A note on the Hartwick rule as a conservation law

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Abstract

Using conservation laws, we provide a new proof of the Hartwick result, i.e. there is intergenerational equity if and only if net investment is constant. Subsequently, the technique is used to show that while the Hartwick result still holds (when shadow prices are used) if consumers value the existence of an essential non-renewable resource, it is no longer a property of a competitive equilibrium.

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1 Introduction

A fundamental question in resource economics is whether, in the presence of exhaustible resources, it is possible to achieve intergenerational equity. Hartwick (1977) shows that in the Dasgupta-Heal-Solow model (Dasgupta & Heal, 1974; Solow, 1974) this is possible along competitive paths provided that net investment in man-made capital and natural resources is zero. When the depletion of natural resources is balanced by investment in man-made capital, this will lead to sustainable levels of consumption. The rule of zero net investment is referred to as the Hartwick rule and the Hartwick result is that following the Hartwick rule leads to intergenerational equity.

Dixit, Hammond, and Hoel (1980) extend the Hartwick result in two directions. First, the Hartwick result holds in a wide variety of economic models. We refer to the general version of the Dasgupta-Heal-Solow model as the Dixit-Hammond-Hoel model (DHH model). Second, they show that constant net investment (i.e. the generalized Hartwick rule) is necessary and sufficient for intergenerational equity. Asheim, Buchholz, and Withagen (2003) give an overview of the literature and discuss the subtleties of the Hartwick rule.

The competitive equilibrium of Hartwick (1977) and Dixit et al. (1980) is closely connected to the optimal allocation by a social planner with a zero discount rate. A distinctive feature of these kind of optimal control problems is that the value of the Hamilton function is a preserved quantity. Sato and Kim (2002) show that in the Dasgupta-Heal-Solow model the generalized Hartwick rule can be derived from a conservation law. In this paper, we show that this can also be done in the more general DHH model.

The paper is organized as follows. Section 2 reviews concepts. In Section 3 the DHH model is introduced and we give a new proof of the Hartwick result. Subsequently, in Section 4, the technique is used to treat an extension of the DHH model. We show that following the Hartwick rule (evaluated at the shadow price of investment) leads to intergenerational equity if consumers value the existence of an essential non-renewable resource, but in the competitive equilibrium welfare is decreasing. Section 5 concludes.

2 Preliminary remarks

Suppose we have the following optimal control problem:

$$\max_{c(\cdot)} \int_0^T u(c,k)\mathrm{d}t \text{ subject to } \dot{k} = f(c,k),$$

where $k(0) = k_0$ is given, $T > 0$, $c(t) \in \mathbb{R}^n$ the control variable and $k(t) \in \mathbb{R}^m$ is the state variable. Let $q \in \mathbb{R}^m$ denote the costate variable. The Pontryagin function is given by[1]

$$\mathcal{P}(c,k,q) = u(c,k) + q' f(c,k)$$

[1] We assume that the optimal control problem is nicely behaved, i.e. the maximum exists and the Hamilton function is differentiable. Moreover all vectors are column vectors and a transpose is indicated by a $'$.

and the Hamilton function by

$$\mathcal{H}(k,q) = \max_c \mathcal{P}(c,k,q).$$

Then the canonical equations, which every optimal path has to satisfy, are:

$$\dot{k}_i = \frac{\partial \mathcal{H}}{\partial q_i} \text{ and } \dot{q}_i = -\frac{\partial \mathcal{H}}{\partial k_i} \text{ for } i = 1, \ldots, m.$$

This is called a Hamiltonian system. A fundamental property of a Hamiltonian system is that the Hamilton function $\mathcal{H}$ is a first integral of this system, i.e. $\mathcal{H}$ is constant along every solution of the system. To show this note that differentiation of $\mathcal{H}$ along trajectories yields:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_{i=1}^{m} \frac{\partial \mathcal{H}}{\partial k_i}\dot{k}_i + \frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i = \sum_{i=1}^{m} \frac{\partial \mathcal{H}}{\partial k_i}\frac{\partial \mathcal{H}}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial \mathcal{H}}{\partial k_i} = 0$$

The system has a conservation law where $\mathcal{H}$ is the preserved quantity. For a textbook treatment of Hamiltonian systems see e.g. Hirsch, Smale, and Devaney (2004, pp.207–210).

Hamiltonian systems are most commonly used in physics. In that context $\mathcal{H}$ is the total energy of a closed system. Conservation laws are a manifestation of the symmetry of nature. Noether's theorem states (roughly) that every symmetry leads to a conservation law (Gowers, Barrow-Green, & Leader, 2008, IV.12 §4.1). For example, the invariance with respect to time reversal gives the law of conservation of energy.

3 The DHH model and the Hartwick result

We mostly follow Asheim et al. (2003) in the presentation of the DHH model.[2] Let $t \geq 0$ denote time. Consumption flows at time t are denoted by a vector $c(t)$. Capital stocks at time t are denoted by $k(t)$ and investment flows by $\dot{k}(t)$. Capital stocks consist of both man-made capital and natural resources. The initial stock of capital $k(0)$ is denoted by k_0. Technology is described by a time-independent set $\mathcal{F}$. The triple $(c(t), k(t), \dot{k}(t))$ is *attainable* if $(c(t), k(t), \dot{k}(t)) \in \mathcal{F}$. A path $\{c(t), k(t), \dot{k}(t)\}_0^T$ is *feasible given* k_0 if $k(0) = k_0$ and for all $t \in [0, T]$ $(c(t), k(t), \dot{k}(t))$ is *attainable*. The time index is mostly suppressed to avoid cluttered notation. We put the following standard assumptions on $\mathcal{F}$:

- $\mathcal{F}$ is smooth, closed and convex,
- Non-negative consumption: $(c, k, \dot{k}) \in \mathcal{F}$ implies $c \geq 0$,
- Non-negative capital stocks: $(c, k, \dot{k}) \in \mathcal{F}$ implies $k \geq 0$,
- Free disposal of investment: $(c, k, \dot{k}) \in \mathcal{F}$ and $\ell \leq \dot{k}$ implies $(c, k, \ell) \in \mathcal{F}$.

[2] We set the discount factor $\mu(t)$ equal to one. Without discounting current prices and present prices are equal. This avoids cumbersome notation without loss of generality. Note that the generalized Hartwick rule only holds in present prices and the value of the present value Hamilton function is preserved.

We assume that population size is constant. Each generation lives for one instance and each generation has the same preferences about the flow of consumption. These preferences are captured by a utility function $u(c)$, which is increasing and strictly concave. Let $p(t) \geq 0$ denote the vector of prices of the consumption flow and let $q(t) \geq 0$ denote the vector of prices of the investment flow. Then profit is given by the sum of revenue, the change in the value of the capital stock and net investment :

$$\pi = p'c + q'\dot{k} + \dot{q}'k.$$

Note that the generalized Hartwick rule is that net investment $q'\dot{k}$ is constant.

Definition (Investment function)**.** *Define* $f(c, k|q) := \arg\max\{q's|(c, k, s) \in \mathcal{F}\}$ *as the optimal amount of investment.*

Remark. Profit maximization and non-negative prices imply that investment choices will be on the boundary of $\mathcal{F}$ and given by $f(c, k|q)$. Given the assumptions on $\mathcal{F}$, this will be a differentiable function.

Definition (Competitive)**.** *Let* $T > 0$*. A path* $\{c(t), k(t), \dot{k}(t)\}_0^T$ *is* competitive during $[0, T]$ *if the path is feasible and both instantaneous utility and profit are maximized along the path given* $p(t)$ *and* $q(t)$*.*

Theorem (The generalized Hartwick result)**.** *Suppose a path* $\{c(t), k(t), \dot{k}(t)\}_0^T$ *is competitive during* $[0, T]$*. Then* $u(c)$ *is constant over time if and only if* $q'\dot{k}$ *is constant.*

Remark. Hartwick (1977) only showed that $q'\dot{k} = 0$ implies that $u(c)$ is constant over time. The generalized Hartwick result is due to Dixit et al. (1980).

The original proof in Dixit et al. (1980) uses the first order conditions that result from the instantaneous utility and profit maximizing. We show that competitive paths are solutions of the canonical equations of an optimal control problem. Moreover, the canonical equations for this optimal control problem form a Hamiltonian system. The result then follows directly from the Hamilton function being a preserved quantity.

Proof. First we have to establish equivalence between the competitive path and solutions of the canonical equations of an optimal control problem. Firms maximize instantaneous profit π given prices $p(t)$ and $q(t)$ and the technological constraint $\dot{k} = f(c, k|q)$. This yields the following two first order conditions:[3]

$$p + (Df_c)q = 0, \tag{1}$$

and

$$\dot{q} = -(Df_k)q. \tag{2}$$

[3]Subscripts attached to Jacobians and gradients denote with respect to which vector we are taking the derivative.

Equation (2) is Hotelling's rule, which guarantees local efficiency. Instantaneous utility maximization means maximization of $u(c) - p'c$ yielding the first-order condition $\nabla_c u = p$. Substituting this into (1) yields:

$$\nabla_c u + (Df_c)q = 0. \tag{3}$$

The evolution of capital stock is given by:

$$\dot{k} = f(c, k|q). \tag{4}$$

We now claim that properties of a competitive path can be found by examining the necessary conditions of the following optimal control problem:

$$\max_{c,\ell} \int_0^T u(c)\mathrm{d}t \text{ such that } \dot{k} = \ell, (c, k, \ell) \in \mathcal{F} \text{ and } k(0) = k_0.$$

The Pontryagin function is

$$\mathcal{P}(c, k, \ell, q) = u(c) + q'\ell,$$

where q the costate. Given c, k and q, optimal investment is given by $\ell^*(c, k|q) = f(c, k|q)$. Substituting this into the Pontryagin function, the first-order condition for optimal consumption is given by (3). The Hamiltonian of this system is

$$\mathcal{H}(k, q) = \max_{(c,\ell)\in\mathcal{F}_k} \mathcal{P}(c, k, \ell, q) = u(c^*) + q'f(c^*, k|q),$$

where $\mathcal{F}_k = \{(c, \ell)|(c, k, \ell) \in \mathcal{F}\}$ and $c^* = \arg\max_c u(c) + q'f(c, k|q)$. The canonical equations are then

$$\dot{q} = -\nabla_k \mathcal{H} \text{ and } \dot{k} = \nabla_q \mathcal{H}.$$

Straightforward calculations show that these are resp. equal to (2) and (4).

Now we can exploit the fact that we have a Hamiltonian system. The value of $\mathcal{H}$ is preserved as shown in Section 2. Hence if $u(c^*)$ is constant over time, then $q'\dot{k} = \mathcal{H} - u(c^*)$ is also constant. And similarly if $q'\dot{k}$ is constant over time, then $u(c^*) = \mathcal{H} - q'\dot{k}$ is also constant. □

This approach is useful for a number of reasons. Foremost, it is a natural approach that allows for properties of the equilibrium to be derived directly from the Hamilton function. Moreover the Hamilton function is easy to interpret: the approach reveals clearly the direct tradeoff between consumption now and additional investment in capital. In fact, the Hamilton function is the Hicksian measure of welfare. Hicksian welfare is an explicitly dynamic measure and is loosely defined as the maximal amount of consumption that will not harm future consumption. Its measurable equivalent is net national product (Weitzman, 1976; Asheim & Weitzman, 2001). Additionally with our approach extending the Hartwick result to other models becomes a straightforward exercise.

4 Application to an extension of the DHH model

In this section, an extension of the DHH model will be discussed. In doing so, we will see that extending the Hartwick result to other models is straightforward using the method proposed in this paper.

For instance, Heal (1998, pp.7–10) criticizes the DHH model — and implicitly the Hartwick rule as a criterion for sustainability — for only valuing natural resources as inputs to production and not as assets in their own right. If we include capital in the utility function, thereby explicitly valuing the existence of certain natural resources, then the Hamilton function becomes $\max_{(c,\ell)\in\mathcal{F}_k} u(c,k) + q'\ell$ and we still have that $u(c,k)$ is constant over time if and only if $q'\dot{k}$ is constant.[4] As long as we use shadow prices instead of market prices, the Hartwick rule will indicate intergenerational equity. However, the Hartwick rule is no longer a property of competitive equilibria: welfare is decreasing over time.

To see this, suppose consumers value the existence of certain essential non-renewable resources: $\nabla_k u \geq 0$. The inequality is strict only for the resources whose existence is valued. If the capital stock with index i is essential and non-renewable, then $\dot{k}_i < 0$. At least one of the non-renewable resources is both essential for production and valued by consumers. Hence $(\nabla_k u)'\dot{k} < 0$. We use the Hamilton function as the measure of welfare.

Theorem (No intergenerational equity with existence value)**.** *If consumers value the existence of an essential non-renewable resource, then in the competitive equilibrium welfare is strictly decreasing over time.*

Proof. Taking the time derivative of the Hamilton function yields:

$$\begin{aligned}\dot{\mathcal{H}} &= (\nabla_k\mathcal{H})'\dot{k} + (\nabla_q\mathcal{H})'\dot{q},\\ &= (\nabla_k u)'\dot{k} + q'(Df_k)'\dot{k} + \dot{q}'\dot{k},\\ &= (\nabla_k u + (Df_k)q + \dot{q})'\dot{k}.\end{aligned}$$

As a consequence of profit maximization, we have $(Df_k)q + \dot{q} = 0$ (cf. eq. 2):

$$\dot{\mathcal{H}} = (\nabla_k u)'\dot{k} < 0,$$

which shows that welfare is decreasing. □

Contrary to Heal's views, it is not that the Hartwick rule fails if resources have existence value — it is just a matter of using the right prices — but there is no competitive equilibrium with the property of intergenerational equity. Then the naive use of the Hartwick rule, where market prices instead of shadow prices are used, could lead one to believe that the economy is on a sustainable path, whereas welfare is declining over time.

[4]This result is also obtained by d'Autume and Schubert (2008) in the context of the Dasgupta-Heal-Solow model. We look at the more general DHH model. Moreover d'Autume and Schubert do not compare the optimal solution to a competitive equilibrium.

5 Concluding remarks

In this paper we have shown that properties of competitive paths can be derived directly from the Hamilton function. This leads to the (generalized) Hartwick rule, where constant net investment is necessary and sufficient for intergenerational equity. As an application of this method, we showed that the Hartwick rule is still a measure of sustainability if natural resources have existence value provided shadow prices are used.

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