“Credit Cycles” in an OLG Economy with Money and Bequest

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Abstract

In the late ’90s Kiyotaki and Moore (KM) put forward a new framework (Kiyotaki and Moore,1997) to explore the Financial Accelerator hypothesis. The original model was framed in an Infinitely Lived Agent context (ILA-KM economy). As in KM we develop a dynamic model in which the durable asset (“land”) is not only an input but also collateralizable wealth to secure lenders from the risk of borrowers’default. In this paper, however, we model an OLG-KM economy whose novel feature is the role of money as a store of value and of bequest as a vehicle of resources to be "invested" in landholding. The dynamics generated by the model are complex. Not only cyclical patterns are routinely generated but the periodicity and amplitude are irregular. A route to chaotic dynamics is open.

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1. Introduction

In the late '90s Kiyotaki and Moore (KM hereafter) put forward a new framework to explore the *Financial Accelerator* hypothesis, i. e. the idea that financial factors affect investment and output (Kiyotaki and Moore, 1997, 2002). In their model the borrowers' *financial constraint* plays a crucial role. The constraint is due to the fact that lenders extend credit only up to the present value of borrowers' future *collateralizable wealth*, which is proxied by land. In a KM economy, in fact, land is at the same time an input and a real asset which can be collateralized to secure lenders from the risk of borrowers'default. The market value of land, in turn, depends on the future price of land, i.e. on the future asset price. The novel and appealing feature of their model therefore is a "*dynamic feedback process* between asset prices and borrowing constraints" (Kasa, 1998, p. 17, emphasis added): booming asset prices relax borrowing constraints and boost economic activity, driving the expansion; the upswing, in turn, affects asset prices.

The KM framework is the natural vehicle to study the transmission of shocks *through net worth (balance sheet channel)*: "The dynamic interaction between credit limits and asset prices turns out to be a powerful transmission mechanism by which the effects of shocks persist, amplify and spread out" (Kiyotaki and Moore, 1997:212). However, since all the variables are measured in real terms – an odd and counterfactual feature for a model of a credit economy – the framework is not suitable to study the effects of nominal shocks, such as a monetary injection.

Thanks to the dynamic feedback of asset prices and borrowing constraints, the KM framework has gained the reputation of being particularly suitable to explore the intertwined dynamics of asset prices and aggregate output. Open economy variants of such a model have been adopted to study the *twin crises*, i.e. the currency and financial crises which hit the Far East (Edison et al. 1998, Kasa, 1998). More recently, the same framework has been applied to an empirical study of the US and Europe (Iacoviello, 2005a,b).

In the original framework, KM consider an economy populated by infinitely lived agents, which will be referred to hereafter as a ILA-KM economy. The reduced form of the model boils down to the law of motion of the borrowers' landholding (which determines aggregate output and asset prices in real terms). KM use a linearized version of the law of motion to assess the dynamic impact of productivity shocks according to the impulse-propagation approach. In a sense, therefore, they are exploring *fluctuations* due to borrowing constraints. The evocative term "credit cycles" in the title of the 1997 paper sounds inappropriate because the model does not yield self-sustained oscillations. In fact the non linear law of motion yields trajectories not significantly different from those obtained by a linear law.

In an appendix, KM sketch the building blocks of an overlapping generations variant of their model along the lines of Blanchard's "finite horizon" framework (Blanchard 1985). This suggestion has been followed by Kasa (1998). The dynamics, however, are not significantly different from that of the original KM framework.

To the best of our knowledge, no other attempt has been made to develop an OLG framework of a KM economy. In this paper we model an OLG-KM economy à la Diamond-Samuelson with money and bequest. In our framework nominal variables play a crucial role. Money is essentially a store of value, which allows to access consumption and leave a bequest when old. The dynamics generated by the OLG-KM model are much richer than the dynamics of the original framework. In the present model, not only cyclical patterns are routinely generated – so that in this context the expression *credit cycles* deserves a mention in the title[1] – but the periodicity and amplitude are irregular. A route to chaotic dynamics is open.

The paper is organized as follows. In section 2 we present a quick refresher course on ILA-KM economies. Section 3 is devoted to the background assumptions concerning our OLG-KM economy. The optimization problems of the farmer and the gatherer are discussed in sections 4 and 5 respectively. The analytical details are confined in the appendix A. A discussion of the economic content of these optimization problems is conducted in section 6. Section 7 describes the trickling down process by which money spreads in the economy and is carried out from one period to the next by means of exchanges and bequests. Section 8 is devoted to the dynamics. Section 9 concludes.

2. A quick refresher course on ILA-KM economies

KM assume that in a principal-agent relationship between borrowers and lenders, characterized by asymmetric information and moral hazard, borrowers face a financing constraint: the loan they get is smaller or equal to the *value of their collateralizable assets*, which play, in this framework a role analogous to that of net worth or the equity base in Greenwald and Stiglitz (1993) and entrepreneurs' savings (internal finance) in Bernanke and Gertler (1989, 1990).

[1] Cordoba e Ripoll (2004) obtain credit cycles in a ILA-KM economy with a cash in advance constraint.

KM assume that infinitely lived agents can be either financially constrained borrowers ("farmers") or lenders ("gatherers"). A farmer is an agent endowed with inalienable human capital. Therefore, he can get from lenders no more than the value of his collateralizable assets. This is the reason of the financing constraint.[2] A gatherer, on the contrary, does not face financing constraints.

There are two types of goods, output ("fruit") and a collateralizable, durable, non-reproducible asset ("land") whose total supply is fixed ($\bar{K}$). Output can be consumed or lent. If lent, each unit of output yields a constant return $R = 1 + r$ where r is the real interest rate. Output is produced by means of a technology which uses land and labour.

By assumption farmers and gatherers have access to different technologies.

The production function of each farmer is: $y_t^F = (a+\bar{c})K_{t-1}^F$ where y_t^F is output of the farmer in t, $a, \bar{c}$ are positive technological parameters and K_{t-1}^F is land of the farmer in $t-1$. $\bar{c}\,K_{t-1}^F$ is the output which deteriorates ("bruised fruit") and is therefore non-tradable.

The technology of the farmer is idiosyncratic in the sense that once production has started only the farmer has the skills to complete the production process successfully, i.e. to make land bears fruit. If the farmer withdrew his labour, production would not be carried out, i.e. land would bear no fruit. As a consequence, if the farmer goes into debt, he may have an incentive to threaten his creditors to withdraw his labour and repudiate debt. Creditors protect themselves against this threat by collateralizing the farmer's land. This is the reason why the farmer faces a *financing constraint:*

$$b_t = \frac{q_{t+1}K_t^F}{R} \tag{1}$$

According to (1), the maximum amount of debt a farmer succeeds to get "today" b_t is such that the sum of principal and interest Rb_t is equal to the value of the farmer's land when the debt is due, i.e. $q_{t+1}K_t^F$ where q_{t+1} is the (real) price of land at time $t+1$.

The farmer faces also a *flow-of-funds constraint*:

$$y_t^F + b_t = q_t(K_t^F - K_{t-1}^F) + Rb_{t-1} + c_t^F \tag{2}$$

where c_t^F is the farmer's consumption. Substituting (1) into (2) we get:

$$c_t^F = (a+\bar{c})K_{t-1}^F - \mu_t K_t^F \tag{3}$$

[2] On this issue see Hart and Moore (1994, 1998).

where $\mu_t = q_t - \dfrac{q_{t+1}}{R}$ is the *downpayment,* i.e. the amount the farmer has to put aside as internal finance to acquire one unit of land.

Preferences are modelled in such a way that farmers consume only non-tradable output, i.e. $c_t^F = \bar{c}\, K_{t-1}^F$. From (3) follows

$$\mu_t K_t^F = a K_{t-1}^F \tag{4}$$

i. e. the revenues obtained by selling (non-bruised) fruit are employed as downpayment. The farmer's demand for land, therefore, is:

$$K_t^F = \frac{1}{\mu_t}\left[(a+q_t)K_{t-1}^F - Rb_{t-1}\right] = \frac{a}{\mu_t}K_{t-1}^F \tag{5}$$

Substituting (5) into (1), we obtain:

$$b_t = \frac{q_{t+1}}{R}\frac{1}{\mu_t}\left[(a+q_t)K_{t-1}^F - Rb_{t-1}\right] \tag{6}$$

The production function of each gatherer is: $y_t^G = G(K_{t-1}^G)$ where y_t^G is output of the gatherer in t , $G(.)$ is a well behaved production function and K_{t-1}^G is land of the gatherer in $t-1$. The gatherer faces only a *flow-of-funds constraint*:

$$y_t^G + Rb_{t-1} = q_t(K_t^G - K_{t-1}^G) + b_t + c_t^G \tag{7}$$

Substituting the production function of the gatherer and the financing constraint of the farmer into (7) and assuming, for the sake of simplicity and without loss of generality, that population consists only of one farmer and one gatherer so that $K_t^F = \bar{K} - K_t^G$ we get the constraint:

$$c_t^G = G(K_{t-1}^G) + \mu_t\left(\bar{K} - K_t^G\right) \tag{8}$$

From maximization of the utility of the gatherer one gets $G'(K_t^G) = R\mu_t$. Since the total amount of land is fixed by assumption, $K_t^F = \bar{K} - K_t^G$, $G'\left(K_t^G\right) = G'\left(\bar{K} - K_t^F\right)$. In the following, in order to save on notation, we will write $G'\left(\bar{K} - K_t^F\right) = g\left(K_t^F\right)$, where $g' = -G'' > 0$. Hence the condition above can be written as

$$\mu_t = \frac{g\left(K_t^F\right)}{R} \tag{9}$$

Substituting this expression into (5) and rearranging we end up with:

$$K_t^F = \frac{Ra}{g\left(K_t^F\right)}K_{t-1}^F \tag{10}$$

(10) is a non-linear difference equation in the state variable K_t^F.

Denoting with a star the steady state value of a variable, plugging the steady state condition $K_t^F = K_{t-1}^F = K^{*F}$ into (5) we obtain $\mu^* = a$. But $\mu^* = q^*\left(1 - \frac{1}{R}\right)$ so that $q^* = a\frac{R}{R-1}$ and $b^* = \frac{q^* K^{*F}}{R}$ so that $b = a\frac{K^{*F}}{R-1}$. Substituting these steady state conditions into (9) we obtain $K^{*F} = g^{-1}(Ra)$. Hence $b^* = \frac{ag^{-1}(Ra)}{R-1}$ and $Rb^* = a\frac{RK^*}{R-1} = q^*K^*$. In the steady state from (2) we infer $s^{*F} = aK^{*F} = rb^*$ where $r = R - 1$. In words, in the steady state the farmer saves and sells all the tradable output to obtain resources which are used to pay the interest on debt. The principal is never paid back. Debt is rolled over period by period in the same amount. The leverage ratio $\frac{b^*}{aK^*} = \frac{1}{r}$ is constant and equal to the reciprocal of the (net) interest rate.

As to the gatherer, from (8) follows that $c_t^G = G(\bar{K} - K^{*F}) + aK^{*F}$ or $s^{*G} = -aK^{*F} = rb^*$. In words, in the steady state the gatherer is dissaving an amount equal to the interest on debt. The flow of interest payments from the farmer allows the gatherer to consume in excess of income.

KM log-linearize (10) in the neighborhood of the steady state and show that small shocks to the technological parameter a can produce large and persistent fluctuations in output and asset prices. In their model, in fact, the durable, non reproducible asset (land) plays the dual role of a factor of production for both constrained and unconstrained agents and of collateralizable wealth for financially constrained agents. Therefore the price of assets affects the borrowers' financing constraint and at the same time, the size of the borrowers' credit limits feeds back on asset prices.

3. An OLG-KM economy: The environment

In a OLG-KM economy in each period there are four classes of agents. In order to simplify matters, we normalize the population in each class to unity so that we will deal in the following with a young farmer (YF), an old farmer (OF), a young gatherer (YG) and an old gatherer (OG).

In the present context we try to reproduce as much as possible the environment originally envisaged by KM. There are two types of goods, output ("fruit") and a non-reproducible asset ("land") whose total supply is fixed ($\bar{K}$). Output is produced by means of a technology which uses land and labour. Each young agent is endowed with one unit of labour. By assumption farmers and gatherers have access to different technologies. The production function of the YF is: $y_t^F = \alpha K_{t-1}^F$ while the production function of the YG is: $y_t^G = G\left(K_{t-1}^G\right) = G\left(\bar{K} - K_{t-1}^F\right)$ and $G(.)$ is increasing, strictly concave and satisfies the Inada conditions. Due to the time lag between land use and production, the agents work when young and obtain the fruit of their effort when old.

For the sake of simplicity we assume that agents work when young and consume when old. Moreover, they leave a bequest to the offspring when old. As usual the bequest motive is rooted in intergenerational altruism. We assume also that money provides specific utility to the young agent. The marginal utility of money is positive due to the amenities it provides. The generic utility function therefore is

$$U^i = U\left(c_{t,t+1}^i, a_{t+1}^i, m_{t,t}^i\right) \qquad i = F, G$$

where $c_{t,t+1}^i$ is consumption of the agent of type i and generation t in $t+1$, a_{t+1}^i is bequest left by the agent of type i and generation t in $t+1$ to his child. $m_{t,t}^i := \dfrac{M_{t,t}^i}{P_t}$ are real money balances of the agent of of type i and generation t in t.

4. The farmer/borrower

For simplicity we assume that the utility function is separable, we adopt a Cobb-Douglas specification for consumption and bequest and a linear specification for money in the utility function. Preferences of the farmer are represented by

$$U^F = \gamma \ln c_{t,t+1}^F + (1-\gamma) \ln a_{t+1}^F + \nu^F m_{t,t}^F \tag{11}$$

where $0 < \gamma < 1$, $\nu^F > 0$.[3]The farmer maximizes (11) subject to three constraints: the flow-of-funds (FF) constraint of the YF, the FF constraint of the OF and the financing constraint. From optimization it turns out that all the constraints are binding (see the appendix A).

The FF constraint of the YF in t (in real terms) is:

$$q_t\left(K_t^F - K_{t-1}^F\right) + m_{t,t}^F = b_t + a_t^F \tag{12}$$

[3]In the case of bequest, the notation is unambiguous. The bequest left by the agent of generation t in t+1 (i.e. when old) to his child can be denoted by $a_{t,t+1}^F$. The bequest received by agent of generation t+1 in t+1 (i.e. when young) is $a_{t+1,t+1}^F$. Of course the two notions amount to the same magnitude, i.e. $a_{t,t+1}^F = a_{t+1,t+1}^F = a_{t+1}^F$.

where $q_t := \dfrac{Q_t}{P_t}$ is the real price of land; b_t is credit and a_t^F is bequest, i.e. wealth inherited by the YF. According to (12), the resources of the YF, of internal or external origin (a_t^F and b_t respectively), can be employed to "invest", $q_t\left(K_t^F - K_{t-1}^F\right)$ – i.e. to change the farmer's landholding – and accumulate money balances.

Since the young does not derive utility from consumption, the YF carries money over from youth to old age in order to use it as a means of payment in the second stage of his life. Notice that the agent can consume (and leave as a bequest) the output that he obtains from working when young – net of interest payments to the gatherer – because it takes one period for land to produce output. Strictly speaking therefore money is not absolutely necessary to make consumption possible when old. Money only allows to increase consumption (and bequest) when old over and above the level made possible by production alone.

The YF borrows from the YG. Being endowed with *inalienable human capital,* the former can get a loan equal at most to the value of collateralizable assets, i.e. the future value of the land he is currently owning. The financing constraint can be expressed as:

$$b_t = \frac{q_{t+1}}{R} K_t^F \tag{13}$$

where R is the real (gross) interest rate and q_{t+1} is the real price of land in the future which we assume is known in advance (perfect foresight).

In t, the YF uses labour and land K_t^F to produce output which will become available in $t+1$. When old, the farmer's resources e_{t+1}^F consist of output $y_{t+1}^F = \alpha K_t^F$ and real money balances $m_{t,t+1}^F = \dfrac{M_{t,t+1}^F}{P_{t+1}}$ less debt service Rb_t i.e. $e_{t+1}^F = \alpha K_t^F + m_{t,t+1}^F - Rb_t$. These resources can be employed to consume and leave a bequest:

$$c_{t,t+1}^F + a_{t+1}^F + Rb_t \leq \alpha K_t^F + m_{t,t+1}^F \tag{14}$$

Substituting (13) into (12) and rearranging one gets

$$\mu_t K_t^F + m_{t,t}^F = a_t^F + q_t K_{t-1}^F \tag{15}$$

where $\mu_t = q_t - \dfrac{q_{t+1}}{R}$ is the downpayment. Equation (15) provides a different interpretation of the FF constraint of the YF. The farmer's wealth $a_t^F + q_t K_{t-1}^F$ can be employed as downpayment or held as money balances to be employed in old age. If the YF did not hold money, he could put aside a higher downpayment, obtain more land and produce more: Money holding has a crowding out effect on investment in land when the farmer is young.

Substituting (13) into (14) and rearranging one gets

$$c^F_{t,t+1} + a^F_{t+1} = (\alpha - q_{t+1})\, K^F_t + m^F_{t,t+1} \tag{16}$$

Money carried over from young age increases resources of the old farmer. The effort to put aside money when young pays off in old age because it adds resources to those already available to the old for consumption and bequest.

The RHS, i.e. $(\alpha - q_{t+1})\, K^F_t + m^F_{t,t+1}$ is the equation of the resources of the old e^F_{t+1} once one takes into account the fact that the financing constraint is binding. Absent money $m^F_{t,t+1}$, the following condition should be imposed: $\alpha > q_{t+1}$. In the presence of money this is not as stringent as before. It suffices to assume

$$\alpha K^F_t + m^F_{t,t+1} > q_{t+1} K^F_t$$

From the focs and the constraints it is easy to conclude that, thanks to the Cobb-Douglas specification of preferences, consumption and bequest are a fraction γ and $1-\gamma$ respectively of the *resources* available in $t+1$ to the OF: $c^F_{t,t+1} = \gamma e^F_{t+1}$ and $a^F_{t+1} = (1-\gamma)\, e^F_{t+1}$. Hence:

$$c^F_{t,t+1} = \gamma \left[(\alpha - q_{t+1})\, K^F_t + m^F_{t,t+1}\right] \tag{17}$$

$$a^F_{t+1} = (1-\gamma) \left[(\alpha - q_{t+1})\, K^F_t + m^F_{t,t+1}\right] \tag{18}$$

Notice now that from (18) follows that the optimal bequest of the OF of generation $t-1$ in t is:

$$a^F_t = (1-\gamma) \left[(\alpha - q_t)\, K^F_{t-1} + m^F_{t-1,t}\right]$$

where $m^F_{t-1,t} = \dfrac{M^F_{t-1,t}}{P_t}$ are real money balances of the OF of generation $t-1$ in t. Substituting this expression into (15) and rearranging one gets:

$$K^F_t = \frac{[(1-\gamma)\,\alpha + \gamma q_t]\, K^F_{t-1} + (1-\gamma)\, m^F_{t-1,t} - m^F_{t,t}}{\mu_t} \tag{19}$$

which is the law of motion of the land of the farmer.

There are two differences with respect to the law of motion of the farmer's landholding in an ILA-KM economy. First, the dependence of K^F_t on K^F_{t-1} is more complicated than in the original framework (compare (19) with (5)). In fact $\left.\dfrac{dK^F_t}{dK^F_{t-1}}\right|_{OLG} = \dfrac{(1-\gamma)\,\alpha + \gamma q_t}{\mu_t}$ while $\left.\dfrac{dK^F_t}{dK^F_{t-1}}\right|_{ILA} = \dfrac{\alpha}{\mu_t}$. The denominator of the two expressions is the same but the numerator is different. In particular, the numerator of $\dfrac{dK_t^F}{dK_{t-1}^F}$ in the OLG-KM case is a weighted average of q_t and α.

Second, in the OLG-KM case the law of motion of K_t^F depends also on money balances $m_{t-1,t}^F$ and $m_{t,t}^F$. Money has two different and contrasting effects on landholding:

1. *given the bequest,* the higher is money of the young $m_{t,t}^F$, the lower landholding: In fact resources of the young (bequest and credit) can be devoted either to money or landholding;

2. the higher is money of the old $m_{t-1,t}^F$, the higher resources available to him and the higher bequest the old leaves to the young. This bequest, in turn, is employed by the young to expand landholding.

Let's focus now on the way in which real money balances change over time. Following the usual modelling procedure, we conceive of money injections as monetized transfers from the public sector to the old agents. Therefore nominal money balances of the i-th agent when old $M_{t,t+1}^i$ are equal to the sum of money carried on from youth $M_{t,t}^i$ and of subsidies T_{t+1}^i. Moreover, we assume that these transfers are proportional to money balances in youth, i.e. $T_{t+1}^i = \eta_{t+1}^i M_{t,t}^i$. Hence $M_{t,t+1}^i = M_{t,t}^i \left(1 + \eta_{t+1}^i\right)$, where η_{t+1}^i is the rate of growth of money supply for the i-th agent. In our framework there are only two agents (a farmer and a gatherer), so that $i = F, G$. It is straightforward to conclude that real money balances when old are

$$m_{t,t+1}^i = \frac{M_{t,t+1}^i}{P_{t+1}} = m_{t,t}^i \left(1 + \eta_{t+1}^i\right) \theta_{t+1} \qquad i = F, G$$

where $\theta_{t+1} := \dfrac{P_t}{P_{t+1}}$ is the real return on money[4]. The expression $\left(1 + \eta_{t+1}^i\right) \theta_{t+1}$ represents the (gross) rate of change of real money balances.

The expression $(1-\gamma)\, m_{t-1,t}^F - m_{t,t}^F$ which shows up in (19) simplifies to: $(1-\gamma)\, m_t^F \left(1 + \eta_{t+1}^i\right) \theta_{t+1} - m_t^F = -\left[1 - (1-\gamma)\left(1 + \eta_{t+1}^i\right) \theta_{t+1}\right] m_t^F$. Hence (19) boils down to:

$$K_t^F = \frac{\left[(1-\gamma)\,\alpha + \gamma q_t\right] K_{t-1}^F - \left[1 - (1-\gamma)\left(1 + \eta_{t+1}^i\right)\theta_{t+1}\right] m_t^F}{\mu_t} \tag{20}$$

[4] Of course $\theta_{t+1} = \dfrac{1}{1 + \pi_{t+1}}$ where π_{t+1} is the inflation rate.

The first effect prevails – so that all in all an increase in money balances brings about a decrease in landholding – iff $\left(1+\eta_{t+1}^{i}\right)\theta_{t+1}<(1-\gamma)^{-1}$. In the following we will characterize the steady state as a situation in which real money balances are constant, i.e. $\left(1+\eta_{t+1}^{i}\right)\theta_{t+1}=1$. In this case, the first effect prevails.

5. The gatherer/lender

Following the same modelling strategy of the previous section, we assume that preferences of the gatherer are represented as follows

$$U^{G}=\gamma\ln c_{t,t+1}^{G}+(1-\gamma)\ln a_{t+1}^{G}+\nu^{G}m_{t,t}^{G} \tag{21}$$

where $c_{t,t+1}^{G}$ and a_{t+1}^{G} are consumption and bequest of the OG, $m_{t,t}^{G}$ are real money balances of the YG, $\nu^{G}>0$. Being unconstrained from the financial point of view, the gatherer maximizes utility subject to the FF constraints of the YG and of the OG. All the constraints are binding (see the appendix A).

The FF constraint of the YG in t is

$$m_{t,t}^{G}+b_{t}+q_{t}\left(K_{t}^{G}-K_{t-1}^{G}\right)=a_{t}^{G} \tag{22}$$

According to (22), the resources of the YG which coincide with bequest (a_{t}^{G}) can be employed to "invest", $q_{t}\left(K_{t}^{G}-K_{t-1}^{G}\right)$, extend credit and hold money balances. If the YG did not put aside some money in order to employ it in the future – i.e. to increase his resources when old – he could invest more in land and produce more or lend more.

In t, the YG uses labour and land K_{t}^{G} to produce output which will become available in $t+1$: $y_{t+1}^{G}=G\left(K_{t}^{G}\right)$. When old, the gatherer's resources consist of output (produced when young), interest payments received from the farmer and money balances: $e_{t+1}^{G}=G\left(K_{t}^{G}\right)+Rb_{t}+m_{t,t+1}^{G}$. These resources can be employed to consume and leave a bequest. Therefore the FF constraint of the OG in $t+1$ in real terms is:

$$c_{t,t+1}^{G}+a_{t+1}^{G}=G\left(K_{t}^{G}\right)+Rb_{t}+m_{t,t+1}^{G} \tag{23}$$

From the First Order Conditions follows $q_{t}=\dfrac{G'\left(K_{t}^{G}\right)}{R}$. Recalling that, in order to save on notation, we write $G'\left(\bar{K}-K_{t}^{F}\right)=g\left(K_{t}^{F}\right)$, where $g'=-G''>0$, we can write

$$q_{t}=\frac{g\left(K_{t}^{F}\right)}{R} \tag{24}$$

We will refer to (24) in the following as the *asset price equation.*

Since the financing constraint is binding, the amount of credit extended by the YG in t is $b_t = \frac{q_{t+1}}{R} K_t^F$. Therefore the resources of the old gatherer are $e_{t+1}^G = G\left(K_t^G\right) + q_{t+1} K_t^F + m_{t,t+1}^G$ once one takes into account the fact that the financing constraint is binding. Using the focs and the constraints it is easy to conclude that $c_{t,t+1}^G = \gamma e_{t+1}^G$ and $a_{t+1}^G = (1-\gamma) e_{t+1}^G$. Thanks to the Cobb-Douglas specification of prefences, consumption and bequest are a fraction γ and $1-\gamma$ respectively of the *resources* available in $t+1$ to the OG:

$$c_{t,t+1}^G = \gamma \left[G\left(K_t^G\right) + q_{t+1} K_t^F + m_{t,t+1}^G\right] \tag{25}$$

$$a_{t+1}^G = (1-\gamma) \left[G\left(K_t^G\right) + q_{t+1} K_t^F + m_{t,t+1}^G\right] \tag{26}$$

6. On money, land and debt

In order to understand the basic features of the present model, it may be useful to go back to the original ILA-KM economy for a comparison. In the original setting, the farmer may be thought of as maximizing a generic intertemporal utility function $U^F = u\left(c_t^F\right) + \beta^F u\left(c_{t+1}^F\right) + ...$ subject to the FF constraint (2), i.e. $y_t^F + b_t \geq q_t(K_t^F - K_{t-1}^F) + Rb_{t-1} + c_t^F$ and the financing constraint (1), i.e. $b_t \leq \frac{q_{t+1}}{R} K_t^F$ for each time period. The focs are

$$\begin{aligned}
u'\left(c_t^F\right) &= \lambda_t^F \\
\beta^F u'\left(c_{t+1}^F\right) &= \lambda_{t+1}^F \\
\lambda_t^F - \lambda_{t+1}^F R &= \phi_t > 0
\end{aligned}$$

where λ^F is the Lagrange multiplier concerning the FF constraint and ϕ is the Lagrange multiplier associated with the financing constraint. Substituting the first and second focs into the third one, we get

$$u'\left(c_t^F\right) > \beta^F u'\left(c_{t+1}^F\right) R \tag{27}$$

If inequality (27) is satisfied, the financing constraint is binding – i.e. $\phi_t > 0$. This condition may be interpreted as follows. Suppose the farmer obtains one unit of fruit from the gatherer as a new loan and increases consumption in period t by the same amount. The marginal utility he experiences is $u'\left(c_t^F\right)$. On the other hand, he has to give back $R > 1$ units of fruit to the gatherer in period $t+1$

to reimburse debt. The marginal utility of future consumption the farmer should give up therefore is $u'\left(c_{t+1}^F\right)R$. If the marginal utility of the increase in current consumption $u'\left(c_t^F\right)$ is greater than the present value of the marginal utility of future consumption $\beta^F u'\left(c_{t+1}^F\right)R$, the farmer has an incentive to borrow as much as possible, i.e. up to the limit established by the lender equal to the present value of collateralizable wealth.

In the original ILA-KM framework, the felicity function $u(.)$ is linear: $U^F = c_t^F + \beta^F c_{t+1}^F$. Hence the condition for a binding financing constraint becomes

$$1 > \beta^F R \tag{28}$$

As suggested by intuition, the borrower pushes indebtedness to the limit if the interest rate is sufficiently low. In this case, it has to be lower than the rate of time preference.

The gatherer maximizes a generic intertemporal utility function $U^G = u\left(c_t^G\right) + \beta^G u\left(c_{t+1}^G\right) + ...$ subject to the FF constraint (7) which, after substitution of $K_t^G = \overline{K} - K_t^F$ and rearrangement, can be written as $y_t^G + Rb_{t-1} + q_t\left(K_t^F - K_{t-1}^F\right) \geq b_t + c_t^G$. Resources consist of output y_t^G, interest payments on loans extended in the past $Rb_{t-1} = q_t K_{t-1}^F$ and the revenues from the sale of land to the farmer which coincide with the farmer's "investment" $q_t\left(K_t^F - K_{t-1}^F\right)$. Funds can be used to consume c_t^G and extend loans b_t.

The focs are

$$\begin{aligned}
u'\left(c_t^G\right) &= \lambda_t^G \\
\beta^G u'\left(c_{t+1}^G\right) &= \lambda_{t+1}^G \\
-\lambda_t^G + \lambda_{t+1}^G R &= 0 \\
-\lambda_t^G q_t + \lambda_{t+1}^G\left[G'\left(K_t^G\right) + q_{t+1}\right] &= 0
\end{aligned}$$

Substituting the first and second focs into the third one, we get

$$u'\left(c_t^G\right) = \beta^G u'\left(c_{t+1}^G\right)R \tag{29}$$

The marginal utility $u'\left(c_t^G\right)$ the gatherer obtains from increasing current consumption by one unit as a consequence of reducing the amount of loans extended to the farmer by the same amount must be equal to the present value of the marginal utility of future consumption the farmer should give up because of forgone interest $\beta^G u'\left(c_{t+1}^G\right)R$.

In the original ILA-KM framework, the felicity function is linear. Hence equation (29) boils down to

$$1 = \beta^G R \tag{30}$$

Considering (28) and (30) simultaneouly we get

$$\beta^F < \beta^G = R^{-1}$$

which is a restatement of the assumption of *preference heterogeneity* characterizing the framework put forward by KM.

Finally substituting the first and second focs into the fourth one we get

$$u'\left(c_t^G\right)\left(q_t - \frac{q_{t+1}}{R}\right) - \beta^G u'\left(c_{t+1}^F\right) G'\left(K_t^G\right) = 0 \tag{31}$$

In order to interpret (31) suppose the gatherer sells one unit of land to the farmer in t at the price q_t. The farmer's landholding increases by one unit so that the loan the gatherer is willing to extend to the farmer goes up by q_{t+1}/R. All in all, the increase of resources available in t to the gatherer is equal to $q_t - (q_{t+1}/R) = \mu_t$ i.e. the downpayment.

The reduction in the gatherer's landholding in t translates into a loss of output $G'\left(K_t^G\right)$ in $t+1$. The loan made in t yields interest payments equal to $R\left(q_{t+1}/R\right) = q_{t+1}$. On the other hand the smaller landholding in t translates into a smaller revenue from the sale of land in $t+1$ equal to q_{t+1}.[5] Interest payments offset the reduction of revenues from the sale of land so that, all in all, the effect of selling one unit of land in t on resources available in $t+1$ to the gatherer boils down to the loss of output $G'\left(K_t^G\right)$. In the optimum, the marginal utility of increased consumption in t $u'\left(c_t^G\right)\left(q_t - \frac{q_{t+1}}{R}\right)$ must be equal to the present value of marginal disutility associated to forgone consumption in $t+1$ $\beta^G u'\left(c_{t+1}^G\right)\left[G'\left(K_t^G\right)\right]$ as stated in (31).

Due to linearity of the felicity function and recalling (30) the condition (31) specializes to

$$\mu_t = \frac{G'\left(K_t^G\right)}{R}$$

Let's consider now a generic OLG-KM economy. In this setting the farmer maximizes $U^F = u\left(c_{t+1}^F, a_{t+1}^F, m_t^F\right)$ subject to the FF constraints when young and

[5] In fact the farmer's landholding goes up by one unit in t so that the farmer's investment in $t+1$ $q_{t+1}\left(K_{t+1}^F - K_t^F\right)$ goes down by the same amount, coeteris paribus. Therefore resources of the gatherer go down by q_{t+1} in $t+1$.

when old and to the financing constraint. The focs are

$$\begin{aligned} u_c &= u_a = \lambda_{t+1}^F \\ u_m - \lambda_t^F + \lambda_{t+1}^F \left(1 + \eta_{t+1}^F\right) \theta_{t+1} &= 0 \\ \lambda_t^F - \lambda_{t+1}^F R &= \phi_t > 0 \end{aligned}$$

where $u_m = \dfrac{\partial u}{\partial m_t^F}; u_c = \dfrac{\partial u}{\partial c_{t+1}^F}; u_a = \dfrac{\partial u}{\partial a_{t+1}^F}$ are the marginal utilities of money, consumption (when old) and bequest (left to the offspring).

Substituting the first and second focs into the third one, we get

$$u_m + u_c \left(1 + \eta_{t+1}^F\right) \theta_{t+1} > u_c R \tag{32}$$

If inequality (32) is satisfied, the financing constraint is binding. In order to interpret this condition, let's simplify the argument by assuming that $\left(1 + \eta_{t+1}^F\right) \theta_{t+1} = 1$, i.e. the rate of inflation is equal to the rate of growth of money supply of the farmer. Therefore real money balances are constant: $m_{t+1}^F = m_t^F$. Thanks to this assumption inequality (32) becomes

$$u_m + u_c > u_c R \tag{33}$$

Suppose the young farmer obtains a new loan consisting of one unit of fruit and increases money holding in period t by the same amount[6]. In our setting additional money yields an increase in utility both in youth and in old age. The first effect is captured by the term u_m, the marginal utility the farmer gets from increasing money holding m_t^F by one unit when young. The second effect is due to the fact that an increase in money balances in youth carried over to old age adds to the resources available to the farmer when old: Storing value by means of money allows the old farmer to increase consumption and bequest. The marginal utility the old farmer gets from increasing money by one unit when young therefore is $\dfrac{\partial u}{\partial c_{t+1}^F} \dfrac{\partial c_{t+1}^F}{\partial m_{t+1}^F} \dfrac{\partial m_{t+1}^F}{\partial m_t^F} = u_c$ since $c_{t,t+1}^F + a_{t+1}^F = y_t^F + m_{t+1}^F - Rb_t$ and $m_{t+1}^F = m_t^F$. Hence the LHS of (33) is the total increase in utility due to an increase in money holding by one unit when the farmer is young.

On the other hand, the farmer has to give back $R > 1$ units of fruit to the gatherer when old to reimburse debt. The marginal utility of future consumption

[6]By assumption consumption when young does not yield utility. Therefore, the additional loan is not consumed.

the old farmer should give up therefore is $\frac{\partial u}{\partial c_{t+1}^F}\frac{\partial c_{t+1}^F}{\partial b_t} = u_c R$. Inequality (33) states that the financing constraint is binding if the marginal utility $u_m + u_c$ the farmer obtains from increasing money holding by one unit as a consequence of increasing debt by the same amount when young is greater than the marginal utility of consumption $u_c R$ the farmer should give up because he has to reimburse debt when old[7]. In this case the farmer has an incentive to get as much debt as he can.

Notice that (33) can be written as

$$u_m > u_c (R-1) \tag{34}$$

where $R-1>0$. If there were no effect on utility of money holding when young, i.e. $u_m = 0$, the condition for a binding financing constraint would never be satisfied because $u_c R > u_c$. An increase of one unit of money when young, in fact, would yield additional utility u_c when old since it would increase resources devoted to consumption (or bequest). This increase in utility, however, would always be smaller that the reduction of utility $u_c R$ the farmer would suffer in order to reimburse debt. In the presence of debt, money would not be held in portfolios even if it allowed to store "value" for future consumption (or bequest) simply because it would be "too expensive" in terms of forgone consumption to get a new loan to hold money.

A necessary condition for money to be held in portfolios by rational farmers when young, therefore, is $u_m > 0$. This is the reason why money shows up in (11). Therefore the present model belongs to the class of OLG models with Money in the Utility Function. To the best of our knowledge there are relatively few models in this class. A remarkable example is the framework put forward by L. Weiss (1980).

In the present setting the utility function specializes to (11) so that $u_m = \nu^F$ and $u_c = \frac{\gamma}{c_{t,t+1}^F} = \frac{1}{e_{t+1}^F}$ since $c_{t+1}^F = \gamma e_{t+1}^F$ [8]. Hence (34) specializes to

$$\nu^F > \frac{R - \left(1+\eta_{t+1}^F\right)\theta_{t+1}}{e_{t+1}^F} = \frac{R - \left(1+\eta_{t+1}^F\right)\theta_{t+1}}{\left(\alpha - q_{t+1}\right)K_t^F + m_t^F \theta_{t+1}} = \hat{\nu}^F$$

[7] For simplicity, the discount factor is implicitly set to unity.

[8] Notice also that $u_c = \frac{\gamma}{c_{t,t+1}^F} = \lambda_{t+1}^F = \frac{1-\gamma}{a_{t+1}^F} = u_a$

In words, the (constant) marginal utility of money should be greater than a threshold level $\hat{\nu}^F$ which in turn is a function, among other things, of the price of land, the farmer's landholding and the young farmer's money balances. In the following we assume that this condition is satisfied.

Let's consider now the gatherer. In an OLG setting the gatherer maximizes $U^G = u\left(c_{t+1}^G, a_{t+1}^G, m_t^G\right)$ subject to the FF constraints when young and when old. The focs are

$$\begin{aligned} u_c &= u_a = \lambda_{t+1}^G \\ -\lambda_t^G q_t + \lambda_{t+1}^G G'\left(K_t^G\right) &= 0 \\ \lambda_t^G - \lambda_{t+1}^G R &= 0 \end{aligned}$$

where $u_c = \dfrac{\partial u}{\partial c_{t+1}^G}; u_a = \dfrac{\partial u}{\partial a_{t+1}^G}$ are the marginal utilities of consumption (when old) and bequest (left to the offspring) for the gatherer.

Substituting the first and third focs into the second one, we get the asset price equation

$$\frac{G'\left(K_t^G\right)}{R} = q_t \tag{35}$$

The interpretation of (35) is as follows. Suppose the young gatherer sells one unit of land in t at the price q_t. The young farmer's landholding increases by one unit so that the loan the gatherer extends to the farmer goes up by q_{t+1}/R. All in all, the increase of resources available to the gatherer when young is equal to $q_t - (q_{t+1}/R) = \mu_t$, i.e. the downpayment. The marginal impact of this increase of resources in t on the objective function is $\lambda_t \mu_t = u_c R\left[q_t - (q_{t+1}/R)\right] = u_c\left(Rq_t - q_{t+1}\right)$.

The reduction in the gatherer's landholding in t translates into a loss of output $G'\left(K_t^G\right)$ in $t+1$. The loan made in t yields interest payments equal to $R\left(q_{t+1}/R\right) = q_{t+1}$. Therefore, selling one unit of land in t leads to a reduction of resources in $t+1$ equal to $G'\left(K_t^G\right) - q_{t+1}$. The marginal impact of this decrease of resources in $t+1$ on the objective function is $u_c\left[G'\left(K_t^G\right) - q_{t+1}\right]$. In the optimum $u_c\left(Rq_t - q_{t+1}\right) = u_c\left[G'\left(K_t^G\right) - q_{t+1}\right]$. From this condition we get (35).

Notice that in (35) the present value of the marginal productivity of land is equal to the asset price and not to the downpayment.

7. Money flows

Since the total amount of land is fixed by assumption an increase of landholding for the farmer can occur only if there is a corresponding decrease of landholding for the gatherer: $K_t^F - K_{t-1}^F = -\left(K_t^G - K_{t-1}^G\right)$. Taking this fact into account, summing side by side the FF constraints of the young agents (12) and (22), one gets:

$$m_{t,t}^F + m_{t,t}^G = a_t^F + a_t^G \tag{36}$$

In words: the total amount of bequest obtained by the young agents is equal to the total amount of *money of the young agents.* In the special case in which the young agents devote internal resources exclusively to money holding $m_{t,t}^F = a_t^F$ and $m_{t,t}^G = a_t^G$, investment of the farmer is financed exclusively by means of credit, i.e. $q_t\left(K_t^F - K_{t-1}^F\right) = b_t$.

The polar opposite case in which investment is financed exlusively by internal resources means that $q_t\left(K_t^F - K_{t-1}^F\right) = a_t^F$ so that $m_{t,t}^F = b_t$.

In any case both money and bequests are necessary ingredients of the model.

Updating (36) we get

$$m_{t+1,t+1}^F + m_{t+1,t+1}^G = a_{t+1}^F + a_{t+1}^G \tag{37}$$

Summing side by side the FF constraints of the old agents (14) and (23) yields:

$$c_{t,t+1}^F + c_{t,t+1}^G + a_{t+1}^G + a_{t+1}^F = y_{t+1}^G + y_{t+1}^F + m_{t,t+1}^G + m_{t,t+1}^F \tag{38}$$

In words: the sum of aggregate output and real *money balances of the old agents* is equal to the sum of aggregate consumption and aggregate bequest.

We assume equilibrium on the goods market, i.e.

$$c_{t,t+1}^F + c_{t,t+1}^G = y_{t+1}^G + y_{t+1}^F \tag{39}$$

Taking (39) into account, (38) boils down to:

$$m_{t,t+1}^F + m_{t,t+1}^G = a_{t+1}^F + a_{t+1}^G \tag{40}$$

i.e. the total amount of bequest left by the old agents is equal to the total amount of *money of the old agents.* From (36) and (40) follows:

$$m_{t,t+1}^F + m_{t,t+1}^G = m_{t+1,t+1}^F + m_{t+1,t+1}^G$$

In words, money of the old agents should be equal to money of the young agents in each period.

In our economy money "trickles down" from one period to the next and from one agent to the other. In fact a network of money transfers is taking place from the pool of monetary resources of one agent to the pool of another agent. In principle we distinguish two types of transfers among private agents:

- "within generations" or *horizontal transfers*, i.e. transfers between agents of the same generation but of different types (farmers and gatherers). Horizontal transfers are the monetary counterpart of transactions between agents of different types concerning goods (fruit) or land. Therefore they are motivated by agents' decisions to consume and invest, i.e. modify landholdings;

- "between generations" or *vertical transfers*, i.e. transfers between agents of different generations but of the same type (old and young agents). Vertical transfers coincides with bequests, which are motivated by intergenerational altruism;

In order to describe the way in which money flows in the economy, let's take a look at table 1. In each row we report the inflows and outflows which show up in the FF constraints of the agents in period $t+1$. The amount in the inflow cell is equal to the amount in the outflow cell. For instance, the first row represents the FF constraint of the YF in $t+1$. In other words, we have rewritten in a suitable form equation (12). The third row is the sum of rows 1 and 2 (concerning young agents), the sixth row is the sum of rows 4 and 5 (concerning old agents). Therefore, the table contains equations (12), (22), (37), (14), (23) and (38).

	inf lows	outflows
YF	$a_{t+1}^F + b_{t+1}$	$q_{t+1}\left(K_{t+1}^F - K_t^F\right) + m_{t+1,t+1}^F$
YG	a_{t+1}^G	$-q_{t+1}\left(K_{t+1}^F - K_t^F\right) + m_{t+1,t+1}^G + b_{t+1}$
$\sum$	$a_{t+1}^G + a_{t+1}^F$	$m_{t+1,t+1}^G + m_{t+1,t+1}^F$
OF	$y_{t+1}^F + m_{t,t+1}^F$	$c_{t,t+1}^F + a_{t+1}^F + Rb_t$
OG	$y_{t+1}^G + m_{t,t+1}^G + Rb_t$	$c_{t,t+1}^G + a_{t+1}^G$
$\sum$	$y_{t+1}^F + m_{t,t+1}^F + y_{t+1}^G + m_{t,t+1}^G$	$c_{t,t+1}^F + a_{t+1}^F + c_{t,t+1}^G + a_{t+1}^G$

(Table 1)

Let's assume that $y_{t+1}^F - c_{t,t+1}^F = s_{t,t+1}^F > 0$, i.e. the OF consumes less than the output he has produced. In a sense he is "saving" the amount $s_{t,t+1}^F$. Market clearing on the goods market implies $s_{t,t+1}^G = -\left(c_{t,t+1}^G - y_{t+1}^G\right) = -s_{t,t+1}^F < 0$ i.e. the OG consumes more than the output he has produced. He is "dissaving" the amount $-\left(c_{t,t+1}^G - y_{t+1}^G\right)$. In other words, the OG has excess consumption $c_{t,t+1}^G - y_{t+1}^G$.

The OF sells $s_{t,t+1}^F$ units of output to the OG in order to let him consume in excess of his output. The OG pays this output by means of money. Therefore, after the transaction, the OF has money balances equal to $m_{t,t+1}^F + \left(c_{t,t+1}^G - y_{t+1}^G\right)$. This money is used to reimburse debt $b_t R$ to the OG and leave the bequest a_{t+1}^F to the YF. Accounts are consistent: In fact $c_{t,t+1}^G - y_{t+1}^G = y_{t+1}^F - c_{t,t+1}^F$ so that $m_{t,t+1}^F + y_{t+1}^F - c_{t,t+1}^F = a_{t+1}^F + b_t R$ which is the FF of the OF.

The YF receives a_{t+1}^F from OF and b_{t+1} from the YG and employs these resources to invest, i.e. $q_{t+1}\left(K_{t+1}^F - K_t^F\right)$ and hold money balances. Notice that, since $a_{t+1}^F = (1-\gamma)\left[(\alpha - q_{t+1}) K_t^F + m_{t,t+1}^F\right]$,

$$m_{t+1,t+1}^F = \left[(1-\gamma)\alpha + \gamma q_{t+1}\right] K_t^F + (1-\gamma) m_{t,t+1}^F - \mu_{t+1} K_{t+1}^F$$

This equation links the money of the young farmer of generation $t+1$ to the money of the old farmer of generation t in period $t+1$. It is (19) rewritten and updated.

Recalling that $m_{t,t+1}^i = \dfrac{M_{t,t+1}^i}{P_{t+1}} = m_{t,t}^i\left(1+\eta_{t+1}^i\right)\theta_{t+1} \qquad i = F, G$

$$\left[m_{t,t}^F\left(1+\eta_{t+1}^F\right) + m_{t,t}^G\left(1+\eta_{t+1}^G\right)\right]\theta_{t+1} = m_{t+1,t+1}^F + m_{t+1,t+1}^G \tag{41}$$

In the steady state $m_{t-1,t-1}^F = m_{t,t}^F$ and $m_{t-1,t}^F = m_{t,t+1}^F$. In words, the farmer of generation t when young (old) holds the same money balances in real terms of the farmer of generation $t-1$ when young (old) or the son has the same money balances of the father when young.Hence $m_{t-1,t-1}^F\left(1+\eta_t^F\right)\theta_t = m_{t,t}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1}$. In this case we can simplify notation writing m_t^F in place of $m_{t,t}^F$. Hence $m_{t-1,t}^F = m_{t-1}^F\left(1+\eta_t^F\right)\theta_t = m_t^F\left(1+\eta_{t+1}^F\right)\theta_{t+1}$. The same applies to the gatherer. In the steady state therefore (41) holds only if

$$\left(1+\eta_{t+1}^i\right)\theta_{t+1} = 1$$

This is true if

$$\eta_{t+1}^i = \pi_{t+1}$$

i.e. the steady state. This case means that (i) the rate of growth of the money supply for each and every agent is the same, i.e. the ratio of the two types of money is constant, (ii) the rate of change of prices is equal to the rate of change of money.

Alternatively one can think that $m^F_{t-1,t} = m^F_{t,t}$. In words, the farmer of generation t when young holds the same money balances in real terms of the farmer of generation t-1 when old or the son has the same money balances of the father. This means that $m^F_{t-1,t-1}\left(1+\eta^F_t\right)\theta_t = m^F_{t,t}$. We can simplify notation writing m^F_t in place of $m^F_{t,t}$. Hence $m^F_{t-1}\left(1+\eta^F_t\right)\theta_t = m^F_t$. Only in case $\eta^i_{t+1} = \pi_{t+1}$ we get the same result as before so that

$$s^F = (R-1)\,b$$

as in KM-ILA. This means that only the case of a positive farmer's saving can be true in the steady state.

Thanks to the Cobb-Douglas specification of the utility function, from the $FOCs$ (see appendix A) one gets:

$$c^i_{t,t+1} = \frac{\gamma}{1-\gamma} a^i_{t+1} \qquad i = F, G \tag{42}$$

Substituting (42) and the market clearing condition (39) into (38) we obtain:

$$m^G_{t,t+1} + m^F_{t,t+1} = \frac{1-\gamma}{\gamma}\left(y^G_{t+1} + y^F_{t+1}\right) \tag{43}$$

Total real money balances are proportional to aggregate output. Equation (43) is a sort of quantity theory of money in this context.

Recalling $m^i_{t,t+1} = \frac{M^i_{t,t+1}}{P_{t+1}} = m^i_{t,t}\left(1+\eta^i_{t+1}\right)\theta_{t+1} \qquad i = F, G$ we get

$$m^F_{t,t+1} + m^G_{t,t+1} = m^F_{t+1,t+1} + m^G_{t+1,t+1}$$

(43) becomes

$$m^F_{t+1,t+1} + m^G_{t+1,t+1} = \frac{1-\gamma}{\gamma}\left(y^G_{t+1} + y^F_{t+1}\right)$$

In the following we will re-write the equation above as:

$$m^F_{t+1,t+1} = \frac{1-\gamma}{\gamma\left(1+\sigma_{t+1}\right)}\left(y^G_{t+1} + y^F_{t+1}\right) \tag{44}$$

where $\sigma_{t+1} := \dfrac{m^G_{t+1,t+1}}{m^F_{t+1,t+1}}$ is the ratio of the money of the gatherer to the money of the farmer (*money ratio* for short). This ratio is constant iff $\eta^G_{t+1} = \eta^F_{t+1}$.

8. Dynamics

The dynamics of the macroeconomy are described by equation (19), i.e the law of motion of the farmer's land, equation (24), i.e. the asset price equation, and equation (43), i.e. the quantity theory of money.

Let's assume $\eta_{t+1}^{i}=\pi_{t+1}$ so that $(1-\gamma)\, m_{t-1,t}^{F}-m_{t,t}^{F}=-\gamma m_{t}^{F}$. We list the equations below for the reader's convenience.

$$K_t^F=\frac{\left[(1-\gamma)\,\alpha+\gamma q_t\right]K_{t-1}^F-\gamma m_t^F}{q_t-\frac{q_{t+1}}{R}}$$
$$q_t=\frac{g\left(K_t^F\right)}{R}$$
$$m_t^F=\frac{1-\gamma}{\gamma\,(1+\sigma)}\left[\alpha K_{t-1}^F+G\left(\bar{K}-K_{t-1}^F\right)\right]$$

Plugging the third equation into the first one, the system boils down to

$$\begin{aligned} K_t^F &= \frac{\left[\frac{(1-\gamma)\,\alpha\sigma}{1+\sigma}+\gamma q_t\right]K_{t-1}^F-\frac{1-\gamma}{1+\sigma}G\left(\bar{K}-K_{t-1}^F\right)}{q_t-\frac{q_{t+1}}{R}} \\ q_t &= \frac{g\left(K_t^F\right)}{R} \end{aligned} \tag{45}$$

Substituting the second equation into the first one and noting that $q_{t+1}=\dfrac{g\left(K_{t+1}^F\right)}{R}$ the system boils down to

$$\left[\frac{g\left(K_t^F\right)}{R}-\frac{g\left(K_{t+1}^F\right)}{R^2}\right]K_t^F-\left[\frac{(1-\gamma)\,\alpha\sigma}{1+\sigma}+\frac{\gamma g\left(K_t^F\right)}{R}\right]K_{t-1}^F+\frac{1-\gamma}{1+\sigma}G\left(\bar{K}-K_{t-1}^F\right)=0 \tag{46}$$

which is a non linear second order difference equation in the state variable K_t^F in implicit form.

Let's assume $\sigma=\sigma_0$. In the steady state $K_t^F=K_{t-1}^F=K^F$ and $q_t=q_{t+1}=q$ so that $\mu_t=\mu$ with $\mu=q\varepsilon$ with $\varepsilon=1-\dfrac{1}{R}$. In the steady state, therefore, the system boils down to

$$\begin{gathered} q=\zeta\left[h\left(K^F\right)-\sigma_0\alpha\right] \\ q=\frac{g\left(K^F\right)}{R} \end{gathered} \tag{47}$$

where

$$\zeta = \frac{1-\gamma}{(\gamma-\varepsilon)(1+\sigma_0)}$$
$$h\left(K^F\right) = \frac{G\left(\bar{K}-K^F\right)}{K^F}$$

and $h'\left(K^F\right) < 0$. (47) is a system of two equations which can be solved for the steady state values of in $K^F$and q. The second equation yields a decreasing relationship between q and K^F. On the other hand, from the first equation it is clear that the corresponding isocline can be either upward sloping or downward sloping on the $\left(K^F, q\right)$ plane depending upon the relative value of γ and ε. In the case $\gamma > \varepsilon$ (respectively: $\gamma < \varepsilon$), the curve is downward (upward) sloping. In both cases, the curve crosses the x-axis when $K_c^F = h^{-1}(\sigma_0\alpha)$. The steady state is unique if the curve is downward sloping; there can be more than one steady state in the opposite case.

In order to assess the properties of the trajectories generated by this system, we have to specify the gatherer's production function. We assume $G\left(\bar{K}-K_t^F\right) = \sqrt{\bar{K}-K_t^F}$.

The system is characterized by 5 parameters: the scale of the economy $\bar{K}$, the preference parameter γ, the productivity of the farmer's land α, the money ratio σ, the real interest rate R. Depending upon the configuration of parameters, we can have different dynamic patterns and properties of equilibria.

In figure 1, left panel, for instance, we have a unique steady state E^*. The white region is the basin of attraction of E^*. Trajectories originating in points in the grey region diverge.

Other things being equal, when σ goes up an attracting closed curve Γ emerges (figure 1 right panel), as a consequence of a supercritical Neimark-Sacker bifurcation of the steady state E^*. Once again, the white region is the basin of attraction of the closed curve Γ. Trajectories originating in points in the grey region diverge. Quasi periodic or aperiodic orbits emerge when the system "circles" along the curve.

As often occurs in nonlinear models, coexistence of attractors and complex dynamics are possible outcomes. A possible route is shown in the next figures. In Fig.2, the attracting closed curve Γ coexists with a stable cycle C of period 6. The light grey points denote the basin of attraction of the cycle C. Observe that the curve Γ is very close to the boundary of its basin of attraction.

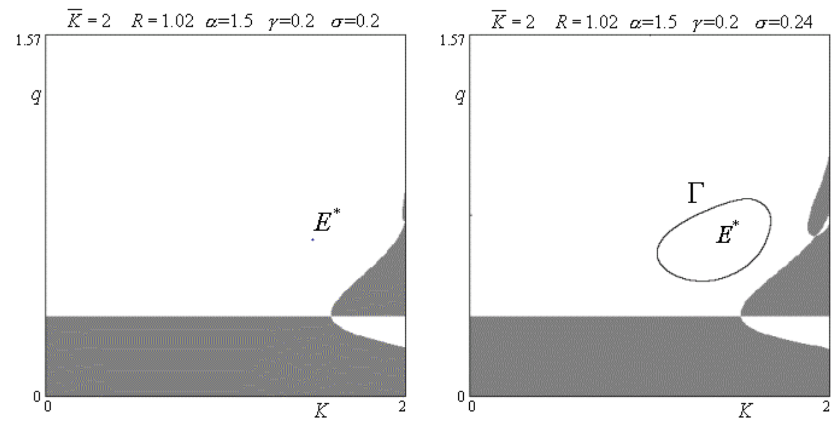

Figure 1: Left panel: unique steady state E^*. Right panel: closed curve Γ

As σ is slightly increased, Γ disappears via contact bifurcation, and the long-run behavior of the bounded trajectories is given by a 6-piece chaotic attractor, obtained through a sequence of period doubling bifurcations of the cycle of period 6.

After a sequence of homoclinic bifurcations, a (one-piece) strange attractor appears (Fig.3). Once more, it is very close to its basin boundary. This means that a further increase of σ will cause a second contact bifurcation, whose effect is the desappearance of the attractor and the divergence of the generic trajectories.

The sequence just illustrated is typical of this model, occurring at different parameter configurations. In the last figure, we show the stability region of E^* in the parameter space (σ, α). We may observe that the Neimark-Sacker bifurcation curve establishes a decreasing relationship between σ and α and that the stability region shrinks as R increases (Fig.4).

In order to interpret the consequences of a change in the stance of monetary policy, let's assume, for the sake of discussion, that the rate of change of money is uniform across agents and equal to the rate of inflation, i.e. $\eta_0^F = \eta_0^G = \eta_0 = \pi_0$.

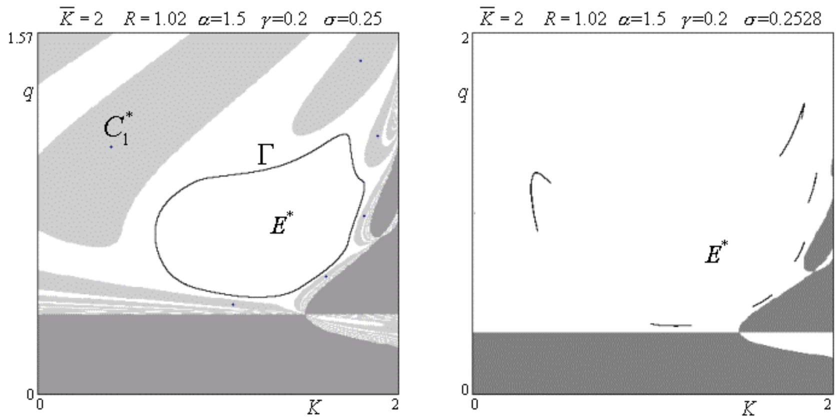

Figure 2: Left panel stable cicle C. Right panel: 6 piece chaotic attractor.

Let the money ratio be σ_0. Suppose now that the rate of change of money supply goes up and is still uniform across agents, i.e. $\eta_1^F = \eta_1^G = \eta_1$. In this case, the money ratio does not change so that also K^F and m^F will not change. The effect of such a move is to increase the rate of inflation to $\pi_1 = \eta_1$. The real interest rate remains unchanged $R = \dfrac{1+i_0}{1+\pi_0} = \dfrac{1+i_1}{1+\pi_1}$ but the nominal interest rate goes up in the same proportion as the inflation rate.

Suppose now that the central bank adopts a *differentiated policy move*. For instance the rate of growth of money of the gatherer becomes $\eta_1^G > \eta_0$ while the rate of growth of money of the farmer remains unchanged $\eta_0^F = \eta_0$. The money ratio goes up to σ_1 and stays there even if the rate of growth of money of the gatherer goes down to η_0 thereafter. After the shock, therefore, the inflation rate goes back to π_0. Due to the (permanent) change of the money ratio σ the dynamics of the model change dramatically as shown above. We can draw therefore the following conclusion

Remark 1. *If a policy move does not change the money ratio* σ*, i.e. if the*

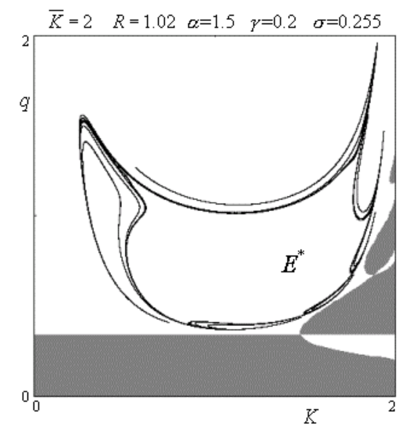

Figure 3: One piece strange attractor

central bank changes the rates of growth of the monetary aggregates of the farmer and the gatherer by the same amount, monetary policy is superneutral, i.e. the allocation of land to the farmer and the gatherer does not change, real variables are unaffected and the only effect of the policy move is an increase in the rate of inflation, which is pinned down to the (uniform) rate of change of money. If, on the other hand, the move is differentiated, i.e. the rates of growth of the two monetary aggregates are heterogeneous – albeit temporarily – σ changes and monetary policy is not superneutral, i.e. the allocation of land changes and real variables are permanently affected, even if the rates of growth of the two aggregates go back to the original value afterwards.

9. Conclusions

In this paper we have presented and discussed an OLG model of an economy characterized by financing constraints *à la* Kiyotaki and Moore. We adopt a

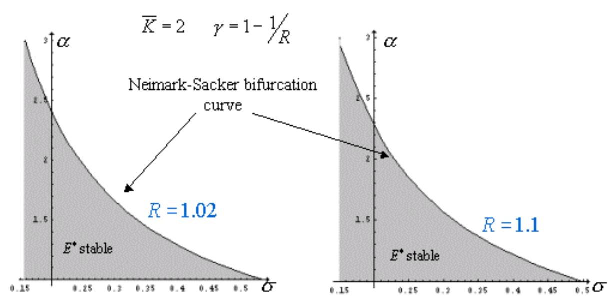

Figure 4: Neimark-Sacker bifurcations

Diamond-Samuelson approach, a strategy not followed in the literature, which has explored only the case of a Blanchard-Yaari framework to model the financial accelerator with overlapping generations (Kasa, 1998).

In this setting we explore the properties of the dynamical two-dimensional system generated by the model. Without imposing ad-hoc non-linearities, we get a straightforward route to complex dynamics.

Changes in the rate of growth of money supply can have real effects if the central bank changes also the allocation of money to the two types of agents. If monetary policy moves do not influence the allocation of money, also the allocation of land – and therefore aggregate output – will be unaffected and money will turn out to be superneutral.

A. Optimization

In the following we will denote magnitudes at current (constant) prices with capital (small) letters.

Let's examine first the optimization problem of the farmer. The young farmer (YF) is endowed at birth with bequest A_t^F. He employs the bequest and credit B_t to invest in land $Q_t\left(K_t^F - K_{t-1}^F\right)$ and hold money balances $M_{t,t}^F$. The flow-of-funds (FF) constraint of the YF in t therefore is:

$$Q_t\left(K_t^F - K_{t-1}^F\right) + M_{t,t}^F \leq B_t + A_t^F$$

Dividing by P_t and rearranging we get:

$$q_t\left(K_t^F - K_{t-1}^F\right) + m_{t,t}^F \leq b_t + a_t^F \tag{48}$$

where $q_t = \dfrac{Q_t}{P_t}$, $m_{t,t}^F = \dfrac{M_{t,t}^F}{P_t}$, $b_t = \dfrac{B_t}{P_t}$, $a_t^F = \dfrac{A_t^F}{P_t}$.

The YF is financially coinstrained. The financing constraint in nominal terms can be expressed as follows:

$$B_t \leq \frac{Q_{t+1}}{1+i_t} K_t^F$$

where i_t is the nominal interest rate. Multiplying and dividing the expression above by P_{t+1} one gets:

$$b_t \leq \frac{q_{t+1}}{R} K_t^F \tag{49}$$

where $R := (1+i_t)/(1+\pi_{t+1})$ is the real (gross) interest rate and $1+\pi_{t+1} := P_{t+1}/P_t$ is the (gross) rate of inflation. As in KM, R is given and constant[9].

In t, the YF uses labour and land K_t^F to produce output y_{t+1}^F which will become available in $t+1$. When old, the farmer employs output and money balances $M_{t,t+1}^F$ to reimburse debt, consume and leave a bequest. Therefore the FF constraint of the old farmer (OF) in $t+1$ in nominal terms is:

$$P_{t+1} c_{t,t+1}^F + A_{t+1}^F + B_t(1+i_t) \leq P_{t+1} y_{t+1}^F + M_{t,t+1}^F$$

Dividing by P_{t+1}, recalling that $y_{t+1}^F = \alpha K_t^F$ and $m_{t,t+1}^F = \dfrac{M_{t,t+1}^F}{P_{t+1}} = m_{t,t}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1}$ – where η_{t+1}^F is the rate of growth of money of the farmer and $\theta_{t+1} := \dfrac{1}{1+\pi_{t+1}}$ so

[9] This assumption holds if the current nominal interest rate is adjusted for future inflation as follows $i_t = (R-1) + R\pi_{t+1}$.

that $\left(1+\eta_{t+1}^F\right)\theta_{t+1}$ is the rate of growth of real money balances of the farmer – we obtain

$$c_{t,t+1}^F + a_{t+1}^F + Rb_t \le \alpha K_t^F + m_{t,t}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1} \tag{50}$$

The farmer maximizes (11) subject to (48), (50) and (49). The Lagrangian is:

$$\begin{aligned}\mathcal{L} &= \gamma \ln c_{t,t+1}^F + (1-\gamma)\ln a_{t+1}^F + \nu^F m_{t,t}^F + \lambda_t^F\left[b_t + a_t^F - q_t\left(K_t^F - K_{t-1}^F\right) - m_{t,t}^F\right] + \\ &\quad + \lambda_{t+1}^F\left[\alpha K_t^F + m_{t,t}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1} - c_{t,t+1}^F - a_{t+1}^F - b_t R\right] + \phi_t\left[\frac{q_{t+1}}{R}K_t^F - b_t\right]\end{aligned}$$

The $FOCs$ are:

$$\begin{aligned}(iF)\quad \frac{\partial \mathcal{L}}{\partial c_{t,t+1}^F} &= 0 \Rightarrow \frac{\gamma}{c_{t,t+1}^F} = \lambda_{t+1}^F \\ (iiF)\quad \frac{\partial \mathcal{L}}{\partial a_{t+1}^F} &= 0 \Rightarrow \frac{1-\gamma}{a_{t+1}^F} = \lambda_{t+1}^F \\ (iiiF)\quad \frac{\partial \mathcal{L}}{\partial m_{t,t}^F} &= 0 \Rightarrow \nu^F - \lambda_t^F + \lambda_{t+1}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1} = 0 \\ (ivF)\quad \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Rightarrow \lambda_t^F - \lambda_{t+1}^F R = \phi_t\end{aligned}$$

From (iF) and (iiF) follows that $\lambda_{t+1}^F = \dfrac{\gamma}{c_{t,t+1}^F} = \dfrac{1-\gamma}{a_{t+1}^F} > 0$. Hence the FF of the OF is binding (see (14)). Taking into account (iF), from (iiiF) follows $\lambda_t^F = \nu^F + \dfrac{\gamma}{c_{t,t+1}^F}\left(1+\eta_{t+1}^F\right)\theta_{t+1} = \nu^F + \dfrac{1-\gamma}{a_{t+1}^F}\left(1+\eta_{t+1}^F\right)\theta_{t+1} > 0$ so that $\lambda_t^F > 0$. Also the FF of the YF is binding (see (12)).

Finally, we assume that

$$\frac{\lambda_t^F}{\lambda_{t+1}^F} > R \tag{51}$$

so that from (ivF) follows that $\phi_t > 0$. Therefore the financing constraint is binding (see (13)).

Using the focs and the constraints it is easy to conclude that $c_{t,t+1}^F = \gamma e_{t+1}^F$ and $a_{t+1}^F = (1-\gamma)e_{t+1}^F$ where $e_{t+1}^F = (\alpha - q_{t+1})K_t^F + m_{t,t}^F\left(1+\eta_{t+1}^F\right)\theta_{t+1}$ are the resources available to the OF – i.e. output and money less interest payments – which we assume to be positive. Hence

$$\lambda_{t+1}^F = \frac{\gamma}{c_{t,t+1}^F} = \frac{1-\gamma}{a_{t+1}^F} = \frac{1}{e_{t+1}^F}$$

In words, the marginal utility of consumption is equal to the marginal utility of bequest and is equal to the reciprocal of the resources available to the OF.

Using the equality above, the Lagrange multiplier λ_t^F turns out to be

$$\lambda_t^F = \nu^F + \frac{\left(1 + \eta_{t+1}^F\right)\theta_{t+1}}{e_{t+1}^F}$$

i.e. it is the sum of the marginal utility of money and the marginal utility of consumption (or bequest) $\frac{1}{e_{t+1}^F}$ times the rate of growth of real money balances of the farmer $\left(1 + \eta_{t+1}^F\right)\theta_{t+1}$.

Substituting the values of the Lagrange multipliers into (51) we can rewrite it as

$$\frac{\lambda_t^F}{\lambda_{t+1}^F} = \nu^F e_{t+1}^F + \left(1 + \eta_{t+1}^F\right)\theta_{t+1} > R$$

or

$$\nu^F > \frac{R - \left(1 + \eta_{t+1}^F\right)\theta_{t+1}}{e_{t+1}^F} = \frac{R - \left(1 + \eta_{t+1}^F\right)\theta_{t+1}}{\left(\alpha - q_{t+1}\right)K_t^F + m_{t,t}^F\left(1 + \eta_{t+1}^F\right)\theta_{t+1}} = \hat{\nu}^F \tag{52}$$

Notice that $R = \theta_{t+1}\left(1 + i_t\right)$ so that $R - \left(1 + \eta_{t+1}^F\right)\theta_{t+1} = \theta_{t+1}\left(i_t - \eta_{t+1}^F\right)$. The condition above – which implies that $\phi_t > 0$ and the financing constraint is binding – will be always satisfied if $i_t \leq \eta_{t+1}^F$ because in this case $\hat{\nu}^F \leq 0$. On the other hand, if $i_t > \eta_{t+1}^F$ the marginal utility of money must be sufficiently high, i.e. higher than a threshold $\hat{\nu}^F > 0$ for (52) to hold. In the steady state $\left(1 + \eta_{t+1}^F\right)\theta_{t+1} = 1$ so that $R - \left(1 + \eta_{t+1}^F\right)\theta_{t+1} = R - 1$. The condition above boils down to

$$\nu^F > r\left[\left(\alpha - q_s\right)K_s^F + m_s^F\right]^{-1}$$

Let's consider now the gatherer's optimization problem. The young gatherer (YG) is endowed at birth with bequest A_t^G which he employs to extend credit B_t, invest in land $Q_t\left(K_t^G - K_{t-1}^G\right) = -Q_t\left(K_t^F - K_{t-1}^F\right)$ and hold money balances $M_{t,t}^G$. The FF constraint of the YG in t is

$$Q_t\left(K_t^G - K_{t-1}^G\right) + B_t + M_{t,t}^G \leq A_t^G$$

Dividing by P_t we get:

$$q_t\left(K_t^G - K_{t-1}^G\right) + b_t + m_{t,t}^G \leq a_t^G \tag{53}$$

In t, the YG uses labour and land $K_t^G = \bar{K} - K_t^F$ to produce output y_{t+1}^G which will become available in $t+1$. When old, the gatherer employs the output, the repayment of the loan extended when young and money to consume and leave a bequest. Therefore the FF constraint of the OG in $t+1$ is:

$$P_{t+1} c_{t,t+1}^G + A_{t+1}^G \leq P_{t+1} y_{t+1}^G + B_t \left(1 + i_t\right) + M_{t,t+1}^G$$

Dividing by P_{t+1} and recalling that $y_{t+1}^G = G\left(K_t^G\right)$ and $\dfrac{M_{t,t+1}^G}{P_{t+1}} = m_{t,t+1}^G = m_{t,t}^G \left(1 + \eta_{t+1}^G\right) \theta_{t+1}$ we get

$$c_{t,t+1}^G + a_{t+1}^G \leq G\left(K_t^G\right) + q_{t+1} K_t^F + m_{t,t}^G \left(1 + \eta_{t+1}^G\right) \theta_{t+1} \tag{54}$$

The gatherer maximizes (21) subject to (53) and (54). The Lagrangian is:

$$\begin{aligned} \mathcal{L} &= \gamma \ln c_{t,t+1}^G + (1-\gamma) \ln a_{t+1}^G + \nu^G m_{t,t}^G + \lambda_t^G \left[a_t^G - q_t \left(K_t^G - K_{t-1}^G\right) - m_{t,t}^G - b_t\right] \\ &\quad + \lambda_{t+1}^G \left[G\left(K_t^G\right) + R b_t + m_{t,t}^G \left(1 + \eta_{t+1}^G\right) \theta_{t+1} - c_{t,t+1}^G - a_{t+1}^G\right] \end{aligned}$$

The $FOCs$ are:

$$\begin{aligned} (iG) \ \frac{\partial \mathcal{L}}{\partial c_{t,t+1}^G} &= 0 \Rightarrow \frac{\gamma}{c_{t,t+1}^G} = \lambda_{t+1}^G \\ (iiG) \ \frac{\partial \mathcal{L}}{\partial a_{t+1}^G} &= 0 \Rightarrow \frac{1-\gamma}{a_{t+1}^G} = \lambda_{t+1}^G \\ (iiiG) \ \frac{\partial \mathcal{L}}{\partial K_t^G} &= 0 \Rightarrow \lambda_{t+1}^G G'\left(K_t^G\right) = \lambda_t^G q_t \\ (ivG) \ \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Rightarrow \lambda_t^G = \lambda_{t+1}^G R \end{aligned}$$

From (iF) and (iiF) follows that $\lambda_{t+1}^G = \dfrac{\gamma}{c_{t,t+1}^G} = \dfrac{1-\gamma}{a_{t+1}^G} > 0$. Hence the FF of the OG is binding (see equation (23)).

From (ivG) follows that

$$\frac{\lambda_t^G}{\lambda_{t+1}^G} = R \tag{55}$$

so that

$$\lambda_t^G = \lambda_{t+1}^G R = \frac{\gamma}{c_{t,t+1}^G} R = \frac{1-\gamma}{a_{t+1}^G} R > 0$$

Hence also the FF of the YF is binding (see equation (22)).

Using the focs and the constraints it is easy to conclude that $c^G_{t,t+1} = \gamma e^G_{t+1}$ and $a^G_{t+1} = (1-\gamma)\, e^G_{t+1}$ where $e^G_{t+1} = G\left(K^G_t\right) + q_{t+1}K^F_t + m^G_{t,t}\left(1+\eta^G_{t+1}\right)\theta_{t+1}$ are the resources available to the OG. Hence $\lambda^G_{t+1} = \frac{\gamma}{c^G_{t,t+1}} = \frac{1-\gamma}{a^G_{t+1}} = \frac{1}{e^G_{t+1}}$. Using the equality above, the Lagrange multiplier λ^G_t turns out to be

$$\lambda^G_t = \frac{R}{e^G_{t,t+1}} = \frac{R}{G\left(K^G_t\right) + q_{t+1}K^F_t + m^G_{t,t}\left(1+\eta^G_{t+1}\right)\theta_{t+1}}$$

Comparing the focs of the optimization problem of the farmer with those of the gatherer we can draw the following conclusion

$$\frac{\lambda^F_t}{\lambda^F_{t+1}} = \nu^F e^F_{t+1} + \left(1+\eta^F_{t+1}\right)\theta_{t+1} > \frac{\lambda^G_t}{\lambda^G_{t+1}} = R$$

This conclusion was true also in KM but with a different meaning. In fact in a KM-ILA economy $\frac{\lambda^F_t}{\lambda^F_{t+1}} = \frac{1}{\beta^F} > \frac{\lambda^G_t}{\lambda^G_{t+1}} = \frac{1}{\beta^G} = R$. Hence this condition reflected preference heterogeneity.

Finally, from (iiiG) and (ivG) follows (24):

$$q_t = \frac{G'\left(K^G_t\right)}{R}$$

Notice that we do not assume that the gatherer chooses money balances optimally. In fact the optimal choice of money balances would imply $\frac{\partial \mathcal{L}}{\partial m^G_{t,t}} = 0$ which yields $\nu^G - \lambda^G_t + \lambda^G_{t+1}\left(1+\eta^G_{t+1}\right)\theta_{t+1} = 0$. Taking (ivG) into account, this condition would translate into $\nu^G - \lambda^G_{t+1}\left[R - \left(1+\eta^G_{t+1}\right)\theta_{t+1}\right] = \nu^G - \frac{R-\left(1+\eta^G_{t+1}\right)\theta_{t+1}}{e^G_{t+1}} = 0$. We do not impose this restriction. In our scenario therefore if $e^G_{t+1} > \frac{R-\left(1+\eta^G_{t+1}\right)\theta_{t+1}}{\nu^G}$, then $\frac{\partial \mathcal{L}}{\partial m^G_{t,t}} > 0$ i.e. the gatherer holds too little money (w.r.t. the optimal money balances) and viceversa.

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