section* of γ is a hypersurface Σ of dimension $n-1$, through a point $\mathbf{x}^* \in \gamma$ and *transverse* all integral curves in a neighborhood of γ. That is when $\mathbf{x} \in \Sigma$ is in a neighborhood U of $\mathbf{x}^*$, $\boldsymbol{\psi}(\mathbf{x})$ is not tangent to Σ. Choose U small enough, such that $\mathbf{x}^*$ is the only intersection point of γ with Σ. The *first return* or *Poincaré map* $P\colon U \to \Sigma$ is defined as follows. Let $\mathbf{x} \in U$, then there is an integral curve through $\mathbf{x}$. Follow this integral curve till one intersects Σ again. $P(\mathbf{x})$ is defined as this second intersection point. So P defines a dynamical system on Σ, however this time in discrete time. A critical point of a dynamical system in continuous time, is stable when all eigenvalues of the Jacobi matrix have negative real parts. A critical point of a dynamical system in discrete time, is stable when all eigenvalues of the Jacobi matrix $J(P)(\mathbf{x}^*)$ have a modulus (absolute value) smaller than one. Notice that although $J(\boldsymbol{\psi})(\mathbf{x}^*)$ is an $n \times n$ matrix, $J(P)(\mathbf{x}^*)$ is $(n-1) \times (n-1)$.

Let n_s be the number of eigenvalues of $J(P)(\mathbf{x}^*)$ with a modulus smaller than one and n_u the number of eigenvalues with a modulus larger than one. When the cycle γ is hyperbolic, there is no eigenvalue with a modulus equal to one. In that case $n_s + n_u = n - 1$. n_s (resp. n_u) is the dimension of the *stable manifold* (resp. *unstable manifold*) of $\mathbf{x}^*$ and $n_s + 1$ (resp. $n_u + 1$) is the dimension of the *stable manifold* (resp. *unstable manifold*) of γ.

Now let $\boldsymbol{\psi}$ defined on U be a Morse Smale vectorfield that point inwards at the boundary of the regular region and such that it has a finite number of critical elements, singular points and cycles, all hyperbolic. Let c_j be the number of critical points with Morse coindex equal to j, let b_j be the number of cycles, for which the unstable manifold has dimension $j+1$, then the Morse inequalities hold for $m_j = c_j + b_j + b_{j+1}$, see [37]. One should realize what this means. Suppose one has a cycle for which the unstable manifold has dimension $j+1$. This cycle contributes 1 to m_i with $i = j$, namely through b_{i+1} and 1 to m_i with $i = j+1$, namely through b_i.

Example, suppose we have a unique attracting cycle γ in the plane $\mathbb{R}^2$. By the Poincaré–Bendixson Theorem, see [16, Chapt. 10], there must be a critical point, a source, in the interior of γ. This critical point is the only point of Morse coindex 2. Hence $c_0 = c_1 = 0$ and $c_2 = 1$. For a point on the cycle we have that $n_s = 1$ and $n_u = 0$. That is $n_u + 1 = 1$. It follows that $b_0 = b_1 = 1$. So finally $m_0 = b_0 = 1$, $m_1 = b_1 = 1$ and $m_2 = c_2 = 1$. This corresponds with our findings for a volcano.

It should be understood that in the most general case $m_i = c_i + a_i + b_i + b_{i+1}$.

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