Is more memory in evolutionary selection (de)stabilizing?

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Abstract

We investigate the effects of memory on the stability of evolutionary selection dynamics based on a multi-nomial logit model in an asset pricing model with heterogeneous beliefs. Whether memory is stabilizing or destabilizing depends in general on three key factors: (1) whether or not the weights on past observations are normalized; (2) the ecology of forecasting rules, in particular the average strength of trend extrapolation and the spread in biased forecasts, and (3) whether or not costs for information gathering of economic fundamentals have to be incurred.

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1 Introduction

Heterogeneous expectations models are becoming increasingly popular in various fields of economic analysis, such as exchange rate models (De Grauwe *et al.*, 1993; Da Silva, 2001; De Grauwe and Grimaldi, 2005; 2006), macro-monetary policy models (Evans and Honkapohja, 2003; Evans and McGough, 2005; Bullard *et al.*, 2008; Anufriev et al., 2009), overlapping-generations models (Duffy, 1994; Tuinstra, 2003; Tuinstra and Wagener, 2007) and models of socio-economic behaviour (Lux, 1995, Brock and Durlauf, 2001; Alfarano *et al.*, 2005). Yet the application with the most systematic and perhaps most promising heterogeneous expectations models seems to be asset price modelling. Contributions of e.g. Brock and Hommes (1998), Lux and Marchesi (1999), LeBaron (2000), Chiarella and He (2002), Brock *et al.* (2005) and Gaunersdorfer *et al.* (2008) demonstrate how a simple standard asset pricing model with heterogeneous beliefs is able to lead to complex dynamics that makes it extremely hard to predict the co-evolution of prices and forecasting strategies in asset markets. The main framework of analysis of such asset pricing models constitutes a financial market application of the evolutionary selection of expectation rules, introduced by Brock and Hommes (1997) and called the adaptive belief system (ABS). See Hommes (2006) and LeBaron (2006) for extensive reviews of agent-based models in finance; recent overviews stressing the empirical and experimental validation of agent-based models are Lux (2009) and Hommes and Wagener (2009).

An important result in asset pricing models with heterogeneous beliefs is that non-rational traders, such as technical analysts extrapolating past price trends, may survive evolutionary competition. These results contradict the hypothesis that irrational traders will be driven out of the market by rational arbitrageurs, who trade against them and earn higher profits and accumulate higher wealth (Friedman, 1953). In most asset pricing models with heterogeneous beliefs, irrational chartists can survive because evolutionary selection is driven by *short run* prof-

itability. The role of memory or time horizon in the evolutionary fitness measure underlying strategy selection has hardly been studied in the literature however.

LeBaron (2001, 2002) are among the few papers that have addressed the role of investor's time horizon in learning and strategy selection in an agent-based financial market. It has been argued that investors' time horizon is related to whether they believe that the world is stationary or non-stationary. In a stationary world agents should use all available information in learning and strategy selection, while if one views the world as constantly in a state of change, then it will be better to use time series reaching a shorter length into the past. One of LeBaron's main findings is that in a world where more agents have a long-memory horizon the volatility of asset price fluctuations is smaller. Stated differently, long-horizon investors make the market more stable, while short-horizon investors contribute to excess volatility and prevent asset prices to converge to the rational, fundamental benchmark.

Another contribution along these lines is Brock and Hommes (1999), who use a simple, tractable asset pricing model with heterogeneous beliefs to investigate the effect of memory in the fitness measure for strategy selection. In contrast to LeBaron (2001, 2002) they find that more memory in strategy selection may destabilize asset price dynamics.

Honkapohja and Mitra (2003) provide analytical results for dynamics of adaptive learning when the learning rule has finite memory. These authors focus on the case of learning a stochastic steady state. Although their work is not done in a heterogeneous agent setting, the results are interesting for our analysis. Their fundamental outcome is that the expectational stability principle, which plays a central role in stability of adaptive learning, as discussed e.g. in Evans and Honkapohja (2001), retains its importance in the analysis of incomplete learning, though it takes a new form. Their main result is that expectational stability guarantees stationary dynamics under learning with finite memory, with unbiased forecasts but higher price volatility than under complete learning with infinite memory.

Chiarella *et al.* (2006) study the effect of the time horizon in technical trading

rules upon the stability in a dynamic financial market model with fundamentalist and chartists. The chartist demand is governed by the difference between the current price and a (long-run) moving average. One of their main results is that an increase of the window length of the moving average rule can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour. The analysis of the corresponding stochastic model was able to explain various market price phenomena, including temporary bubbles, sudden market crashes, price resistance and price switching between different levels.

The aim of our paper is thus to study the role of memory or time horizon in evolutionary strategy selection in a simple, analytically tractable asset pricing model with heterogeneous beliefs. We shall analyze the effects of additional memory in the fitness measure on evolutionary adaptive systems and the consequences for survival of technical trading strategies. By complementing the stability analysis with local bifurcation theory (see Kuznetsov (2004) for an extensive mathematical treatment), we will be able to analyze the effects of adding different amounts of memory to the fitness measure on stability in a standard asset pricing model with heterogeneous beliefs.

The outline of the paper is as follows. In Chapter 2 an adaptive belief system is presented in its general form. In Chapter 3 an ABS with two types of agents and costs for information gathering is examined. In Chapter 4 we investigate the stability of the fundamental steady state in a more generalized framework without information costs. In Chapter 5 our theoretical findings with respect to memory are examined numerically in an example with three strategies. The final section concludes and proofs are collected in an appendix.

2 Adaptive Belief Systems

An adaptive belief system is a standard discounted value asset pricing model derived from mean-variance maximization with heterogeneous beliefs about future

asset prices. We shall briefly recall the model as in Brock and Hommes (1998); for a recent more detailed discussion see e.g. Hommes and Wagener (2009).

2.1 The asset pricing model

Agents can either invest in a risk free asset or in a risky asset. The risk free asset is in infinite elastic supply and pays a fixed rate of return r; the risky asset is in fixed supply z^s and pays uncertain dividend. Let p_t be the price per share of the risky asset at time t, y_t the stochastic dividend process of the risky asset and z_t be the number of shares of risky assets purchased at date t. Then wealth dynamics is given by

$$W_{t+1} = (1+r)W_t + (p_{t+1} + y_{t+1} - (1+r)p_t)\, z_t. \tag{2.1}$$

There are H different types of trading strategies. Let $\mathbb{E}_{ht}$ and $\mathbb{V}_{ht}$ denote forecasts of trader type h, with $h = 1, ..., H$, about conditional expectation and conditional variance, which is based on a publicly available information set of past prices and past dividends. Demand $z_{h,t}$ of a trader of type h for the risky asset is derived from myopic mean-variance maximization, i.e.

$$\max_{z_t} \left\{ \mathbb{E}_{ht}\left[W_{t+1}\right] - \frac{a}{2}\mathbb{V}_{ht}\left[W_{t+1}\right] \right\}, \tag{2.2}$$

where a is the risk aversion parameter. Then the demand $z_{h,t}$ is given by

$$z_{h,t} = \frac{\mathbb{E}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\mathbb{V}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}. \tag{2.3}$$

Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let $n_{h,t}$ denote the fraction of type h at date t. Then equality of the

demand and the supply in the market equilibrium implies

$$\sum_{h=1}^{H} n_{ht} \frac{\mathbb{E}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\mathbb{V}_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]} = z^s. \tag{2.4}$$

We shall assume the conditional variance $V_{h,t} = \sigma^2$ to be constant and equal for all types[1], thus the equilibrium pricing equation is given by

$$(1+r)p_t = \sum_{h=1}^{H} n_{h,t}\mathbb{E}_{h,t}[p_{t+1} + y_{t+1}] - a\sigma^2 z^s. \tag{2.5}$$

As in Brock and Hommes (1998) we focus on the case of zero outside supply, i.e. $z^s = 0$. It is well known that, if all agents are rational, the asset price is given by the discounted sum of expected future dividends

$$p_t^* = \sum_{k=1}^{\infty} \frac{\mathbb{E}_t[\mathbf{y_{t+k}}]}{(1+r)^k}. \tag{2.6}$$

The price p_t^* is called the *fundamental price*. The properties of p_t^* depend upon the stochastic dividend process y_t. We focus on the case of IID dividend process y_t with constant mean $\bar{y}$, for which the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{2.7}$$

It will be convenient to work with the deviation from the fundamental price

$$x_t = p_t - p^*. \tag{2.8}$$

Beliefs of type h satisfy the following assumptions

[B1] $\mathbb{V}_{h,t}[\mathbf{p_{t+1}} + \mathbf{y_{t+1}} - (1+r)p_t] = \sigma^2$,

[B2] $\mathbb{E}_{h,t}[\mathbf{y_{t+1}}] = \mathbb{E}_t[\mathbf{y_{t+1}}] = \bar{y}$,

[1]Gaunersdorfer (2000) investigates the case with time varying beliefs about variances and shows that the asset price dynamics are quite similar. Chiarella and He (2002, 2003) investigate the model with heterogeneous risk aversion coefficients.

[B3] $\mathbb{E}_{h,t}[\mathbf{p_{t+1}}] = \mathbb{E}_t[\mathbf{p^*_{t+1}}] + f_h(x_{t-1}, ..., x_{t-L}) = p^* + f_h(x_{t-1}, ..., x_{t-L})$.

Assumption [B1] says that beliefs about conditional variance are equal and constant for all types. According to assumption [B2] expectations about future dividends y_{t+1} are the same and correct for all trader types. According to assumption [B3], traders of type h believe that in a heterogeneous world the price may deviate from its fundamental value p^*_t by some function $f_h = f_h(x_{t-1}, ..., x_{t-L})$ of past deviations. The function f_h represents agent type h's view of the world.

Brock and Hommes (1998) investigated evolutionary competition between simple linear forecasting rules with only one lag

$$f_{h,t} = g_h x_{t-1} + b_h, \tag{2.9}$$

where g_h is the *trend* and b_h is the *bias* of trader type h. If $b_h = 0$ we call an agent h a pure *trend chaser* if $g_h > 0$ and a *contrarian* if $g_h < 0$. In the special case $g_h = 0$ and $b_h = 0$ trader of type h is a *fundamentalist*, believing that price returns to its fundamental value.

An important and convenient consequence of the assumptions [B1]-[B3] is that the heterogeneous agent market equilibrium (2.5) can be reformulated in deviations from the fundamental price. The fact that the fundamental price satisfies $(1 + r)p^* = \mathbb{E}_t[p_{t+1} + y_{t+1}]$ yields the equilibrium equation in deviations from the fundamental value

$$(1 + r)x_t = \sum_{h=1}^{H} n_{h,t} f_{h,t}. \tag{2.10}$$

2.2 Evolutionary fitness with memory

The evolutionary part of the model describes how beliefs are updated, i.e. how the fractions $n_{h,t}$ of trader types in the market evolve over time. Fractions are updated according to an *evolutionary fitness measure* $U_{h,t}$. The fractions of agents choosing

strategy h are given by the *multi-nomial logit* probabilities

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}. \tag{2.11}$$

The *intensity of choice* parameter $\beta \geq 0$ measures how sensitive the traders are to selecting the optimal prediction strategy. The extreme case $\beta = 0$ corresponds to the case where agents do not switch and all fractions are fixed and equal $1/H$. The other extreme case $\beta = \infty$ corresponds to the case where all traders immediately switch to the optimal strategy. An increase in the intensity of choice β represents an increase in the degree of rationality with respect to evolutionary selection of trading strategies. One of the main results of Brock and Hommes (1998) is that a rational route to randomness occurs, that is, as the intensity of choice increases the fundamental steady state becomes unstable and a bifurcation route to complicated, chaotic asset price fluctuations arises. The key question to be addressed in this paper is whether more memory is stabilizing or destabilizing. In particular, we are interested in the question how memory in the fitness measure affects the primary bifurcation towards instability and how it affects the rational route to randomness.

A natural candidate for evolutionary fitness is a weighted average of current realized profits π_{ht} and last period fitness $U_{h,t-1}$

$$\begin{aligned} U_{h,t} &= \gamma \pi_{h,t} + w U_{h,t-1} \\ &= \gamma \left[(p_t + y_t - R p_{t-1}) \frac{\mathbb{E}_{h,t-1}[\mathbf{p_t} + \mathbf{y_t} - R p_{t-1}]}{a\sigma^2} - C_h \right] + w U_{h,t-1}, \end{aligned} \tag{2.12}$$

where $R = 1 + r$, $C_h \geq 0$ is an average per period cost of obtaining forecasting strategy h, and $w \in [0, 1)$ is a memory parameter measuring how quickly past realized fitness is discounted for strategy selection. The parameter γ in (2.12) has been introduced to distinguish between two important cases in the literature. Brock and Hommes (1998) proposed the case $\gamma = 1$, implying that the weights given to past profits decline exponentially, more precisely realized profits $k-$periods ago get

weight w^k; Brock and Hommes (1998) however, as well as almost all subsequent literature, focus the analysis on the case without memory, i.e., $w = 0$, with fitness equal to current realized profit[2]. An advantage of this case $\gamma = 1$ is that $w = 1$ corresponds to the benchmark where fitness equals the accumulated excess profit of the risky asset over the risk free asset[3]. A disadvantage however is that for $\gamma = 1$ the weights are not normalized, but rather sum up to $1/(1-w)$. The second case studied in the literature assumes $\gamma = 1 - w$, corresponding to the case where the weights are normalized to 1. Note that for $w = 1/T$ and $\gamma = 1 - 1/T$, this case reduces to a $T-$period average (see e.g. LeBaron (2001) and Diks and van der Weide (2005)). We will refer to the case $\gamma = 1$ as *cumulative fitness* and to the case $\gamma = 1 - w$ as *normalized fitness*[4]. An important difference between these two cases is the fact that in the case with cumulative fitness, the current realized profits π_{ht} (getting weight 1) always get more weight than past fitness $U_{h,t-1}$ (which gets weight $(0 \leq w \leq 1)$, regardless of the memory level w. In contrast, in the case with *normalized fitness* high memory ($w > 0.5$) gives more weight (w) to past fitness $U_{h,t-1}$ than to current profits π_{ht} (which gets weight $1 - w$). Notice that the two different fitness measures lead to the same distribution of the relative weights over past profits. Stated differently, the relative contribution of current profits to overall fitness is the same for both fitness measures. The difference between the fitness measures however lies in the direct, absolute effect of current realized profits on strategy selection. In the case of normalized fitness, the absolute weight given to current realized profit $(1 - w)$ vanishes as memory w approaches 1. In contrast, in the case of cumulative fitness, the direct, absolute effect of current realized profits on strategy selection remains non-neglible as memory w approaches 1. As

[2]It is interesting to note that Anufriev and Hommes (2009) fit an evolutionary selection model to data from laboratory experiments and use a memory parameter $w = 0.7$.

[3]There is a large related literature on wealth-driven selection models with heterogeneous investors, with fractions of each type determined by relative wealth. See e.g. Anufriev (2008) and Anufriev and Bottazzi (2006) for some recent contributions and Chiarella et al. (2009) and Hens and Schenk-Hoppé (2009) for extensive up to date reviews.

[4]This terminology is similar to that used in the experience-weighted attraction (EWA) learning in games literature (e.g. Camerer and Ho (1999) and Camerer (2003)), where a parameter moves from 0 to 1 between the extremes of cumulative and average reinforcement.

we will see, these differences will lead to different stability results for evolutionary selection[5].

Fitness (2.12) can be rewritten in deviations from the fundamental as

$$U_{h,t} = \gamma \left[(x_t - Rx_{t-1} + \delta_t) \left(\frac{g_h x_{t-3} + b_h - Rx_{t-1}}{a\sigma^2} \right) - C_h \right] + wU_{h,t-1}, \quad (2.13)$$

where $\delta_t = p_t^* + y_t - \mathbb{E}_{t-1}[\mathbf{p_t^*} + \mathbf{y_t}]$ is a martingale difference sequence, which represents intrinsic uncertainty about economic fundamentals. The Adaptive Belief System (ABS) with linear forecasting rules, in deviations from the fundamental, is given as

$$(1+r)x_t = \sum_{h=1}^{H} n_{h,t} \left(g_i x_{t-1} + b_i \right) + \varepsilon_t, \quad (2.14)$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}, \quad (2.15)$$

$$U_{h,t} = \gamma \left[(x_t - Rx_{t-1} + \delta_t) \left(\frac{g_h x_{t-3} + b_h - Rx_{t-1}}{a\sigma^2} \right) - C_h \right] + wU_{h,t-1}, \quad (2.16)$$

where an additional noise term ε_t, e.g. representing a small fraction of noise traders, has been added to the pricing equation and will be used in some stochastic simulations below. A special case, the *deterministic skeleton*, arises when all noise terms are set to zero. In order to understand the properties of the general stochastic model it is important to understand the properties of the deterministic skeleton.

[5]The difference between cumulative fitness versus normalized fitness as expressed through the weighting coefficients $\gamma = 1$ versus $\gamma = 1 - w$ is related to the more general issue of whether one should use a normalization of the fitness measure $U_{h,t}$. An advantage of normalization is that one can compare the magnitude of the intensity of choice parameter across different fitness measures and market settings. In general it is not clear however, how exactly a fitness measure should be normalized, especially when the fitness (such as realized profits) may attain (arbitrarily large) positive as well as negative values. The normalization itself may affect e.g. the primary bifurcation towards instability. The cases $\gamma = 1$ and $\gamma = 1 - w$ of the fitness measure in (2.12) may be seen as two simple parameterizations of a cumulative and a normalized fitness measure.

3 Two types of agents and information costs

Consider an Adaptive Belief System (ABS) with two types of traders and the following forecasting rules

$$\begin{cases} f_{1,t} = g_1 x_{t-1}, & 0 \leq g_1 < 1, \\ f_{2,t} = g_2 x_{t-1}, & 1 < g_2. \end{cases} \tag{3.1}$$

Type 1 believes in *mean reversion*, that the price will converge to its fundamental value. In the special case $g_1 = 0$, type 1 becomes a pure fundamentalists, as in Brock and Hommes (1998). In contrast, type 2 believes that any price deviation from the fundamental will increase[6]. The dynamics in deviations from the fundamental is described by the following system

$$Rx_t = n_{1,t} g_1 x_{t-1} + n_{2,t} g_2 x_{t-1}, \tag{3.2}$$

$$n_{h,t} = \frac{\exp\left(\beta U_{h,t-1}\right)}{\sum_{h=1}^{2} \exp\left(\beta U_{h,t-1}\right)}, \tag{3.3}$$

$$U_{h,t-1} = \gamma \left[(x_{t-1} - Rx_{t-2}) \left(\frac{g_h x_{t-3} - Rx_{t-2}}{d} \right) - C_h \right] + wU_{h,t-2}, \tag{3.4}$$

where $C_2 = 0$, but $C_1 = C > 0$ is the information gathering costs for fundamentalists that agents of type 1 must pay per period. These costs reflect the effort investors incur to collect information about economic fundamentals[7].

[6] Boswijk et al. (2007) estimated this ABS with two types of investors using yearly S&P 500 data and found coefficients of $g_1 \approx 0.8$ and $g_2 \approx 1.15$, thus suggesting behavioral heterogeneity.

[7] In our formulation of the model in deviations from the fundamental it may seem that both predictors make use of knowledge of the fundamental. However, this example is equivalent to the case where type 1 has a mean reversion forecast towards a known fundamental, while type 2 uses a linear forecast, with trend parameter $g_2 > R$, not related to the fundamental.

We can rewrite the system above as a five-dimensional map

$$\begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ U_{1,t-2} \\ U_{2,t-2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{R}(n_{1,t}g_1 + n_{2,t}g_2)x_{t-1} \\ x_{t-1} \\ x_{t-2} \\ \gamma\pi_{1,t-1} + wU_{1,t-2} \\ \gamma\pi_{2,t-1} + wU_{2,t-2} \end{pmatrix}. \tag{3.5}$$

The following theorem describes the results concerning existence and stability of the steady states (see Appendix A for the proof).

Theorem 3.1. ***(Existence and stability of the steady states)*** *Let us denote the fundamental steady state as $x_f = 0$, and non-fundamental steady states as $x_+ = x^* > 0$ and $x_- = -x^* < 0$, where*

$$x^* = \sqrt{\frac{C - \frac{1-w}{\gamma\beta}\log(\frac{R-g_1}{g_2-R})}{(R-1)\frac{g_2-g_1}{a\sigma^2}}}. \tag{3.6}$$

Let

$$\beta^* = \frac{1-w}{C\gamma}\log\frac{R-g_1}{g_2-R}. \tag{3.7}$$

Then three cases are possible:

(i) $1 < g_2 < R$*: the fundamental steady state x_f is the unique steady state and it is globally stable;*

(ii) $R \leq g_2 < 2R - g_1$*, the system displays a pitchfork bifurcation at $\beta = \beta^*$ such that*

- *for $0 < \beta < \beta^*$ x_f is unique and stable;*
- *for $\beta > \beta^*$ there are three steady states: x_f, x_+ and x_-; the fundamental steady state x_f is unstable;*

(iii) $g_2 \geq 2R - g_1$*: there are always three steady states:* x_f, x_+ *and* x_-*; the fundamental steady state* x_f *is unstable.*

When the trend chasers extrapolate only weakly, i.e. $1 < g_2 < R$, the fundamental steady state $x_f = 0$ is globally stable. If $C = 0$ then the two types of agents are equally represented in the market, i.e. $n_1 = n_2 = 1/2$ for any value of β, because the difference in fitnesses $U_2 - U_1 = 0$ at $x = 0$. If agents on average extrapolate very strongly, i.e. $(g_1 + g_2)/2 > R$, the fundamental steady state is unstable and there are always two additional non-fundamental steady states $x = x_+ > 0$ and $x = x_- < 0$, even when there are no information costs. The case with strongly extrapolating trend chasers, i.e. $R < g_2 < 2R - g_1$, is the most interesting. If there are no information costs, $C = 0$, the fundamental steady state is stable for all values of β and agents are equally distributed over the two types due to equality of profits. But when $C > 0$ the fundamental steady state is stable only if the agents are not too sensitive to switch the prediction strategy, i.e. for $\beta < \beta^*$. As the intensity of choice increases ($\beta > \beta^*$) most of the agents switch to use the cheap prediction rule, because if the price is in a small neighborhood of its fundamental value then due to information costs the first type of agents have lower profits and for large β a majority of agents switches to the trend extrapolating strategy.

It can be seen immediately from expressions (3.6) and (3.7) how memory affects the primary bifurcation of the system. In the case with normalized fitness ($\gamma = 1 - w$) memory does *not* affect the stability. However, in the case of accumulated profits ($\gamma = 1$) and information gathering costs for fundamentalists, memory indeed affects the stability and in fact it *destabilizes* the system, i.e. with more memory the primary bifurcation occurs earlier.

3.1 Simulation 2 type example

As a typical example consider an ABS with the following two prediction rules

$$f_{1,t} = 0.5x_{t-1}, \tag{3.8}$$

$$f_{2,t} = 1.2x_{t-1}. \tag{3.9}$$

Traders of the first type believe that the next period deviation of the price from the fundamental will be two times less than in the current period, whereas traders of the second type predict an increase in deviation of the price from fundamental.

It follows from Theorem 3.1 that the fundamental steady state $x_f = 0$ is unique and stable for $\beta \in (0, \beta^*)$, with $\beta^*(w) = 1.79(1-w)/\gamma$. When the parameter β passes the critical value β^*, the fundamental steady state looses stability due to a pitchfork bifurcation and two new stable equilibria of the price dynamics appear.

Next consider the cases with two different specifications of the fitness measure: cumulative and normalized fitness.

Cumulative fitness ($\gamma = 1$). In the case with accumulated profits, i.e. when $\gamma = 1$, the pitchfork bifurcation curve is given by $\beta^*(w) = 1.79(1-w)$, which is declining with respect to the memory parameter. It means that memory destabilizes the price dynamics: the larger w the earlier the primary bifurcation occurs.

Fig. 1 illustrates the dynamics without memory ($w = 0$, left panel) and with memory ($w = 0.5$, right panel). In both cases a rational route to randomness, that is, a bifurcation route to complicated dynamics as the intensity of choice increases, occurs. Notice that, with memory in the fitness measure, the temporary bubbles and crashes in the price series occur less frequently, but when they occur they last longer with much larger deviations from the fundamental benchmark.

Normalized fitness ($\gamma = 1 - w$). In the case with normalized fitness, i.e. when $\gamma = 1-w$, the pitchfork bifurcation curve is given by $\beta^*(w) = 1.79$. Hence, memory

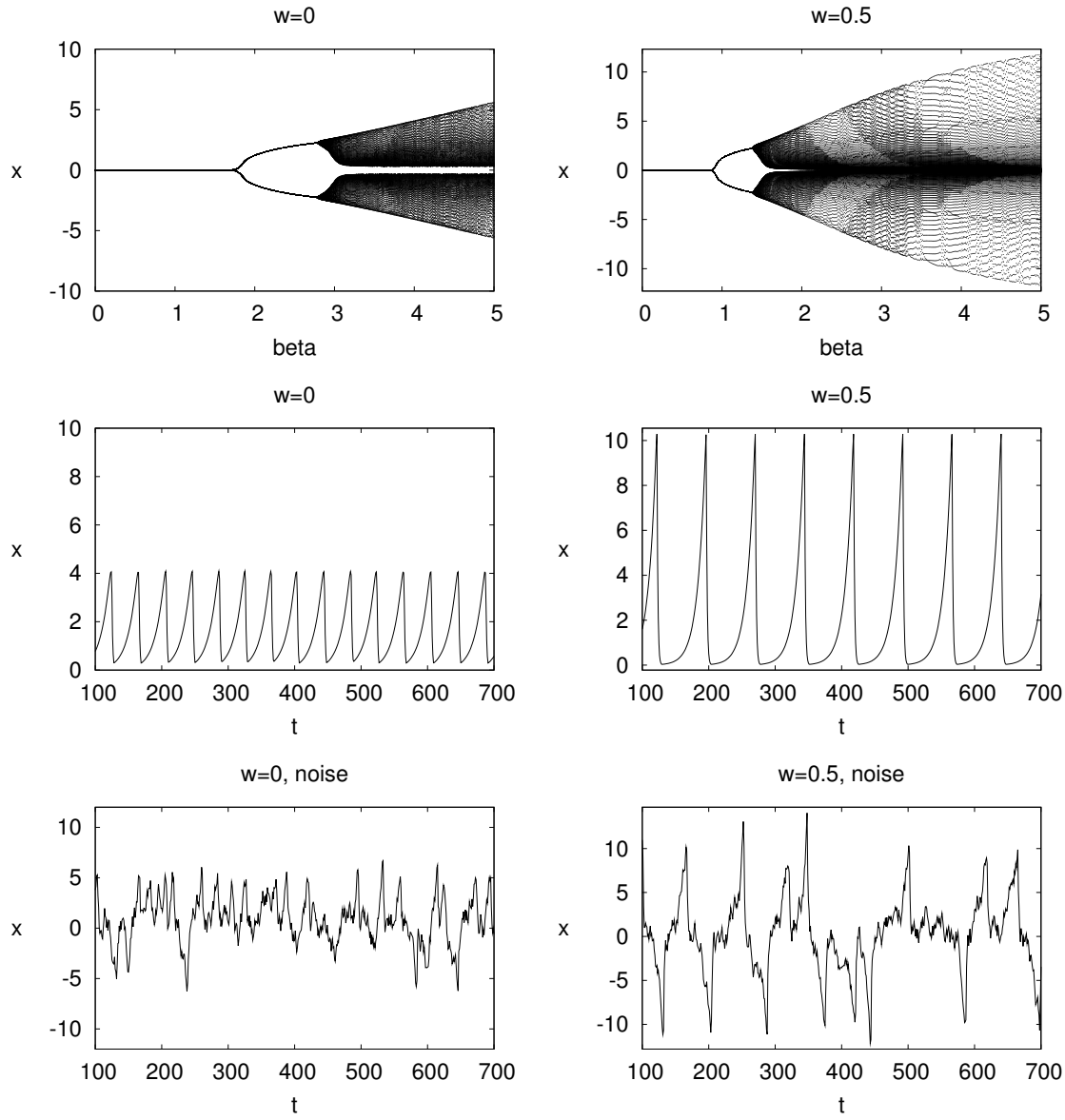

Figure 1: The case of two types of prediction rules and accumulated profits ($\gamma = 1$). The left column corresponds to $w = 0$, the right column corresponds to $w = 0.5$. Upper figures display bifurcation diagrams with respect to β. Time series of the price deviation are represented by the middle figures (without noise) and the lower figures (with noise). Belief parameters are: $g_1 = 0.5$ and $g_2 = 1.2$; the other parameters are: $\beta = 4$, $R = 1.1$, $C = 1$ and $d = 1$.

does not affect the stability of the fundamental steady state. Fig. 2 illustrates the dynamics without memory ($w = 0$, left panel) and with memory ($w = 0.8$, right panel). Although less pronounced, memory has a similar effect on price fluctuations: with memory in the fitness measure, the temporary bubbles and

crashes in the price series occur less frequently, but once started bubbles last longer with larger swings away from the fundamental benchmark.

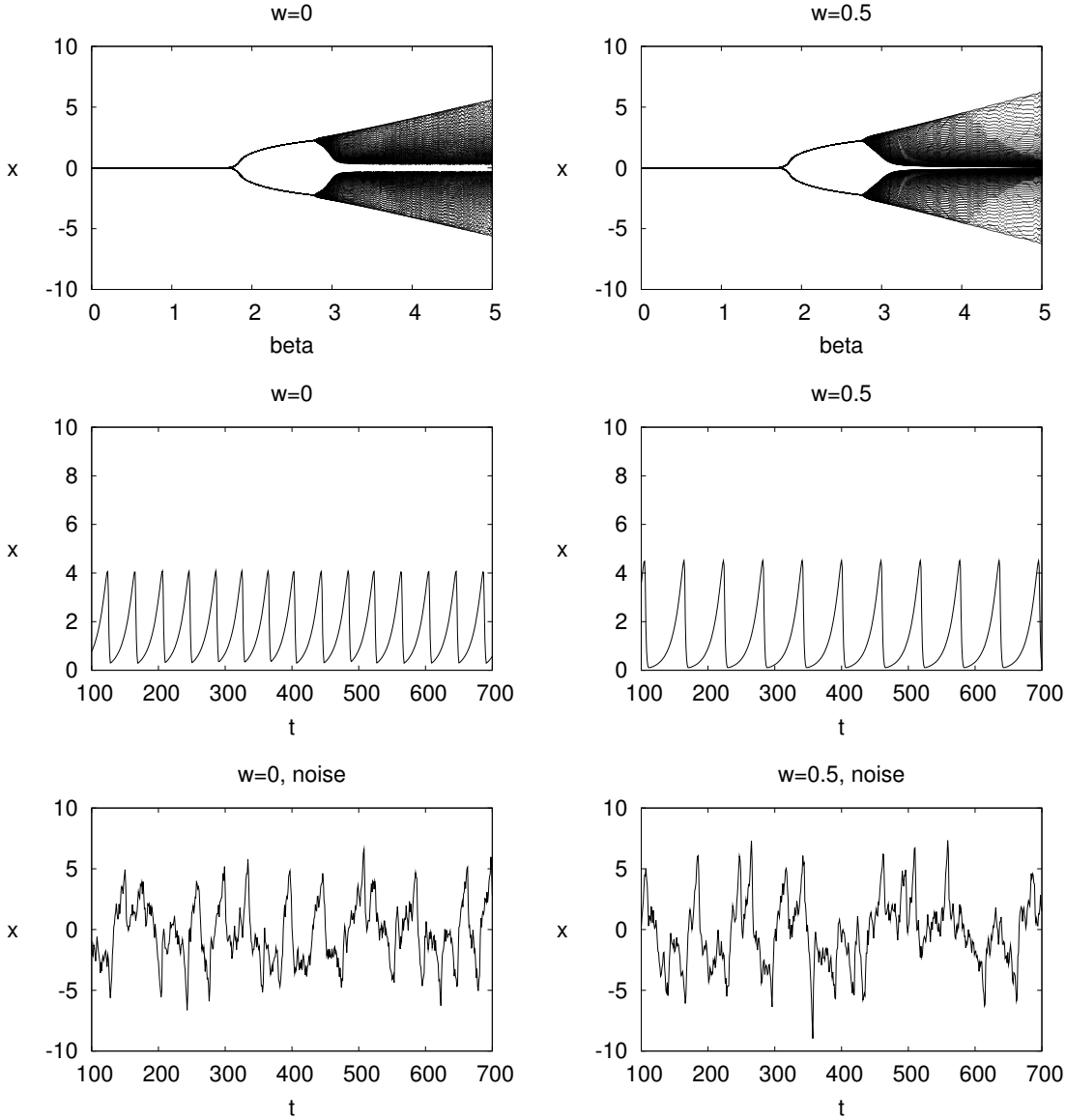

Figure 2: The normalized fitness measure case ($\gamma = 1 - w$): time series of the price deviation from its fundamental value for different levels of the memory. Belief parameters are: $g_1 = 0.5$ and $g_2 = 1.2$; the other parameters are: $\beta = 4$, $R = 1.1$, $C = 1$ and $d = 1$.

4 Stability in a more general framework

Brock and Hommes (1998) stressed the importance of *simple* forecasting rules, because it is unlikely that enough traders will coordinate on a complicated rule for it to have an impact in real markets. The learning to forecast laboratory experiments of Hommes et al. (2005) also show that simple, linear forecasting rules with only a few lags describe individual forecasting behavior surprisingly well. In this section, we investigate the role of memory in an ABS with an arbitrary number H of linear forecasting rules with one lag, i.e.

$$f_{i,t} = g_i x_{t-1} + b_i, \quad g_i, b_i \in \mathbb{R}, \quad i = 1, \ldots, H. \tag{4.1}$$

The co-evolution prices and beliefs is described by the following difference equation

$$Rx_t = \sum_{h=1}^{H} n_{h,t} \left(g_h x_{t-1} + b_h \right), \tag{4.2}$$

$$n_{h,t} = \frac{\exp\left(\beta U_{h,t-1}\right)}{\sum_{h=1}^{H} \exp\left(\beta U_{h,t-1}\right)}, \tag{4.3}$$

$$\begin{aligned} U_{h,t-1} &= \gamma \left[(x_{t-1} - Rx_{t-2}) \left(\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{d} \right) \right] + wU_{h,t-2} \\ &= \gamma \pi_{h,t} + wU_{h,t-2}. \end{aligned} \tag{4.4}$$

with $d = a\sigma^2$. Equation (4.2) can be rewritten as a $(H+3)$-dimensional map

$$\begin{pmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ U_{1,t-2} \\ \cdots \\ U_{H,t-2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{R} \sum_{h=1}^{H} n_{h,t}(g_h x_{t-1} + b_h) \\ x_{t-1} \\ x_{t-2} \\ \gamma \pi_{1,t-1} + wU_{1,t-2} \\ \cdots \\ \gamma \pi_{H,t-1} + wU_{H,t-2} \end{pmatrix}. \tag{4.5}$$

The following theorem describes the results concerning existence and stability of the fundamental steady state (see Appendix B for the proof).

Theorem 4.1. ***(Existence and stability of the fundamental steady state)*** *Assume that*

1. *The average bias equals zero, i.e.,* $\sum_{i=1}^{H} b_i = 0$*;*
2. *There is at least one non-zero bias, i.e.* $V = \frac{1}{H}\sum_{i=1}^{H} b_i^2 > 0$*;*
3. *The mean trend is not too strong, i.e.* $|\bar{g}| = |\frac{1}{H}\sum_{i=1}^{H} g_i| < R$*.*

Then the fundamental price $x_f = 0$ *is a steady state of (4.5). The fundamental steady state is stable for* $0 \leq \beta < \beta_{NS}$*, where*

$$\beta_{NS} = \frac{a\sigma^2(R - \bar{g}w)}{RV\gamma} > 0. \tag{4.6}$$

At the value $\beta = \beta_{NS}$ *the steady state loses stability due to a Neimark-Sacker bifurcation. For* $\beta > \beta_{NS}$ *the fundamental steady state is unstable*[8]*.*

The assumptions that the average bias is zero seems reasonable, as there is no a priori reason why the average bias would be negative or positive[9]. The other two assumptions, that there is at least one non-zero bias and that the average trend over all rules is not too strong, also seem plausible. The theorem says that, under these assumptions, the dynamic behavior of the price of the risky asset is independent of the number of agent's strategies, but rather depends on the mean value $\bar{g}$ of the trend extrapolating coefficients g_h and the spread V of the biases b_h. The larger the absolute average trend $|\bar{g}|$, the lower β_{NS} and the earlier the primary bifurcation occurs; if the trend chasers on average extrapolate more heavily away from the fundamentals, the system destabilizes faster. Similarly, the greater the variance V in biases, the lower β_{NS} and the bifurcation again occurs earlier; if there is more variability among biased traders, the price dynamics becomes unstable earlier. Note

[8]Note that in the special case $V = 0$ all biases equal zero, and if $|\bar{g}| < R$ the fundamental steady state is stable for all values of β and w.

[9]If the average bias is non-zero and close to 0, the fundamental price is not a steady state but the system has a steady state close to the fundamental. In that case, a stability analysis becomes much more cumbersome however.

that for the special case $\bar{g} = 0$ and $\gamma = 1$, memory does not affect the stability of the fundamental steady state, since $\beta_{NS} = a\sigma^2/V$ (*cf.* Brock and Hommes, 1998).

Role of the parameter γ. In the case $\gamma = 1$, i.e. in the case of cumulative fitness, the Neimark-Sacker bifurcation curve (4.6) becomes a straight line

$$\beta_{NS} = \frac{a\sigma^2}{V}\left(1 - \frac{\bar{g}}{R}w\right), \tag{4.7}$$

as illustrated in Figure 3 (left panel). The slope of the line depends on the sign of $\bar{g}$. If agents on average extrapolate positively, then the line is decreasing and the bifurcation w.r.t. β comes earlier with more memory. The intuition is that positive trend extrapolation reinforces market movements away from the fundamentals and the system destabilizes faster. On the other hand, if agents on average are contrarians extrapolating negatively, then (4.7) is an increasing line and the bifurcation w.r.t. β comes later with more memory. Here the intuition is that contrarian behavior counter-balances market movements away from the fundamentals and the system destabilizes slower.

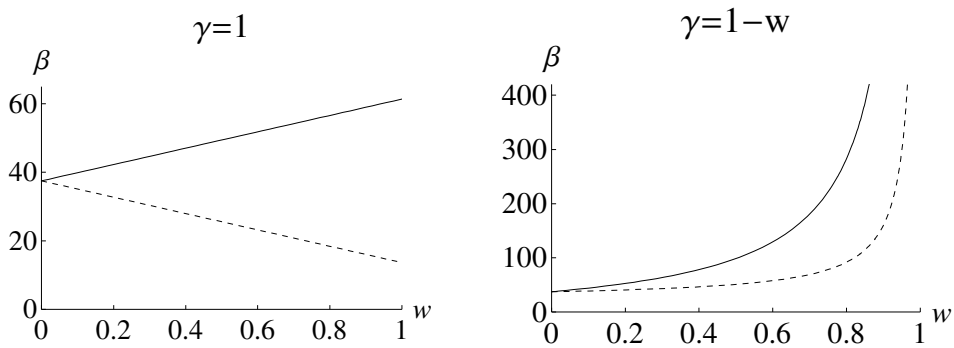

Figure 3: Neimark-Sacker bifurcation curves β_{NS} in (4.6) for different values of the parameters γ and $\bar{g}$: dotted lines correspond to the case $\bar{g} > 0$, while solid lines correspond to the case $\bar{g} < 0$. For the case with $\gamma = 1$ (left panel) the bifurcation curves are straight lines, whereas for $\gamma = 1 - w$ (right panel) they are hyperbolas. In the case $\gamma = 1$ (left panel) and $\bar{g} > 0$ memory has a destabilizing effect on the dynamics, i.e. the bifurcation w.r.t. β comes earlier. In contrast, in the case $\gamma = 1 - w$ (right panel) more memory always has a stabilizing effect.

In the case with normalized fitness, $\gamma = 1 - w$, memory is always stabilizing.

The Neimark-Sacker bifurcation curve (4.6) becomes a hyperbola for both positive and negative values of $\bar{g}$ (see Figure 3, right panel):

$$\beta_{NS} = \frac{a\sigma^2(R - \bar{g}w)}{RV(1-w)}. \tag{4.8}$$

A higher memory strength means more weight on cumulative past fitness and less weight on current realized profits. Hence, by increasing the level of memory in the system the contemporaneous destabilizing trend extrapolation is not powerful enough any more to prevail, irrespective of the direction of the average trend extrapolation $\bar{g}$, and the system stabilizes.

5 Numerical simulation of a 3-type example

In this section we discuss a simple, but typical ABS with three types of traders in order to illustrate the differences in impact of the memory strength on the stability of the fundamental price in the two cases of cumulative fitness ($\gamma = 1$) and normalized fitness ($\gamma = 1 - w$).

Consider the ABS with the following three types of prediction rules

$$f_{1,t} = 0, \tag{5.1}$$

$$f_{2,t} = 1.2x_{t-1} - 0.2, \tag{5.2}$$

$$f_{3,t} = 0.9x_{t-1} + 0.2. \tag{5.3}$$

The second and the third types are symmetrically opposite biased positive trend extrapolators, the first type are fundamentalists. The remaining parameters are fixed at: $R = 1.1$, $a\sigma^2 = 1$. Since $\bar{g} = 0.7 < R$, $V = 0.08/3 \neq 0$ and biases sum up to zero, according to Theorem 4.1, the fundamental steady state looses stability in a Neimark-Sacker bifurcation at $\beta = \beta_{NS}$,

$$\beta_{NS} = \frac{37.5 - 23.9w}{\gamma}. \tag{5.4}$$

The case $\gamma = 1$. In the case with cumulative fitness, i.e. when $\gamma = 1$, the Neimark-Sacker bifurcation curve is a declining straight line:

$$\beta_{NS} = 37.5 - 23.9w. \tag{5.5}$$

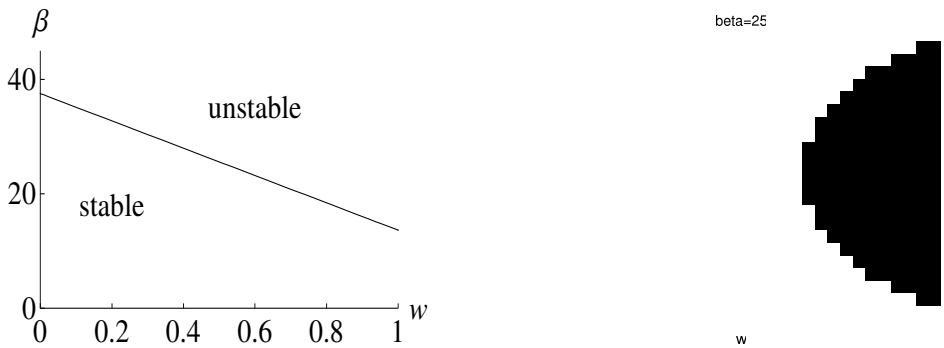

Figure 4: Neimark-Sacker bifurcation curve (left panel) and bifurcation diagram with respect to the memory parameter w (right panel) for the model with three types of agents and fitness given by accumulated profits, i.e. $\gamma = 1$. Belief parameters are: $g_1 = 0, b_1 = 0$; $g_2 = 1.1, b_2 = -0.2$; and $g_3 = 0.9, b_3 = 0.2$; other parameters are: $R = 1.1, a\sigma^2 = 1$ and $\beta = 25$ (for the right panel). The Neimark-Sacker bifurcation curve divides the $(w, \beta)-$plane into two regions; for the parameter values in the upper region the fundamental steady state is unstable, while for the parameter values in the lower region it is stable.

As can be seen from Figure 4, in this case memory destabilizes the price dynamics; with higher memory strength the bifurcation occurs earlier, i.e. for smaller values of β. Since both non-fundamentalist agents extrapolate positively, and thus the average trend extrapolation is also positive, in accordance with our findings from Section 4, the extrapolation of trend reinforces markets movements away from the fundamentals and the bifurcation line is thus decreasing. In addition, it can be observed in the bifurcation diagram of Figure 4 (right panel) how, for a fixed β-value, the fundamental steady state becomes unstable and complicated, chaotic price movements arise as the memory parameter w increases. Figure 4 (right panel) also illustrates that the amplitude of price fluctuation increases as memory increases, in accordance with our earlier finding that bubbles last longer with more memory.

The case $\gamma = 1 - w$. In the case with normalized fitness, i.e. when $\gamma = 1 - w$, the Neimark-Sacker bifurcation curve (5.4) becomes a hyperbola:

$$\beta_{NS} = \frac{37.5 - 23.9w}{1 - w}. \tag{5.6}$$

As can be seen from Figure 5 (left panel), more memory now stabilizes the price

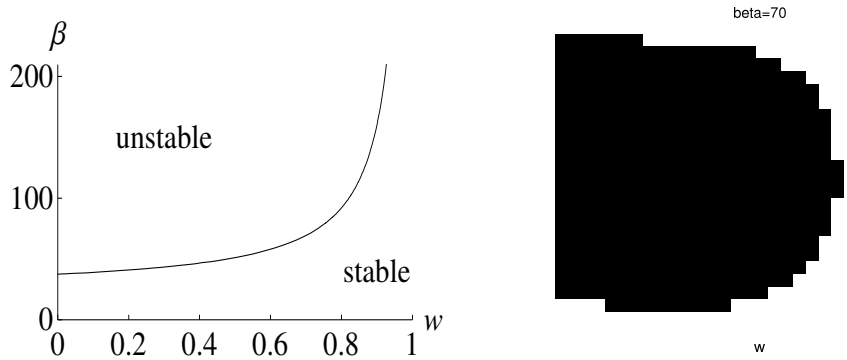

Figure 5: Neimark-Sacker bifurcation curve (left) and bifurcation diagram with respect to the memory (right) for the model with three types of agents' strategies and normalized fitness measure, i.e. $\gamma = 1 - w$. Belief parameters are: $g_1 = 0, b_1 = 0$; $g_2 = 1.1, b_2 = -0.2$; and $g_3 = 0.9, b_3 = 0.2$; other parameters are: $R = 1.1, d = 1$ and $\beta = 70$ (for the right figure). The Neimark-Sacker bifurcation curve divides the $(w, \beta)-$plane into two regions; for the parameter values in the upper region the fundamental steady state is unstable, while for the parameter values in the lower region it is stable.

dynamics; an increase in the memory strength makes the bifurcation occur later, i.e. for larger values of β. Even when the traders are on average positive trend extrapolators (with some bias), if the weight on cumulative past fitness (the memory strength w) is high enough compared to the weight on current realized profits ($\gamma = 1 - w$), the dynamics is stable. Indeed it can be observed in the bifurcation diagram in Figure 5 (right panel) that, for a given β, the dynamics stabilizes from chaotic movements (interspersed with stable cycles) for low values of the memory parameter w to a stable fundamental steady state when memory w is sufficiently large.

6 Conclusion

We have investigated how memory affects the stability of evolutionary selection dynamics in a simple, analytically tractable asset pricing model with heterogeneous beliefs. By complementing the stability analysis with local bifurcation theory, we were able to analyze the effects of adding different amounts of memory to the fitness measure on the stability of the fundamental steady state. Whether memory is stabilizing or destabilizing depends in general on three key factors: (1) whether we have a fitness measure of cumulative profits or a normalized fitness measure; (2) the ecology of forecasting rules, in particular the average strength of trend extrapolation and the spread in biased forecasts, and (3) whether or not costs for information gathering of economic fundamentals have to be incurred.

When there are costs for gathering fundamental information, more memory in the fitness measure does *not* stabilize the dynamics. In the case with normalized fitness, due to the information gathering costs, memory has no effect on stability; in the case of cumulative fitness, when there are information gathering costs for fundamentalists, more memory is destabilizing.

We have also studied the model with an arbitrary number of linear forecasting rules with one lag and no costs for information gathering. The stability depends critically on the ecology of forecasting rules. In particular, the system may become unstable more easily when the average trend parameter and or the variability of biased forecasts become larger. How memory affects the stability of the fundamental steady state depends again on whether we have cumulative fitness or normalized profits. In the case of cumulative fitness, the effect of memory on the stability further depends on the direction of average trend extrapolation. If agents on average are contrarians, extrapolating negatively, more memory stabilizes the system; if on the other hand agents on average extrapolate positively, memory destabilizes the system. In contrast, in the case with a normalized fitness measure more memory is always stabilizing.

Our theoretical results show that the stability of evolutionary selection depends critically on behavioral assumptions of how exactly agents switch between different strategies. In particular, it is critical how much weight agents put on recent realized profits compared to past accumulated profits. The more weight they put on the most recent observation, the more easily the system may destabilize. Future research with laboratory experiments with human subjects may shed light on which behavioral assumptions fit individual behavior in strategy selection more closely and, in particular, how much weight individuals put on most recent observations.

A Proof of Theorem 3.1

The steady states of the map (3.5) satisfy the following equation

$$Rx = x\left(\frac{g_1}{1+\exp(\beta\Delta)} + \frac{g_2}{1+\exp(-\beta\Delta)}\right) \tag{A.1}$$

where $\Delta = \frac{\gamma}{1-w}\left[(1-R)\left(\frac{g_2-g_1}{d}\right)x^2 + C\right]$.

It is easy to see that the fundamental steady state $x_f = 0$ always exists. The other (non-fundamental) steady state is a solution of the equation

$$\exp\left[\beta\frac{\gamma}{1-w}\left((1-R)\frac{g_2-g_1}{d}x^2 + C\right)\right] = \frac{R-g_1}{g_2-R}. \tag{A.2}$$

Note that if $(R-g_1)/(g_2-R) \leq 0$ there are no solutions for this equation. If we take into account that $g_1 < 1$ then we can conclude that for $1 < g_2 < R$ the map (3.2)-(3.4) is contracting and has a unique globally stable steady state $x_f = 0$.

Assume now that $g_2 > R$, then we can obtain non-fundamental steady states from the equation

$$x^2 = \frac{C - \frac{1-w}{\beta\gamma}\ln\frac{R-g_1}{g_2-R}}{(R-1)\frac{g_2-g_1}{d}}, \tag{A.3}$$

which has solutions $x = \pm x^*$, when its right hand side is positive. It is satisfied for

$\beta > \beta^*$ in (3.7) if $R \leq g_2 < 2R - g_1$, and for any positive β if $g_2 \geq 2R - g_1$. Now the statements about existence of equilibria in (i), (ii) and (iii) are proved.

In order to explore the stability of the fundamental steady state we need to compute eigenvalues of the Jacobian matrix

$$J(x_f) = \begin{pmatrix} \frac{g_1 + g_2 \exp(\frac{C\beta\gamma}{1-w})}{\left(1+\exp(\frac{C\beta\gamma}{1-w})\right)R} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & w \end{pmatrix}. \tag{A.4}$$

The characteristic equation is given by

$$(w-\lambda)^2\lambda^2 \left(g_1 \exp\left(\frac{-C\gamma\beta}{1-w}\right) + g_2 - R\lambda \left(1 + \exp\left(\frac{-C\gamma\beta}{1-w}\right)\right)\right) \tag{A.5}$$

and thus

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = w, \quad \lambda_5 = \frac{g_1 \exp\left(\frac{-C\gamma\beta}{1-w}\right) + g_2}{R\left(1 + \exp\left(\frac{-C\gamma\beta}{1-w}\right)\right)} > 0. \tag{A.6}$$

Note that all eigenvalues are real and non-negative, so the only bifurcation that may occur is a pitchfork bifurcation, which happens if

$$\lambda_5 = 1 \Leftrightarrow \beta = \beta^*. \tag{A.7}$$

This means that if $g_2 \in [R, 2R - g_1)$ for $\beta \in (0, \beta^*)$ there exists a unique stable fundamental steady state, and at the critical parameter value $\beta = \beta^*$ two non-fundamental steady states occur due to a pitchfork bifurcation. $\square$

B Proof of Theorem 4.1

Note that at the fundamental steady state all fitnesses are equal to zero, i.e. $U_h^* = 0$ for $h = 1, .., H$, which implies that all fraction are equal, $n_h^* = 1/H$. Therefore the steady state price satisfies the following equation

$$Rx^* = \frac{1}{H}\sum_{h=1}^{H}(g_h x^* + b_h) \tag{B.1}$$

and thus

$$x^*(R - \bar{g}) = \frac{1}{H}\sum_{h=1}^{H} b_h. \tag{B.2}$$

It is clear that the fundamental steady state exists if and only if $\sum_{h=1}^{H} b_h = 0$.

The Jacobian of (4.5) computed at the fundamental steady state is given by

$$\begin{pmatrix} \frac{d\bar{g}+V\gamma\beta}{d} & -\frac{V\gamma\beta}{d} & 0 & J_{1,1} & \cdots & & & J_{1,H} \\ 1 & 0 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & 0 & \cdots & & & 0 \\ \frac{b_1\gamma}{d} & -\frac{b_1 R\gamma}{d} & 0 & w & 0 & \cdots & & 0 \\ \frac{b_2\gamma}{d} & -\frac{b_2 R\gamma}{d} & 0 & 0 & w & 0 & \cdots & 0 \\ \vdots & \ddots & & & & & & \\ \frac{b_H\gamma}{d} & -\frac{b_H R\gamma}{d} & 0 & 0 & \cdots & & 0 & w \end{pmatrix}$$

where $d = a\sigma^2$ and

$$J_{1,s} = -\frac{b_s w\beta}{HR}, \quad s = 1, \ldots, H.$$

The characteristic equation for the fundamental steady state is given by

$$\lambda^2(w-\lambda)^{H-1}\underbrace{\left[dw\bar{g} + R\beta V\gamma + (-d(\bar{g}+Rw) - \beta V\gamma)\lambda + dR\lambda^2\right]}_{p(\lambda)} = 0. \tag{B.3}$$

The characteristic equation (B.3) has $H+3$ roots, where $H+1$ of them are inside

the unit circle; $\lambda_3 = \lambda_4 = 0$ and $\lambda_5 = \ldots = \lambda_{H+3} = w < 1$, while the other two are roots of the polynomial $p(\lambda)$ and thus they determine stability of the steady state. If $p(\lambda)$ has at least one root outside of the unit circle, the steady state is unstable. We denote roots of $p(\lambda)$ as λ_1 and λ_2.

Let us now explore three cases when one or two roots of $p(\lambda)$ are crossing a unit circle:

1. $\lambda_1 = 1$, *pitchfork bifurcation,*

$$p(1) = 9d(R - \bar{g})(1 - w) + 9V(R - 1)\gamma\beta.$$

If $V = 0$ then $p(1) > 0$ for $w \in [0, 1)$ and $|\bar{g}| < R$. If $V > 0$ then

$$p(1) = 0 \Leftrightarrow \beta = \frac{d(1 - w)(\bar{g} - R)}{V(R - 1)\gamma} < 0 \text{ for } \bar{g} < R, \tag{B.4}$$

which means that this type of bifurcation cannot occur in the system.

2. $\lambda_1 = -1$, *period doubling bifurcation,*

$$p(-1) = 9d(R + \bar{g})(1 + w) + 9V(R + 1)\gamma\beta.$$

If $V = 0$ then $p(-1) > 0$ for $w \in [0, 1)$ and $|\bar{g}| < R$. If $V > 0$ then

$$p(-1) = 0 \Leftrightarrow \beta = \beta_{PD} = -\frac{4(\bar{g} + R)(1 + w)}{V(1 + R)(1 - w)} < 0,$$

which means that this type of bifurcation can not occur in the system either.

3. $\lambda_{1,2} = \mu_1 \pm \mu_2 i$, where $\mu_2 > 0$ and $\mu_1^2 + \mu_2^2 = 1$, *Neimark-Sacker bifurcation.*

Using Vieta's Formula we get

$$\mu_1^2 + \mu_2^2 = \lambda_1\lambda_2 = \frac{d\bar{g}w + RV\beta\gamma}{dR} = 1. \tag{B.5}$$

If $V = 0$, the equation (B.5) does not have solutions for $w \in [0, 1)$ and

$|\bar{g}| < R$. Therefore all eigenvalues corresponding to the fundamental steady state are inside the unit circle and thus the steady state is stable for $w \in [0, 1)$ and $\beta \geq 0$.

If $V > 0$, we obtain from (B.5) the equation of the Neimark-Sacker bifurcation curve

$$\beta_{NS} = \frac{d(R - w\bar{g})}{RV\gamma}. \tag{B.6}$$

We have to make sure that $\mu_2 \neq 0$ or equally $\mu_2^2 > 0$. Since $\mu_1^2 + \mu_2^2 = 1$ the latter inequality holds if $\mu_1^2 < 1$. Using again the Vieta's Formula we have

$$\mu_1 = \frac{\lambda_1 + \lambda_2}{2} = \frac{d(\bar{g} + Rw) + \beta V\gamma}{2dR} > 0.$$

To make sure that $\mu_1^2 < 1$ we need to check the inequality

$$\frac{d(\bar{g} + Rw) + V\beta\gamma}{2dR} < 1.$$

Together with (B.6) it implies

$$w(R^2 - \bar{g}) < R(2R - 1 - \bar{g}), \tag{B.7}$$

which is satisfied for $|\bar{g}| < R$ and any value of $w \in [0, 1)$.

Our analysis shows that the Neimark-Sacker bifurcation is the only bifurcation that occurs in the system. It happens for $\beta = \beta_{NS}$ as in (B.6) and leads to a loss of stability of the fundamental steady state. □

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