Bifurcations of optimal vector fields in the shallow lake system

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Abstract

The shallow lake problem is a nonconvex production-pollution dynamic optimisation problem whose solution structure depends nonlinearly on the system parameters. We perform a bifurcation analysis to investigate the consequences of varying the relative cost of pollution and the discount rate.

keywords: resource economics, shallow lake, bifurcations, Skiba points

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1 Introduction

Dynamic optimisation problems may feature "Skiba" or indifference states, which are initial states to more than one optimal solution. In problems with a one-dimensional state space, Skiba states are always boundary points of basins of attraction of long-term optimal steady states; their knowledge gives therefore information about the complete solution structure of the problem. Analogously to the situation in dynamical systems, we can investigate the qualitative changes in this structure by performing a bifurcation analysis.

In this paper the main ideas of the appropriate bifurcation methodology are sketched, which can be applied to a wide range of problems. These ideas are used here to analyse the effects of varying the discount rate and the social costs of pollution in the shallow lake pollution problem [1, 3, 4].

2 The shallow lake model

We begin by describing the shallow lake pollution problem. The state variable in this model is the phosphorus concentration $x > 0$ in a shallow lake. There is an inflow u of phosphorus due to agricultural activities. The state dynamics are modelled as

$$\dot{x} = u - bx + \frac{x^2}{x^2+1}, \tag{1}$$

where b is the natural decay rate. The highly stylised nonlinear term captures the main qualitative characteristic of shallow lakes: for small decay rates and an initially clean lake, increasing the loading u sufficiently slowly such that the system can always settle down to a steady state, there is a critical level u^* at which the system shifts catastrophically to a polluted state.

The instantaneous utilities from farming and using the lake are modelled as $\log u$ and $-cx^2$ respectively, leading to a social utility functional

$$J = \int_0^\infty \mathrm{e}^{-\rho t} \left(\log u - cx^2\right) \,\mathrm{d}t,$$

where $c > 0$ is the relative economic weight of the lake and $\rho > 0$ the discount rate.

We consider the problems $\mathcal{P}(x_0)$ to maximise J given the dynamics (1) and the initial condition $x(0) = x_0$.

3 Optimal vector fields

The solution procedure for problems of the type $\mathcal{P}(x_0)$ is well-known: introducing a costate variable y, we form the Pontryagin[1] function

$$P(x, y, u) = \log u - cx^2 + y\left(u - bx + \frac{x^2}{x^2+1}\right);$$

[1] Also called Hamilton or pre-Hamilton function.

the maximum principle then requires that $\frac{\partial P}{\partial u} = 0$, yielding $u(y) = -1/y$. Defining the Hamilton function $H(x, y) = P(x, y, u(y))$, the state–costate pair of an optimal trajectory has to satisfy the equations

$$\dot{x} = \frac{\partial H}{\partial y} = -\frac{1}{y} - bx + \frac{x^2}{x^2 + 1}, \tag{2}$$

$$\dot{y} = \rho y - \frac{\partial H}{\partial x} = 2cx + y\left(\rho + b - \frac{2x}{(x^2 + 1)^2}\right), \tag{3}$$

together with the initial condition $x(0) = x_0$ and the transversality condition

$$\lim_{t \to \infty} \mathrm{e}^{-\rho t} y = 0,$$

which has to hold for all solutions that are bounded away from 0 (the boundary of our state space).

A solution to $\mathcal{P}(x_0)$ consist of a set of initial costates $Y(x_0) \subset \mathbb{R}$ which is such that if $y_0 \in Y(x_0)$, then $(x(0), y(0)) = (x_0, y_0)$ is initial condition to an optimal trajectory; the costate y_0 is the initial shadow price. The set-valued function Y is called the *optimal costate rule*. For problems with one-dimensional state spaces the points (x_0, y_0) with $y_0 \in Y(x_0)$ are usually situated on the stable manifold of a steady state of the state–costate system. If $Y(x_0)$ contains more than one element, the state x_0 is an indifference state. It follows from the principle of optimality that $Y(x(t))$ is single-valued for all $t > 0$.

The multivalued *optimal vector field* is given as

$$X^o(x) = \frac{\partial H}{\partial y}(x, Y(x)).$$

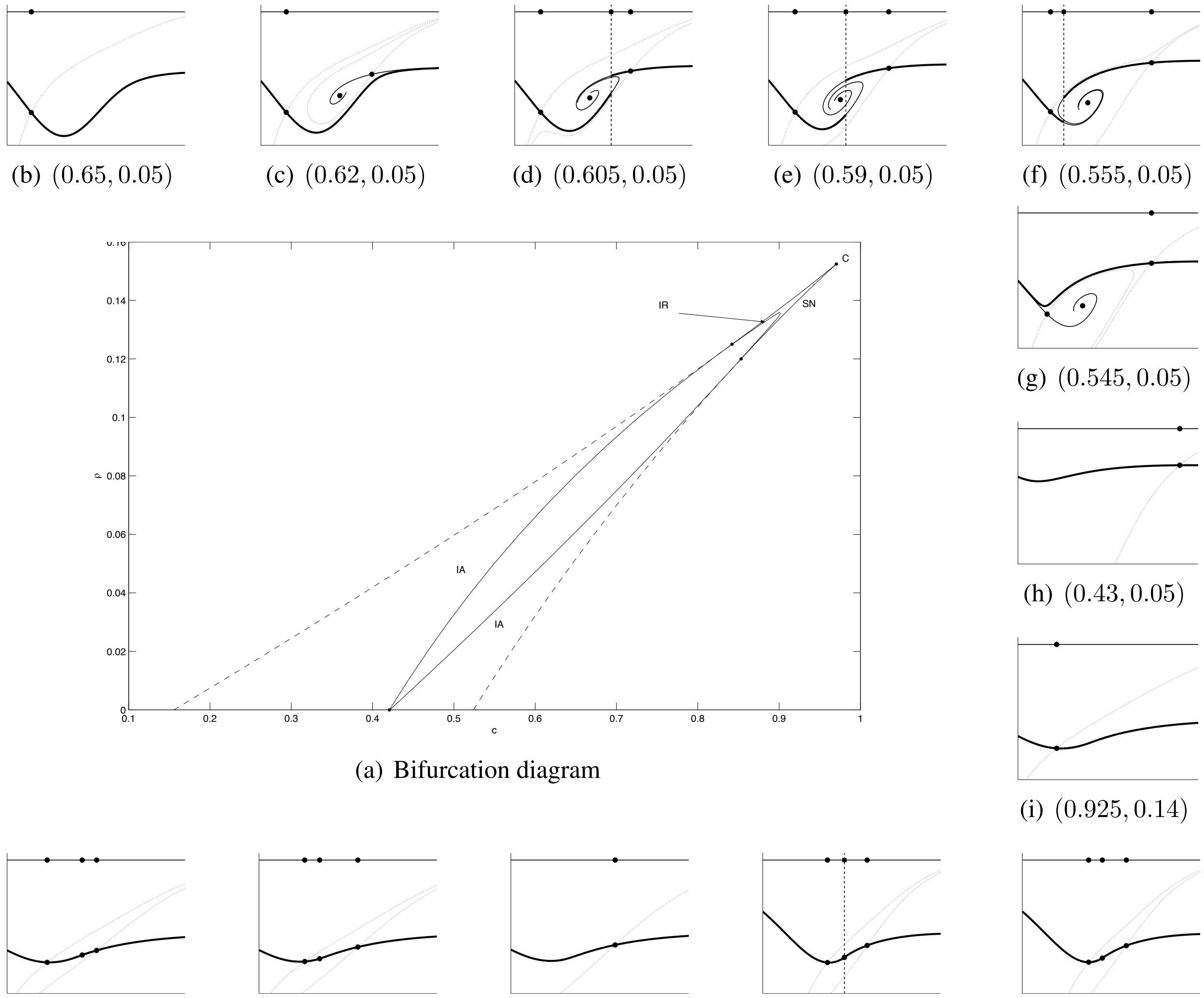
An optimal state trajectory is then a solution of

$$\dot{x}(t) = X^o(x(t)) \quad \text{if } t > 0, \qquad x(0) \in X^o(x_0).$$

4 Bifurcations

The main result of this paper is the bifurcation diagram given in figure 1(a). It is related to the bifurcation diagram given in [4], but in that article only indifference-attractor and saddle-node curves were given. Figure 1(a) additionally shows indifference-repeller curves, of whose existence the second author was not yet aware. The terms “indifference-attractor” and “indifference-repeller” bifurcation are explained in this section. The reader is referred to [2] for proofs.

We have fixed the decay rate to $b = 0.675$ and we have taken the relative economic weight of pollution c and the discount rate ρ as bifurcation parameters.



(b) $(0.65, 0.05)$ (c) $(0.62, 0.05)$ (d) $(0.605, 0.05)$ (e) $(0.59, 0.05)$ (f) $(0.555, 0.05)$

(g) $(0.545, 0.05)$

(h) $(0.43, 0.05)$

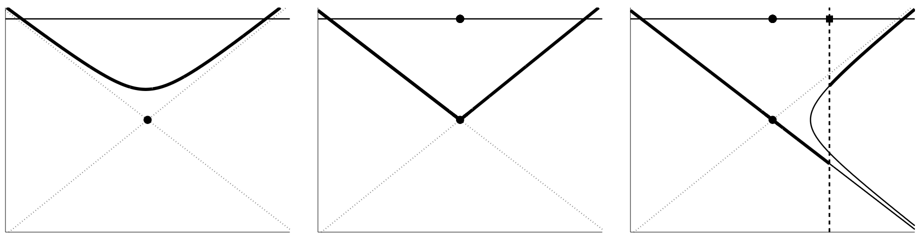
(a) Bifurcation diagram

(i) $(0.925, 0.14)$

(j) $(0.922, 0.14)$ (k) $(0.916, 0.14)$ (l) $(0.91, 0.14)$ (m) $(0.8932, 0.134)$ (n) $(0.90, 0.136)$

Figure 1: *Figure 1(a) shows the bifurcation diagram of the shallow lake system in the (c, ρ)-parameter space. Phase portraits and optimal vector fields are given for selected values of (c, ρ). Solid curves represent stable manifolds, dotted curves unstable manifolds. Thick lines are graphs of optimal costate rules Y; dashed vertical lines indicate indifference points.*

4.1 Indifference-attractor bifurcation. The lines indicated "IA" in figure 1(a) are indifference-attractor bifurcations, occurring for instance between 1(f) and 1(g). When passing through these bifurcation curves, an attractor and an indifference point of the optimal vector field are created. In the state–costate system the unstable manifold of the left saddle and the stable manifold of the right saddle change their relative position, going through a heteroclinic bifurcation in the process [4]. The situation is illustrated



(a) Before bifurcation (b) At bifurcation (c) After bifurcation

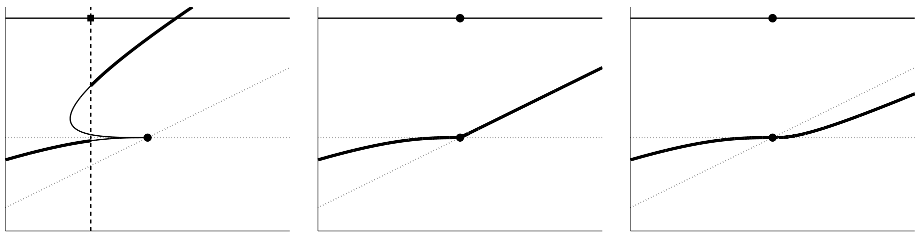
Figure 2: *The indifference-attractor bifurcation.*

in figure 2, showing the behaviour of the state–costate system in the vicinity of the left saddle s_ℓ. Before the bifurcation, the stable manifold $W^s_{s_r}$ of the right saddle s_r "covers" s_ℓ, which is then not optimal. After the bifurcation, the state s_ℓ is optimal, and all optimal trajectories starting in an open interval converge towards it. This interval is bounded on the right by an indifference point, from which the system can optimally go to either s_ℓ or s_r.

4.2 Indifference-repeller bifurcation. At an indifference-repeller bifurcation, denoted "IR", an unstable steady state of the optimal vector field changes into an indifference point, as between 1(m) and 1(n). In figure 3 we depict the scenario in a neighbourhood of the central repelling steady state $r = (x_r, y_r)$ of the state–costate system. This state has two positive eigenvalues $0 < \lambda^u < \lambda^{uu}$. To the largest eigenvalue, a unique one-dimensional manifold is associated, the so-called strong unstable manifold W^{uu}_r: this is the set of all points that approach r at the rate $\mathrm{e}^{\lambda^{uu}t}$ as $t \to -\infty$.

At the bifurcation, the relative positions of the strong unstable manifold W^{uu}_r and the stable manifold $W^s_{s_r}$ change. Again, before the bifurcation the manifold $W^s_{s_r}$ "covers" r, which is then again not optimal. But $W^s_{s_r}$ does not cover all state space and there is an indifference point close to x_r. After the bifurcation the state r, though repelling, is optimal. The corresponding state x_r, while not an indifference point any more, is still a *threshold* point: in every neighbourhood of x_r there are optimal trajectories tending to separate long-term steady states.

4.3 Other bifurcations. Optimal vector fields can also undergo saddle-node "SN" and cusp "C" bifurcations, which are associated to saddle-node and cusp bifurcations



(a) Before bifurcation (b) At bifurcation (c) After bifurcation

Figure 3: *The indifference-repeller bifurcation.*

of the state–costate system (see figures 1(i)–1(l)). However, not all such bifurcations of the canonical system imply bifurcations of the optimal vector field (see figures 1(b)–1(c)); these are inessential to the optimal vector field.

More complicatedly, we can have the situation that at a saddle-node bifurcation of the state–costate system, the strongly unstable manifold of the bifurcating steady state coincides with the stable manifold of one of the saddles. This is called a indifference-saddle-node bifurcation; it is a codimension two bifurcation, typically occurring only if there are two or more system parameters. From such a point indifference–attractor, indifference–repeller, saddle–node and inessential saddle node curves emanate (see figure 4 for the bifurcation diagram). Two such points occur in figure 1(a).

Another codimension two bifurcation is the double indifference-repeller bifurcation; we leave it aside here, as it only produces kinks in the indifference-repeller curve.

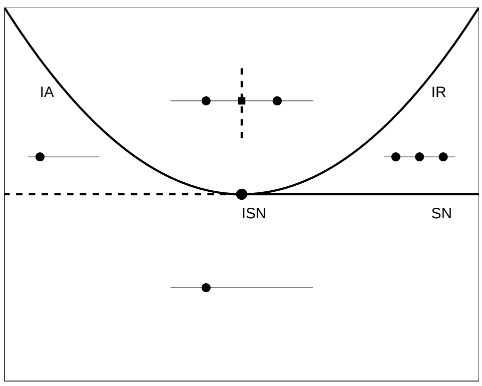


Figure 4: *The indifference-repeller bifurcation.*

5 Effects of varying the parameters

The bifurcation diagram amounts to a "comparative dynamics" analysis of the system, as it indicates how the totality of the solutions change with the parameters.

The bifurcation curves, the solid curves in figure 1(a), divide the parameter space into three separate regions. In the outer region, the optimal vector field X^o has a unique global attractor. For parameters taking values in the lower inner region, X^o has two attractors, separated by an indifference point. In the small upper inner region, there are again two attractors but separated by an repeller. All three steady states are engaged in the cusp bifurcation which marks the point with the largest value $\bar{\rho}$ of ρ in the inner region.

The union of the two inner regions is the region where there are multiple long-term steady states: we might call it also the region of history-dependence. Consider what happens when we fix $\rho = 0.05$ and decrease c from $c = 1$ towards $c = 0$. If c is large, it is always optimal to steer the lake towards a clean ("oligotrophic") long-term steady state. Then at $c \approx 0.61$, we enter the region of history-dependence: if the lake is initially sufficiently clean, it is still optimal to steer it towards a clean state. However, if the lake is initially already too polluted, this is not worthwhile any more. Finally, at $c \approx 0.54$, the basin of attraction of the oligotrophic state collapses, and we enter the region where there is again a single long-term optimal steady state, but now a polluted ("eutrophic") one.

Increasing the discount rate ρ has the same effect as decreasing the economic weight of the lake c. This is according to our intuition, since increasing the discount rate decreases the importance attached to long-term effects.

For parameters in the region $\rho > \bar{\rho}$, there is always only a single, globally attracting steady state, changing continuously with c: since the future is discounted so heavily, the optimal policy does for some values of c reach a steady state that would be highly disadvantageous if ρ were small.

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