

CONTRACTS, COST SHARING AND CONSISTENCY

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ABSTRACT. Under a *contract*, agents are not only held to honor the allocation as prescribed by a cost sharing mechanism but also a full description of allocated units and costs once production falls short. For agents leaving the cost sharing problem by taking their demanded units and prepaying the corresponding bill, a contract allows for a reformulation of the cost sharing problem to serve the remaining agents. *Consistency* requires invariance of cost shares relative to any such *reduced cost sharing problem*. Under consistency, the proportional mechanisms uniquely satisfy *additivity* and *positivity of cost shares*. Exchanging positivity by equal treatment characterizes the set of mechanisms which propose proportional shares for only those agents in the maximal indifference set for some preordering on the rest of non-negative numbers. This includes egalitarian and average cost sharing. The latter is further characterized by the properties *linearity*. Under *R-consistency*, a mechanism is supported by at least one *reasonable contract* which meets *upperbounds*. The class of *additive* and *R-consistent* mechanisms is isomorphic to the class of consistent and monotonic rationing methods. Consequently serial cost sharing is R-consistent, whereas Shapley-Shubik is not. Examples are given how the extensive literature on consistent monotonic rationing can be inferred to study and characterize cost sharing mechanisms.

Keywords: Cost sharing; Consistency; Additivity; Constant returns; Rationing; (Quasi-) Proportional mechanism; Egalitarian mechanism; Serial mechanism; Shapley-Shubik mechanism

JEL-Classification: C70, D63, D70

1. INTRODUCTION

This paper fits in the stream of axiomatic cost sharing literature (see Thomson (2001), Moulin (2002)) which discusses structural and characterizing properties of solutions. I will consider homogeneous cost sharing problems; a group of agents N have demands $q = (q_i)_{i \in N}$ for a single perfectly divisible output and corresponding production costs $C(\sum_{i \in N} q_i)$ have to be shared. Central property is that of *consistency*, which is studied in various fields in economics and social choice theory as a fundamental principle to decision problems pertaining to varying

Acknowledgements: I thank Hervé Moulin, Justin Leroux, Juan Moreno-Ternero and two anonymous referees for many useful comments. The main ideas for this paper were developed when I was at NPS, Monterey, and Rice University. Financial support by these institutes is gratefully acknowledged.

sets of agents. In the cost sharing context, the basic idea is the following: Suppose that agent i leaves the cost sharing problem, by taking away the demanded units q_i and prepaying his cost share $\mu_i(q, C)$. In line with much of the literature then a solution μ is called *consistent* if the calculated cost shares for the remaining agents in $N \setminus i$ in the *reduced cost sharing problem* $(q_{N \setminus i}, C^*)$ are the same, or $\mu_{N \setminus i}(q, C) = \mu(q_{N \setminus i}, C^*)$; the reduced cost function C^* ideally presents the cost structure after i has gone, thereby using the information contained in the model given by (q, C) and μ . Following Young (1985, p19), this self-similarity property envisions an idea of *fairness* of solutions at all levels of cooperation, for any subgroup of agents, according to which 'no subgroup should want to "re-contract"'.

Where for some models in the distributive justice literature, like the rationing model, the concept of consistency is transparent and unambiguous, for the (homogeneous) cost sharing problem it is not. The problems do not arise in choosing the reduced demand profile, which is obviously $q_{N \setminus i}$, but in choosing an appropriate *reduced cost function* C^* . In the literature several proposals have been made, each serving its own purpose. Below I will argue that there is still a clear issue with defining consistency in homogeneous cost sharing problems, and contribute to the literature with a new proposal.

First I will extend the notion of a cost sharing solution by that of a *contract*, which also specifies the way produced units and corresponding costs are shared in case demand is not fully met. Each contract is used to define a reduced cost function, and a reduced cost sharing problem in particular. I will call a solution consistent if for all possible contracts the agents in the reduced cost sharing problem need not renegotiate since the solution calculates the same cost shares as before. A solution is *weakly consistent* if there exists at least one contract supporting it accordingly. So the basic feature of a consistent solution – under the weak or strong notion – is that it offers the agents an opportunity to agree to a contract which rationalizes the cost shares at hand, by implementing the same solution for all contingent reductions based on this contract. If no such contract exists, there is the intrinsic problem of agents holding different opinions to which reduced problem the solution must be applied.

1.1. Relation to the literature. Basically the efforts to properly define consistency in cost sharing problems can be divided in two branches, each of which originated from notions of consistency for cooperative games due to Davis and Maschler (1965), Sobolev (1973), and Hart and Mas-Colell (1989). The fundamental difference is the way that reduced costs are modeled. First, the residual cost put forward by the literature à la Hart and Mas-Colell is the total cost minus the total amount the complement subgroup of remaining agents would have to pay under the cost sharing solution, where potential *revision* of the demands of the remaining agents is incorporated. Leroux (2007) suggests a definition in this fashion for the model discussed here, but needs solutions fit for heterogeneous cost functions. Sudhölter (1998)

applies the original ideas to cooperative games associated with cost sharing problems. However, in light of contingent externalities it is not clear which transformation of a cost sharing problem into a cost game fits best.

The second branch of literature, based on ideas of Davis and Maschler (1965), treats the residual costs as the total cost minus what the complement coalition has already paid for. Examples are Moulin and Shenker (1994), Kolpin (1994), Tijs and Koster (1998), and Albizuri and Zarzuelo (2007). Again, these papers show the richness of the model allows for many types of reduction, each expressing a different notion of consistency.

This paper contributes to the literature in the following ways. First, the arbitrariness in choosing the appropriate reduction is to some degree tackled by requiring invariance of cost shares to sets of reductions. Second, although one could argue the paper belongs to the Hart and Mas-Collel type of literature, the notion of a contract takes into account a continuum of hypothetical cost sharing problems.

1.2. Overview of the paper and results. Section 2 provides the basic model of homogeneous cost sharing, as well as the notion of solution in these contexts. Section 3 introduces the basic notions of contract, reduced cost sharing problem and consistency. In Section 4 we derive an integral representation of additive cost sharing mechanisms which extends the isomorphism of Moulin and Shenker (1994) concerning the additive mechanisms with the constant returns property. Within the class of additive mechanisms the *proportional* mechanisms are the only consistent ones, assuming that positive demands always reward a positive share of total costs. Exchanging positivity by the weak fairness property *equal treatment*, we also end up with mechanisms billing in a proportional fashion. But now a preordering is used to preselect the real contributors as those with demands that are in its maximal indifference set. Egalitarian and average cost sharing belong to the mentioned class; the latter is further characterized by the property *linearity*, which weakens the standard *constant returns*. Adding positivity again, we obtain the class of *quasi-proportional* mechanisms.

In Section 5 is devoted to relaxations of the consistency property, *weak consistency* and *regular consistency*. The additive cost sharing mechanisms characterized by a consistent rationing method are *regularly consistent*, and, vice versa, each such cost sharing mechanism defines a consistent rationing method. In particular, the existence of these perfectly linked up notions of consistency provide a positive answer to the question raised in Moulin (2002, p. 328). We now may conclude that the serial mechanism is *R*-consistent, whereas the Shapley-Shubik mechanism is not. The characterization of parametric rationing methods of Young (1987) is extended to the context of cost sharing using this isomorphism. Finally we discuss *converse consistency*, the requirement that a solution for 2 agent problems be consistently extendible to problems with variable number of agents.

2. THE MODEL AND PRELIMINARIES

Consider a production facility for some perfectly divisible good Y , of which the technology is summarized by a *cost* function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$; $C(y)$ denotes the minimal (monetary) input to generate y units of Y . It will be assumed that C is nondecreasing and there are no fixed costs, so that $C(0) = 0$. In addition we shall assume *absolutely continuous* cost functions¹. This technical condition implies that a cost function is differentiable almost everywhere with respect to the Lebesgue measure λ . With slight abuse of notation C' is the *marginal cost function*, i.e., it coincides with the derivative of C whenever the latter exists, and assumes the value 0 otherwise. In particular, C' is λ -integrable and costs for output level y may be expressed as $C(y) = \int_0^y C'(t) dt$.² The set of all cost functions is denoted by \mathcal{C} . Special cost functions are the *base functions* B_t defined for $t \geq 0$ by $B_t(y) = \min\{t, y\}$, *slant functions* $\Lambda_t(y) = \max\{y - t, 0\}$ and the identity $C_{\text{id}}(y) = y$.

Let \mathcal{N} be the set of nonempty and finite subsets of the natural numbers; \mathcal{N} is interpreted as the set of potential agents in the cost sharing problem. A *cost sharing problem* for the set of agents $N \in \mathcal{N}$ is an ordered pair $P = (q, C) \in \mathbb{R}_+^N \times \mathcal{C}$. The interpretation of P is that the agents in N jointly own the production facility, and $q = (q_i)_{i \in N}$ summarizes the individual demands of the agents for good Y ; then $q(N) = \sum_{i \in N} q_i$ is produced and cost $C(q(N))$ has to be shared. The set of all cost sharing problems for N is denoted \mathcal{P}^N . The demand profile for N containing only zeroes is denoted $\mathbf{0}_N$.

For $(q, c) \in \mathcal{P}^N$, $y \in \mathbb{R}_+^N$ is called *vector of cost shares* if $y(N) = C(q(N))$. A *cost sharing mechanism* assigns to each cost sharing problem a vector of cost shares. Reknown examples of mechanisms include average cost sharing, serial cost sharing (Moulin and Shenker (1992)), and the Shapley-Shubik cost sharing mechanism (Shubik (1962)). Concise definitions of these mechanisms are in the text below. The class of mechanisms with properties P_1, \dots, P_m will subsequently be denoted $M(P_1, \dots, P_m)$.

3. CONTRACTS AND CONSISTENCY

Assume that for each cost sharing problem $(q, C) \in \mathcal{P}^N$ not only it is specified how much an agent is billed but also how much he would consume and pay in case the actual production level were $y \leq q(N)$. A complete description for all contingent cost sharing problems will be called a *contract*. Formally, a *contract* for mechanism μ is an ordered pair (ξ, γ) where for all cost

¹For such functions it holds that for all intervals $[a, b] \subset \mathbb{R}_+$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for every finite collection of pairwise disjoint intervals $(a_k, b_k) \subset [a, b]$, $k = 1, 2, \dots, n$ with $\sum_{k=1}^n (b_k - a_k) < \delta$, we have $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$.

²This follows by the Fundamental Theorem in Lebesgue (1904).

sharing problems $(q, C) \in \mathcal{P}^N$ the mapping $\xi(\cdot, q, C)$ describes the allocation of units available for consumption such that for all $y \in [0, q(N)]$

- $\sum_{i \in N} \xi_i(y, q, C) = y$, and
- $\xi(q(N), q, C) = q$.

The mapping $\gamma(\cdot, q, C)$ assigns to each production level y vectors of cost shares such that

- $\sum_{i \in N} \gamma_i(y, q, C) = C(y)$, and
- $\gamma(q(N), q, C) = \mu(q, C)$.

I will make further assumptions. First, the mappings $y \mapsto \xi(y, q, C)$ and $y \mapsto \gamma(y, q, C)$ are non-decreasing. Second, for all $(q, C) \in \mathcal{P}^N, i \in N, k^* - \xi_i(k^*, q, C) = k' - \xi_i(k', q, C)$ implies $C(k^*) - \gamma_i(k^*, q, C) = C(k') - \gamma_i(k', q, C)$. This condition expresses that if i is the sole consumer of the last $k^* - k'$ units then he pays for the corresponding incremental cost $C(k^*) - C(k')$.

Suppose that agent i wants to leave group N . Then, as long as his demand is not met and his cost share not paid, agent i leaves with a *contract* which stipulates that if a quantity $k \leq q(N)$ of the good is produced, he will receive a quantity $\xi_i(k, q, C)$ and pay a bill $\gamma_i(k, q, C)$. The agents in $N \setminus i$ are obliged to honor the very same contract and may use it to calculate the production cost of y units for their own consumption as follows: first they find the level k such that $y = k - \xi_i(k, q, C)$, then they produce k , give the quantity $\xi_i(k, q, C)$ to i , collect $\gamma_i(k, q, C)$ from him and end up paying the remainder of the cost, $C(k) - \gamma_i(k, q, C)$. This is taken as definition of the *reduced cost* $C_{(\xi, \gamma), -i}(y)$ for production level $y \leq q(N)$.

Definition 3.1 Let $\varphi = (\xi, \gamma)$ be a contract for mechanism μ . For $(q, C) \in \mathcal{P}^N, i \in N$ the *reduced cost function* $C_{\varphi, -i}$ is defined by

$$\begin{aligned} y < q(N) : \quad & C_{\varphi, -i}(y) = C(k) - \gamma_i(k, q, C) \\ & \text{where } k \leq q(N) \text{ solves } \sum_{j \in N \setminus i} \xi_j(k, q, C) = y, \\ y \geq q(N) : \quad & C_{\varphi, -i}(y) = C(y) - \mu_i(q, C). \end{aligned}$$

Note that $C_{\varphi, -i}$ indeed specifies an element in \mathcal{C} : a solution to $\sum_{j \in N \setminus i} \xi_j(k, q, C) = y$ exists by continuity of $y \mapsto \xi(y, q, C)$. Moreover, $C_{\varphi, -i}$ is non-decreasing: $\sum_{i \in N} \gamma_i(y, q, C) = C(y)$ so for almost all y , $\sum_{i \in N} \frac{d}{dy} \gamma_i(y, q, C) = C'(y)$, hence $\frac{d}{dy} \gamma_i(y, q, C) \leq C'(y)$ by monotonicity.

Definition 3.2 Mechanism μ is *consistent* if for all contracts φ for μ , all $i \in N$, and all $(q, C) \in \mathcal{P}^N$

$$(1) \quad \mu_{N \setminus i}(q, C) = \mu(q_{N \setminus i}, C_{\varphi, -i}).$$

Example 3.3 The average cost sharing mechanism μ^{AV} defined for $q \neq \mathbf{0}_N$ by

$$\mu^{\text{AV}}(q, C) = q \cdot \frac{C(q(N))}{q(N)}$$

is consistent (see Theorem 4.8) below. Now consider the *serial mechanism* μ^{SR} introduced by Moulin and Shenker (1992). Take $(q, C) \in \mathcal{P}^{\{1,2,\dots,n\}}$ and assume the demands are ordered such that $q_i \leq q_j \Leftrightarrow i \leq j$. Then with $x^0 = 0, x^i := q_1 + q_2 + \dots + q_{i-1} + (n+1-i)q_i$ for all $i = 1, 2, \dots, n$

$$(2) \quad \mu_i^{\text{SR}}(q, C) = \sum_{j=1}^i \frac{C(x^j) - C(x^{j-1})}{n+1-j}.$$

Consider the contract $\varphi = (\xi, \gamma)$ defined by $\xi(y, q, C) = \frac{q}{q(N)} \cdot y, \gamma(y, q, C) = \frac{\mu^{\text{SR}}(q, C)}{C(q(N))} C(y)$. Then for $(q, C) \in \mathcal{P}^N$ with $q = (1, 2, 3), C(y) = y^2, \mu^{\text{SR}}(q, C) = (3, 11, 22)$ and $C_{\varphi, -1}(y) = \frac{33}{25}y^2$ so that $\mu_{\{2,3\}}(q_{\{2,3\}}, C_{\varphi, -1}) = (\frac{264}{25}, \frac{561}{25}) \neq \mu_{\{2,3\}}(q, C)$. So μ^{SR} is not consistent. \triangleleft

4. CONSISTENT AND ADDITIVE COST SHARING

As a first exercise I will study for the consistent members among the additive mechanisms, the class of mechanisms which accomodates most of the popular cost sharing mechanisms such as $\mu^{\text{AV}}, \mu^{\text{SR}}, \mu^{\text{SS}}$.

Additivity

For all $C_1, C_2 \in \mathcal{C}$, all $q \in \mathbb{R}_+^N, \mu(q, C_1 + C_2) = \mu(q, C_1) + \mu(q, C_2)$.

Additivity is a standard property in the cost sharing literature which is usually motivated as an accounting convention. I will exploit the fact that the class of additive mechanisms allows for a transparent integral expression as in Moulin and Shenker (1994).³

³A precise statement in Moulin and Shenker (1994) is missing, so below I will refer to Moulin (2002), Th. 2.2.

4.1. Production assignment and rationing. A *production sharing problem* for agents in $N \in \mathcal{N}$ consists of a pair $(q, t) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $t \in [0, q(N)]$. Here q is usually interpreted as a vector of claims, and t the amount to be divided. A *production assignment* ρ associates to each production sharing problem a vector $\rho(q, t) \in \mathbb{R}_+^N$ such that $\sum_{j \in N} \rho_j(q, t) = t$. Such ρ is called *rationing method* if $\rho(q, t) \leq q$. In that case nobody is granted more than her claim while the available amount is fully distributed. The prevalent symmetric rationing methods are the *proportional* rationing method r^P and *uniform gains* method r^{UG} . These are defined by $r^P(q, t) = q/q(N)t$, and $r_i^{UG}(q, t) = \min\{q_i, \omega\}$, where ω solves $\sum_{j \in N} \min\{q_j, \omega\} = t$. Consider a permutation σ of N . Priority rationing relative to σ is defined as $r_i^\sigma(q, t) = \max\{\min\{t - \sum_{j: \sigma(j) < \sigma(i)} q_j, q_i\}, 0\}$. Random priority is defined by averaging the priority methods, $r^{RP}(q, t) = \frac{1}{n!} \sum_\sigma r^\sigma(q, t)$. See, e.g., Moulin (2002) and Thomson (2003) for overviews.⁴

Production assignment ρ is *monotonic* whenever $t \leq t'$ implies $\rho(q, t) \leq \rho(q, t')$ for all $t, t', q \in \mathbb{R}_+^N$; then ρ defines for all $q \in \mathbb{R}_+^N$ a monotonic (and continuous) path $t \mapsto \rho(q, t)$ from 0 to q .

Any monotonic production assignment ρ induces an additive cost sharing mechanism μ^ρ by

$$\mu^\rho(q, C) = \int_0^{q(N)} C'(t) d\rho(q, t).$$

We may also reverse the statement, since each additive mechanism may be expressed in this way by choosing a suitable production assignment :

Theorem 4.1 $\mu \in M(ADD)$ if and only if $\mu = \mu^\rho$ for some monotonic production assignment ρ .

Proof. The ‘if’ part is clear, which leaves us to show the ‘only if’ part. Take $\mu \in M(ADD)$. Define a production assignment ρ by $\rho(q, t) := \mu(q, B_t)$. Then it is easy to see that ρ is monotonic. Take $q \in \mathbb{R}_+^N$ and $x, y \in [0, q(N)]$, $x \leq y$. Then $B_y - B_x \in \mathcal{C}$ and

$$\rho(q, y) - \rho(q, x) = \mu(q, B_y) - \mu(q, B_x) = \mu(q, B_y - B_x) \geq 0.$$

Now consider the function $\Lambda_x(y) := B_{q(N)}(y) - B_x(y) = \max\{y - x, 0\}$. Then

$$\begin{aligned} \mu^\rho(q, \Lambda_x) &= \int_0^{q(N)} \Lambda'_x(t) d\rho(q, t) = \int_x^{q(N)} 1 d\rho(q, t) = \rho(q, q(N)) - \rho(q, x) \\ &= \mu(q, B_{q(N)}) - \mu(q, B_x) = \mu(q, B_{q(N)} - B_x) = \mu(q, \Lambda_x). \end{aligned}$$

Then we may conclude that $\mu = \mu^\rho$ by Lemma 1 in Moulin and Shenker (1994). ■

Constant returns

For any $\alpha \geq 0$, $\mu(q, \alpha C_{id}) = \alpha q$.

⁴Moulin (2000) focuses on discrete formulation of the problem and asymmetric priority rules.

Corollary 4.2 (Moulin and Shenker (1994)) $\mu \in M(ADD, CR) \Leftrightarrow \mu = \mu^r$ for some monotonic rationing method r .

Equal Treatment $q_i = q_j$ implies $\mu_i(q, C) = \mu_j(q, C)$.

Similarly, any production assignment ρ satisfies equal treatment if for all production sharing problems (q, t) , $q_i = q_j$ implies $\rho_i(q, t) = \rho_j(q, t)$.

Theorem 4.3 $\mu \in M(ADD, ET)$ if and only if $\mu = \mu^\rho$ and ρ satisfies ET.

Proof. We only show the 'only if' part. ADD implies the functional representation as in Theorem 4.1. Now suppose there is $q \in \mathbb{R}_+^N$ such that $q_i = q_j$ but not $\rho_i(q, t) = \rho_j(q, t)$ for all $t \in [0, q(N)]$. Assume that not $\rho_i(q, t) \geq \rho_j(q, t)$ for all $t \in [0, q(N)]$. Then there must be an interval U such that $\frac{d}{dt}\rho_i(q, t) < \frac{d}{dt}\rho_j(q, t)$ for all $t \in U$. Consider the indicator function \mathbb{I}_U with respect to U and define $C \in \mathcal{C}$ by $C(y) = \int_0^y \mathbb{I}_U(t) dt$ for all $y \in \mathbb{R}_+$. Then $\mu_i(q, C) = \int_0^{q(N)} \mathbb{I}_U(t) d\rho_i(q, t) < \int_0^{q(N)} \mathbb{I}_U(t) d\rho_j(q, t) = \mu_j(q, C)$, contradicting ET. ■

Within the demand sharing context the idea of consistency is transparent and intuitive: a production assignment ρ is called *consistent* if for all demand sharing problems (q, x) among agents in N , all $j \in N$,

$$(3) \quad \rho_{N \setminus j}(q, x) = \rho(q_{N \setminus j}, x - \rho_j(q, x)).^5$$

Hence, consistency states that with removing an agent from the cooperative N , and taking all the resources that are allocated to this agent, renewed allocation of the remaining pieces within the reduced society does not make a difference as long as ρ is used. As Moulin (2000) puts it within the context of rationing, 'changing the status of an agent from *active participant* to *passive expense of resources* does not alter the overall distribution'.

Proposition 4.4 If μ is additive and consistent then $\mu = \mu^\rho$ with consistent and monotonic ρ .

Proof. If μ is additive then take ρ as is guaranteed by Theorem 4.1. Define the contract $\varphi = (\xi, \gamma)$ by $\xi(y, q, C) = \rho(q, y)$ and $\gamma(y, q, C) = \int_0^y C'(t) d\rho(q, t)$. Then for $x \in [0, q(N)]$, $y \in [0, x]$ we have $\xi(y, q, B_x) = \rho(q, y) = \gamma(y, q, B_x)$. Hence, for $i \in N$, $(B_x)_{\varphi, -i} = B_{x - \rho_i(q, x)}$, and thus, by

⁵In fact the notion is usually defined in terms of general sets of agents leaving, but is derived from repeated application of this statement.

consistency

$$\begin{aligned}\rho_{N \setminus i}(q, x) &= \mu_{N \setminus i}(q, B_x) = \mu(q_{N \setminus i}, (B_x)_{\varphi, -i}) = \int_0^{q(N \setminus i)} (B_{x - \rho_i(q, x)})'(t) \, d\rho(q_{N \setminus i}, t) \\ &= \int_0^{x - \rho_i(q, x)} d\rho(q_{N \setminus i}, t) = \rho(q_{N \setminus i}, x - \rho_i(q, x)).\end{aligned}$$

■

Example 4.5 The converse does not hold: r^{UG} defines a consistent production assignment but $\mu^{\text{SR}} = \mu^{r^{\text{UG}}}$ is not consistent. ◁

Example 4.6 μ^{SS} is not consistent since $\mu^{\text{SS}} = \mu^{r^{\text{RP}}}$ and r^{RP} is not a consistent rationing method. ◁

Null Agent If $q_i = 0$ then $\mu_i(q, C) = 0$.

Lemma 4.7 If $\mu \in M(\text{CONS})$ then $\mu \in M(\text{NULL})$.

Proof. Let φ be a contract for $\mu \in M(\text{CONS})$ and take a cost sharing problem $P = (q, C) \in \mathcal{P}^N$ with $q_i = 0$ for some agent $i \in N$. Denote $N \setminus i = \{i_1, i_2, i_3, \dots, i_{n-1}\}$. Define iteratively cost sharing problems $P^k = (q^k, C^k)$ for the agents $N^k = \{i, i_k, i_{k+1}, \dots, i_{n-1}\}$ for $k = 1, 2, \dots, n-1$ so that $P^1 = P$ and $q^k := q_{i, i_k, i_{k+1}, \dots, i_{n-1}}$ and $C^k = (C^{k-1})_{\varphi, -i_{k-1}}$ for $k \geq 2$. By consistency we have

$$\mu_i(P) = \mu_i(P^1) = \dots = \mu_i(P^{n-1}) = \mu_i(q_i, C^{n-1}).$$

Now, since $C^{n-1} \in \mathcal{C}$, we obtain $C^{n-1}(0) = 0$ which in turn implies $\mu_i(q_i, C^{n-1}) = 0$. ■

Positive cost for positive demand

For any $C \in \mathcal{C}$, any $i \in N$, and any $q \in \mathbb{R}_+^N$, if $q_i > 0$ while $C(q(N)) > 0$ then $\mu_i(q, C) > 0$.

Theorem 4.8 Assume $n \geq 3$. Mechanism μ satisfies additivity, consistency, and positive cost for positive demand if and only if for each $i \in N$ there is a function $f_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f_i(y) = 0 \Leftrightarrow y = 0$ and for all nontrivial $(q, C) \in \mathcal{P}^N$

$$(4) \quad \mu_i(q, C) = \frac{f_i(q_i)}{\sum_{j \in N} f_j(q_j)} C(q(N)).$$

Proof. First we show the implication \Leftarrow . Suppose μ takes the form as in (4). Consider the contract (ξ, γ) for μ defined for $(q, C) \in \mathcal{P}^N$ by

$$\xi_i(y, q, C) = \frac{q_i}{q(N)}y, \gamma_i(y, q, C) = \frac{f_i(q_i)}{\sum_{k \in N} f_j(q_j)}C(y).$$

Take $i \in N$, then for each $y \geq 0$ we obtain

$$k - \xi_i(k, q, C) = y \implies k = \frac{q(N)}{q(N \setminus i)}y.$$

Then

$$\begin{aligned} C_{(\xi, \gamma), -i}(y) &= C\left(\frac{q(N)}{q(N \setminus i)}y\right) - \gamma_i\left(\frac{q(N)}{q(N \setminus i)}y, q, C\right) \\ &= C\left(\frac{q(N)}{q(N \setminus i)}y\right) - \frac{f_j(q_j)}{\sum_{k \in N} f_k(q_k)}C\left(\frac{q(N)}{q(N \setminus i)}y\right) \\ &= \frac{\sum_{j \in N \setminus i} f_j(q_j)}{\sum_{j \in N} f_j(q_j)}C\left(\frac{q(N)}{q(N \setminus i)}y\right). \end{aligned}$$

Hence for $j \in N \setminus i$:

$$\mu_j(q_{N \setminus i}, C_{(\xi, \gamma), -i}) = \frac{f_j(q_j)}{\sum_{k \in N \setminus i} f_i(q_k)}C_{(\xi, \gamma), -i}(q(N \setminus i)) = \frac{f_j(q_j)}{\sum_{k \in N} f_k(q_k)}C(q(N)) = \mu_j(q, C).$$

‘ \Rightarrow ’ Assume $\mu \in M(\text{ADD}, \text{CONS})$. By Theorem 4.1 there is ρ such that $\mu = \mu^\rho$. Define the following contract for μ , $\xi(y, q, C) = \frac{q}{q(N)}y, \gamma(y, q, C) = \frac{\rho(q, q(N))}{c(q(N))}C(y)$ for $(q, C) \in \mathcal{P}^N$ such that $q(N) > 0$. Now take $(q, C) \in \mathcal{P}^N$, and assume without loss of generality that $q_i > 0, q(N \setminus i) > 0$. Consistency implies

$$C_{\varphi, -i}(y) = \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N)}C\left(\frac{q(N)}{q(N \setminus i)}y\right).$$

In particular,

$$\begin{aligned} (B_t)_{\varphi, -i}(y) &= \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N)}B_t\left(\frac{q(N)}{q(N \setminus i)}y\right) = \\ &= \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N)} \cdot \frac{q(N)}{q(N \setminus i)}B_{\frac{q(N \setminus i)}{q(N)}t}(y) = \\ &= \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N \setminus i)} \cdot B_{\frac{q(N \setminus i)}{q(N)}t}(y). \end{aligned}$$

Again by consistency and additivity

$$\begin{aligned}
 (5) \quad \rho_{N \setminus i}(q, t) &= \mu_{N \setminus i}^\rho(q, B_t) = \mu^\rho\left(q_{N \setminus i}, (B_t)_{\varphi, -i}\right) = \\
 (6) \quad &= \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N \setminus i)} \cdot \mu^\rho\left(q_{N \setminus i}, B_{\frac{q(N \setminus i)}{q(N)}t}(y)\right) = \\
 (7) \quad &= \frac{\sum_{j \in N \setminus i} \rho_j(q, q(N))}{q(N \setminus i)} \cdot \rho\left(q_{N \setminus i}, \frac{q(N \setminus i)}{q(N)}t\right).
 \end{aligned}$$

This equality has two consequences:

(A) By taking sums on the left and right we get, for $t \in [0, q(N)]$

$$\begin{aligned}
 t - \rho_i(q, t) &= \sum_{k \in N \setminus i} \rho_k(q, t) = \frac{q(N) - \rho_i(q, q(N))}{q(N \setminus i)} \cdot \sum_{k \in N \setminus i} \rho_k\left(q_{N \setminus i}, \frac{q(N \setminus i)}{q(N)}t\right) = \\
 &= \frac{q(N) - \rho_i(q, q(N))}{q(N \setminus i)} \cdot \frac{q(N \setminus i)}{q(N)}t = t - \frac{\rho_i(q, q(N))}{q(N)}t.
 \end{aligned}$$

So for $t \in [0, q(N)]$,

$$\rho_i(q, t) = \frac{\rho_i(q, q(N))}{q(N)}t.$$

Hence

$$(8) \quad \mu^\rho(q, C) = \frac{\rho(q, q(N))}{q(N)}C(q(N)).$$

(B) For $q \gg 0$, we get by substituting $t = q(N)$ for all $j, k \neq i$ and positive cost shares for positive demands that

$$(9) \quad \frac{\rho_k(q, q(N))}{\rho_j(q, q(N))} = \frac{\rho_k(q_{N \setminus i}, q(N \setminus i))}{\rho_j(q_{N \setminus i}, q(N \setminus i))}.$$

So the left-hand fraction is actually independent from the value q_i . By varying over all triples i, j, k , the conclusion is that the following expression is well-defined

$$h_{ij}(q_i, q_j) := \frac{\rho_i(q, q(N))}{\rho_j(q, q(N))} \text{ for all } i, j \in N.$$

Then

$$(10) \quad h_{ij}(q_i, q_j) h_{jk}(q_j, q_k) h_{ki}(q_k, q_i) = 1 \text{ for different } i, j, k \in N,$$

and, in particular, we find that any expression of type $h_{jk}(q_j, q_k) h_{ki}(q_k, q_i)$ is independent from q_k . I claim that there are functions $f_j : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}_+$ such that

$$h_{ij}(q_i, q_j) = \frac{f_i(q_i)}{f_j(q_j)}.$$

To see this, fix $k \in N$ and define, invoking the independence resulting from (10),

$$(11) \quad \begin{aligned} f_i(q_i) & : = h_{ki}(1, q_i) \text{ for all } i \neq k, \\ f_k(q_k) & : = h_{k\ell}(1, y) h_{k\ell}(q_k, y) \text{ for some } \ell \neq k, y \neq 0. \end{aligned}$$

Then, finally, we are able to assert our claim since for $i, j \neq k$:

$$\frac{f_i(q_i)}{f_j(q_j)} = \frac{h_{ki}(1, q_i)}{h_{kj}(1, q_j)} = \frac{1}{h_{ik}(q_i, 1) h_{kj}(1, q_j)} = h_{ij}(q_i, q_j),$$

and by choosing $\ell = i, y = q_i$ in (11):

$$\frac{f_i(q_i)}{f_k(q_k)} = \frac{h_{ki}(1, q_i)}{h_{ki}(1, q_i) h_{ki}(q_k, q_i)} = h_{ki}(q_k, q_i).$$

So I established the claim, as for all $i, j \in N$

$$(12) \quad \frac{\rho_j(q, q(N))}{\rho_i(q, q(N))} = \frac{f_j(q_j)}{f_i(q_i)}.$$

It is not hard to see that (12) is true for any $q \in \mathbb{R}_+^N$ such that $q_i > 0$ and all $j \in N$. Proof by induction on the number of zero demanders in $q, Z(q)$. Suppose the statement is correct for all q with $q_i > 0$ and $Z(q) \leq k - 1$ for $1 \leq k \leq n - 1$. Now take q with $q_i > 0, Z(q) = k$. For h with $q_h = 0$ and $j \neq h, \epsilon > 0$

$$\frac{\rho_j(q, q(N))}{\rho_i(q, q(N))} = \frac{\rho_j(q_{N \setminus h}, q(N \setminus h))}{\rho_i(q_{N \setminus h}, q(N \setminus h))} = \frac{\rho_k((q_{N \setminus h}, \epsilon), q(N \setminus h) + \epsilon)}{\rho_j((q_{N \setminus h}, \epsilon), q(N \setminus h) + \epsilon)} = \frac{f_j(q_j)}{f_i(q_i)}.$$

Here, the last equality is due to the induction hypothesis.

Now, by combining (A) and (B) we are finally done since for all $q_i > 0$

$$\begin{aligned} \mu_i^\rho(q, C) & = \frac{\rho_i(q, q(N))}{q(N)} C(q(N)) = \frac{\rho_i(q, q(N))}{\sum_{j \in N} \rho_j(q, q(N))} C(q(N)) \\ & = \frac{\rho_i(q, q(N))}{\frac{\rho_i(q, q(N))}{f_i(q_i)} \sum_{j \in N} f_j(q_j)} C(q(N)) = \frac{f_i(q_i)}{\sum_{j \in N} f_j(q_j)} C(q(N)). \end{aligned}$$

■

Well-known examples in the class of all such mechanisms are the proportional rule with $f(y) = y$, the egalitarian rule with $f(y) = 1$ whenever $y > 0$. A few examples that have become classic in queuing theory and network performance evaluation are *limited (discrimanatory) processor sharing* $f_i(y) = w_i \min\{y, \ell_i\}$ and *coupled processor allocation* $f_i(y) = \ell_i \mathbb{I}_{\mathbb{R}_{++}}(y)$.

Remark At first glance consistency may seem a very strong notion, as a solution should offer stability with respect to varying sets of agents relative to the unrestricted domain of contracts. However, note that in Theorem 4.8 only very special contracts are used to arrive at the formula (8).

The following examples point out that the three axioms additivity, consistency and positive cost for positive demand are logical independent.

Example 4.9 Define the following mechanism μ . Select two agents $i, j \in N$. Define, for any $C \in \mathcal{C}$, any $q \in \mathbb{R}_+^N$, $\mu(q, C) = \mu^{\text{AV}}(q, C)$ if $\{k \in N \mid q_k = 0\} = \{i, j\}$, and else $\mu_{\{i, j\}}(q, C) := (0, 0)$, $\mu_{N \setminus \{i, j\}}(q, C) = \frac{q_{N \setminus \{i, j\}}}{q(N \setminus \{i, j\})} C(q(N))$. Then μ is additive and consistent, but fails to charge the agents i, j a positive share of positive costs. \triangleleft

Example 4.10 For any $(q, C) \in \mathcal{P}^N$, $j \in N$ define $f_j(q, C) = C(q(N) + q_j) - C(q(N)) + q_i$. Define a cost sharing mechanism μ by

$$\mu_i(q, C) = \frac{f_i(q, C)}{\sum_{j \in N} f_j(q, C)} C(q(N)).$$

Then μ is consistent, since any reduction leaves the values of f_j unchanged. Moreover, since $f_i(q, C) = 0$ if and only if $q_i = 0$, μ satisfies positive cost for positive demand. But clearly μ is not additive. \triangleleft

Example 4.11 Below we will see that the Shapley-Shubik mechanism fails to be weakly consistent, and therefore it does not constitute a consistent solution. But the mechanism is additive and does satisfy positive cost for positive demand. \triangleleft

Adding ET to the characterization in Theorem 4.8 leads to a characterization of the family of *quasi-proportional* mechanisms:⁶

Corollary 4.12 *Assume $n \geq 3$. Then $\mu \in M(\text{ADD}, \text{CONS}, \text{POS}, \text{ET})$ if and only if there is a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(x) = 0 \Leftrightarrow x = 0$ such that*

$$(13) \quad \mu_i(q, C) = \frac{f(q_i)}{\sum_{j \in N} f(q_j)} C(q(N)) \text{ for all } i \in N, (q, C) \in \mathcal{P}^N, q \neq \mathbf{0}.$$

Proof. Take $\mu \in M(\text{ADD}, \text{CONS}, \text{POS}, \text{ET})$, $(q, C) \in \mathcal{P}^N$ and functions $\{f_j\}_{j \in N}$ as in Theorem 4.8. Then by ET, for arbitrary $i, j \in N$, $q_i = q_j \implies f_i(q_i) = f_j(q_j)$ and so we obtain $f_i = f_j$ for all i, j . \blacksquare

⁶The name is borrowed from Moulin and Sprumont (2005) who discuss the same family, only for a discrete framework.

Example 4.13 Positivity can not be left out. Define $\mu_i(q, C)$ for $(q, C) \in \mathcal{P}^N$ as follows. If $q_i \leq 1$ for all $i \in N$, then define $\mu(q, C) = \mu^{\text{AV}}(q, C)$, else let $S = \{i \in N \mid q_i > 1\}$ and put $\mu_{N \setminus S}(q, C) = 0$ and $\mu_S(q, C) = \frac{q_S}{q(S)}C(q(N))$. Then μ satisfies all but the property POS. \triangleleft

For a given preordering \preceq of \mathbb{R}_+ , any $q \in \mathbb{R}_+^N$, denote by $A(q, \preceq) \subseteq N$ the set of agents for which the demands in q are in the maximal indifference set of \preceq , i.e., $i \in A(q, \preceq)$ if there is no $j \in N$ such that $q_i \prec q_j$.

Theorem 4.14 *Assume $n \geq 3$. Then $\mu \in M(\text{ADD}, \text{CONS}, \text{ET})$ if and only if there is a preordering \preceq of \mathbb{R}_+ and a function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for each $q \in \mathbb{R}_+^N, C \in \mathcal{C}$*

$$(14) \quad \mu_i(q, C) = \begin{cases} \frac{f(q_i)}{\sum_{j \in A(q, \preceq)} f(q_j)} C(q(N)) & \text{if } i \in A(q, \preceq), \\ 0 & \text{if } i \notin A(q, \preceq). \end{cases}$$

Proof. We will only show the ‘only if’ part. Let $\mu \in M(\text{ADD}, \text{CONS}, \text{ET})$ and take a production assignment ρ such that $\mu = \mu^\rho$ as is guaranteed by Theorem 4.1. Below I will use (5) for 3-agent sets $U \in \mathcal{N}$: for any $q \in \mathbb{R}_+^U$, and all pairs $u, v \in U$ we have $\rho_{\{u, v\}}(q, q(U)) = 0_{\{u, v\}}$ if $q_{\{u, v\}} = 0_{\{u, v\}}$, and

$$(15) \quad \rho_{\{u, v\}}(q, q(U)) = \frac{\rho_u(q, q(U)) + \rho_v(q, q(U))}{q_u + q_v} \rho(q_{\{u, v\}}, q_u + q_v).$$

otherwise.

Define a complete ordering \preceq on \mathbb{R}_+ by

$$q_i \preceq q_j \text{ if } \rho((q_i, q_j), q_i + q_j) \gg 0 \text{ or } \rho_i((q_i, q_j), q_i + q_j) = 0.$$

Claim: \preceq defines a preordering.

1) \preceq is reflexive, i.e., $q_i \preceq q_i$ for all $q_i \in \mathbb{R}_+$. By ET it holds $\rho_i((q_i, q_i), 2q_i) = q_i$, hence if $q_i = 0$ then $\rho_i((q_i, q_i), 2q_i) = 0$ and $\rho_i((q_i, q_i), 2q_i) = (q_i, q_i) \gg 0$ otherwise.

2) \preceq is transitive, i.e., if $q_i \preceq q_j$ and $q_j \preceq q_k$, then $q_i \preceq q_k$.

Take $q_i, q_j, q_k \in \mathbb{R}_+$ and assume $q_i \preceq q_j$ and $q_j \preceq q_k$, $q_i \succ q_k$. Then $q_i > 0$ by NULL and $\rho_k(q_{\{i, k\}}, q_i + q_k) = 0$. Now apply (15) to $\{u, v\} = \{i, k\}$ and conclude that $\rho_k(q, q(U)) = 0$.

Distinguish between the two cases:

(A) $q_j \prec q_k$. Then $\rho_j(q_{\{j, k\}}, q_j + q_k) = 0$, hence by (15) with $\{u, v\} = \{j, k\}$, $\rho_j(q, q(U)) = 0$. Then $\rho_i(q, q(U)) = q(U) > 0$. Now by (15) for $\{u, v\} = \{i, j\}$, $\rho_j(q_{\{i, j\}}, q_i + q_j) = 0$ and $\rho_i(q_{\{i, j\}}, q_i + q_j) > 0$, contradicting $q_i \preceq q_j$.

(B) $q_j \sim q_k$. Without loss of generality assume $q_j \neq q_k$. Then $\rho(q_{\{j, k\}}, q_j + q_k) \gg 0$ and by application of (15) to $\{u, v\} = \{j, k\}$ we obtain $\rho_j(q, q(N)) + \rho_k(q, q(U)) = 0$, so $\rho_j(q, q(N)) = 0$. Then, like in case (A), we obtain $\rho_i(q, q(U)) = q(U) > 0$, which leads to contradiction with

$q_i \preceq q_j$. In both the cases (A) and (B) the assumption $q_k \prec q_i$ leads to a contradiction, so $q_i \preceq q_k$. This proves transitivity.

Take $N \in \mathcal{N}, q \in \mathbb{R}_+^N$ and suppose for $i, j \in N$ that $q_i \prec q_j$. Let $N \setminus \{i, j\} = \{k_1, k_2, \dots, k_{n-2}\}$ and define $T_0 = \emptyset, T_\ell = \{k_1, k_2, \dots, k_\ell\}$. Then by successive removal of the agents k_1, k_2, \dots, k_{n-2} by CONS and (5),

$$\rho_{\{i,j\}}(q, q(N)) = \prod_{\ell=0}^{n-2} \sum_{t \in N \setminus T_\ell} \frac{\rho_t(q_{N \setminus T_\ell}, q(N \setminus T_\ell))}{q(N \setminus T_\ell)} \cdot \rho(q_{\{i,j\}}, q_i + q_j),$$

which shows that $\rho_i(q, q(N)) = 0$. In particular, the only agents making positive contributions are those with demands in the maximal indifference class of \preceq . If $q \neq \mathbf{0}_N$, the only interesting case, then $A(q, \preceq) \cap A(\mathbf{0}_N, \preceq) = \emptyset$ which means that for all $i \in A(q, \preceq), \rho_i(q, q(N)) > 0$. Then this suffices to be able to repeat the reasoning as in the second part of the proof of Theorem 4.8. The same logic applies to derive values $f_i(q_i)$ for agent $i \in A(q, \preceq)$, and by ET the functions f_i are all the same so that we may put $f(q_i) = f_i(q_i)$ for all $i \in A(q, \preceq)$. Then basically this shows how to derive the values of the desired function f on the maximal indifference class corresponding to q . We only need to be cautious where the induction hypothesis is applied: ϵ can not be any positive value, but needs to be in the maximal indifference class determined by q . Then finally, complete the definition of f by varying q so that f gets values on all the indifference classes. ■

Linearity

For all $\alpha, \beta \in \mathbb{R}_+$, it holds $\mu_i((q_{-i}, \beta q_i), \alpha C_{\text{id}}) = \beta \mu_i(q, \alpha C_{\text{id}})$.

This property is similar to that discussed in Trudeau (2009). In absence of externalities it imposes that a raise or decrease of ones individual demand with a factor β will magnify the bill in the cost sharing problem accordingly. Notice that it is weaker than the constant returns property.

Theorem 4.15 *Assume $n \geq 3$. Then $\mu \in M(\text{ADD}, \text{CONS}, \text{LIN}, \text{ET})$ if and only if $\mu = \mu^{\text{AV}}$.*

Proof. We only show the 'only if' part. Suppose that $\mu \in M(\text{ADD}, \text{CONS}, \text{LIN}, \text{ET})$. We have $\mu = \mu^\rho$ and by (8):

$$\mu^\rho(q, C) = \frac{\rho(q, q(N))}{q(N)} C(q(N)) \text{ for all } (q, C) \in \mathcal{P}^N \text{ with } q(N) > 0.$$

Now fix $N = \{1, 2, \dots, n\}, q \in \mathbb{R}_+^N$. If $q(N) = 0$ then by nonnegativity of cost shares $\mu(q, C) = \mathbf{0}_N = \mu^{\text{AV}}(q, C)$. So we may assume $q \neq \mathbf{0}_N$ and, without loss of generality, that $q_1 > 0$. We

will show that $\rho(q, q(N)) = q$ for all $q_{N \setminus 1} \in \mathbb{R}_+^{N \setminus 1}$ by (increasing) induction on the number of agents with demands unequal to q_1 , i.e. the size of $S(q) = \{j \in N \setminus 1 \mid q_j \neq q_1\}$. First of all, if $|S(q)| = 0$, then ET implies $\rho(q, q(N)) = q$. Now suppose $\rho(q, q(N)) = q$ if $|S(q)| \leq k$ for some $k \leq n - 1$. Take a demand profile q with $|S(q)| = k + 1$. For $i \in S(q)$, by application of linearity and the induction hypothesis

$$\rho_i(q, q(N)) = \rho_i\left(\left(q_{-i}, \frac{q_i}{q_1} \cdot q_1\right), q(N)\right) = \frac{q_i}{q_1} \cdot \rho_i((q_{-i}, q_1), \sum_{j \neq i} q_j + q_1) = \frac{q_i}{q_1} \cdot q_1 = q_i.$$

In addition, by ET it holds for all $i \notin S(q)$

$$\rho_i(q, q(N)) = \frac{q(N) - \sum_{j \in S(q)} \rho_j(q, q(N))}{|N \setminus S(q)|} = \frac{|N \setminus S(q)| q_1}{|N \setminus S(q)|} = q_1.$$

Then we have shown that $\rho(q, q(N)) = q$. ■

Example 4.16 μ^E satisfies all of the above properties except LIN. ◁

5. RELAXING CONSISTENCY, CONSISTENT RATIONING

Consistency stipulates that a mechanism should be able to take care of all kinds of contracts – even the less sensible ones. Then this property seems overly strong when the agents are willing to accept any contract in agreement with the idea of preserving the cost shares in the induced reduced cost sharing problems.

Definition 5.1 Mechanism μ is *weakly consistent* if a contract φ for μ exists, such that for all $i \in N$, and all $(q, C) \in \mathcal{P}^N$, $\mu_{N \setminus i}(q, C) = \mu(q_{N \setminus i}, C_{\varphi, -i})$.

Theorem 5.2 *If $\mu = \mu^\rho$ and ρ is a consistent production assignment, then μ is weakly consistent.*

Proof. Assume ρ is a consistent and monotonic production assignment and $\mu = \mu^\rho$. Define the contract $\varphi = (\xi, \gamma)$ for μ by $\xi(y, q, C) = \rho(q, y)$ and $\gamma(y, q, C) = \int_0^y C'(t) d\rho(q, t)$. For $i \in N, y \in [0, q(N)]$

$$\begin{aligned} C_{\varphi, -i}(y - \rho_i(q, y)) &= C_{\varphi, -i}(y - \xi_i(y, q, C)) \\ &= C(k) - \gamma_i(k, q, C) = \int_0^k \left(1 - \frac{d}{dt} \rho_i(q, t)\right) C'(t) dt. \end{aligned}$$

where k is such that $y = k - \xi_i(k, q, C) = k - \rho_i(q, k)$. Substituting the latter expression for y and using the chain rule we obtain for almost all $y \in [0, q(N)]$ ⁷

$$\left(1 - \frac{d}{dy} \rho_i(q, y)\right) (C_{(\xi, \gamma), -i})'(y - \rho_i(q, y)) = \left(1 - \frac{d}{dy} \rho_i(q, y)\right) C'(y).$$

So whenever $\frac{d}{dy} \rho_i(q, y) \neq 1$ it holds $(C_{(\xi, \gamma), -i})'(y - \rho_i(q, y)) = C'(y)$. This is exactly what we need since now

$$\begin{aligned} \mu_{N \setminus i}^\rho(q, C) &= \int_0^{q(N)} C'(t) \, d\rho_{N \setminus i}(q, t) = \\ &= \int_0^{q(N)} C'(t) \, d\rho(q_{N \setminus i}, t - \rho_i(q, t)) = \\ &= \int_0^{q(N)} (C_{(\xi, \gamma), -i})'(t - \rho_i(q, t)) \, d\rho(q_{N \setminus i}, t - \rho_i(q, t)) = \\ &= \int_0^{q(N \setminus i)} (C_{(\xi, \gamma), -i})'(t) \, d\rho(q_{N \setminus i}, t) = \mu^\rho(q_{N \setminus i}, C_{(\xi, \gamma), -i}). \end{aligned}$$

■

Weak consistency as a property for mechanism μ is as appealing as the contract supporting it. Recall that there are no severe built-in requirements on contracts, so that we may find mechanisms weakly consistent although there is in practice no good reason to assume that agents will agree to the underlying contract. So weak consistency as a relaxation of consistency takes too extreme a position. Now we will look for a more restrictive notion than weak consistency by demanding the existence of special types of contracts under which solutions to the induced reductions are stable.

Denote for $C \in \mathcal{C}$ the right derivative at $t \in \mathbb{R}_+$ by $D^+C(t)$.

Upperbound

Contract (ξ, γ) for μ satisfies *upperbounds* if for all $(q, C) \in \mathcal{P}^N, y \in [0, q(N)], i \in N$

$$(16) \quad \gamma_i(y, q, C) \leq \xi_i(y, q, C) \cdot \sup_{t \in [0, q(N)]} D^+C(t).^8$$

The idea is that a contract conflicting with upperbounds will make individual agents want to revoke it. Call contract (ξ, γ) for μ *regular* if it satisfies upperbounds. Note that μ satisfies

⁷Here I use the fact that a monotonic rationing method ρ is Lipschitz-continuous and – in particular – absolutely continuous in the resource component t .

⁸Here other ideas come to mind. Also that $\gamma(y, q, C)$ be contained in the pessimistic imputation set $I^*(\xi(y, q, C))$ for all $(q, C) \in \mathcal{P}^N, y \in [0, q(N)]$. I will use the property only for the case $C = B_x$, and here both types of upperbound lead to the same conclusions.

upperbounds as well, by substitution of $y = q(N)$ in (16). Clearly any such μ satisfies constant returns.

Regular consistency

There is at least one regular contract φ for μ such that $\mu_{N \setminus i}(q, C) = \mu(q_{N \setminus i}, C_{\varphi, -i})$ for all $(q, C) \in \mathcal{P}^N, i \in N$.

Lemma 5.3 *Consider a regular contract (ξ, γ) for mechanism μ . Then for all $x \in \mathbb{R}_+, i \in N$*

$$(B_x)_{(\xi, \gamma), -i} = B_{x - \gamma_i(q(N), q, B_x)}.$$

Proof. Take $x \in [0, q(N)]$. We have by definition of the reduced cost function

$$(17) \quad (B_x)_{(\xi, \gamma), -i}(k - \xi(k, q, B_x)) = \begin{cases} k - \gamma_i(k, q, B_x) & \text{if } k \leq x, \\ x - \gamma_i(k, q, B_x) & \text{if } k \in (x, q(N)]. \end{cases}$$

Firstly, in order to qualify as a cost function we must have that on $(x, q(N)]$ the mapping $k \mapsto x - \gamma_i(k, q, B_x)$ is non-decreasing. But at the same time $k \mapsto \gamma_i(k, q, B_x)$ is non-decreasing. Then $k \mapsto x - \gamma_i(k, q, B_x)$ must be constant on $(x, q(N)]$, and in particular $x - \gamma_i(k, q, B_x) = x - \gamma_i(q(N), q, B_x)$ for all $k \in (x, q(N)]$.

Secondly, regularity implies for $y \leq x, \gamma(y, q, B_x) \leq \xi(y, q, B_x)$. Also by definition of γ and ξ we have $\sum_{j \in N} \gamma_j(y, q, B_x) = B_x(y) = y = \sum_{j \in N} \xi_j(y, q, B_x)$, hence $\gamma(y, q, B_x) = \xi(y, q, B_x)$.

Then (17) turns into

$$(18) \quad (B_x)_{(\xi, \gamma), -i}(y) = \begin{cases} y & \text{for } y \in [0, x - \gamma_i(q(N), q, B_x)], \\ x - \gamma_i(q(N), q, B_x) & \text{if } y > x - \gamma_i(q(N), q, B_x). \end{cases}$$

■

Within the context of Theorem 4.1, regular consistency smoothly transfers from the cost sharing context to the rationing model and back. The mappings $r \mapsto \mu^r$ and $\mu \mapsto r^\mu$ with $r^\mu(q, t) = \mu(q, B_t)$ define an isomorphism between the set of consistent rationing methods and $M(\text{ADD}, \text{RCONS})$.

Theorem 5.4 *$\mu \in M(\text{ADD}, \text{RCONS})$ if and only if $\mu = \mu^r$ and r is a consistent monotonic rationing method.*

Proof. Take $(q, B_x) \in \mathcal{P}^N, i \in N$. Since $\mu = \mu^r$ is consistent there is a regular contract (ξ, γ) such that $\mu_{N \setminus i}(q, B_x) = \mu(q_{N \setminus i}, (B_x)_{(\xi, \gamma), -i})$. In particular $\gamma_i(q(N), q, B_x) = \mu_i(q, B_x) = r_i(q, x)$. By application of Lemma 5.3 $(B_x)_{(\xi, \gamma), -i} = B_{x - \gamma_i(q(N), q, B_x)} = B_{x - r_i(q, x)}$. So

$$\mu(q_{N \setminus i}, (B_x)_{(\xi, \gamma), -i}) = \mu(q_{N \setminus i}, B_{x - r_i(q, x)}) = r(q_{N \setminus i}, x - r_i(q, x)).$$

Then it holds $r_{N \setminus i}(q, x) = \mu_{N \setminus i}(q, B_x) = \mu(q_{N \setminus i}, (B_x)_{(\xi, \gamma), -i}) = r(q_{N \setminus i}, x - r_i(q, x))$. By variation over all rationing problems (q, x) we conclude that r is consistent.

Now the other way, assume r is a consistent and monotonic rationing method and let $\mu = \mu^r$. Define a regular contract (ξ, γ) for μ by $\xi(y, q, C) = r(q, y)$ and $\gamma(y, q, C) = \int_0^y C'(t) dr(q, t)$ and continue the proof as in Theorem 5.2. \blacksquare

The result shows in particular that under additivity, RCONS is the same as the consistency notion discussed in Koster (2007).

Interestingly, μ^{SR} may be not consistent, but it is regularly consistent as it relates to a consistent rationing method r^{UG} . Also the priority rationing methods r^σ are consistent and so μ^σ is regularly consistent. Compare this with μ^{SS} , which is not regularly consistent, since its natural counterpart in rationing, r^{RP} , is not.

Conjecture 5.5 $M(\text{ADD}, \text{RCONS}) = M(\text{ADD}, \text{CR}, \text{WCONS})$

Example 5.6 The properties RCONS and ADD are logically independent. Tijs and Koster (1998) relate to each $P = (q, C) \in \mathcal{P}^N$ a *pessimistic cost sharing problem* (q, C_P^*) , where the *pessimistic cost function* C_P^* is defined as the concave rearrangement of C on $[0, q(N)]$,

$$(19) \quad C_P^*(y) = \begin{cases} \sup \left\{ \int_T C'(t) dt \mid T \subseteq [0, q(N)], \lambda(T) = y \right\} & \text{if } y \in [0, q(N)], \\ C(y) & \text{if } y > q(N), \end{cases}$$

Tijs and Koster (1998) shows that for any $P = (q, C) \in \mathcal{P}^N$ there is a family $\{T_P(x)\}_{x \in [0, q(N)]}$ of increasing and λ -measurable subsets such that $\lambda(T(x)) = x$ and $\int_{T(x)} C'(t) dt = C_P^*(x)$ for all $x \in [0, q(N)]$. Now consider the pessimistic marginal mechanism $\mu^{\sigma, *}$, defined for $P = (q, C) \in \mathcal{P}^N$ by $\mu^{\sigma, *}(q, C) := \mu^\sigma(q, C_P^*)$. Define $U_i = T(q_i + \sum_{j: \sigma(j) < \sigma(i)} q_j) \setminus T(\sum_{j: \sigma(j) < \sigma(i)} q_j)$ and a contract (ξ, γ) for $\mu^{\sigma, *}$ by $\xi_i(y, q, C) = \int_0^y \mathbb{1}_{U_i}(t) dt$, $\gamma_i(y, q, C) = \int_0^y C'(t) \mathbb{1}_{U_i}(t) dt$. Then this contract may be used to show that $\mu^{\sigma, *}$ is regularly consistent. Clearly $\mu^{\sigma, *}$ is not additive. \triangleleft

Having established the tight relationship between consistent rationing and regularly consistent cost sharing, we may find other characterizations of cost sharing mechanisms by known characterizations of corresponding rationing methods. I will give one such an example, based on the characterization of parametric rationing by Young (1987).

A rationing method r is called *continuous* if it is jointly continuous in both arguments, i.e., $(q, t) \mapsto r(q, t)$ is continuous for all rationing problems (q, t) . Such r is then robust against small changes in the parameters defining the rationing problem. For cost sharing mechanisms

the approach is similar; a mechanism will be called *continuous* if small changes in demands and B_t cause only small changes in cost shares. More specifically,

Continuity (CONT) The mapping $(q, t) \mapsto \mu(q, B_t)$ is continuous on $\mathbb{R}_+^S \times \mathbb{R}_+$, for all $S \subseteq N$.⁹

Theorem 5.7 $\mu \in M(\text{ADD}, \text{RCONS}, \text{ET}, \text{CONT})$ if and only if $\mu = \mu^r$ for some continuous parametric r .

Proof. Take μ with the listed properties. Then the rationing method r defined by $r(q, t) := \mu(q, B_t)$ satisfies continuity (by definition), consistency, and equal treatment. Since we have an infinite pool \mathcal{N} of agents we can apply Theorem 1 in Young (1987), which implies that r is continuous and parametric. Moulin and Shenker (1994) show that $\mu = \mu^r$. Now take a continuous parametric r . By definition μ^r is continuous. Moreover, since r satisfies equal treatment, so does μ^r by Theorem 4.3. Finally, r is consistent by Theorem 1 in Young (1987), so we are done by referring to Theorem 5.4. ■

5.1. Consistent extensions. Given a cost sharing mechanism defined for only 2-agent problems, it is natural to ask whether it is possible to extend it in a consistent way to a mechanism for arbitrarily finite sets of agents. The related problem for rationing methods has been partially solved. Dagan and Volij (1997) show that any rationing method for 2-agent problems allows for a unique *average consistent* method. And, moreover, whenever a consistent extension exists, it must coincide with the average consistent method. In particular this shows that there is at most one possibility to extend a rationing method for 2-agent problems in a consistent way. An example of a rationing method for 2-agent problems which cannot be extended consistently is the claims-truncated proportional rule (see Thomson (2008)). Then Theorem 5.4 implies:

Theorem 5.8 Let $\bar{\mu}$ be an additive mechanism with the property CR defined for 2-agent cost sharing problems only. Then there is at most one way to extend $\bar{\mu}$ to $\mu \in M(\text{ADD}, \text{RCONS})$.

For instance, this shows that there is at most one mechanism $\mu \in M(\text{ADD}, \text{RCONS})$ extending serial- or average cost sharing for 2-agent problems, whereas there is no consistent extension of the mechanism defined through the truncated claims proportional method.

In principle, consistent cost sharing mechanisms can be constructed using the techniques of Dagan and Volij (1997). Once the rationing method underlying the mechanism for 2-agent problems is known, calculate the corresponding average consistent extension and – if at all – the consistent mechanism results from Theorem 5.4.

⁹In order to avoid the hybrid character of CONT one may consider the replacement by two requirements, continuity of the mappings $t \mapsto \mu(q, B_t)$ and $q \mapsto \mu(q, B_t)$.

6. DISCUSSION AND OPEN PROBLEMS

Instead of taking cost sharing mechanisms as the basic tool to share costs, I could have started with contracts as the primitive notion of solution; all could have been written in terms of contracts, since any contract defines a cost sharing mechanism in an unambiguous way. I have chosen not to do so in order to stick to study the notion of consistency in the standard framework.

I reflected on the fact that consistency may be valued contingent on the quality of the underlying contracts. I started off with the two diametrically opposed notions of consistency, and discussed regular consistency as a reasonable intervening notion. The idea is that mechanisms with many regular contracts are desirable, and that for the same reason it may be easier for these to comply with regular consistency. However, another way to proceed is to demand for consistency with respect to *all* regular contracts. Notice the tension between the pluriformity of regular contracts and the corresponding notion of consistency: on the one hand it will be easier to meet existence of regular contracts, but on the other hand, the more regular contracts exist, the more demanding consistency gets. The situation becomes even more complex if revision of the idea of a regular contract is considered; the definition is rather ad-hoc now. Further study on these refinements is needed to shed more light on the structure of the space of solutions meeting these new standards.

Also this may lead to a better understanding so that a couple of open problems arising in this paper can be answered. First there is Conjecture 5.5, which states that under additive cost sharing and constant returns weak consistency is equivalent to regular consistency. Another open problem is whether a nonadditive and consistent mechanism μ can be found which meets *independence from irrelevant cost*, i.e., $\mu(q, C_1) = \mu(q, C_2)$ if only $C_1(y) = C_2(y)$ for all $y \geq q(N)$. If not, then this would lead to the conclusion that consistency and independence form irrelevant cost together imply additivity, see, e.g., example 4.10.

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