On the Stability of the Cournot Equilibrium: An Evolutionary Approach

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Abstract

We construct an evolutionary version of Theocharis (1960)'s seminal work on the stability of equilibrium in multi-player quantity-setting oligopolies. Two sets of behavioral heuristics are investigated under fixed and endogenously evolving fractions: (myopic) Cournot firms vs. Nash firms and Cournot firms vs. rational firms. The analysis with evolutionary competition between these heuristics nests the famous Theocharis instability threshold, $n = 3$, as a special case and shows that Theocharis' result is quite robust. For evolutionary competition between the Cournot and the rational heuristic, both the *existence* and the *magnitude* of this threshold depend on the information costs associated with the rational heuristic. When information costs are positive a bifurcation route to chaos occurs as the number of firms increases. The evolutionary model therefore exhibits perpetual but bounded fluctuations, a feature not present in Theocharis' original model.

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1 Introduction

The seminal paper of Theocharis (1960) shows that, in a quantity-setting game with firms using the Cournot (1838) adjustment process,[1] the Cournot-Nash equilibrium becomes unstable as the number of firms increases.[2] In fact, with linear demand and constant marginal costs, the Cournot-Nash equilibrium loses stability and bounded but perpetual oscillations arise already for triopoly ($n = 3$). For more than three firms oscillations grow unbounded, but they are limited once the non-negativity price and demand constraints bind. This remarkable result has sparked a substantial literature on the stability of the Cournot-Nash equilibrium. Fisher (1961) and McManus (1964), for example, show that with linear demand and quadratic costs stability can be recovered when adjustment toward the best response is not instantaneous and/or marginal costs are increasing. Moreover, for the *continuous* adjustment process the Cournot-Nash equilibrium is always locally stable for this specification. Hahn (1962) extends this latter result by deriving *global* stability conditions for general demand and cost structures.[3] In Okuguchi (1970) both the actual output and expectations of rivals' output follow a continuous adjustment process. Restrictions on the speeds of adjustment are derived in relation to the rate of marginal cost increase such that stability is regained for an arbitrary number of players. In an oligopoly game where players follow a discrete-time best-response to adaptively formed expectations about rivals' output, Szidarovszky, Rassenti, and Yen (1994)

[1] Firms employ a Cournot adjustment process (or display Cournot behavior) whenever they take the *last* period's aggregate output of their rivals as a predictor for the *current* period choices of those rivals and best-respond to it. This is sometimes also referred to as best-reply dynamics, and we will use the two concepts interchangeably.

[2] Puu (2008) argues that this argument was already made, in Swedish and some 20 years before Theocharis (1960), by the Swedish economist Tord Palander (see Palander (1939)).

[3] Al-Nowaihi and Levine (1985) identify and correct an error in the proof of Hahn (1962)'s global stability result and show that it only holds for $n \leq 5$, where n is the number of firms.

find restrictions on the coefficients of adaptive expectations such that the Cournot-Nash equilibrium is stable.

As an alternative to this stream of models of learning and adaptation one can consider models of introspection. In these models firms have full knowledge of the demand function and of their own and their opponents' cost functions. Common knowledge of rationality then allows firms to derive and coordinate on the Cournot-Nash equilibrium through deductive reasoning.[4] The Cournot-Nash equilibrium is also supported by more sophisticated learning models. For instance, fictitious play converges to Nash equilibrium play for fairly general oligopolies and for an arbitrary number of players (see Thorlund-Petersen (1990)).

There has been some dissatisfaction in the literature with the assumption of Cournot behavior (i.e. the expectation that rivals will not revise their output from the last period) as a reasonable quantity-adjustment process because it is, in fact, continuously invalidated outside equilibrium (see e.g. Seade (1980), Al-Nowaihi and Levine (1985)). On the other hand, the assumption of rational play has been criticized extensively because it demands too much from the cognitive capabilities of the players. In this paper we introduce a model that presents a middle ground between adaptation and introspection.

In our evolutionary model of Cournot competition firms switch between different heuristics (or equivalently, between different expectation rules concerning aggregate output of rivals) on the basis of past performance, as in e.g. Brock and Hommes (1997) and Droste, Hommes, and Tuinstra (2002). We consider the interaction between Cournot behavior and more sophisticated, but costly, play. The latter may for example correspond to fully rational play. Thus, we relax the assumption of homogenous expectations, while preserving the Cournot adjustment dynamics of Theocharis (1960). We find that Theocharis (1960)'s classical instability is quite robust: it persists under evolutionary selection of heterogenous heuristics. The threshold number of players that triggers instability varies with the costs of the sophisticated rule and with evolutionary pressure. As the number of firms increases, a period-doubling route to

[4] However, coordination problems may emerge when the Cournot-Nash equilibrium is not unique.

chaos may arise, and the model might exhibit complicated but bounded dynamics, a feature not present in the original model of Theocharis (1960).

Section 2 briefly reviews the general n-player Cournot model with homogenous rational play along with the Theocharis (1960) instability threshold under the original Cournot adjustment process. Section 3 introduces Cournot population games with two sets of heterogenous learning rules – Cournot vs. Nash and Cournot vs. rational – with a fixed distribution of the population of firms over the different types. Switching between learning rules based on past performance is allowed in Section 4, whereas Section 5 illustrates the global dynamics of this evolutionary model for the Cournot oligopoly game with linear inverse demand – constant marginal costs (i.e. Theocharis (1960)'s specification). Analytical and numerical results are reported along with the key (in)stability thresholds of the number of players for the two sets of heuristics. Finally, we discuss our results in Section 6.

2 The Cournot model

Consider a homogeneous Cournot oligopoly model with n firms. Inverse demand is given by a twice continuously differentiable inverse demand function $P(Q)$ with $P(Q) \geq 0$ and $P'(Q) \leq 0$ for every Q. Here $Q = \sum_{i=1}^{n} q_i$ is aggregate output, with q_i production of firm i. The cost function $C(q_i)$ is twice continuously differentiable and the same for every firm. Moreover, $C(q_i) \geq 0$ and $C'(q_i) \geq 0$ for every q_i.

Each firm wants to maximize instantaneous profits $P(Q_{-i} + q_i) q_i - C(q_i)$, where $Q_{-i} = \sum_{j \neq i} q_j = Q - q_i$. This gives the first order condition:

$$P(Q_{-i} + q_i) + q_i P'(Q_{-i} + q_i) - C'(q_i) = 0, \qquad (1)$$

with second order condition for a local maximum given by $2P'(Q_{-i} + q_i) + q_i P''(Q_{-i} + q_i) - C''(q_i) \leq 0$.

The first order condition (1) implicitly defines the *best-response correspondence* or *reaction curve*:

$$q_i = R(Q_{-i}). \tag{2}$$

We assume that a symmetric Cournot-Nash equilibrium q^*, that is, the solution to $q^* = R((n-1)q^*)$, exists and is unique. Aggregate production is then given by $Q^* = nq^*$.

From the first order condition (1) we find that:

$$\frac{dq_i}{dQ_{-i}} = R'(Q_{-i}) = -\frac{P'(Q) + q_i P''(Q)}{2P'(Q) + q_i P''(Q) - C''(q_i)}. \tag{3}$$

Note that the second order condition for a local maximum implies that the denominator, evaluated at the Cournot-Nash equilibrium, is negative. Typically the numerator is also negative (although this is not necessarily the case if the inverse demand function is sufficiently convex), and therefore we generally have $R'\left(Q^*_{-i}\right) < 0$.

The key question now is: how does firm i learn to play q^* and, more specifically, what does i believe about Q_{-i} at the time when the production decision has to be made?

2.1 Rational play

One approach is to assume complete information, rational firms, and common knowledge of rationality. In that case each firm can derive the Cournot-Nash equilibrium, and firms implicitly coordinate on playing that equilibrium. Note that this implies that firms form correct expectations about opponents' choices, i.e.

$$Q^e_{-i,t} = Q_{-i,t}.$$

As an alternative to rational play, one may consider long-memory learning processes such as fictitious play, which are known to generate convergence to the Cournot-Nash equilibrium under rather general demand-cost oligopoly structures.[5] Introduced originally as an algorithm for computing Nash equilibria (see Brown (1951)), fictitious play adjustment asserts that each player best-responds to the *empirical distribution* of the opponents' past record of play. For Cournot games with a linear inverse demand function this boils down to playing against the time average of the opponents' past quantities, that is playing against

$$Q^e_{-i,t} = \frac{1}{t}\sum_{s=0}^{t-1} Q_{-i,s}.$$

We will employ Nash or "equilibrium" play as a proxy for this belief-based learning process, i.e.:

$$Q^e_{-i,t} = Q^*_{-i},$$

where Q^*_{-i} is the opponents' aggregate Nash equilibrium quantity.[6] Note that decisions under rational play may deviate from equilibrium play if not all firms are rational (see the discussion of rational play in a heterogeneous environment in Subsection 3.2).

2.2 Learning rules

In a dynamic, boundedly rational environment, one possible reading of the reaction curves (2) is that each player best-responds to *expectations* about the other players' strategic choices:

$$q_{it} = R\left(Q^e_{-i,t}\right).$$

A number of 'expectation formation' or 'learning' rules are encountered in the literature. One of them, fictituous play, was already discussed above. Another well-known classic expectation formation rule is *adaptive expectations* where current expectations about aggregate

[5] Linear two-player Cournot games (see Deschamp (1975)) and nonlinear, unimodal n-player Cournot games (see Thorlund-Petersen (1990)) display the Fictitious Play Property (FPP), i.e. fictitious play converges to the Cournot-Nash equilibrium.

[6] Note that the 'Nash firms' in Droste, Hommes, and Tuinstra (2002) correspond to rational firms in the terminology of the current paper (and not to our Nash firms, who always play the Cournot-Nash equilibrium).

output of the other firms is a weighted average of previous period's expectations and the last observed aggregate output of the other firms:

$$Q^e_{-i,t} = \alpha Q^e_{-i,t-1} + (1-\alpha)\, Q_{-i,t-1}.$$

That is, for $0 < \alpha < 1$, a firm adapts its expectations in the direction of the last observed aggregate output of the other firms.[7] For $\alpha = 0$ we obtain naive expectations:

$$Q^e_{-i,t} = Q_{-i,t-1}. \tag{4}$$

In the next subsection we will investigate the dynamics under expectation rule (4) in more detail.

2.3 The Theocharis instability threshold

If all firms use the Cournot adjustment heuristic, that is, best respond to naive expectations about aggregate output of the other $n-1$ firms, quantities evolve according to the following system of n first order difference equations

$$\begin{aligned} q_{1,t} &= R\left(q_{2,t-1} + q_{3,t-1} + \ldots + q_{n,t-1}\right), \\ q_{2,t} &= R\left(q_{1,t-1} + q_{3,t-1} + \ldots + q_{n,t-1}\right), \\ &\vdots \\ q_{n,t} &= R\left(q_{1,t-1} + q_{2,t-1} + \ldots + q_{n-1,t-1}\right). \end{aligned} \tag{5}$$

Local stability of the Cournot-Nash equilibrium is determined by the eigenvalues of the Jacobian matrix of (5), evaluated at that Cournot-Nash equilibrium. This Jacobian matrix is

[7] See Fisher (1961) for an adaptive *behavior* process, where the *quantity* produced is adapted in the direction of the best response to previous period's aggregate output, that is, $q_t = \alpha q_{t-1} + (1-\alpha)\, R\left(Q_{-i,t-1}\right)$.

given by

$$\begin{pmatrix} 0 & R'\left(Q^*_{-1}\right) & \cdots & R'\left(Q^*_{-1}\right) \\ R'\left(Q^*_{-2}\right) & 0 & & \vdots \\ \vdots & & \ddots & R'\left(Q^*_{-(n-1)}\right) \\ R'\left(Q^*_{-n}\right) & \cdots & R'\left(Q^*_{-n}\right) & 0 \end{pmatrix}. \tag{6}$$

Note that, because no firm responds to its own previous decision, all diagonal elements are 0, and that the off-diagonal elements in row i are all equal to $R'\left(Q^*_{-i}\right)$, since individual production levels only enter through aggregate production of the other firms. Moreover, at the symmetric Cournot-Nash equilibrium we have $Q^*_{-i} = (n-1)\,q^*$ for all $i = 1, \ldots, n$, and therefore all off-diagonal elements of (6) are the same and equal to $R'\left((n-1)\,q^*\right)$. The Jacobian matrix (6) then has $n-1$ eigenvalues equal to $-R'\left((n-1)\,q^*\right)$ and one eigenvalue equal to $(n-1)\,R'\left((n-1)\,q^*\right)$ which, in absolute value, is the largest one. From this it follows immediately that the symmetric Cournot-Nash equilibrium is stable whenever

$$\xi_n\left(q^*\right) \equiv (n-1)\left|R'\left((n-1)\,q^*\right)\right| < 1. \tag{7}$$

It is also illustrative to consider what happens when the initial quantities are symmetric, that is $q_{i,0} = q_0$ for all i. In that case quantities will be symmetric, $q_{it} = q_t = \frac{Q_t}{n}$, for every future period t. The dynamics then reduce to the following single difference equation[8]

$$Q_t = \sum_{i=1}^{n} R\left(Q_{-i,t-1}\right) = \sum_{i-1}^{n} R\left(\frac{n-1}{n} Q_{t-1}\right) = nR\left(\frac{n-1}{n} Q_{t-1}\right),$$

for which the fixed point Q^* is obviously stable when (7) holds.

One would expect that $\xi_n\left(q^*\right) = (n-1)\left|R'\left((n-1)\,q^*\right)\right|$ increases with n and that therefore the Cournot-Nash equilibrium always becomes unstable for n large enough. However, since in general q^* depends on n, in principle a market structure may exist such that the term $\left|R'\left((n-1)\,q^*\right)\right|$ decreases in n *faster* than $\frac{1}{n}$, which could make $\xi_n\left(q^*\right)$ a decreasing function

[8] This could alternatively be expressed as $q_t = R\left((n-1)\,q_{t-1}\right)$.

of n. This seems to be unlikely, however, and to the best of our knowledge a market structure with that property has not been considered in the literature.[9] From now on, therefore, we assume that $\xi_n(q^*)$ increases monotonically in n and a threshold value $\hat{n}$ of n exists such that $\xi_{\hat{n}-1}(q^*) < 1 \leq \xi_{\hat{n}}(q^*)$.

Leading Example *As a leading example we consider the original specification of Theocharis (1960), where inverse demand is linear, $P(Q) = a - bQ$, and marginal costs are constant: $C_i(q_i) = cq_i$, for all i, with $a > c \geq 0$ and $b > 0$. The unique and symmetric Cournot-Nash equilibrium is given by $q^* = \frac{a-c}{b(n+1)}$, with aggregate output given by $Q^* = \frac{n}{n+1}\frac{a-c}{b}$. The reaction function – abstracting from non-negativity constraints for the moment – equals:*

$$R(Q_{-i}) = \frac{a-c}{2b} - \frac{1}{2}Q_{-i} = q^* - \frac{1}{2}(Q_{-i} - (n-1)q^*). \tag{8}$$

We therefore obtain $R'(Q_{-i}) = -\frac{1}{2}$. That is, an increase in aggregate output of the other firms by one unit would lead to a decrease in output of firm i by $\frac{1}{2}$.[10] *Also note that if the other firms produce on average more than the Cournot-Nash equilibrium quantity, firm i reacts by producing less than that quantity (and the other way around). From stability condition (7) it follows immediately that the Cournot-Nash equilibrium is stable under Cournot behavior or best-response dynamics for this specification only when $n = 2$, and that it is unstable with exploding fluctuations for $n > 3$ (and neutrally stable, resulting in bounded oscillations, for $n = 3$). The reason for this instability is 'overshooting': if, for example, aggregrate output is above the Cournot-Nash equilibrium quantity, firms*

[9] For the specification of Theocharis (1960), with linear inverse demand function and constant marginal costs the reaction curve is linear with a constant slope that is independent of n (see our Leading Example below). For an iso-elastic inverse demand function and constant marginal costs the slope of the reaction curve, evaluated at the Cournot-Nash equilibrium does depend upon n. In this case $\xi_n(q^*) = \frac{1}{2}(n-2)$ and the Cournot-Nash equilibrium is neutrally stable under the Cournot adjustment heuristic for $n = 4$ and unstable for $n > 5$ (see Ahmed and Agiza (1998) and Puu (2008)). Puu (2008) provides an example for which the Cournot dynamics do remain stable when n increases, but he assumes that the cost function of each firm depends directly upon the number of firms n, reducing capacity of an individual firm and increasing its marginal costs, when the number of firms increases.

[10] In general, the decrease in firm i's output might even be *larger* than the aggregate increase in other firms outputs, leading to an aggregate decrease in production. This happens when $R'(Q_{-i}) < -1$ or, as can be easily checked, when $C''(q_i) > P'(Q)$, that is, when marginal costs are decreasing faster than price (for an example, see e.g. Droste, Hommes, and Tuinstra (2002)).

respond by reducing output. For $n > 3$ the aggregate reduction in output is so large that the resulting deviation of aggregate output from the equilibrium quantity is larger than before, and so on.

3 Heterogeneity in behaviour in Cournot oligopolies

In this and the following sections we study the Cournot model as a population game, in order to facilitate studying the aggregate behavior of a heterogeneous set of interacting quantity-setting heuristics. We consider a large population of firms from which in each period groups of n firms are sampled randomly and matched to play the one-shot n-player Cournot game. We assume that a fixed fraction $\rho \in [0, 1]$ of the population uses one heuristic, and a fraction $1 - \rho$ uses another. After each one-shot Cournot game, the random matching procedure is repeated, leading to new combinations of the two types of firms. The distribution of possible samples follows a binomial distribution with parameters n and ρ. The assumption of fixed ρ will be relaxed in Section 4. Below we will discuss two stylized examples with heterogenous heuristics:[11] Cournot vs. Nash firms and Cournot vs. rational firms.

3.1 Cournot firms vs. Nash firms

Suppose that a fraction ρ of the population of firms produces the Nash equilibrium quantity q^* and a fraction $1 - \rho$ observes the population-wide average quantity $\overline{q}_{t-1}$ produced in the previous period and best responds to it, $q_t = R\left((n-1)\,\overline{q}_{t-1}\right)$, where q_t is the quantity produced by each Cournot firm in period t. Making use of the law of large numbers, the average quantity played in the previous period can be expressed as:

$$\overline{q}_{t-1} = \rho q^* + (1-\rho)\, q_{t-1}.$$

[11] See Ochea (2010) for more examples, including simulations, for the duopoly model with Cournot vs. fictitious play, with similar qualitative results.

Therefore we obtain:

$$q_t = R\left((n-1)\left(\rho q^* + (1-\rho)\, q_{t-1}\right)\right). \tag{9}$$

Note that $q_t = q^*$ is a fixed point of this difference equation. The following result follows from straightforward differentiation of (9).

Proposition 1 *The Cournot-Nash equilibrium q^* is a locally stable fixed point of the model with exogenous fractions of Nash and Cournot firms if and only if*

$$(n-1)(1-\rho)\left|R'\left((n-1)\, q^*\right)\right| < 1, \tag{10}$$

where ρ is the fraction of Nash firms.

In general an increase in ρ pulls the average output $\overline{q}_{t-1}$ towards the Cournot-Nash equilibrium quantity q^* and thereby weakens the tendency of overshooting which causes instability in the original model of Theocharis (1960). An increase in ρ will therefore tend to stabilize the dynamics around q^*. Note that, obviously, for $\rho = 0$ condition (10) reduces to condition (7).

Leading Example (continued) *For the linear inverse demand – constant marginal cost model, which has $R'\left((n-1)\, q^*\right) = -\frac{1}{2}$, condition (10) reduces to $(n-1)(1-\rho) < 2$. From this it follows that for any value of $\rho < 1$ there exists a market size n^N such that the dynamics become unstable whenever $n > n^N$. This critical threshold value of n is given by*

$$n^N = \frac{3-\rho}{1-\rho}. \tag{11}$$

For example, if the population of firms is evenly spread between Nash and Cournot firms, that is $\rho = \frac{1}{2}$, we obtain $n < 5$ as stability condition (and of course we obtain $n < 3$ for $\rho = 0$).

3.2 Cournot firms vs. rational firms

Now consider the setting where the polulation consists of rational firms and Cournot firms, where the fraction of rational firms is equal to ρ. A fully rational firm is assumed to know the fraction of Cournot firms in the population. Moreover, it knows exactly how much these Cournot firms will produce. However, we assume that it does not know the composition of firms in its market (or has to make a production decision before observing this). The rational firm therefore maximizes expected profits, that is, it solves[12]

$$\max_{q_i} E\left[P\left(Q_{-i}+q_i\right)q_i - C\left(q_i\right)\right].$$

It forms expectations over all possible mixtures of heuristics resulting from independently drawing $n-1$ other players from a large population, each of which is either a Cournot or a rational firm. Rational firm i therefore chooses quantity q_i such that the objective function

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k\left(1-\rho\right)^{n-1-k}\left[P\left(\left(n-1-k\right)q_t + kq^r + q_i\right)q_i - C\left(q_i\right)\right],$$

is maximized. Here q^r is the (symmetric) output level of each of the other rational firms, and q_t is the output level of each of the Cournot firms. The first order condition for an optimum is characterized by equality between marginal cost and *expected* marginal revenue. Typically, marginal revenue in the realized market will differ from marginal costs.

Given the value of q_t, all rational firms coordinate on the same output level q^r. This gives equilibrium condition

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k\left(1-\rho\right)^{n-1-k}\times$$
$$\left[P\left(\left(n-1-k\right)q_t+\left(k+1\right)q^r\right)+q^rP'\left(\left(n-1-k\right)q_t+\left(k+1\right)q^r\right)-C'\left(q^r\right)\right]=0. \qquad (12)$$

[12]Note that the solution to this problem is typically different from the solution to $\max_{q_i}\left[P\left(Q^e_{-i}+q_i\right)q_i - C\left(q_i\right)\right]$, unless $P\left(\cdot\right)$ is a linear function.

Let the solution to (12) be given by $q^r = H(q_t, \rho)$. It is easily checked that if the Cournot firms play the Cournot-Nash equilibrium quantity q^*, or if all firms are rational, then rational firms will play the Cournot-Nash equilibrium quantity, that is $H(q^*, \rho) = q^*$, for all ρ and $H(q_t, 1) = q^*$ for all q_t. Moreover, if a rational firm is certain it will only meet Cournot firms, that is $\rho = 0$, it plays a best response to *current* aggregate output of these Cournot firms, that is $H(q_t, 0) = R((n-1) q_t)$, for all q_t. In the remainder we will denote by $H_q(q, \rho)$ and $H_\rho(q, \rho)$ the partial derivatives of $H(q, \rho)$ with respect to q and ρ, respectively.

Cournot firms play a best-response to the *average* quantity played at time $t-1$, where a rational firm's choice at time $t-1$ is $H(q_{t-1}, \rho)$. Cournot firms therefore produce

$$q_t = R\left((n-1)\left(\rho H(q_{t-1}, \rho) + (1-\rho) q_{t-1}\right)\right), \tag{13}$$

whereas the output of a rational firm in period t is given by $q_t^r = H(q_t, \rho)$. We have the following result.

Proposition 2 *The Cournot-Nash equilibrium is a locally stable fixed point for the model with exogenous fractions of rational and Cournot firms if and only if*

$$|H_q(q^*, \rho)| < 1, \tag{14}$$

where ρ is the fraction of rational firms.

Proof. Straightforward differentiation of (13) gives stability condition

$$\left|(n-1)\left(\rho H_q(q^*, \rho) + (1-\rho)\right) R'\left((n-1) q^*\right)\right| < 1. \tag{15}$$

In order to determine $H_q(q^*, \rho)$ we totally differentiate the first order condition (12), which gives

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} \left[(n-1-k)\left(P'(Q^*) + q^* P''(Q^*)\right)\right] dq_t +$$
$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} \left[(k+1)\left(P'(Q^*) + q^* P'(Q^*)\right) + P'(Q^*) - C''(q^*)\right] dq^r = 0.$$

Using $\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} = 1$ and $\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} k = \rho(n-1)$, and rearranging we find that

$$\frac{dq^r}{dq_t} = H_q(q^*, \rho) = -\frac{(1-\rho)(n-1)\left(P'(Q^*) + q^* P''(Q^*)\right)}{\rho(n-1)\left(P'(Q^*) + q^* P'(Q^*)\right) + 2P'(Q^*) + q^* P'(Q^*) - C''(q^*)}. \quad (16)$$

Substituting (16) into stability condition (15) and rearranging gives

$$(n-1)\left(\rho H_q(q^*, \rho) + (1-\rho)\right) R'\left((n-1) q^*\right) =$$
$$\frac{(1-\rho)(n-1)\left(P'(Q) + q^* P''(Q)\right)}{\rho(n-1)\left(P'(Q^*) + q^* P''(Q^*)\right) + 2P'(Q^*) + q^* P''(Q^*) - C''(q^*)} = H_q(q^*, \rho),$$

from which stability condition (14) follows immediately. ■

The following result shows that, under the familiar and relatively innocuous assumption that inverse demand is "not too convex", the Cournot-Nash equilibrium will be locally stable when at least half of the firms is rational.

Corollary 3 *Let inverse demand satisfy $P'(Q^*) + q^* P''(Q^*) < 0$. Then we have*

$$(n-1) R'\left((n-1) q^*\right) \le H_q(q^*, \rho) \le 0,$$

for all $\rho \in [0,1]$. Moreover, if there are at least as many rational firms as Cournot firms, $\rho \ge \frac{1}{2}$, the Cournot-Nash equilibrium is locally stable for model (13).

Proof. From the proof of Proposition 2 we obtain

$$H_q(q^*,\rho) = -\frac{(1-\rho)(n-1)(P'(Q^*)+q^*P''(Q^*))}{\rho(n-1)(P'(Q^*)+q^*P'(Q^*))+2P'(Q^*)+q^*P'(Q^*)-C''(q^*)}.$$

By assumption $P'(Q^*)+q^*P''(Q^*)<0$, and by the second order condition for a (local) maximum, $2P'(Q^*)+q^*P'(Q^*)-C''(q^*)\leq 0$, it follows that $H_q(q^*,\rho)\leq 0$. Furthermore, $H_q(q^*,\rho)$ is increasing in ρ, taking its lowest value at $\rho=0$, which gives (see (3)) $H_q(q^*,0) = (n-1)R'((n-1)q^*)$.

Moreover, since $2P'(Q^*)+q^*P''(Q^*)-C''(q^*)\leq 0$, the numerator is certainly larger (in absolute value) than the denominator when $\rho\geq\frac{1}{2}$, implying that the Cournot-Nash equilibrium is locally stable for those values of ρ. ■

Rational firms respond to Cournot firms by selecting a high (low) production level when the Cournot firms choose a low (high) production level q_t in that period, taking into account that there are other rational firms – typically their output level will be somewhere between q^* and $R((n-1)q_t)$. Rational firms therefore partly neutralize the deviation of Cournot firms from the Cournot-Nash equilibrium quantity. Hence, their stabilizing effect on the dynamics is stronger than that of Nash firms. However, if the number of rational firms is sufficiently small, the Cournot-Nash equilibrium may still be unstable, as the following example illustrates.

Leading Example (continued) *Substituting the linear inverse demand function and linear cost function into first order condition (12) and solving for q^r we find*

$$q_t^r = H(q_t,\rho) = \frac{a-c-b(n-1)(1-\rho)q_t}{b(2+(n-1)\rho)} = q^* - \frac{(1-\rho)(n-1)}{2+(n-1)\rho}(q_t-q^*). \qquad (17)$$

Note that for $\rho\in(0,1)$ and $q_t\neq q^*$ either $R((n-1)q_t) < H(q_t,\rho) < q^* < q_t$ or

$R((n-1)q_t) > H(q_t,\rho) > q^* > q_t$. Moreover, it is straightforward to check that

$$\begin{aligned} q_t &= R\left((n-1)\left(\rho H\left(q_{t-1},\rho\right)+(1-\rho)\, q_{t-1}\right)\right) \\ &= q^* - \frac{(1-\rho)(n-1)}{2+(n-1)\rho}\left(q_{t-1}-q^*\right) \\ &= H\left(q_{t-1},\rho\right) = q_{t-1}^r, \end{aligned}$$

that is – in the special case where inverse demand is linear – the Cournot firms produce the same quantity in period t as the rational firms did one period earlier. This clearly illustrates the advantage the latter have over the former.

Stability condition (14) reduces to $(1-2\rho)(n-1) < 2$. Clearly, for $\rho \geq \frac{1}{2}$ this stability condition is always satisfied. For $\rho < \frac{1}{2}$, however, the Cournot-Nash equilibrium becomes unstable for a high enough market size n. In fact, for $\rho < \frac{1}{2}$ and $n > n^R$ with

$$n^R = \frac{3-2\rho}{1-2\rho}, \tag{18}$$

the Cournot-Nash equilibrium is unstable (but comparing (18) with (11) reveals that, for equal ρ, $n^R > n^N$, that is, rational firms present a stronger stabilizing force than Nash firms).

4 Evolutionary competition between heuristics

In this section we develop an evolutionary version of the model outlined in Section 3. As before, in every period t, groups of n firms are drawn from a large population to play the n-player Cournot game, where each firm uses one of two heuristics. In contrast to the previous section we now assume that firms may switch between different heuristics according to a general, monotone selection dynamic, capturing the idea that heuristics that perform relatively better are more likely to spread through the population.[13]

Before we can specify this evolutionary updating we need to determine profits generated by each of the two available heuristics, taking into account that a firm using one heuristic can be matched up with either 0, 1 up to $n-1$ firms using that same heuristic, with the remaining firms using the alternative heuristic. That is, if we let q_1 and q_2 be the two quantities prescribed by heuristics 1 and 2, and if we let ρ be the fraction of firms using heuristic 1, expected profits Π_1 for heuristic 1 are given by

$$\Pi_1 = F(q_1, q_2, \rho) = \sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} \left[P\left((k+1) q_1 + (n-1-k) q_2\right) q_1 - C(q_1) \right], \tag{19}$$

with expected profits for heuristic 2 given by $\Pi_2 = F(q_2, q_1, 1-\rho)$. If the population of firms and the number of groups of n firms drawn from that population are large enough average profits will be approximated quite well by these expected profits, which we will use as a proxy for average profits from now on.

There might be a substantial difference in sophistication between different heuristics. Consequently, some heuristics may require more information or effort to implement than others. Therefore we allow for the possibility that heuristics involve an *information* or *deliberation* cost, $\kappa_i \geq 0$, that may differ across heuristics. Performance of a heuristic is then given by the difference between average profits generated in the market game and these information costs, $V_i = \Pi_i - \kappa_i$.

We are now ready to introduce evolutionary updating. Let the fraction of firms using the first heuristic be given by ρ_t in period t. This fraction evolves endogenously according to an evolutionary dynamic which is an increasing function of the performance differential between

[13] One such an updating mechanism was investigated in Droste, Hommes, and Tuinstra (2002) for an evolutionary competition between the Cournot and rational heuristics in a Cournot duopoly game and extended in Ochea (2010) to a wide selection of learning heuristics.

the two heuristics, that is

$$\rho_t = G\left(V_{1,t-1} - V_{2,t-1}\right) = G\left(\Pi_{1,t-1} - \Pi_{2,t-1} - \kappa\right).$$

Here $\kappa \equiv \kappa_1 - \kappa_2 \geq 0$ is the difference in information or deliberation costs, which we assume to be nonnegative without loss of generality.[14] The map $G : \mathbb{R} \to [0,1]$ is a continuously differentiable, monotonically increasing function with $G(0) = \frac{1}{2}$, $G(x) + G(-x) = 1$, that is, it is symmetric around $x = 0$, $\lim_{x\to-\infty} G(x) = 0$ and $\lim_{x\to\infty} G(x) = 1$. Note that it is straightforward to generalize this approach to allow for more heuristics, or to let evolution depend upon performance of the heuristics from earlier periods.

In the next two subsections we will discuss the evolutionary versions of the two scenarios from Section 3: Cournot firms versus Nash firms and Cournot firms versus rational firms, respectively.

4.1 Cournot firms vs. Nash firms

Let the (time-varying) fraction of Nash firms in period t be given by ρ_t. As discussed in Section 3 Nash firms choose the Cournot-Nash equilibrium quantity and Cournot firms best-respond to the population-wide average quantity played in the previous period. The quantity dynamics of the model can therefore be represented by (see equation (9)):

$$q_t = R\left((n-1)\left(\rho_{t-1} q^* + \left(1 - \rho_{t-1}\right) q_{t-1}\right)\right). \tag{20}$$

Let $\Pi_{N,t} = \Pi_N\left(q_t, \rho_t\right) = F\left(q^*, q_t, \rho_t\right)$ and $\Pi_{C,t} = \Pi_C\left(q_t, \rho_t\right) = F\left(q_t, q^*, 1 - \rho_t\right)$, where $F(\cdot)$ is given by equation (19), denote the average payoffs accruing to the Nash and Cournot firms in period t, respectively. The fraction of Nash firms ρ_t then evolves endogenously according

[14] Note that κ does not necessarily have to represent the difference in information or deliberation costs; it could also present some predisposition or bias towards one of the heuristics.

to

$$\rho_t = G\left(\Pi_{N,t-1} - \Pi_{C,t-1} - \kappa\right). \tag{21}$$

Given that at the Cournot-Nash equilibrium both types of firms choose the same quantity q^* we necessarily have that $\Pi_N^* = \Pi_C^*$. Let $\rho_\kappa = G(-\kappa)$ denote the equilibrium steady state fraction of Nash firms. Recall that, in absence of information costs, the population of firms spreads evenly over the two heuristics, that is: $\rho_0 = G(0) = \frac{1}{2}$.

Proposition 4 *The equilibrium (q^*, ρ_κ) of the model with evolutionary competition between the Cournot heuristic and the Nash heuristic, given by equations (20) and (21), is locally stable if and only if the following condition holds:*

$$\left|(n-1)(1-\rho_\kappa) R'((n-1)q^*)\right| < 1. \tag{22}$$

Proof. The dynamical system, consisting of equations (20) and (21), can be written as

$$\begin{aligned} q_t &= \Psi_1\left(q_{t-1}, \rho_{t-1}\right) \equiv R\left((n-1)\left(\rho_{t-1}q^* + \left(1-\rho_{t-1}\right)q_{t-1}\right)\right), \\ \rho_t &= \Psi_2\left(q_{t-1}, \rho_{t-1}\right) \equiv G\left(\Pi_N\left(q_{t-1}, \rho_{t-1}\right) - \Pi_C\left(q_{t-1}, \rho_{t-1}\right) - \kappa\right). \end{aligned} \tag{23}$$

To evaluate local stability of the equilibrium (q^*, ρ_κ) we need to determine the eigenvalues of the Jacobian matrix, evaluated at that equilibrium, of system (23).

In order to do so, first note that

$$\frac{\partial \Psi_1}{\partial \rho_{t-1}} = (n-1)\left(q^* - q_{t-1}\right) R'\left((n-1)\left(\rho_{t-1}q^* + \left(1-\rho_{t-1}\right)q_{t-1}\right)\right),$$

which, when evaluated at (q^*, ρ_κ), equals zero because in equilibrium $q_{t-1} = q^*$.

Second, observe that $\Pi_{N,t-1} - \Pi_{C,t-1}$ can be written as

$$\triangle\Pi_N = \Pi_{N,t-1} - \Pi_{C,t-1} = \sum_{k=0}^{n-1} A_k\left(\rho_{t-1}\right) B_k\left(q_{t-1}, q^*\right),$$

where $A_k\left(\rho_{t-1}\right) = \binom{n-1}{k}\rho_{t-1}^k\left(1-\rho_{t-1}\right)^{n-1-k}$ is independent of q_{t-1} and

$$B_k\left(q_{t-1}, q^*\right) = P\left((n-1-k)\, q_t + (k+1)\, q^*\right) q^* - C\left(q^*\right) - \left[P\left((n-k)\, q_t + kq^*\right) q_t - C\left(q_t\right)\right]$$

is independent of ρ_{t-1}. We then have

$$\frac{\partial \triangle \Pi_N}{\partial q_{t-1}} = \sum_{k=0}^{n-1} A_k\left(\rho_{t-1}\right) \frac{\partial B_k\left(q_{t-1}, q^*\right)}{\partial q_{t-1}} \text{ and } \frac{\partial \triangle \Pi_N}{\partial q_{t-1}} = \sum_{k=0}^{n-1} \frac{\partial A_k\left(\rho_{t-1}\right)}{\partial \rho_{t-1}} B_k\left(q_{t-1}, q^*\right).$$

It follows that $B_k\left(q^*, q^*\right) = 0$ and $\left.\frac{\partial B_k(q_{t-1},q^*)}{\partial q_{t-1}}\right|_{q^*} = -q^* P'\left(Q^*\right) - P\left(Q^*\right) + C'\left(q^*\right)$ which equals 0 by the first order condition for a Cournot-Nash equilibrium. Evaluated at the equilibrium (q^*, ρ_κ), we therefore have $\frac{\partial \triangle\Pi}{\partial q_{t-1}} = \frac{\partial\triangle\Pi}{\partial \rho_{t-1}} = 0$ and this gives

$$\left.\frac{\partial\Psi_2}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = G'\left(-\kappa\right) \left.\frac{\partial \triangle \Pi_N}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = 0 \text{ and } \left.\frac{\partial\Psi_2}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)} = G'\left(-\kappa\right) \left.\frac{\partial \triangle \Pi_N}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)} = 0.$$

Summarizing the above results, it follows that the Jacobian matrix of dynamical system (23), evaluated at the equilibrium (q^*, ρ_κ), is given by

$$\begin{pmatrix} \left.\frac{\partial\Psi_1}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} & 0 \\ 0 & 0 \end{pmatrix}.$$

This Jacobian matrix has eigenvalues $\lambda_1 = \left.\frac{\partial\Psi_1}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = (n-1)\left(1-\rho_\kappa\right) R'\left((n-1)\, q^*\right)$ and $\lambda_2 = 0$, and the equilibrium is therefore locally stable if and only if $|\lambda_1| < 1$, that is, if condition (22) is satisfied. ■

Note the similarity between the stability condition for the model with evolutionary competition between the two heuristics (condition (22)), and the stability condition for the model where the number of firms using each heuristic is fixed (condition (10)). Moreover, even in the absence of information costs, $\kappa = 0$, the introduction of Nash players may fail to stabilize the Cournot-Nash equilibrium. For $\kappa = 0$ the stability condition (22) reduces to $|(n-1) R'((n-1) q^*)| < 2$, which typically will not hold for n large enough (although the condition for local stability is weaker than under pure Cournot behavior, see condition (7)).

4.2 Cournot vs. rational firms

Now consider evolutionary competition between the Cournot and rational heuristic. Again the fraction of firms employing the rational heuristic will be determined by the payoff differential between the two heuristics, that is:

$$\rho_t = G\left(\Pi_{R,t-1} - \Pi_{C,t-1} - \kappa\right), \tag{24}$$

where $\kappa \geq 0$ are the information costs that firms using the rational heuristic incur, and $\Pi_{R,t} = \Pi_R(q_t, \rho_t) = F(H(q_t, \rho_t), q_t, \rho_t)$ and $\Pi_{C,t} = \Pi_C(q_t, \rho_t) = F(q_t, H(q_t, \rho_t), 1 - \rho_t)$ denote the average profits of the rational and Cournot firms, respectively. Given the fraction of rational firms ρ_t, the quantity produced by the Cournot firms evolves as

$$q_t = R\left((n-1)\left(\rho_{t-1} H\left(q_{t-1}, \rho_{t-1}\right) + \left(1 - \rho_{t-1}\right) q_{t-1}\right)\right). \tag{25}$$

The following proposition holds:

Proposition 5 *The equilibrium (q^*, ρ_κ) of the model with evolutionary competition between the Cournot heuristic and the rational heuristic, given by equations (24) and (25), is locally stable if and only if the following condition holds:*

$$\left|H_q\left(q^*, \rho_\kappa\right)\right| < 1. \tag{26}$$

Proof. The full model, consisting of equations (24) and (25), is given as

$$q_t = \Phi^1\left(q_{t-1}, \rho_{t-1}\right) \equiv R\left((n-1)\left(\rho_{t-1} H\left(q_{t-1}, \rho_{t-1}\right) + \left(1-\rho_{t-1}\right) q_{t-1}\right)\right), \tag{27}$$
$$\rho_t = \Phi^2\left(q_{t-1}, \rho_{t-1}\right) \equiv G(\Pi_{R,t-1} - \Pi_{C,t-1} - \kappa).$$

In order to determine local stability of (q^*, ρ_κ) we need to determine the Jacobian matrix of system (27), evaluated at that equilibrium.

First, let us determine the partial derivatives of Φ^2 with respect to q_{t-1} and ρ_{t-1}, respectively. To that end, note that we can write the profit differential as

$$\triangle\Pi_R = \Pi_{R,t-1} - \Pi_{C,t-1} = \sum_{k=0}^{n-1} A_k\left(\rho_{t-1}\right) D_k\left(q_{t-1}, \rho_{t-1}\right)$$

with $A_k\left(\rho_{t-1}\right) = \binom{n-1}{k}\rho_{t-1}^k\left(1-\rho_{t-1}\right)^{n-1-k}$, which does not depend upon q_{t-1}, and

$$\begin{aligned} D_k\left(q_{t-1}, \rho_{t-1}\right) &= P\left((n-1-k)\, q_{t-1} + (k+1)\, q_{t-1}^r\right) q_{t-1}^r - C\left(q_{t-1}^r\right) \\ &\quad - \left[P\left((n-k)\, q_{t-1} + k q_{t-1}^r\right) q_{t-1} - C\left(q_{t-1}\right)\right], \end{aligned}$$

which depends upon ρ_{t-1} through $q_{t-1}^r = H\left(q_{t-1}, \rho_{t-1}\right)$. Note that $D_k\left(q^*, \rho_\kappa\right) = 0$. Moreover, the partial derivatives of $D_k\left(q_{t-1}, \rho_{t-1}\right)$, evaluated at the equilibrium (q^*, ρ_κ) are given by

$$\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial q_{t-1}}\right|_{(q^*, \rho_\kappa)} = \left[P\left(Q^*\right) + q^* P'\left(Q^*\right) - C'\left(q^*\right)\right]\left(H_q\left(q^*, \rho_\kappa\right) - 1\right) = 0,$$
$$\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{(q^*, \rho_\kappa)} = \left[P\left(Q^*\right) + q^* P'\left(Q^*\right) - C'\left(q^*\right)\right] H_\rho\left(q^*, \rho_\kappa\right) = 0.$$

where the second equalities follows from the fact that $P\left(Q^*\right) + q^* P'\left(Q^*\right) = C'\left(q^*\right)$ is the first order condition of any firm in a Cournot-Nash equilibrium. Using this it follows immediately that

$$\left.\frac{\partial \Phi^2}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = G'(-\kappa)\left.\frac{\partial \triangle \Pi_R}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = G'(-\kappa)\sum_{k=0}^{n-1} A_k(\rho_\kappa)\left.\frac{\partial D_k\left(q_{t-1},\rho_{t-1}\right)}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} = 0$$

and

$$\begin{aligned}\left.\frac{\partial \Phi^2}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)} &= G'(-\kappa)\left.\frac{\partial \triangle \Pi_R}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)} \\ &= G'(-\kappa)\sum_{k=0}^{n-1}\left[\left.\frac{\partial A_k\left(\rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{\rho_\kappa} D_k(q^*,\rho_\kappa) + A_k(\rho_\kappa)\left.\frac{\partial D_k\left(q_{t-1},\rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)}\right] \\ &= 0.\end{aligned}$$

The Jacobian matrix of (27), evaluated in the equilibrium (q^*,ρ_κ), therefore has the following structure

$$\begin{pmatrix} \left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)} & \left.\frac{\partial \Phi^1}{\partial \rho_{t-1}}\right|_{(q^*,\rho_\kappa)} \\ 0 & 0 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = \left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)}$ and $\lambda_2 = 0$. Consequently, the equilibrium is locally stable when

$$\begin{aligned}|\lambda_1| &= \left|\left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{(q^*,\rho_\kappa)}\right| = \left|(n-1)\left(\rho_\kappa H_q(q^*,\rho_\kappa) + (1-\rho_\kappa)\right) R'\left((n-1)\,q^*\right)\right| \\ &= \left|H_q(q^*,\rho_\kappa)\right| < 1,\end{aligned}$$

where the last equality follows from the proof of Proposition 2. ■

Again, note the similarity between the stability condition for the evolutionary model (condition (26)) with that of the model with fixed fractions (condition (14)). The next result follows immediately from Corollary 3 and Proposition 5 and the fact that $\rho_0 = G(0) = \frac{1}{2}$.

Corollary 6 *Suppose $P'(Q^*) + q^*P''(Q^*) < 0$ and that there are no information costs for the rational heuristic, $\kappa = 0$. Then the equilibrium (q^*,ρ_0) of the model of evolutionary* *competition between the Cournot and the rational heuristic is locally stable.*

5 Global dynamics and perpetual bounded fluctuations

In this section we study the global dynamical behavior of the two evolutionary models discussed in Section 4, by means of numerical simulations. In order to do so we need to specify the inverse demand and cost structures, as well as the evolutionary process. We will use the inverse linear demand curve, $P(Q) = a - bQ$, and constant marginal costs, $C_i(q_i) = cq_i$, from our leading example and Theocharis (1960). Recall that this gives rise to reaction curve

$$q_i = R_i(Q_{-i}) = \frac{a-c}{2b} - \frac{1}{2}Q_{-i}, \tag{28}$$

and a unique and symmetric Cournot-Nash equilibrium given by

$$q^* = \frac{a-c}{b(n+1)}. \tag{29}$$

For the evolutionary process we use the logit dynamics, as for example discussed in Brock and Hommes (1997):[15]

$$G(\Pi_{1,t-1} - \Pi_{2,t-1} - \kappa) = \frac{1}{1 + \exp[-\beta(\Pi_{1,t-1} - \Pi_{2,t-1} - \kappa)]}. \tag{30}$$

Here parameter $\beta \geq 0$ measures *evolutionary pressure*: for a higher value of β firms are more likely to switch to the more successful heuristic from the previous period.

The assumption of linear inverse demand and constant marginal costs allows us to rewrite

[15] The logit dynamics can alternatively be expresed as

$$\rho_t = \frac{\exp[\beta(\Pi_{1,t-1} - \kappa)]}{\exp[\beta(\Pi_{1,t-1} - \kappa)] + \exp[\beta\Pi_{2,t-1}]},$$

which is a more common but equivalent formulation.

expected payoffs for heuristic 1, using $\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} k = (n-1)\rho$, as follows

$$\Pi_1 = F(q_1, q_2, \rho) = q_1 \left(a - b\left[q_1 + (n-1)\overline{q}\right] - c\right), \tag{31}$$

where $\overline{q} = \rho q_1 + (1-\rho) q_2$. That is, expected payoffs for heuristic 1 are equal to payoffs obtained when playing once against the average production level in the population.

5.1 Global evolutionary dynamics for Cournot vs. Nash firms

First we consider the model of evolutionary competition between the Cournot and the Nash heuristic. Using (31), the payoff differential between Nash and Cournot firms can be written as

$$\Pi_{N,t} - \Pi_{C,t} = b\left(1 + (1-\rho_t)(n-1)\right)(q_t - q^*)^2. \tag{32}$$

Note that, outside of equilibrium and not taking into account information costs κ, average payoffs of Nash firms will always be higher than those of Cournot firms. Moreover, the differences in payoffs increases with the deviation of q_t from the Cournot-Nash equilibrium, and with the fraction of Cournot firms. The intuition for the higher profitability of Nash firms is the following. Suppose $q_t > q^*$ (a similar argument holds for $q_t < q^*$). The profit maximizing output then is to produce *less* than q^*. Nash firms are then closer to the optimal quantity than Cournot firms and therefore (given that the profit function is concave) have higher payoffs.

Using (8) and (32) the full evolutionary model can now be written as

$$\begin{aligned} q_t &= q^* - \frac{1}{2}\left(1 - \rho_{t-1}\right)(n-1)\left(q_{t-1} - q^*\right), \\ \rho_t &= \frac{1}{1 + \exp\left[-\beta\left(b\left(1 + \left(1-\rho_{t-1}\right)(n-1)\right)\left(q_{t-1} - q^*\right)^2 - \kappa\right)\right]}. \end{aligned} \tag{33}$$

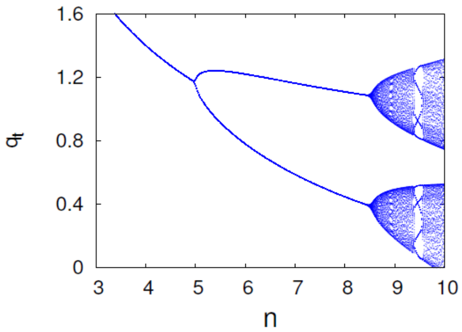
The equilibrium of this dynamical system is given by (q^*, ρ_κ), where $\rho_\kappa = \left[1 + \exp\left[\beta\kappa\right]\right]^{-1} \leq \frac{1}{2}$. Note that in equilibrium Cournot firms perform better than Nash firms, because $\Pi_C^* = \Pi_N^*$

and Nash firms need to pay (higher) information costs. The equilibrium fraction of Nash firms is determined by the product of β and κ. This fraction will be lower if information costs for the Nash heuristic are higher, and it will be lower, for the same information costs, if evolutionary pressure (as measured by β) is higher. Note however that outside of equilibrium the effects of β and κ differ.

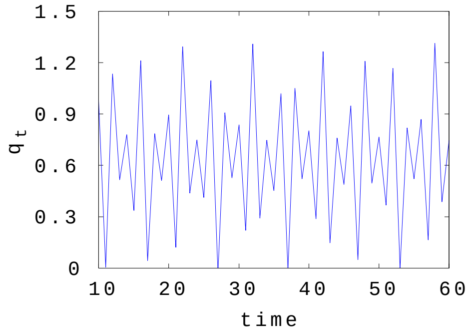
The equilibrium is locally stable for $(n-1)(1-\rho_\kappa) < 2$ (see condition (22) from Proposition 4). From $\rho_\kappa \leq \frac{1}{2}$ it follows immediately that the equilibrium is always unstable when $n > 5$, even in the absence of information costs, and independent of the value of β. Moreover, if $\rho_\kappa < \frac{1}{3}$ (that is, if $\beta \times \kappa > \ln 2$) the equilibrium is unstable even for $n = 4$ (note however that the equilibrium is always locally stable for $n = 3$, provided $\beta \times \kappa$ is finite, whereas the case $n = 3$ is only neutrally stable in Theocharis (1960)). It may seem counterintuitive that even a costless Nash heuristic, which always gives higher expected payoffs than the Cournot heuristic, is unable to stabilize the dynamics for higher values of n. The reason is the following. Outside of equilibrium the Nash heuristic indeed outperforms the Cournot heuristic, and firms switch from the latter to the former, thereby increasing ρ. This stabilizes the dynamics and drives q_t to q^*. At the equilibrium, however, payoffs for both heuristics are the same, and there is no evolutionary pressure against the Cournot heuristic. Firms are therefore indifferent between the two heuristics and half of them will choose the Cournot heuristic, destabilizing the Cournot-Nash equilibrium again. Note that qualitatively similar dynamics will emerge if ρ_t would be determined not just by last period's average payoffs, but by, say average profits of the last T periods.

These dynamics are illustrated by Figure 1 below, that shows numerical simulations for the model with $a = 17$, $b = 1$, $c = 10$, $\beta = 2.8$ and no information costs for Nash firms, $\kappa = 0$. Panel (a) shows a bifurcation diagram, where for convenience the number of firms, n, is treated as a continuous variable (which, although difficult to interpret economically, is mathematically no problem, see equation (33)). The number of firms varies from $n = 3$ to $n = 10$ and the vertical axis shows the values of q_t to which the dynamics converges. For $n = 4$

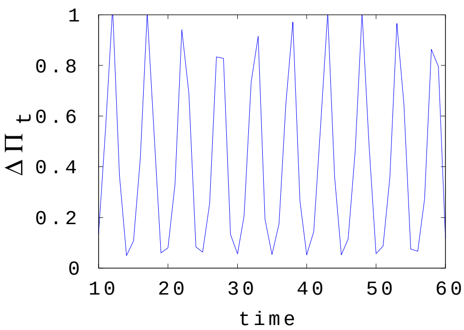
quantities converge to the Cournot-Nash equilibrium level of $q^* = \frac{7}{4}$, but at $n = 5$ a so-called *period-doubling bifurcation* occurs leading to a stable period two cycle for n equal to 6, 7 or 8. Along such a cycle quantities and fractions oscillate between two levels. As n increases even further more complicated dynamic behavior occurs. Panel (b) shows the dynamics of q_t for $n = 10$, which indeed is quite erratic. Panels (c) and (d) illustrate the dynamics in more detail. Panel (c) shows the difference in average profits, $\Delta\Pi = \Pi_{N,t} - \Pi_{C,t}$, which – as argued above – is always positive, implying that $\rho_t \geq \frac{1}{2}$ for all t. Starting with equal shares of heuristics, quantities start to diverge, leading the profit difference and subsequently the fraction of Nash firms to increase. When the fraction of Nash firms is larger than $\frac{7}{9}$ (indicated by the horizontal dashed line in panel (d)) the quantities start to converge again, which decreases the profit differential and the number of Nash firms, and so on. Panel (e) shows the attractor in (q_t, ρ_t) −space for $n = 10$ and panel (f) presents, for $n = 10$, numerical evidence (a strictly positive largest Lyapunov exponent) for chaos, in certain regions of the intensity of choice parameter β.



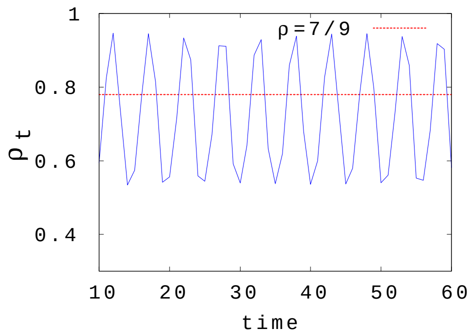
(a) Bifurcation diagram (q_t, n)



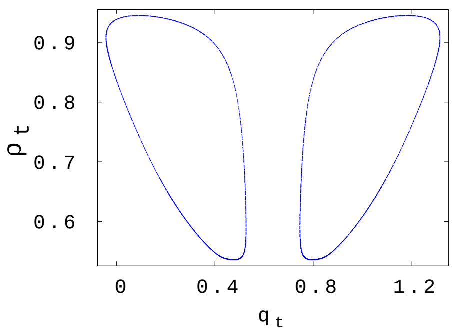
(b) Time path Cournot choice



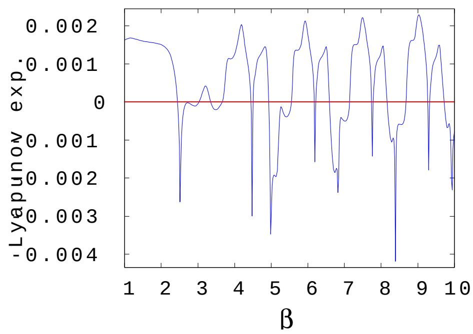
(c) Nash profit differential



(d) Nash players share



(e) Phase plot



(f) Largest Lyapunov exp.

Figure 1: Linear n-player Cournot game with Cournot vs. Nash firms. Panel (a) depicts a period-doubling route to complex quantity dynamics as the number of firms n increases. Instability sets in already for the quintopoly game. Panel (b)-(d) display oscillating time series of the quantity chosen by the Cournot type, profit differential between Nash and Cournot firms and fraction of Nash firms, respectively. The threshold fraction of Nash players $\rho = 7/9$ for which the dynamics become stable is also marked in Panel (d). A typical phase portrait is shown in Panel (e) while Panel (f) plots the largest Lyapunov exponent for increasing β. Game and behavioral parameters: $n = 10, a = 17, b = 1, c = 10, \kappa = 0, \beta = 2.8$.

These numerical findings demonstrate, to a certain extent, that the original result by Theocharis (1960) is robust. For higher values of n the dynamics is unstable. There is also an important difference. The best response dynamics in Theocharis (1960) constitute a system of linear differential equations which diverge from the equilibrium when $n > 3$, until the non-negativity constraints on price and/or production become binding. The endogenously evolving fraction of Nash firms in our model introduces a nonlinearity in the model that gives rise to perpetual endogenous fluctuations, without non-negativity constraints playing a role: quantities and prices remain strictly positive.

5.2 Global evolutionary dynamics for Cournot vs. rational firms

We now replace the Nash heuristic from the previous section by the rational heuristic. We know that, in period t, the rational firms play (see (17)):

$$q_t^r = H(q_t, \rho_t) = q^* - \frac{(1-\rho_t)(n-1)}{2+(n-1)\rho_t}(q_t - q^*)$$

Using (31) we obtain the following payoff difference

$$\Pi_{R,t} - \Pi_{C,t} = b\left(\frac{n+1}{2+(n-1)\rho_t}\right)^2 (q_t - q^*)^2 .$$

Note that, disregarding information costs κ for the moment, payoffs for the rational players are always higher, on average, than those of the Cournot players, as was to be expected. This difference is increasing in the deviation of the best reply output from the Cournot-Nash equilibrium value and decreasing in the number of rational firms. The full model with evolutionary competition between the rational and Cournot heuristics now becomes

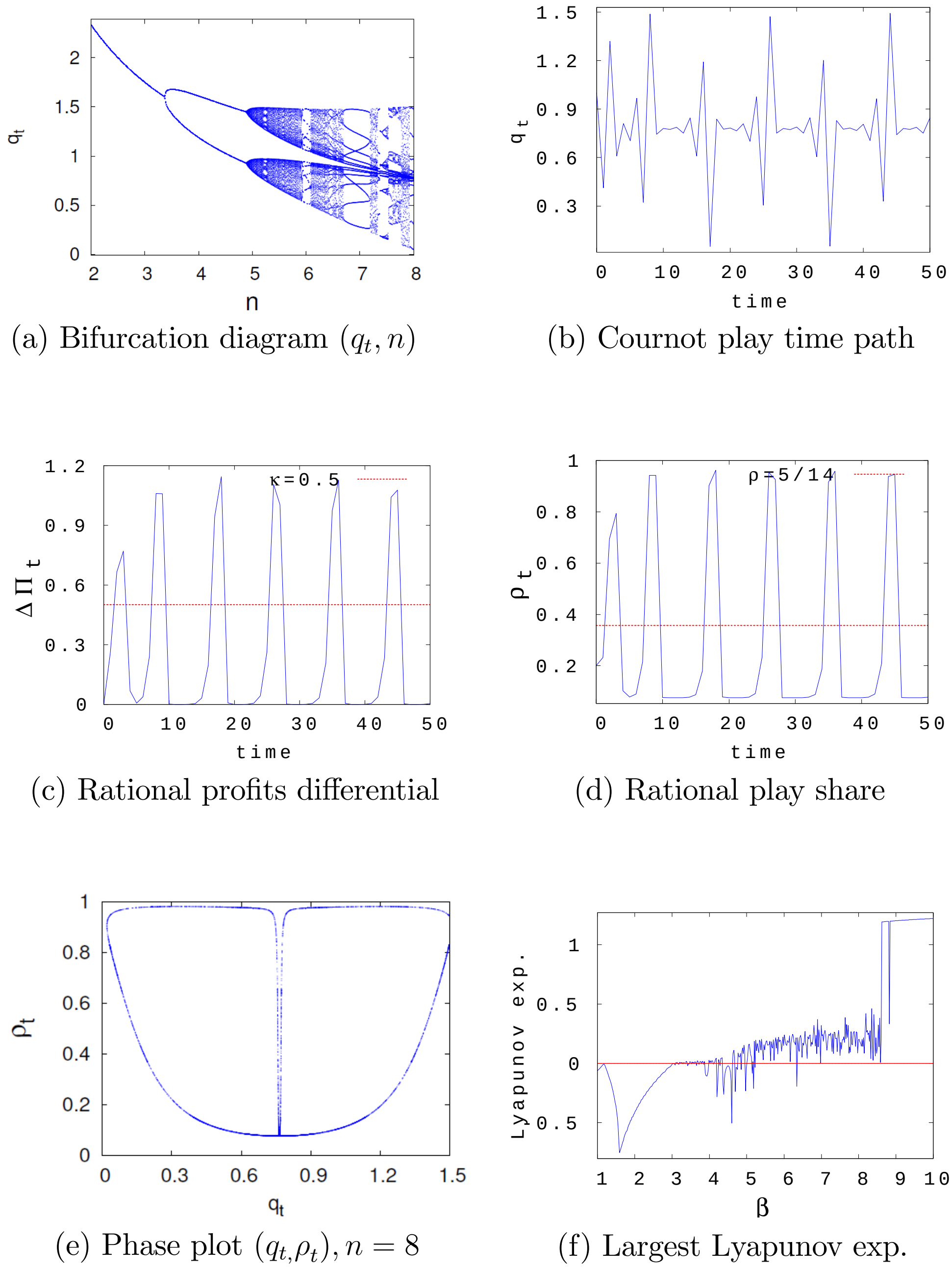
$$\begin{aligned} q_t &= H\left(q_{t-1}, \rho_{t-1}\right) = q^* - \frac{\left(1-\rho_{t-1}\right)(n-1)}{2+(n-1)\rho_{t-1}}\left(q_{t-1}-q^*\right), \\ \rho_t &= \frac{1}{1+\exp\left[-\beta\left(b\left(\frac{n+1}{2+(n-1)\rho_{t-1}}\right)^2\left(q^*-q_{t-1}\right)^2-\kappa\right)\right]}. \end{aligned} \tag{34}$$

The equilibrium of dynamical system (34) is given by $(q^*, \rho_\kappa) = \left(\frac{a-c}{b(n+1)}, [1+\exp[\beta\kappa]]^{-1}\right)$ and it is locally stable for $(1-2\rho_\kappa)(n-1) < 2$, that is (see local stability condition (26) from Proposition 5 and condition (18) for this specific example)

$$\rho_\kappa > \frac{1}{2}\frac{n-3}{n-1}.$$

Clearly, in the absence of information costs ($\kappa = 0$), $\rho_0 = \frac{1}{2}$ and the equilibrium is locally stable, independent of the other parameters (n and β). On the other hand, if $\kappa > 0$ and evolutionary pressure β becomes very large (so that ρ_κ becomes arbitrarily small) the fraction of rational firms in equilibrium will be so small that the equilibrium is unstable for all $n > 3$, as in the original contribution of Theocharis (1960). In fact, the model is unstable for $n \geq 4$ when $\rho_\kappa < \frac{1}{6}$, that is whenever $\beta \times \kappa > \ln 5 \approx 1.609$.

Figure 2 shows the results of some numerical simulations of the model with $a = 17$, $b = 1$, $c = 10$, $\beta = 5$ and $\kappa = \frac{1}{2}$. Note that in this case $\rho_\kappa = \left[1+\exp\left[\frac{5}{2}\right]\right]^{-1} \approx 0.076$, from which it follows immediately that the dynamics will be unstable for any $n > 3$. Panel (a) shows a bifurcation diagram for $n = 2$ to $n = 8$, establishing that a stable period two cycle exists for $n = 4$ and more complicated behavior emerges for larger values of n. Panels (b)-(d) show the dynamics of quantities, profit differences and fractions for $n = 8$, respectively. Note that close to the equilibrium (in fact, when $|q_t - q^*| < \frac{1}{9}\sqrt{2}\left(1+\frac{7}{2}\rho_t\right)$) Cournot firms do better than rational firms because they do not have to pay information costs and the difference in average profits is relatively small. This decreases the number of rational firms, which destabilizes the quantity dynamics. As the dynamics moves away from the equilibrium, eventually the rational heuristic outperforms the Cournot heuristic and firms switch back to the former, increasing ρ_t. Now, when $\rho_t > \frac{5}{14}$ (the horizontal dashed line in panel (d)) the quantity dynamics stabilizes again and quantities converge to their Cournot-Nash equilibrium level, and the whole story repeats. Panel (f) shows that, for $n = 8$, the largest Lyapunov exponent is strictly positive if the evolutionary pressure β is high enough, indicating chaotic behavior in our model.



(a) Bifurcation diagram (q_t, n) (b) Cournot play time path

(c) Rational profits differential (d) Rational play share

(e) Phase plot $(q_t,\rho_t), n = 8$ (f) Largest Lyapunov exp.

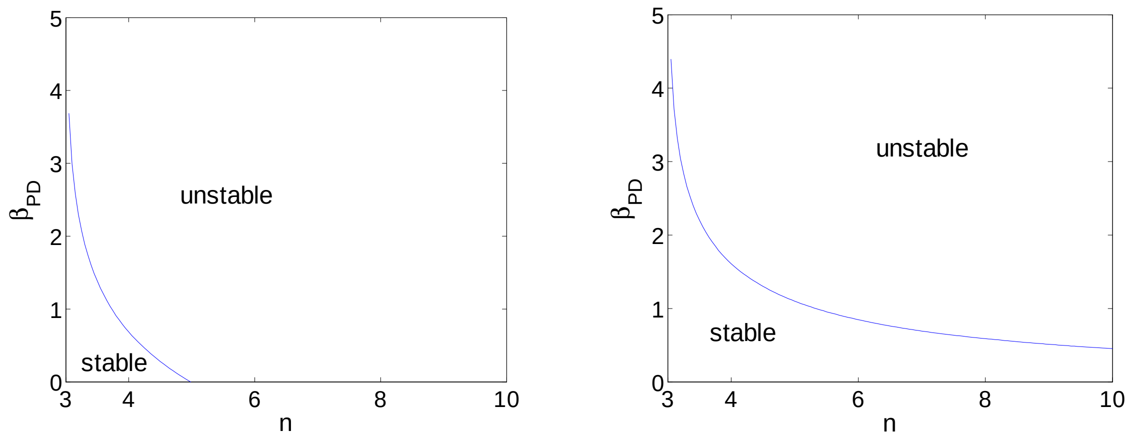
Figure 2: Linear n-player Cournot game with Cournot vs. rational firms. Panel (a) depicts a sequence of period-doubling bifurcations as the number of players n increases. Instability sets in already for the triopoly game. Panel (b)-(d) display oscillating time series of the quantity chosen by the Cournot firm, rational type profit differential (net of predictor costs $\kappa = 0.5$) and fraction of rational firms, respectively. The threshold fraction of rational firms $\rho = 5/14$ for which the dynamics become stable is also marked in Panel (d). A typical phase portrait is shown in Panel (e) while Panel (f) plots the largest Lyapunov exponent for increasing β. Game and behavioral parameters: $n = 8, a = 17, b = 1, c = 10, \kappa = 0.5, \beta = 5$.

6 Discussion and concluding remarks

In the paper we investigated and generalized Theocharis (1960)'s seminal work on the stability of the Cournot-Nash equilibrium in multi-player, quantity-setting games. Theocharis (1960) considered Cournot's adjustment process where each firm selects the quantity that would maximize its profit, under the assumption that the other firms produce the same quantities as they did in the previous period. He found that, in a specification with linear inverse demand and constant marginal costs, the Cournot-Nash equilibrium is neutrally stable for three firms, and unstable when there are more than three firms.

In this paper we relax the assumption that all firms use the Cournot adjustment process. Instead, we develop a model in which firms choose between the Cournot heuristic and a more sophisticated, but possible costly, heuristic. Firms choose a heuristic on the basis of past profitability of the different rules. The two well-known sophisticated heuristics we consider are the Nash heuristic, where firms always play the Cournot-Nash equilibrium, and the rational heuristic, where firms take into account the behavior of the Cournot firms as well as the behavior of the other rational firms in determining their optimal quantity. We derive local stability conditions for our evolutionary model and find that introducing heterogeneity in heuristics tends to stabilize the dynamics, but that perpetual fluctuations are still very well possible, making the result of Theocharis (1960) quite robust.

In particular, for his specification with a linear inverse demand function and constant marginal costs we find the following. Whereas in the original model of Theocharis (1960) the dynamics are unstable for $n > 3$, the dynamics are unstable in the model with evolutionary competition between the Cournot heuristic and the Nash heuristic for $n > 5$. Moreover, if information costs for the Nash heuristic are positive, and evolutionary pressure (as measured by the parameter β) is strong enough, the dynamics are also unstable for $n = 5$ and $n = 4$ (that is, for all $n > 3$). The results for evolutionary competition between the Cournot heuristic and the rational heuristic are different. Most importantly, in absence of information costs for the rational heuristic, the Cournot-Nash equilibrium is locally stable for any $n \geq 2$. With strictly positive information costs, which seems to be the more relevant case for the rational heuristic, the Cournot-Nash equilibrium becomes unstable if either the number of firms n, or the evolutionary pressure, as measured by β, increases. In particular, for β high enough, the equilibrium will be unstable for all $n > 3$. In Figure 3 we plot the *bifurcation curves* (β^{PD}, n) for the two sets of heuristics discussed in this paper (where, for convenience, we interpret n as a continuous variable again).[16]



(a) Cournot vs.Nash firms, $\kappa = 0$ (b) Cournot vs. Rational firms, $\kappa > 0$

Figure 3: Period-doubling bifurcation curves in (β, n) space. When a period-doubling curve is crossed from below the interior Cournot-Nash equilibrium loses stability and a two-cycle is born. Panel (a): Cournot vs. Nash firms; Panel (b): Cournot vs. rational firms. Information costs for the rational firms are set to $\kappa = 1$.

Figure 3 nicely illustrates the relationship between evolutionary pressure and market size, as measured by β and n, respectively. It also shows that in the absence of information costs, $\kappa = 0$, which in terms of local stability is equivalent with $\beta = 0$, the equilibrium is unstable under evolutionary competition between the Cournot and Nash heuristics for $n > 5$ and locally stable for evolutionary competition between the Cournot and rational heuristics for

[16]For a discussion on these period-doubling thresholds for more general learning rules, i.e. adaptive expectations and fictitious play heuristics, see Chapter IV in Ochea (2010).

all n.

The analysis provided in this paper can be extended by considering other behavioral heuristics, but we believe this will lead to qualitatively similar results. We decided to focus on the Cournot, Nash and rational heuristics, since they are the most commonly considered heuristics. In particular, there is evidence from laboratory experiments with paid human subjects that the Cournot heuristic is quite relevant. Cox and Walker (1998), for example, study a laboratory experiment on a Cournot duopoly model with a linear inverse demand function and a quadratic cost function. They show that, when the Cournot-Nash equilibrium is unstable under the Cournot adjustment dynamics (which happens when marginal costs are decreasing sufficiently quickly), participants' quantity choices fail to converge to the (interior) Cournot-Nash equilibrium, which suggests that stability under the Cournot adjustment heuristic may have some predictive power. Furthermore, Rassenti, Reynolds, Smith, and Szidarovszky (2000) present a laboratory experiment on a Cournot oligopoly model with a linear inverse demand function, constant (but asymmetric) marginal costs, and five suppliers. In this setting the Cournot-Nash equilibrium is unstable under the Cournot heuristic. They indeed find that aggregate output oscillates around the equilibrium without converging to it over time. Individual behavior however is not explained very well by Cournot behavior. Finally, Huck, Normann, and Oechssler (2002) discuss a linear (and symmetric) Cournot oligopoly experiment with four suppliers. They do not find that quantities explodes, as the Theocharis (1960) model predicts. Instead the time average of quantities converges to the Cournot-Nash equilibrium quantity, although there is substantial volatility around the Cournot-Nash equilibrium quantity throughout the experiment. Interestingly, Huck, Normann, and Oechssler (2002) find that a process where participants mix between the Cournot adjustment heuristic and imitating the previous period's average quantity gives the best description of their behavior. This supports our model of heterogeneous heuristics. In fact, we could easily use our framework to study evolutionary competition between an imitation heuristic and the Cournot heuristic. For the linear setup and in absence of information costs for either heuristic the condition for local stability then becomes $n \leq 6$.[17]

In addition to extending the literature on stability of the Cournot-Nash equilibrium that originated from Theocharis (1960), our paper also contributes to a separate but related literature on complicated dynamics and endogenous fluctuations in Cournot oligopoly. This literature typically considers Cournot duopolies with non-monotonic reaction functions that are postulated ad hoc (Rand (1978)), derived from iso-elastic demand functions together with substantial asymmetries in marginal costs (Puu (1991)) or derived from cost externalities (Kopel (1996)), and shows that the Cournot adjustment process might exhibit periodic cycles and chaotic behavior. For these models with non-monotonic reaction curves complicated behavior might also arise for other adjustment or learning processes (see e.g. Agiza, Bischi, and Kopel (1999), Bischi, Naimzada, and Sbragia (2007)). Although non-monotonic reaction curves cannot be excluded on economic grounds[18] complicated behavior in our model seems to emerge in a much more natural fashion and perpetual but bounded fluctuations occurs even for linear reaction curves.

[17] When a fraction ρ_t of the population imitates last period's average and a fraction of $1-\rho_t$ uses the Cournot adjustment heuristic, then the average quantity produced evolves as $\overline{q}_t = \rho_t \overline{q}_{t-1} + (1-\rho_t) R\left((n-1)\overline{q}_{t-1}\right)$. It can be shown that the equilibrium (q^*, ρ_κ) is stable in the evolutionary model if and only if $|\rho_\kappa + (1-\rho_\kappa)(n-1) R'((n-1) q^*)| < 1$. In absence of information costs ($\kappa = 0$ and $\rho_0 = \frac{1}{2}$) and with linear inverse demand and constant marginal costs (so that $R'((n-1) q^*) = -\frac{1}{2}$) we obtain that the Cournot-Nash equilibrium is localy stable in this setting for $n \leq 7$ (incidentally, Huck, Normann, and Oechssler (2002) find that behavior is best described by $\rho \approx 0.43$, for which the Cournot-Nash equilibrium will also be locally stable for all $n \leq 6$). Note that the equilibrium is more stable than when there is evolutionary competition between the Cournot and Nash heuristic, with no information costs for the latter. The intuition is that if the current average output is, for example, higher than the equilibrium output level, Cournot players will produce less than the equilibrium quantity in the next period, whereas imitators will produce more. The behavior of imitators therefore has a stronger stabilizing effect upon the dynamics than the behavior of Nash players.

[18] Corchon and Mas-Colell (1996) show that any type of behavior can emerge for continuous time gradient (or best-reply) dynamics in heterogeneous oligopoly, although Furth (2009) argues that for homogeneous Cournot oligopoly there are certain restriction as to what behavior can arise. Relatedly, Dana and Montrucchio (1986) show that in a duopoly model where firms maximize their discounted stream of future profits and play Markov perfect equilibria – and therefore have rational expectations – any behavior is possible for small discount factors.

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