

# Does eductive stability imply evolutionary stability?

*Cars Hommes and Florian Wagener*

September 8, 2009

## Abstract

This note presents a simple example of a model in which the unique rational expectations (RE) steady state equilibrium is eductively stable in the sense of Guesnerie (2002), but where evolutionary learning, as introduced in Brock and Hommes (1997), does not necessarily converge to the RE steady state price. The example is a Muthian cobweb model where producers have heterogeneous expectations and select forecasting strategies based upon recent realized profits. By means of a simple three types example we show that a locally stable RE fundamental steady state may co-exists with a locally stable two-cycle. We also study the Muthian model with a large number of different producer types, and investigate conditions under which an evolutionary adaptive learning process based upon recent realized profits enforces global convergence to the stable RE steady state and when persistent periodic price fluctuations can arise.

**Acknowledgements:** We would like to thank Buz Brock for stimulating discussions, and Roger Guesnerie and an anonymous referee for helpful comments and suggestions on an earlier draft. Remaining imperfections are of our own making. Financial support from the Netherlands Organization for Scientific Research (NWO) by a NWO-MaGW Pionier grant, a NWO-MaGW VIDI grant and by the EU 7<sup>th</sup> framework collaborative project “*Monetary, Fiscal and Structural Policies with Heterogeneous Agents (POLHIA)*”, grant no. 225408, is gratefully acknowledged.

**Corresponding author:** Cars Hommes, Center for Nonlinear Dynamics in Economics and Finance (CeNDEF), Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, NL-1018 WB Amsterdam, The Netherlands, e-mail: C.H.Hommes@uva.nl

# 1 Introduction

In a recent paper, Robert Guesnerie presented a unified methodology to investigate expectational coordination. In particular, he discusses the concept of *strong rationality* of equilibria and how common knowledge (CK) may enforce expectational coordination on such equilibria. He argues that if a rational expectations (RE) equilibrium  $E^*$  is *not* locally strongly rational (LSR), after a small perturbation the equilibrium cannot be self-enforcing. We quote (Guesnerie, 2002, pp. 446-447):

*... the assertion: “it is CK that the equilibrium is exactly  $E^*$ ” is always consistent. But a small “hesitation” or “perturbation” may transform the assertion into: “it is CK that the actual state is very close to (in a small neighbourhood of)  $E^*$ .” Whenever the considered equilibrium is not LSR, the just made “trembling assertion” cannot be self enforcing, whatever the associated “tremble” neighbourhood – at least if it is of nonempty interior. In a sense, the assertion is inconsistent. On theoretical grounds, such an inconsistency seems to be a particularly undesirable property of an equilibrium prediction claiming to be grounded in “rationality”.*

Guesnerie (1993), following the game theoretical terminology of Binmore (1987), called this type of reasoning an *eductive* justification of RE. Strong rationality or *eductive stability* may be seen as a necessary condition for an equilibrium to qualify as “rational”.

An important related question is whether evolutive learning processes based upon past observations and past experience necessarily converge to a RE equilibrium. Guesnerie (2002) presents results that, for a certain class of models, eductive stability of a steady state RE implies that (Guesnerie, 2002, p. 473):

*“reasonable” adaptive learning processes are asymptotically stable.*

This class of learning processes includes for example frequently used adaptive learning schemes such as ordinary least squares (OLS) learning. See Evans and Honkapohja (2001) for a recent and extensive treatment of adaptive learning in economics; see also Milgrom and Roberts (1990), who made a general case for the correspondence between eductive stability and evolutive stability when learning has the “best response” property.

In this note, we address the following question: *does eductive stability always imply evolutive stability?* In fact, we show by means of simple examples that the answer is negative and that eductive stability is not a sufficient condition for adaptive learning processes to converge. In particular, we show that in a Muthian cobweb model which is eductively globally stable, the evolutionary learning processes, as proposed by Brock and Hommes (1997), need not converge to the RE steady state. In these adaptive evolutionary systems, the common knowledge of rationality ad infinitum assumption is not satisfied. Instead, agents are boundedly rational and choose simple strategies according to their performance in the recent past, as measured by realized profits.

All our examples are Muthian cobweb models as discussed by Guesnerie (2002); see Hommes (2009) for a recent overview of adaptive learning in the cobweb setting. Our first and simplest example of a RE steady state which is eductively stable but not evolutionary stable occurs when producers can choose between three strategies: optimistic, fundamentalist and pessimistic. Optimists predict prices to be at a certain level above their steady state value, pessimists predict them to be at another level below the steady state, whereas fundamentalists predict exactly the steady state value. Producers choose the forecasting strategy adaptively, according to its past realized profits.

Although this example is eductively stable in the sense of Guesnerie, the adaptive evolutionary learning process of Brock and Hommes may lock into a stable two-cycle. The intuition behind this example is quite simple. When the price is above its steady state value and optimists dominate the market, next period's realized market price will be below the steady state value. Pessimists will have earned higher profits than optimists. If the initial price is far enough away from the steady state, pessimists' earnings will be higher even than earnings of fundamentalists. In the adaptive evolutionary model, most producers will then switch to the pessimistic strategy, causing the next realized market price to be above the steady state value, with optimists earning highest profits, and the story repeats. When the intensity of choice for strategy selection is high, that is, when producers are sensitive to small differences in evolutionary fitness, the majority of traders switches quickly between optimistic and pessimistic beliefs, and prices will lock into a stable two-cycle at some positive distance from the fundamental steady state.

Notice that in this three type evolutionary model the fundamental steady state may be locally stable: with an initial price close enough to the fundamental, fundamentalists earn higher profits than optimists and pessimists. More producers will switch to the fundamentalists' strategy, enforcing prices to converge to the (locally stable) steady state. This implies that, although the model is globally eductively stable, the evolutionary system allows for two different long run outcomes: a steady state or a two-cycle.

In this example, eductive stability fails to ensure evolutionary stability of the system because of the following: though the strategies of both optimists and pessimists are non-rational in terms of common knowledge rationality, they represent the best available choice of the agents. It therefore seems to be the case that the number and distribution of available strategies is important; this leads to the question whether the co-existence of different stable evolutions is robust when the number of strategy types increases. For example, in the three type case discussed above adding more types might cause the amplitude of the two-cycle to become smaller, or even cause the two-cycle to disappear.

Brock, Hommes, and Wagener (2005) recently introduced the notion of large type limit (LTL) to study evolutionary heterogeneous market systems with *many* different strategy types drawn from a certain initial distribution at the start of the economy. Compare also the section on large type limits in the survey Hommes and Wagener (2009). In Anuvrief et al. (2008) the notion of large type limit is applied to a macroeconomic interest rate model.

In this note, the LTL-framework will be applied to the Muthian cobweb model to study how the number of strategy types influences the global stability of the rational expectations equilibrium.

It turns out that decisive roles are played by the initial distribution of strategies as well as the intensity of choice.

More precisely, we show that two conditions have to be met. The first of these conditions requires the set of admissible strategies to be an open interval, and the initial density of strategy types to be strictly positive on the set of admissible strategies. The second requirement is that the intensity of choice to switch strategies is sufficiently large. Under these conditions, an evolutionary Muthian model with many producer types is likely to be globally stable, and eductive stability implies evolutive stability. However, we also show that if either of these conditions is violated, then there may be stable two–cycles coexisting with the stable rational expectations equilibrium.

This note is organised as follows. Section 2 recalls the concept of strong rationality or eductive stability as discussed in Guesnerie (2002). Section 3 presents the Muthian cobweb model with heterogeneous beliefs and evolutionary learning. In section 4 an example with three strategies is analysed and a numerical example with five strategies is presented. Evolutionary systems with many different strategies are studied in section 5, and sufficient conditions for global evolutive stability as well as examples with co-existing stable two–cycles are given. Section 6 concludes and all proofs are contained in the appendix.

## 2 Eductive stability

Guesnerie (2002) summarises the principles of his approach to expectational coordination as follows:

1. pick a rational expectations equilibrium and call it  $E^*$ ;
2. introduce a Common Knowledge (CK) restriction that places an exogenous bound on the state space and that describes restricted, but CK, beliefs of the agents on the possible states of the system;
3. analyse the consequences of the combination of the CK assumption and the CK restriction on the states of the system.

In his paper, he illustrates this methodology with a simple Muthian cobweb model. Since our subsequent analysis is based on this model, we briefly sketch it.

Economic agents are farmers producing a certain crop. Based on his expectation  $E(p, i)$  of the price  $p$  for next period's crop, farmer  $i$  chooses a supply level  $S(p, i)$  that maximises his expected profit. The aggregate supply  $\mathcal{S}(p)$  of the economy is then given by  $\int S(p, i) di$ ; the corresponding aggregate consumer demand is denoted  $\mathcal{D}(p)$ . As usual, we will assume that  $\mathcal{D}$  is decreasing and  $\mathcal{S}$  is increasing.

In the absence of noise there is a unique rational expectations price  $E^*$ , which is equal to the perfect foresight equilibrium price  $p^*$ , and which satisfies

$$\int S(p^*, i) di = \mathcal{S}(p^*) = \mathcal{D}(p^*).$$

It is assumed to be common knowledge amongst the farmers that the price of the crop (which is here the state variable) has to take values in a neighbourhood  $V(E^*) = [E^* - \varepsilon, E^* + \varepsilon]$  of the rational expectations equilibrium. Note that  $\varepsilon > 0$  is not assumed to be small.

Farmer  $i$  expects that next period's price will be in  $V(E^*)$ . Moreover, he assumes that the other farmers have the same expectation; he infers that each of them will play a best response  $S(p, j)$  to some price  $p$  in  $V(E^*)$ . The aggregate supply will then be within  $[\mathcal{S}(p^* - \varepsilon), \mathcal{S}(p^* + \varepsilon)]$ , and knowing the demand function, he obtains a set

$$\Gamma(V(E^*)) = [\mathcal{D}^{-1}(\mathcal{S}(p^* + \varepsilon)), \mathcal{D}^{-1}(\mathcal{S}(p^* - \varepsilon))]$$

of possible prices. If  $\Gamma(V(E^*))$  is strictly contained in  $V(E^*)$  this reasoning can be iterated, on the assumption that all other farmers reason in the same manner. Using in this way the knowledge of rationality, the knowledge of the knowledge of rationality and so on, farmer  $i$  constructs smaller and smaller intervals  $\Gamma^n(V(E^*))$  of possible prices. If these intervals converge to  $E^*$ , then the rational expectations equilibrium  $E^*$  is called *strongly rational* or *eductively stable* with respect to the restriction  $V(E^*)$ .

Guesnerie (2002) shows that for the Muthian farmer model the rational expectations equilibrium  $p^*$  is (locally) eductively stable if the familiar cobweb stability condition  $\mathcal{S}'(p^*)/\mathcal{D}'(p^*) > -1$  is satisfied. If in addition demand and supply are linear, say  $\mathcal{S}(p) = sp$  and  $\mathcal{D}(p) = A - dp$ , with  $(0 <)s/d < 1$ , then  $p^*$  is even globally eductively stable.

### 3 The Muth model with evolutionary learning

In this section we recall the Muth farmer model with heterogeneous beliefs and evolutionary learning, as introduced in Brock and Hommes (1997).

Producers are expected profit maximisers; they solve

$$\text{Max}_q \Pi = \text{Max}_q (p^e q - c(q)), \quad (1)$$

where  $p^e$  is the expected price and  $c(q)$  are costs from producing quantity  $q$ . Assuming a quadratic cost function  $c(q) = q^2/(2s)$ , the first order condition  $\Pi' = 0$  yields the linear supply curve

$$S(p^e) = sp^e. \quad (2)$$

Consumer demand is assumed to be linearly decreasing in the market price, that is

$$D(p) = A - dp. \quad (3)$$

The rational expectations steady state price  $p^*$ , for which demand equals supply, is given by

$$p^* = \frac{A}{s + d}. \quad (4)$$

We will refer to  $p^*$  as the *fundamental* price.

Producers can choose from  $H$  different forecasting rules. Let  $n_{ht}$  denote the fraction of producers using rule  $h$  at date  $t$ ; this rule gives the forecast  $p_{ht}^e$  for  $p_t$ . Note that every forecast rule is equivalent to a production strategy. Heterogeneous market equilibrium is given by

$$D(p_t) = \sum_{h=1}^H n_{ht} S(p_{ht}^e). \quad (5)$$

The market equilibrium equation (5) represents the first part of the model. We now turn to the evolutionary part of the model describing how the fractions of the different producer types change over time. The basic idea of evolution is that fractions are updated according to past performance, given by realized profits in the recent past. In period  $t$ , producer type  $h$  realises the net profit

$$\begin{aligned} \pi_{ht} &= p_t S(p_{ht}^e) - c(S(p_{ht}^e)) = p_t s p_{ht}^e - \frac{s^2 (p_{ht}^e)^2}{2s} \\ &= \frac{s}{2} p_{ht}^e (2p_t - p_{ht}^e). \end{aligned} \quad (6)$$

A natural choice for the fitness or performance measure is a weighted sum of realized profits

$$U_{ht} = \pi_{ht} + w U_{h,t-1}, \quad (7)$$

where the weight parameter  $w$  measures the memory strength. According to this fitness measure, realized profits further in the past contribute with exponentially declining weights. In the case of infinite memory,  $w = 1$ , fitness equals accumulated wealth. In this note, to keep the model tractable, we focus on the other extreme case  $w = 0$ , with fitness equal to the most recently realized net profit.

The fractions  $n_{ht}$  of belief types are updated according to a discrete choice model:

$$n_{ht} = \exp(\beta U_{h,t-1}) / Z_{t-1}, \quad (8)$$

where  $Z_{t-1} = \sum \exp(\beta U_{h,t-1})$  is a normalisation factor. This evolutionary mechanism for prediction rules has been proposed by Brock and Hommes (1997). It can be derived from a random utility model, where the fitness of all strategies is publically known, but subject to noise or error. If the noise terms are IID across agents and across types and drawn from a double exponential distribution, as the number of agents tends to infinity, the probability of selecting strategy  $h$  converges to the discrete choice fraction  $n_{ht}$ . The parameter  $\beta$  is called the intensity of choice. It is inversely proportional to the noise level and  $(1/\beta)$  is sometimes referred to as the

propensity to err. In the extreme case  $\beta = 0$ , corresponding to noise of infinite variance, agents do not switch strategies at all and all fractions are fixed and equal to  $1/H$ . The other extreme case  $\beta = \infty$  corresponds to the case of no noise, where agents do not make errors and all agents use the best predictor each period. Of course, in this evolutionary setting the ‘best’ predictor may change over time.

The beliefs of the producers are assumed to be of the following simple form:

$$p_{ht}^e = p^* + b_h, \quad (9)$$

where  $b_h$  does not depend on  $t$  or on past realised prices. The forecasting rule (9) represents type  $h$ 's ‘‘model of the market’’: it indicates strategy type  $h$  belief how prices will deviate from the fundamental price; this fundamental is not necessarily known to the agents, but (9) is just a mathematically convenient way to represent type  $h$ 's belief. Types with positive bias  $b_h$  are called optimistic, those with negative bias pessimistic.

We rewrite the model in terms of the *deviation*  $x_t$  from the fundamental price, that is,

$$x_t = p_t - p^*. \quad (10)$$

In terms of deviations, the market equilibrium equation (5) takes the form

$$x_t = -\frac{s}{d} \sum_{h=1}^H n_{ht} b_h. \quad (11)$$

The evolutionary fitness measure  $U_{ht} = \pi_{ht}$  can also be written in deviations form. Indeed, observe that the discrete choice fractions  $n_{ht}$  are independent of the profit level, that is, they remain the same when subtracting the same term from all profits  $\pi_{ht}$ . In particular, subtracting the profit  $\pi_{Rt} = (s/2)p_t^2$  that would be earned by a rational agent, that is an agent with perfect foresight, yields

$$\begin{aligned} \pi_{ht} - \pi_{Rt} &= \frac{s}{2} p_{ht}^e (2p_t - p_{ht}^e) - \frac{s}{2} p_t^2 = -\frac{s}{2} (p_t - p_{ht}^e)^2 \\ &= -\frac{s}{2} (x_t - b_h)^2. \end{aligned} \quad (12)$$

Hence, for this model fitness as measured by the most recent realized profits is equal to negative squared forecast errors, up to a common factor. The discrete choice probabilities based upon last period's realized profit reduce to

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}} = \frac{\exp(-\frac{\beta s}{2} (x_{t-1} - b_h)^2)}{Z_{t-1}}, \quad (13)$$

where as before  $Z_{t-1} = \sum_{h=1}^H \exp(-\frac{\beta s}{2} (x_{t-1} - b_h)^2)$  is a normalisation factor. The Muthian model with evolutionary learning and  $H$  competing forecasting rules  $f_h = p^* + b_h$ , written in

deviations from the RE fundamental price, is thus given by (11) and (13). Substituting (13) into (11) yields

$$x_t = -\frac{s}{d} \sum_{h=1}^H n_{ht} b_h = -\frac{s}{d} \frac{\sum_{h=1}^H b_h \exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)}{\sum_{h=1}^H \exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)} = g_H(x_{t-1}). \quad (14)$$

The evolutionary dynamics with  $H$  belief types  $b_h$  is thus described by a one-dimensional map  $g_H$ . This dynamics depends upon the number of types  $H$ , the initial distribution of types  $b_h$ , the market (in)stability ratio  $s/d$  and the intensity of choice  $\beta$ , measuring the sensitivity to differences in evolutionary fitness.

## 4 Examples with few belief types

Brock and Hommes (1997) considered a Muthian cobweb model where producers could choose between a cheap, simple forecasting rule and a more sophisticated but costly forecasting rule. In particular, they considered a two type example where producers either use a freely available naive forecasting rule or the rational expectations forecasting rule at positive per period information costs. They showed that if the cobweb dynamics are *unstable* under naive expectations, then increasing the intensity of choice also destabilises the evolutionary learning model, and chaotic price fluctuations may arise with producers switching between the simple, cheap, destabilising naive forecasting strategy and the sophisticated, costly and stabilising rational strategy. However, if the Muthian model is *stable* under the naive rule, evolutionary learning enforces prices to convergence to the RE fundamental price. Hence, if the Muthian model is eductively stable in the sense of Guesnerie, the Brock and Hommes (1997) two type example of naive expectations versus costly rational expectations will be (globally) stable under evolutionary learning.

From now on we restrict our attention to the case that the Muthian model is stable under naive expectations, that is, the case that the slopes of demand and supply satisfy the familiar ‘cobweb theorem’ (Ezekiel, 1938):

$$(0 <) \frac{s}{d} < 1. \quad (15)$$

Under this assumption the Muthian model is eductively stable in the sense of Guesnerie (2002). In this section, we show that for three producer types evolutionary learning does not always converge to the RE fundamental steady state price, but may “lock into” a stable two-cycle.

Let  $b > 0$  be a given positive bias. Producers can choose from three different forecasting rules:

$$p_{1t}^e = p^*, \quad (16)$$

$$p_{2t}^e = p^* + b, \quad (17)$$

$$p_{3t}^e = p^* - b. \quad (18)$$

Type 1 are fundamentalists, believing that prices will always be at their fundamental value (or equivalently, expected deviations  $x_{ht}^e$  from the fundamental will always be zero). Type 2 are



optimists, expecting that the price of the good will always be above the fundamental price, whereas type 3 agents are pessimists, always expecting prices below the fundamental price. Notice that this example is symmetric in the sense that the optimistic and the pessimistic strategy are exactly balanced around the fundamental price, but this is not essential in what follows. We have the following:

**Theorem A.** Let  $\frac{1}{2} < \frac{s}{d} < 1$ . The Muthian model with evolutionary learning given by (11) and (13), with three producer types given by (16-18) has the following properties:

- (i) for all  $\beta > 0$ , the fundamental steady state is the unique steady state and it is locally stable;
- (ii) for  $\beta$  sufficiently large, there exists a locally stable two-cycle;
- (iii) for  $\beta = \infty$  the locally stable two-cycle is given by  $\{x_1, x_2\} = \{-bs/d, +bs/d\}$ , with corresponding fractions of optimists and pessimists switching from 0 to 1 along the two-cycle.

The formal proof of the theorem is given in the appendix. Its underlying ideas are however very simple and the main intuition was already given in the introduction. Consider the case where the intensity of choice  $\beta = \infty$  first. Equation (13) then implies that the fraction  $h$  dominates, that is, that  $n_{ht} = 1$ , if  $x_{t-1}$  is closest to the belief  $b_h$ . To be concrete, let us say that optimists dominate. The price dynamic equation (11) then implies that the next deviation  $x_t$  is equal to  $-(s/d)b_h$ ; now, if  $s/d > 1/2$ , then  $x_t$  is closest to  $-b_h$ , that is the type at the other extreme, in this case the pessimists, will dominate and the two-cycle starts evolving. This demonstrates point (iii) of the theorem. As “having a locally stable two-cycle” is a persistent dynamical property under small regular perturbations, point (ii) of the theorem states merely that changing  $\beta$  from infinity to a very large value amounts to performing a small regular perturbation. When the intensity of choice  $\beta$  is large, we thus obtain a locally stable 2-cycle with the market switching between states where the optimists respectively the pessimists dominate the market.

A similar result holds for evolutionary systems with more than three belief types; these can be chosen in such a way that multiple stable two-cycles co-exist. Figure 1 illustrates this for an evolutionary Muthian model with *five* belief types  $b_h \in \{-2, -1, 0, +1, +2\}$ , where  $b_h$  is the belief concerning the deviation from the fundamental steady state as before. In this example, for small intensity of choice,  $0 < \beta < \beta_1 \approx 5$  the fundamental steady state is globally stable (Figure 1, left plot). As  $\beta$  increases two stable two-cycles are created by saddle-node bifurcations, the first for  $\beta = \beta_1 \approx 5$  and the second one for  $\beta = \beta_2 \approx 7$  (Figure 1, right plot). For  $\beta > \beta_2$  two (locally) stable two-cycles co-exist with the locally stable fundamental steady states, separated by two unstable two-cycles (Figure 1, middle plot). Obviously, since a stable steady state and a stable two-cycle are structurally stable, the results also hold for slightly asymmetric cases.

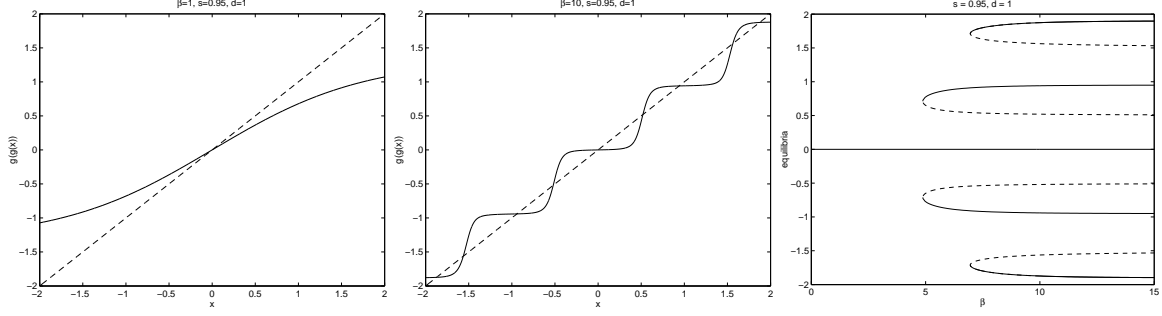


Figure 1: *Dynamics of a Muthian cobweb model under evolutionary dynamics with five strategy types. Shown is the graph of the second iterate  $g(g(x_{t-1}))$ ; intersections (other than at the origin) of this graph with the diagonal are period-2 cycles. Left: low intensity of choice; middle: high intensity of choice. The rightmost graph shows the  $(\beta, x)$ -bifurcation diagram: solid curves indicate branches of stable 2-cycles, dashed curves unstable 2-cycles.*

## 5 Evolutionary dynamics with many belief types

In the previous section we have seen that, in general, for the Muthian model global eductive stability does *not* imply evolutionary stability. In the simple three type evolutionary learning example considered, price fluctuations may settle down to a stable two-cycle with supplier types switching between optimistic and pessimistic strategies. As indicated in the introduction, this behaviour might be the consequence of the fact that there are not “enough” strategies available.

In this section we therefore analyse evolutionary dynamics in the Muthian model with *many* strategy types. Recall that the evolutionary dynamics with  $H$  belief types  $b_h$  is given by (14). We assume that at the beginning of the market, at date 0, a large number  $H$  stochastic belief parameters  $b = b_h \in \mathbb{R}$  are drawn from a common initial distribution with density function  $\varphi(b)$ . Thereafter these  $H$  beliefs compete against each other according to the evolutionary dynamics specified in (14).

Evolutionary systems with many different belief types are difficult to handle analytically. Brock, Hommes, and Wagener (2005) have recently introduced the notion of *large type limit (LTL)* to approximate evolutionary systems with many belief types; see also Diks and van der Weide (2005) for a related approach. The LTL of the Muth model can be derived as follows. Divide both numerator and denominator of (14) by  $H$  to obtain

$$x_t = -\frac{s}{d}g_H(x_t) = \left(-\frac{s}{d}\right) \frac{\frac{1}{H} \sum_{h=1}^H b_h e^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}}{\frac{1}{H} \sum_{h=1}^H e^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}}. \quad (19)$$

The LTL is obtained by replacing sample averages by sample means in both the numerator and

the denominator, yielding

$$x_t = g(x_{t-1}) = \left(-\frac{s}{d}\right) \frac{\int b e^{-\frac{\beta s}{2}(x_{t-1}-b)^2} \varphi(b) db}{\int e^{-\frac{\beta s}{2}(x_{t-1}-b)^2} \varphi(b) db}. \quad (20)$$

Applying a uniform law of large numbers, Brock, Hommes, and Wagener (2005) have shown that if the number of strategies  $H$  is sufficiently large, the LTL dynamical system is a good approximation of the evolutionary dynamical system with  $H$  belief types. More precisely, they proved that as the number of strategies  $H$  tends to infinity, the H-type map  $g_H$  (given by (14)) and its derivatives converge almost surely to the LTL-map  $g$  (given by (20)) and its derivatives, respectively. An important corollary of the LTL-theorem is that all generic and persistent dynamical properties (such as steady states, periodic cycles, local stability of steady states and periodic points, bifurcations, and even chaos and strange attractors) of the LTL also occur with probability arbitrarily close to one in the system with  $H$  belief types if the number of types  $H$  is sufficiently large. The evolutionary dynamics with many trader types can thus be studied using the LTL-dynamics (20).

The next lemma describes the LTL map  $g$  for large intensity of choice. To announce this lemma, introduce the function

$$h_\beta(x) = \frac{\int b e^{-\frac{\beta s}{2}(x-b)^2} \varphi(b) db}{\int e^{-\frac{\beta s}{2}(x-b)^2} \varphi(b) db}.$$

**Lemma 1.** Let  $J$  be the interior of the support of the distribution  $\varphi$  from which the belief parameters  $b_h$  are drawn at date 0, that is,  $J = \text{int}\{b \mid \varphi(b) \geq 0\}$ .

For all  $x \in J$ :

$$\lim_{\beta \rightarrow \infty} h_\beta(x) = x \quad \text{and} \quad \lim_{\beta \rightarrow \infty} h'_\beta(x) = 1,$$

uniformly on all compact subsets  $K$  of  $J$ .

This lemma implies that, in the interior of the support of the distribution  $\varphi$  from which the belief parameters  $b_h$  are drawn, the LTL-map  $g(x) = -\frac{s}{d}h_\beta(x)$  approaches a linear map as  $\beta$  becomes large, while its derivative approaches the constant function  $-s/d$ . The following theorem is an immediate consequence of this lemma:

**Theorem B.** (Global stability of evolutionary systems with many trader types.)

For any strictly positive initial distribution of beliefs  $\varphi$  and for  $\beta$  sufficiently large, if  $0 < s/d < 1$  the LTL-dynamics (20) are globally stable. If  $s/d > 1$ , all orbits of (20), except the steady state, exhibit unbounded oscillations.

Theorem B is a statement about stability of the LTL-system, but immediate consequences for evolutionary systems with  $H$  trader types when  $H$  is large are obtained, by applying the

LTL theorem of Brock, Hommes, and Wagener (2005). Let us discuss the implications of the case  $0 < s/d < 1$ : if there are sufficiently many strategies (i.e.  $H$  is large), the probability of finding a strategy in any open interval is positive (i.e.  $\varphi(b) > 0$ ), and the intensity of choice  $\beta$  to switch strategies is large enough, then the fundamental steady state is globally stable with a probability that goes to 1 as  $H$  increases over all bounds. In that case, eductive stability implies global stability in a heterogeneous market with evolutionary learning.

Recall that the Muthian model is eductively stable if and only if  $0 < s/d < 1$ . Theorem B therefore essentially states that, if all types occur with positive probability (i.e.  $\varphi(b) > 0$ , for all  $b$ ) and if agents are highly sensitive to differences in evolutionary fitness (i.e. if  $\beta$  is large) eductive stability and evolutionary stability are equivalent. Since the intensity of choice  $\beta$  is inversely related to the propensity to err, we can also reformulate this as follows. If all types occur with positive probability (i.e.  $\varphi(b) > 0$ , for all  $b$ ) and if agents only make small errors in evaluating evolutionary fitness, eductive stability and evolutionary stability are equivalent.

### Many trader types examples with two-cycles

Our final theorem shows that both conditions in theorem B (i.e.  $\varphi(b) > 0$  for all  $b$  and  $\beta$  sufficiently large) are *necessary* conditions for equivalence of eductive and evolutionary stability.

#### Theorem C

1. There is an initial distribution of beliefs  $\varphi$  such that for all  $\beta$  sufficiently large, the LTL-dynamics (20) have a locally stable 2-cycle.
2. For any given value of  $\beta > 0$ , and for any  $s$  and  $d$  such that  $s/d < 1$  is sufficiently close to 1, there exists a strictly positive distribution of beliefs  $\varphi$  such that the dynamics (20) have a locally stable 2-cycle.

Theorem C describes examples of LTL systems with at least one locally stable two-cycle. This implies that, even though the Muthian model is eductively stable (i.e.  $s/d < 1$ ), there are evolutionary systems with many trader types where a locally stable two-cycle occurs with probability arbitrarily close to 1. The two subcases of Theorem C are illustrated in figure 2. Theorem C1 shows that if there are intervals of “unavailable” strategies, that is, if the distribution function  $\varphi(b)$  is zero on certain intervals, global stability of the fundamental steady state may fail, even when the intensity of choice is arbitrarily large. Theorem C2 asserts that for any fixed given value of  $\beta$  (large enough) there is an initial distribution of strategies with everywhere positive density such that there are many coexisting stable two-cycles; again global stability of the fundamental steady state fails in this case. Having everywhere positive distributions is therefore in itself not sufficient for global stability when  $\beta$  is large but finite<sup>1</sup>.

---

<sup>1</sup>Note that the result in Theorem C2 implies that the density function  $\varphi = \varphi_\beta$  depends on the parameter  $\beta$ , as can also be seen from the proof. From theorem B we infer the existence of  $\beta' \gg \beta$  such that the system with strictly positive density  $\varphi_\beta$  and intensity of choice  $\beta'$  has a globally stable fundamental state.

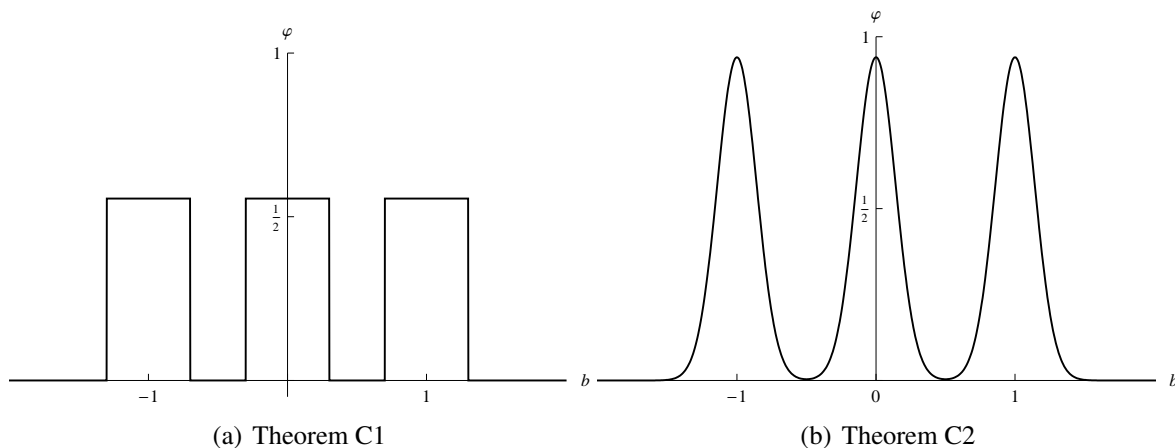


Figure 2: Illustrations to theorem C.

## 6 Concluding Remarks

If a RE steady state is not eductively stable in the sense of Guesnerie (2002), expectational coordination on this RE steady state seems unlikely, since the belief in a small deviation from the steady state may trigger an even larger realized deviation. Eductive stability can thus be seen as a necessary condition for expectational coordination. In this note we have shown that global eductive stability however is not a sufficient condition for evolutionary learning to enforce convergence to the RE steady state. We have presented simple examples of Muthian cobweb models with heterogeneous beliefs where the RE fundamental steady state is globally eductively stable in the sense of Guesnerie (2002), but evolutionary learning in the sense of Brock and Hommes (1997) does not enforce convergence to the unique RE steady state. In particular, we have presented examples of evolutionary systems with three or five belief types which can lock into a stable two-cycle, co-existing with the stable steady state, with up and down price fluctuations and the majority of agents switching between optimistic and pessimistic strategies.

An important issue related to the stability of an evolutionary system is the number of belief types. As shown in this note, if the number of belief types is small, “evolutionary cycles” can occur with the majority of traders switching constantly from one biased belief to another. An increase in the number of types, e.g. by “invasion” of new types, may destroy these “evolutionary cycles” however, and can possibly enforce convergence to a stable RE steady state. In particular, we have shown that if the initial distribution function of beliefs is strictly positive and the intensity of choice to switch strategies is large enough, an evolutionary system with many trader types is likely to be globally stable. In that case eductive stability and evolutionary stability coincide. Both conditions are necessary, that is, if the initial distribution of strategies is not strictly positive everywhere and/or if the intensity of choice is finite (so that agents are only boundedly rational and make errors) for a globally eductive stable Muthian model the corresponding evolutionary system with many trader types may lock into a locally stable two-cycle.

The study of the stability of evolutionary systems with many trader types in various market

settings and with more complicated strategies remains an important topic for future work.

## Appendix: Proofs of the theorems

### Proof of Theorem A.

We follow a strategy of proof similar to Brock and Hommes (1998), section 4.2, lemma 8, for the asset pricing model with  $H$  purely biased beliefs  $b_h$ . When there is no memory in the fitness measure, that is when  $w = 0$ , the fraction  $n_{ht}$  of type  $h$  is given by

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}} = \frac{\exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)}{Z_{t-1}}, \quad (21)$$

where  $Z_{t-1}$  is a normalisation factor. Since the fractions  $n_{ht}$  are independent of the fitness level, we may add the common term  $(\beta s/2)x_{t-1}^2$  to all  $U_{h,t-1}$  to obtain

$$n_{ht} = \frac{\exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2))}{Z_{t-1}}, \quad (22)$$

where the normalisation factor has been redefined as

$$Z_{t-1} = \sum_{h=1}^H \exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2)).$$

If  $w = 0$ , past fitnesses  $U_{h,t-1}$ , and consequently the fractions  $n_{ht}$ , only depend on  $x_{t-1}$ . Then equation (11) implies that the deviation  $x_t$  from the fundamental price depends only on  $x_{t-1}$ , as it is of the form

$$x_t = -\frac{s}{d} \sum_{h=1}^H n_{ht} b_h = g(x_{t-1}). \quad (23)$$

We claim that the one-dimensional map  $g$  is decreasing. Writing  $e^y = \exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2))$ , the derivative  $g'$  is given by

$$\begin{aligned}
g'(x_{t-1}) &= -\frac{s}{d} \sum_{h=1}^H b_h \frac{dn_{ht}}{dx_{t-1}} \\
&= -\frac{s}{d} \sum_{h=1}^H \left\{ b_h \frac{Z_{t-1} e^y \cdot \beta s b_h - e^y \cdot (\sum_h e^y \beta s b_h)}{Z_{t-1}^2} \right\} \\
&= -\frac{\beta s^2}{d} \sum_{h=1}^H \left\{ b_h \frac{e^y \cdot b_h - n_{ht} (\sum_h e^y b_h)}{Z_{t-1}} \right\} \\
&= -\frac{\beta s^2}{d} \sum_{h=1}^H \left\{ n_{ht} b_h^2 - n_{ht} b_h (\sum_h n_{ht} b_h) \right\} \\
&= -\frac{\beta s^2}{d} \left[ \sum_{h=1}^H n_{ht} b_h^2 - \left( \sum_{h=1}^H n_{ht} b_h \right) \left( \sum_h n_{ht} b_h \right) \right] \\
&= -\frac{\beta s^2}{d} [\langle b_h^2 \rangle - \langle b_h \rangle^2] < 0,
\end{aligned} \tag{24}$$

where  $\langle b_h \rangle = \sum_h n_{ht} b_h$  and  $\langle b_h^2 \rangle = \sum_h n_{ht} b_h^2$  are averages. The last inequality follows because the last term between square brackets can be interpreted as the *variance* of a stochastic process  $b_t$ , taking values  $b_h$  with probability  $n_{ht}$ . We conclude that the map  $g$  is decreasing; hence the system (23) has a unique steady state.

If the beliefs are exactly balanced, that is if for each belief  $b_h$  its opposite belief  $-b_h$  is also present in the market, then the unique steady state  $x^*$  coincides with the fundamental:  $x^* = 0$ .

We now restrict our attention to the system with three belief types (16-18). It has exactly balanced beliefs, and hence the fundamental steady state is the unique steady state.

To investigate the stability of the steady state, observe that the steady state fractions are given by

$$n_1^* = \frac{1}{1 + 2 \exp(-\frac{\beta s}{2} b^2)}, \quad n_2^* = n_3^* = \frac{\exp(-\frac{\beta s}{2} b^2)}{1 + 2 \exp(-\frac{\beta s}{2} b^2)}.$$

The local stability of the fundamental steady state is governed by  $g'(0)$ . Applying (24) yields that

$$g'(0) = -\frac{\beta s^2}{d} 2b^2 \frac{\exp(-\frac{\beta s}{2} b^2)}{1 + 2 \exp(-\frac{\beta s}{2} b^2)} = -\frac{2s}{d} h(z),$$

where  $z = \beta s b^2 / 2$  and  $h(z) = 2z e^{-z} / (1 + 2e^{-z})$ . A straightforward computation shows that  $h(z) = p(z)n(z)$ , where  $p(z) > 0$  for all  $z$  and  $n(z) = 1 - z + 2e^{-z}$ . Taylor's expansion yields  $e^{-1/2} \leq 1 - 1/2 + 1/8 = 5/8$ , and hence

$$n\left(\frac{3}{2}\right) = 1 - \frac{3}{2} + 2e^{-3/2} \leq \frac{125}{256} - \frac{1}{2} < 0.$$

As  $n(1) = 2/e > 0$  and as  $n(z)$  is continuous and strictly decreasing, it has a unique zero  $z_m$  in the interval  $(1, 3/2)$ , and the function  $h$  takes its global maximum at this point. We have  $h(z_m) = z_m - 1 < 1/2$ . Since  $1/2 < s/d < 1$  by assumption, it follows that  $-1 < g'(0) (< 0)$ , implying local stability of the steady state  $x^* = 0$ . This proves assertion (i) of theorem A.

Now turn to assertion (ii), the existence of a stable two-cycle for sufficiently large  $\beta$ . Let  $s/d - 1/2 = \varepsilon$ ; by assumption  $\varepsilon > 0$ . Moreover, let  $\delta = \frac{b}{2}\varepsilon$  and let  $U = \{x \mid x \geq b/2 + \delta\}$ . Then for  $\beta > 0$  large enough, and  $x \in U$ ,

$$\begin{aligned} g(x) &= -\frac{s}{d} \sum n_{hi} b_h = -\frac{s}{d} \frac{b - b e^{-2\beta s b x}}{1 + e^{-\frac{\beta s}{2} b (2x-b)} + e^{-2\beta s b x}} \\ &< -b \frac{s}{d} \frac{1 + \varepsilon}{1 + 2\varepsilon} = -b \left( \frac{1}{2} + \varepsilon \right) \frac{1 + \varepsilon}{1 + 2\varepsilon} = -\frac{b}{2} (1 + \varepsilon) = -\frac{b}{2} - \delta \end{aligned}$$

uniformly in  $x$ . Since  $g$  is monotonous, this implies that  $g(U) \subset -U$  and  $g^2(U) = g(g(U)) \subset U$ . Since  $g^2$  monotonically increasing and bounded, all points in  $U$  converge to a fixed point of  $g^2$  in  $U$ . Since the only fixed point of  $g$  is the origin, the fixed points of  $g^2$  are two-cycles.

In the special case  $\beta = +\infty$ , the system reads as

$$g(x) = \begin{cases} bs/d & \text{if } x < -\frac{1}{2}b, \\ 0 & \text{if } -\frac{1}{2}b < x < \frac{1}{2}b, \\ -bs/d & \text{if } \frac{1}{2}b < x. \end{cases}$$

It can be verified directly that that the system has a locally stable two-cycle  $\{x_1, x_2\} = \{-bs/d, bs/d\}$ , with corresponding fractions of optimists (and pessimists) switching from 0 to 1 along the two-cycle. This proves (iii) and completes the proof of the proposition.

## Proof of Lemma 1 and Theorem B

Let  $\varphi(b)$  be a fixed continuous density function, that is, let  $\varphi(b) \geq 0$  for all  $b$ ,  $\int \varphi db = 1$ . Introduce the probability density function

$$\psi_{x,\beta,s}(b) = \frac{e^{-\frac{\beta s}{2}(x-b)^2} \varphi(b)}{Z},$$

where  $Z$  is such that  $\int \psi_{x,\beta,s}(b) db = 1$ . Let

$$h(x) = \int b \psi_{x,\beta,s}(b) db.$$

Let  $J$  be the interior of the support of  $\varphi$ , that is,  $J = \text{int}\{b \mid \varphi(b) \geq 0\}$ . We will show that

**Lemma 1** For all  $x \in J$ :

$$\lim_{\beta \rightarrow \infty} h(x) = x \quad \text{and} \quad \lim_{\beta \rightarrow \infty} h'(x) = 1,$$

uniformly on all compact subsets  $K$  of  $J$ .



The LTL-map  $g$  in (20) satisfies  $g(x) = -(s/d)h(x)$ , and Lemma 1 is thus equivalent to the lemma in section 5. Hence, if  $J = \mathbb{R}$  and  $0 < s/d < 1$ , then for every compact interval  $K$  enclosing 0, there is a  $\beta_0 > 0$  such that all points in  $K$  tends to the globally stable fixed point 0 under iteration of  $g$ . If  $J = \mathbb{R}$  and  $s/d > 1$ , then the lemma implies that  $g$  has an unstable fixed point 0 and all other solutions exhibit unbounded oscillations. Therefore the lemma implies the statements of theorem B.

We proceed to prove the lemma. We shall first show that as  $\beta \rightarrow \infty$ , the expectation  $\mathbb{E}_\psi b = h(x)$  tends to  $x$ .

By the coordinate transformation  $b = x + y/\sqrt{\beta s}$ , we find that

$$d(x) = \mathbb{E}_\psi b - x = \frac{\int (b-x) e^{-\frac{\beta s}{2}(b-x)^2} \varphi(b) db}{\int e^{-\frac{\beta s}{2}(b-x)^2} \varphi(b) db} = \frac{1}{\sqrt{\beta s}} \frac{\int y e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy}{\int e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy}.$$

Let  $M = \sup_{b \in \mathbb{R}} \varphi(b)$ . Since  $\varphi$  is continuous and  $\varphi(x) > 0$ , there are  $\delta > 0$ ,  $0 < \varepsilon < \varphi(x)/2$  such that  $|\varphi(b) - \varphi(x)| < \varepsilon$  for all  $b \in (x - \delta, x + \delta)$ . Set  $A = \delta\sqrt{\beta s}$ . Then

$$|d(x)| \leq \frac{1}{\sqrt{\beta s}} \frac{M}{\varphi(x) - \varepsilon} \frac{\int |y| e^{-y^2/2} dy}{\int_A e^{-y^2/2} dy} \rightarrow 0 \quad \text{as } \beta \rightarrow \infty;$$

note that the convergence is uniform if  $x$  is restricted to a compact subset  $K$  of  $\mathbb{R}$ . This implies the first part of the lemma.

The derivative of  $h$  is obtained by differentiating under the integral (compare equation (24)):

$$\begin{aligned} h'(x) &= \beta s \int b(b-x)\psi db - \beta s \int b\psi db \cdot \int (b-x)\psi db \\ &= \beta s \int b^2\psi db - \beta s \left( \int b\psi db \right)^2 = \beta s \text{Var}_\psi b = \beta s \text{Var}_\psi(b-x). \end{aligned}$$

Here  $\text{Var}_\psi b$  is the variance of a stochastic variable  $b$  distributed according to the probability distribution  $\psi$ . The last equality holds since  $b$  and  $b-x$  have the same variance.

The derivative  $h'(x)$  can be estimated by

$$h'(x) = \beta s \mathbb{E}_\psi(b-x)^2 - \beta s (\mathbb{E}_\psi(b-x))^2 = \beta s \text{Var}_\psi(b-x).$$

As above, we have that

$$h'(x) = \frac{\int y^2 e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy}{\int e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy} - \left( \frac{\int y e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy}{\int e^{-y^2/2} \varphi(x + y/\sqrt{\beta s}) dy} \right)^2.$$

Let  $A, M, \delta, \varepsilon > 0$  be as above. Then

$$h'(x) \leq \frac{\varphi(x) + \varepsilon \int_{-A}^A y^2 e^{-y^2/2} dy + M/\varepsilon \left( \int_{-\infty}^{-A} + \int_A^{\infty} e^{-y^2/2} dy \right)}{\varphi(x) - \varepsilon \int_{-A}^A e^{-y^2/2} dy}$$

Fix  $\alpha > 0$  arbitrarily; then there exists  $\beta_1$  such that if  $\beta > \beta_1$ , it follows that

$$h'(x) \leq \frac{\varphi(x) + \varepsilon}{\varphi(x) - \varepsilon} (1 + \alpha).$$

Likewise, for  $\alpha > 0$  there exists  $\beta_2 > 0$  such that for  $\beta > \beta_2$ :

$$\begin{aligned} h'(x) &\geq \frac{\varphi(x) - \varepsilon \int_{-A}^A y^2 e^{-y^2/2} dy - (M/\varepsilon) \int_{[-A,A]^c} e^{-y^2/2} dy}{\varphi(x) + \varepsilon \int_{-A}^A e^{-y^2/2} dy} \\ &\quad - \left( \frac{\varepsilon \int_{-A}^A |y| e^{-y^2/2} dy + M \int_{[-A,A]^c} |y| e^{-y^2/2} dy}{(\varphi(x) - \varepsilon) \int_{-A}^A e^{-y^2/2} dy} \right)^2 \\ &\geq \frac{\varphi(x) - \varepsilon}{\varphi(x) + \varepsilon} (1 - \alpha) - C\varepsilon^2, \end{aligned}$$

where  $C$  does only depend on  $\beta_2$ . Since  $\alpha$  and  $\varepsilon$  were arbitrary, it follows that

$$h'(x) \rightarrow 1$$

as  $\beta \rightarrow \infty$ . Note that, as before, the convergence is uniform if  $x$  is restricted to a compact subset  $K$  of  $\mathbb{R}$ . This implies the second part of the lemma, and completes the proof of theorem B.

## Proof of theorem C.

**Proof of C1.** Let  $0 < \delta < 1/8$  be a positive constant; and let  $D_1, D_2$  and  $D_3$  be intervals

$$D_1 = [-1 - \delta, -1 + \delta], \quad D_2 = [-\delta, \delta], \quad D_3 = [1 - \delta, 1 + \delta].$$

Moreover, let  $D = D_1 \cup D_2 \cup D_3$ . Densities  $\varphi$  and  $\psi$  are defined as follows:

$$\begin{aligned} \varphi(b) &= \begin{cases} (6\delta)^{-1} & \text{if } t \in D \\ 0 & \text{otherwise,} \end{cases} \\ \psi(b) &= e^{-\frac{\beta s}{2}(b-x)^2} \varphi(b)/Z, \end{aligned}$$

where the normalisation factor  $Z$  is determined by  $\int_D \psi(b) db = 1$ . As in the proof of Theorem B, a function  $h(x)$  is introduced by setting  $h(x) = -(d/s)g(x)$ . Recall that

$$h(x) = \mathbb{E}_\psi b \quad \text{and} \quad h'(x) = \beta s \text{Var}_\psi b.$$

The relevant properties of  $h$  are given in the following lemma.

**Lemma 2** *Take  $x$  in the compact interval  $K = [\frac{1}{2} + \delta, 1 - 2\delta]$ . Then for every  $\varepsilon > 0$  there is a  $\beta_0 > 0$ , such that for  $\beta > \beta_0$  we have that*

$$|h(x) - (1 - \delta)| < \varepsilon \quad \text{and} \quad h'(x) < \varepsilon, \quad (25)$$

*uniformly in  $x$ .*

Assuming the truth of the lemma, the theorem is proved as follows. Let  $s$  and  $d$  be fixed, such that

$$\frac{\frac{1}{2} + \delta}{1 - 2\delta} < \frac{s}{d} < 1 - 2\delta.$$

Choose  $\varepsilon = \delta$ . If  $\beta > \beta_0$ , with  $\beta_0$  obtained from the lemma, it follows for all  $x \in K$  that

$$-(1 - 2\delta) < g(x) < -\left(\frac{1}{2} + \delta\right).$$

As  $g$  is continuous, by construction of  $K$  there is a point  $x_* \in K$  such that  $g(x_*) = -x_*$ ; symmetry of  $g$  yields consequently  $g(g(x_*)) = x_*$ , so that  $x_*$  is a period 2 point.

Moreover, we have for  $x \in K$  that

$$0 \geq g'(x) = \left(-\frac{s}{d}\right) h'(x) > -\frac{s}{d}\delta > -\delta > -1.$$

We conclude that the period-2 point  $x_*$  is attracting.

**Proof of the lemma.** Introduce  $\ell_j$  and  $u_j$  as the respective lower and upper endpoints of the interval  $D_j$ . By partial integration, we obtain

$$h(x) = \frac{1}{Z} \sum_j \int_{\ell_j}^{u_j} b e^{-\frac{\beta s}{2}(b-x)^2} dbx - \frac{1}{Z\beta s} \sum_j e^{-\frac{\beta s}{2}(b-x)^2} \Big|_{\ell_j}^{u_j} \quad (26)$$

In order to obtain information on  $Z$ , the following well-known asymptotic expansion (see for instance Polya and Szegő, 1970) is used:

$$e^{\frac{1}{2}a^2} \int_a^\infty e^{-\frac{1}{2}b^2} db = \frac{1}{a} - \frac{1}{a^3} + \mathcal{O}\left(\frac{1}{a^5}\right) \quad \text{as } a \rightarrow \infty \quad (27)$$

As usual the notation  $f(x) = \mathcal{O}(g(x))$  is taken to mean that there is a constant  $C > 0$  such that  $|f(x)| \leq Cg(x)$ .

Recall that  $Z = \sum_j \int_{\ell_j}^{u_j} \exp(-(\beta s/2)(b-x)^2) db$ . If  $c_j$  denotes the element of  $D_j$  closest to  $x$ , then the expansion (27) yields for  $x$  in the complement of  $D$ :

$$Z = \sum_j \left( \frac{1}{\beta s |x - c_j|} - \frac{1}{(\beta s)^2 |x - c_j|^3} + \mathcal{O}((\beta s)^{-3}) \right) e^{-\frac{\beta s}{2}(x-c_j)^2} \quad (28)$$

Let  $x$  be restricted to the open interval  $I = (\frac{1}{2} + \delta, 1 - \delta)$ , then the point in  $D$  closest to  $x$  is  $\ell_3$ , and any point  $y$  in  $D_1 \cup D_2$  satisfies  $|y - x| \geq |\ell_3 - x| + 2\delta$ . To simplify the expansion of  $Z$ , we multiply equation (28) by  $\exp(\beta s(x - \ell_3)^2/2)$ , and we note that the other exponentials can be bounded from above by  $\exp(-\beta s\delta)$ . This yields

$$Z = \left( \frac{1}{\beta s |x - \ell_3|} - \frac{1}{(\beta s)^2 |x - \ell_3|^3} + \mathcal{O}((\beta s)^{-3}) \right) e^{-\frac{\beta s}{2}(x-\ell_3)^2}.$$

Substitution in equation (26), and recalling that  $|x - \ell_3| = \ell_3 - x$ , yields

$$h(x) = \ell_3 + \frac{1}{\beta s(\ell_3 - x)} + \mathcal{O}((\beta s)^{-2})$$

for  $x \in I$ . Now the first half of (25) follows.

Recall that  $h'(x) = \beta s \text{Var}_\psi T$ . We have

$$\begin{aligned} \text{Var}_\psi b &= -h(x)^2 + \frac{1}{Z} \sum_j \int_{\ell_j}^{u_j} b^2 e^{-\frac{\beta s}{2}(b-x)^2} db \\ &= -h(x)^2 + \frac{1}{Z} \sum_j \int_{\ell_j}^{u_j} \frac{1}{\beta s} (b \cdot \beta s(b-x) + x \cdot \beta s(b-x) + \beta s x^2) e^{-\frac{\beta s}{2}(b-x)^2} db \\ &= -h(x)^2 + x^2 + \frac{1}{\beta s} - \frac{1}{Z\beta s} \sum_j (b+x) e^{-\frac{\beta s}{2}(b-x)^2} \Big|_{\ell_j}^{u_j}. \end{aligned}$$

Using the same kind of asymptotic arguments as before, this expression can be expanded to

$$\begin{aligned} \text{Var}_\psi b &= -h(x)^2 + x^2 + \frac{1}{\beta s} + \frac{\ell_3 + x}{Z\beta s} e^{-\frac{\beta s}{2}(\ell_3-x)^2} + \mathcal{O}((\beta s)^{-2}) \\ &= -h(x)^2 + x^2 + \frac{1}{\beta s} + (\ell_3 + x)(\ell_3 - x) \left( 1 + \frac{1}{\beta s(\ell_3 - x)^2} \right) + \mathcal{O}((\beta s)^{-2}) \\ &= \mathcal{O}((\beta s)^{-2}). \end{aligned}$$

Hence

$$h'(x) = \beta s \text{Var}_\psi b = \mathcal{O}\left(\frac{1}{\beta s}\right).$$

The second half of (25) follows; this concludes the proof of the lemma.

**Proof of C2.** The previous result shows that there exist LTL systems which have for some fixed  $\beta > 0$  the same behaviour as an evolutionary system with finitely many,  $n$ , types. However, in constructing this example, the underlying type distribution had compact support. To construct an example of an LTL support of the type distribution equal  $\mathbb{R}$  having a stable two-cycle, the distribution necessarily has to depend on  $\beta$  because of theorem B.

Take for instance the following  $2n + 1$ -modal distribution:

$$\varphi(b) = \frac{1}{2n + 1} \sum_{h=-n}^n \frac{1}{\sqrt{2\pi}\sigma_h} e^{-\frac{(b-b_h)^2}{2\sigma_h^2}}.$$

We make a couple of specifications. First, the  $b_h$  are assumed to be distributed symmetrically around 0:  $b_{-h} = -b_h$ . Moreover, they are assumed to be arranged as follows:

$$b_h = 4^h b_1, \quad \text{if } h > 0.$$

Finally, all  $\sigma_h$  are taken equal to  $\sigma_h = \sqrt{2/\beta s}$ .

The LTL can be computed in this case. It reads as:

$$x_t = g(x_{t-1}) = \left(-\frac{s}{d}\right) \left( \frac{1}{2} x_{t-1} + \frac{1}{2} \frac{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h - x_{t-1})^2} b_h}{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h - x_{t-1})^2}} \right).$$

Note that  $g(x)$  is an odd function.

Fix  $i > 0$ : then  $b_i \geq 0$  (the case  $b_i < 0$  necessitates only minor modifications of the following). We may write

$$x_t = g(x_{t-1}) = \left(-\frac{s}{d}\right) \left( \frac{1}{2} (x_{t-1} + b_i) + r \right),$$

with

$$r = \frac{1}{2} \frac{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h - x_{t-1})^2} (b_h - b_i)}{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h - x_{t-1})^2}}.$$

Let  $\delta$  and  $\Delta$  denote, respectively,

$$\delta = \min_{h \neq i} |b_h - b_i|, \quad \text{and} \quad \Delta = \max_{h \neq i} |b_h - b_i|.$$

Let  $U$  denote the open interval  $U = (b_i - \delta/3, b_i + \delta/3)$ .

We claim that if  $\frac{6}{7} < s/d \leq 1$  then for  $\beta$  sufficiently large there is a point  $x_* \in U$  such that

$$g(x_*) = -x_*. \tag{29}$$

It then follows by the oddness of  $g$ ,

$$g(g(x_*)) = g(-x_*) = -g(x_*) = x_*,$$

that is, the point  $x_*$  is a periodic point of period 2. The claim is first shown for

$$g_0(x) = \left(-\frac{s}{d}\right) \left(\frac{1}{2}x + \frac{1}{2}b_i\right);$$

afterwards it is extended to  $g = g_0 + r$  by a perturbation argument.

Note that the equation  $g_0(x) = -x$  has solution

$$x_0 = \frac{s/d}{2 - s/d} b_i.$$

Hence, since  $\frac{6}{7} < s/d \leq 1$ , it follows that  $\frac{3}{4}b_i < x_0 \leq b_i$ , and consequently that

$$|x_0 - b_i| < \frac{1}{4}b_i \leq \frac{\delta}{3}.$$

where we used that  $b_{i-1} = \frac{1}{4}b_i$ . The inequality implies that  $x_0 \in U$ .

To apply a perturbation argument to equation (29), the magnitudes  $|r(x)|$  and  $|r'(x)|$  have to be estimated. First

$$\sup_{x \in U} |r(x)| \leq \frac{1}{2} \frac{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h-x)^2} |b_h - b_i|}{\sum_{h=-n}^n e^{-\frac{\beta s}{4}(b_h-x)^2}} \leq \frac{2n\Delta}{2} \frac{e^{-\frac{\beta s}{4} \frac{4\delta^2}{9}}}{e^{-\frac{\beta s}{4} \frac{\delta^2}{9}}} \leq n\Delta e^{-\frac{\beta s}{12}\delta^2}.$$

The derivative  $|r'(x)|$  is estimated along the same lines; a little computation yields that

$$\sup_{x \in U} |r'(x)| \leq 2n^2 \Delta^2 \beta s e^{-\frac{\beta s}{12}\delta^2}.$$

Hence, by choosing  $\beta$  sufficiently large, both  $|r(x)|$  and  $|r'(x)|$  can be made arbitrarily small, uniformly in  $x$ .

By the intermediate value theorem, it follows that if  $\beta$  is sufficiently large, equation (29) has a solution  $x_*$  in  $U$ . Taking  $\beta$  even larger if necessary, it follows that  $-1 < f' < 0$  in  $U$ . Hence the solution  $x_*$  of equation (29) is unique in  $U$ . As in the proof of B2, it follows that  $x_*$  is an attracting period two cycle.

Hence, we constructed an LTL system with  $n$  attracting periodic 2-cycles, together with an attracting fixed point at 0.

Note that as these cycles are hyperbolic, the distribution  $\varphi$  might be perturbed slightly to a distribution that is not symmetric around  $b = 0$ , while all the 2-cycles persist. The existence of multiple periodic 2-cycles is an ‘open’ property, enjoyed by an open set of systems (open with respect to some specific function topology).

## References

- Anufriev, M., Assenza, T., Hommes, C.H., Massaro, D., 2008. Interest rate rules with heterogeneous expectations. CeNDEF Working Paper 08-08.
- Binmore, K., 1987. Modeling rational players: part 1. *Economics and Philosophy*, 3, 179–214.
- Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. *Econometrica*, 65, 1059–1095.
- Brock, W.A., Hommes, C.H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22, 1235–1274.
- Brock, W.A., Hommes, C.H., Wagener, F.O.O., 2005. Evolutionary dynamics in markets with many trader types. *Journal of Mathematical Economics*, 41, 7–42.
- Diks, C.G.H. and van der Weide, R. (2005), Herding, a-synchronous updating and heterogeneity in memory in a CBS *Journal of Economic Dynamics and Control* 29, 741-763.
- Evans, G., Honkapohja, S., 2001. *Learning and expectations in macroeconomics*. Princeton: Princeton University Press.
- Ezekiel, M., 1938. The cobweb theorem. *Quarterly Journal of Economics*, 52(2), 255–280.
- Guesnerie, R., 1993. An exploration of the educative justifications of the rational expectations hypothesis. *American Economic Review*, 82, 1254–1278.
- Guesnerie, R., 2002. Anchoring economic predictions in common knowledge. *Econometrica*, 70, 439–480.
- Hommes, C.H., (2009), Bounded Rationality and Learning in Complex Markets, In: *Handbook of Research on Complexity*, Edited by J. Barkley Rosser, Jr., Cheltenham: Edward Elgar, pp. 87–123.
- Hommes, C.H., Wagener, F.O.O., 2009. Complex evolutionary systems in behavioral finance. In: Hens, T., Schenk-Hoppé, R. (Eds.). *Handbook of Financial Markets: Dynamics and Evolution*. Elsevier. 217–276.
- Milgrom, P., Roberts, J., 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58, 1255–1277.
- Polya, G., Szegő, G., 1970. *Aufgaben und Lehrsätze aus der Analysis I & II*. Heidelberger Taschenbücher, Heidelberg: Springer.