Heterogeneous Expectations in Monetary DSGE Models

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Abstract

This paper derives a general New Keynesian framework consistent with heterogeneous expectations by explicitly solving the micro-foundations underpinning the model. The resulting reduced form is analytically tractable and encompasses the representative rational agent benchmark as a special case. We specify a setup in which some agents, as a result of cognitive limitations, make mistakes when forecasting future macroeconomic variables and update their beliefs as new information becomes available, while other agents have rational expectations. We then address determinacy issues related to the use of different interest rate rules and derive policy implications for a monetary authority aiming at stabilizing the economy in a dynamic feedback system in which macroeconomic variables and heterogeneous expectations co-evolve over time.

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1 Introduction

Over the past decade the New Keynesian model has become increasingly popular in the analysis of monetary policy. This model is built under the hypothesis of rational expectations (RE) and assumes a representative agent structure. Although adaptive learning has become increasingly important as an alternative approach to modeling private sector expectations, most of these models still assume a representative agent who is learning about the economy (see e.g. Evans and Honkapohja (2001) and Sargent (1999) for extensive overviews). Carroll (2003), Mankiw, Reis, and Wolfers (2003), Branch (2004), Pfajfar and Santoro (2010) and Pfajfar (2008) recently provided empirical evidence in support of heterogeneous expectations using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010), Assenza, Heemeijer, Hommes, and Massaro (2011), and Hommes (2011) find evidence for heterogeneity in learning to forecast laboratory experiments with human subjects.[1]

In the light of the empirical evidence, the primary interest of this paper is to incorporate heterogeneous beliefs into the micro-foundations of the New Keynesian framework. The first contribution of our work is thus the development of a micro-founded framework for monetary policy analysis consistent with heterogeneous, possibly boundedly rational, expectations.

In our model agents solve infinite horizon decision problems.[2] The RE hypothesis requires that agents make fully optimal decisions given their beliefs and that agents know the true equilibrium distribution of variables that are beyond their control. Achieving the standard rationality requirements of RE models is especially difficult in a setting with heterogeneous agents. Individuals need to gather and process a substantial amount of information about the economy, including de-

[1]Some recent examples of models with heterogeneous expectations in macroeconomics include Brock and de Fontnouvelle (2000), Evans and Honkapohja (2003, 2006), Branch and McGough (2009, 2010), Berardi (2007), Tuinstra and Wagener (2007), Brazier, Harrison, King, and Yates (2008), Anufriev, Assenza, Hommes, and Massaro (2012), and De Grauwe (2010).

[2]Models with this kind of approach have been studied by Marcet and Sargent (1989) and Sargent (1993) among others.

tails about other agents in the market and their expectations, in order to derive the objective probability distributions of aggregate variables.

Starting from this observation, we slightly depart from the standard RE benchmark and assume that a fraction of agents use simple prediction rules (heuristics) to forecast aggregate variables. However, while we assume cognitive limitations of individual understanding,[3] we fully maintain the assumption that individual choices are made optimally given subjective, possibly non-rational, forecasts. Agents' beliefs determine aggregate outcomes and are subsequently updated based upon recent performances when new public information becomes available in a sort of Bayesian updating mechanism. Co-evolution of subjective expectations with observed macroeconomic outcomes emerges due to the ongoing evaluation of predictors in a dynamic feedback system (Diks and van der Weide (2005)).

As underscored by Preston (2005), when the micro-foundations underpinning the New Keynesian model are solved under the non-rational expectations assumption, the predicted aggregate dynamics depend on long horizon forecasts. Hence, in making current decisions about spending and pricing of their output, agents take into account forecasts of macroeconomic conditions over an infinite horizon. This is a direct consequence of the fact that individuals are assumed to only have knowledge of their own objectives and of the constraints they face, and they do not have a complete economic model of determination of aggregate variables. We derive aggregate demand and supply equations consistent with heterogeneous expectations by explicitly aggregating individual decision rules. The resulting reduced form model is analytically tractable and encompasses the representative rational agent benchmark as a special case. In fact, when RE agents are present in the economy, it is possible to reduce the aggregate equations depending on long horizon forecasts to a system where only one-period ahead forecasts matter by using the law of iterated expectations. When all agents have RE, the model with forecasts over the infinite horizon reduces to the standard New Keynesian model.

[3]Cognitive limitations of individuals have been abundantly documented by psychologists and brain scientists. For recent surveys see Kahneman and Thaler (2006) and Della Vigna (2007).

Within a general framework of heterogeneous expectations, we test the desirability of standard policy recommendations. We find that bounded rationality represents an important source of instability and indeterminacy. In fact, monetary policy rules that lead to determinacy in a world with a representative rational agent may destabilize the economy even when only a small fraction of boundedly rational agents is added to the system. Our results thus confirm the concerns for private sector expectations of policy makers such as Bernanke (2004) and show the importance of taking bounded rationality into account when designing monetary policies.

Closely related to our results is the parallel paper by Branch and McGough (2009) who also introduce heterogeneous expectations in a New Keynesian framework. Differently from our work, Branch and McGough (2009) start from the assumption that agents with subjective (non-rational) expectations choose optimal plans that satisfy the associated Euler equations instead of looking at the intertemporal budget constraint.[4] The consequence of this behavioral approach is that the authors had to place some restrictive assumptions on beliefs in order to obtain aggregation and derive macro dynamics where only one-period ahead forecasts matter. In our setting where individual decision rules as well as aggregate equations depend on long horizon forecasts, instead, we do not impose restrictions on the set of possible forecasting rules and this represents the main difference between the two frameworks.

The paper is organized as follows. The general framework consistent with heterogeneous expectations is derived in Section 2. Section 3 presents an application to monetary policy by considering determinacy issues in an economy with a continuum of boundedly rational beliefs together with perfectly rational expectations. Finally, Section 4 concludes.

[4] See Preston (2005) and Evans, Honkapohja, and Mitra (2003) for a discussion about the two different modeling approaches.

2 The model with heterogeneous expectations

This section develops a version of the New Keynesian model extended to include heterogeneous agents with *possibly* non-rational expectations.

2.1 Households

Following Woodford (2003) we consider a continuum of households in the interval $[0, 1]$. In our setting households have the same utility function and they only differ because of their subjective expectations. Therefore agents using the same rule to form expectations will make identical choices. We will thus index households according to their expectation type i. The fraction of households using the same forecasting rule i will be denoted by n_i^h. Households preferences are defined over a composite consumption good $c_{i,t}$ and time devoted to market employment $h_{i,t}$. Households of type i maximize the expected present discounted value of utility:

$$\max \widetilde{E}_{i,t} \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{c_{i,s}^{1-\sigma}}{1-\sigma} - \chi \frac{h_{i,s}^{1+\gamma}}{1+\gamma} \right),$$

where $\beta \leq 1$ is the discount factor and $\widetilde{E}_{i,t}$ denotes type i subjective expectations at time t. The composite consumption good consists of differentiated products produced by monopolistically competitive final good producers. There is a continuum of goods indexed by $j \in [0, 1]$. The composite consumption good that enters the households' utility function and the aggregate price index for consumption are the usual CES aggregators, defined as

$$c_{i,t} = \left(\int_0^1 c_{i,t}(j)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} \quad \text{and} \quad P_t = \left(\int_0^1 P_t(j)^{1-\eta} dj \right)^{\frac{1}{1-\eta}}, \tag{2.1}$$

where the parameter η governs the price elasticity of demands for individual goods. We follow Woodford (2003) in assuming a cashless economy so that the real budget constraint of households i is given by

$$c_{i,t} + b_{i,t} \leq w_t h_{i,t} + R_{t-1}\pi_t^{-1} b_{i,t-1} + d_t,$$

where $b_{i,t}$ represents holdings of one-period bonds, w_t is the real wage, R_t is the (gross) nominal interest rate, $\pi_t \equiv P_t/P_{t-1}$ is the inflation between period t and period $t-1$ and d_t are dividends received from firms. We assume that each household consists of a continuum of agents which are employed across firms and share proceeds across the household. This allows our agents to hedge against the Calvo risk. Moreover, we assume that bonds are zero in net supply.

The first order conditions for the problem can be written in log-linear terms as

$$\widehat{c}_{i,t} = \widetilde{E}_{i,t}\widehat{c}_{i,t+1} - \sigma^{-1}\left(\widehat{R}_t - \widetilde{E}_{i,t}\widehat{\pi}_{t+1}\right) \quad (2.2)$$

$$\widehat{h}_{i,t} = \gamma^{-1}\left(\widehat{w}_t - \sigma\widehat{c}_{i,t}\right) \quad (2.3)$$

together with the budget constraint

$$\widetilde{b}_{i,t} - \beta^{-1}\widetilde{b}_{i,t-1} - b\beta^{-1}\left(\widehat{R}_{t-1} - \widehat{\pi}_t\right) - \left(1-\eta^{-1}\right)\left(\widehat{w}_t + \widehat{h}_{i,t}\right) - \eta^{-1}\widehat{d}_t + \widehat{c}_{i,t} = 0, \quad (2.4)$$

where $\widehat{x}_t \equiv \ln(x_t/x)$ denotes the log deviation of x_t from its steady state value x, while $\widetilde{x}_t \equiv x_t - x$ is just the difference of variable x_t from its steady state x. As shown in Appendix A, after some algebraic manipulations we can derive the following consumption rule for agents i:

$$\widehat{c}_{i,t} = \zeta_b\widetilde{b}_{i,t-1} + \widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\zeta_w\widehat{w}_s + \zeta_d\widehat{d}_s\right) - \frac{\beta}{\sigma}\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\widehat{R}_s - \widehat{\pi}_{s+1}\right), \quad (2.5)$$

where $\zeta_b \equiv \dfrac{\eta\gamma}{\beta\eta\gamma + \beta(\eta-1)\sigma}$, $\zeta_w \equiv \dfrac{(1-\beta)(\eta-1)(1+\gamma)}{\eta(\gamma+\sigma)-\sigma}$, and $\dfrac{\gamma-\beta\gamma}{\eta(\gamma+\sigma)-\sigma}$. The individual consumption rule (2.5) can be interpreted in the spirit of the canonical consumption model. The first three terms reflect the basic insight that current consumption depends on the expected future discounted wealth, while the last term arises from the assumption of a time-varying real interest rate, and represents deviations from the equilibrium level $R = \beta^{-1}$ due to either variations in the nominal interest rate or inflation.

2.2 Firms

We assume a Calvo (1983) staggered price setting mechanism. Firms in monopolistic competition must pre-commit to prices that can be reset with probability $1-\omega \in (0,1)$ each period. Good j is produced using a single labor input $h(j)$ according to the relation $y_t(j) = h_t(j)$. We assume that there is a continuum of firms of each production type j, and the same proportion of firms of each production type has subjective expectations $\widetilde{E}_{i,t}$ of type i. Given that each firm hires labor from the same integrated economy-wide labor market, the prices chosen by the firms that can re-optimize in period t will only differ because of their subjective forecasts. We will therefore index firms according to their expectation type i and let n_i^f denote the fraction of firms using predictor i. The aggregate price level thus evolves according to the relation

$$P_t = \left(\omega P_{t-1}^{1-\eta} + (1-\omega)(P_t^*)^{1-\eta}\right)^{\frac{1}{1-\eta}}, \tag{2.6}$$

where $P_t^* \equiv \int_i P_{i,t} di$ is the average price set by firms optimizing at time t.

Firms that reset prices maximize expected discounted profits, which are given by

$$\max \widetilde{E}_{i,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s \left(\frac{P_{i,t}}{P_s} - w_s\right)\left(\frac{P_{i,t}}{P_s}\right)^{-\eta} c_s,$$

where Q_s is the stochastic discount factor given by $\beta^{s-t}\left(c_s/c_t\right)^{-\sigma}$ and w_s are real marginal costs. Defining $p_{i,t} \equiv P_{i,t}/P_t$ and log-linearizing the first order conditions of this problem around a zero inflation steady state delivers the pricing equation

$$\widehat{p}_{i,t} = (1-\omega\beta)\, \widetilde{E}_{i,t} \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{w}_s + \omega\beta \widetilde{E}_{i,t} \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{\pi}_{s+1}. \tag{2.7}$$

Equation (2.7) shows that a firm setting its price must be concerned with future marginal costs and future inflation because it may be unable to adjust its price for several periods.

2.3 Aggregation of individual decision rules

We assume for simplicity that households are running firms (or CEOs are appointed by shareholders so they are discounting profits the same way shareholders would do) so expectations of households and firms will be the same. This means that the fractions n_i^h and n_i^f will coincide. In this way we can also justify the fact that firms are using the stochastic multiplier $Q_s = \beta^{s-t} (c_s/c_t)^{-\sigma}$ to discount future profits.

Start from households consumption rule (2.5) and integrate over i to get

$$\widehat{c}_t = \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \zeta_w \widehat{w}_s + \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \zeta_d \widehat{d}_s - \frac{\beta}{\sigma} \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\widehat{R}_s - \widehat{\pi}_{s+1} \right), \tag{2.8}$$

where $\widehat{c}_t = \int_i \widehat{c}_{i,t} di$, $\widetilde{E}_t = \int_i \widetilde{E}_{i,t} di$, and we used that $\int_i \widetilde{b}_{i,t-1} di = 0$ by market clearing.

We now want to derive an IS curve in terms of output and real interest rate. The optimality condition (2.3) can be rewritten in terms of the real wage, which is taken as parametric by the individuals, as $\widehat{w}_t = \gamma \widehat{h}_{i,t} + \sigma \widehat{c}_{i,t}$. Aggregating over individuals we have that $\widehat{w}_t = \gamma \widehat{h}_t + \sigma \widehat{c}_t$. The market clearing condition implies $\widehat{c}_t = \widehat{y}_t$, so we can write $\widehat{w}_t = \gamma \widehat{h}_t + \sigma \widehat{y}_t$. In order to eliminate the term $\widehat{h}_t$ we can use the production function $\widehat{y}_t = \widehat{h}_t$, so that

$$\widehat{w}_t = (\gamma + \sigma) \widehat{y}_t. \tag{2.9}$$

Moreover we can log-linearize the expression for total dividends $d_t = y_t - w_t h_t$, using relation (2.9) and steady state values, in order to get

$$\widehat{d}_t = (1 - (\eta - 1)(\gamma + \sigma))\,\widehat{y}_t. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) and using $\widehat{c}_t = \widehat{y}_t$ we get[5]

$$\widehat{y}_t = \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left((\gamma + \sigma)\left((1-\eta)\zeta_d + \zeta_w\right) + \zeta_w\right) \widehat{y}_s - \frac{\beta}{\sigma} \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\widehat{R}_s - \widehat{\pi}_{s+1}\right), \tag{2.11}$$

which can be simplified to obtain the *Heterogeneous Expectations IS* equation (HE-IS)

$$\widehat{y}_t = (1-\beta)\,\widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \widehat{y}_s - \frac{\beta}{\sigma} \widetilde{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\widehat{R}_s - \widehat{\pi}_{s+1}\right). \tag{2.12}$$

Consider now the supply side of the economy. Using that $\widehat{p}_t^* = \int_i \widehat{p}_{i,t} di$ we can log-linearize equation (2.6) to get

$$\widehat{\pi}_t = \frac{(1-\omega)}{\omega} \widehat{p}_t^*. \tag{2.13}$$

Substituting (2.13) in firms' pricing rule (2.7) yields

$$\widehat{\pi}_t = \frac{(1-\omega)(1-\omega\beta)}{\omega} \widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{w}_s + \frac{(1-\omega)\omega\beta}{\omega} \widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{\pi}_{s+1}, \tag{2.14}$$

where again $\widetilde{E}_t = \int_i \widetilde{E}_{i,t} di$. Using that $\widehat{w}_t = (\sigma + \gamma)\,\widehat{y}_t$, we can rewrite (2.14) in terms of output as

$$\widehat{\pi}_t = \frac{(1-\omega)(1-\omega\beta)}{\omega} (\sigma + \gamma)\, \widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{y}_s + \frac{(1-\omega)\omega\beta}{\omega} \widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t}\, \widehat{\pi}_{s+1}$$

to obtain the *Heterogeneous Expectations New Keynesian Phillips Curve* (HE-

[5]It is consistent to replace $\widehat{w}_t = (\gamma + \sigma)\widehat{y}_t$ and $\widehat{d}_t = (1 - (\eta - 1)(\gamma + \sigma))\,\widehat{y}_t$ in equation (2.8) because we know that (ex post) these equalities will hold in each period t.

NKPC)

$$\widehat{\pi}_t = k\widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t} \widehat{y}_s + (1-\omega)\beta\widetilde{E}_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t} \widehat{\pi}_{s+1}, \tag{2.15}$$

where the constant k is defined as $k \equiv \dfrac{(\sigma+\gamma)(1-\omega)(1-\omega\beta)}{\omega}$.

The benchmark RE model as a special case

It is easy to see that, under the hypothesis of homogeneous rational agents, the HE-IS equation (2.12) and the HE-NKPC relation (2.15) can be reduced to the standard IS and NKPC relations. Let's consider the aggregate supply equation (2.15) as an example. Leading (2.15) one period ahead and taking rational expectations gives

$$\begin{aligned} E_t\widehat{\pi}_{t+1} &= kE_tE_{t+1} \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t-1} \widehat{y}_s + (1-\omega)\beta E_tE_{t+1} \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t-1} \widehat{\pi}_{s+1} \\ E_t\widehat{\pi}_{t+1} &= kE_t \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t-1} \widehat{y}_s + (1-\omega)\beta E_t \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t-1} \widehat{\pi}_{s+1}, \end{aligned}$$

where the second equality makes use of the law of iterated expectations. Rewriting (2.15) as

$$\widehat{\pi}_t = k\widehat{y}_t + (1-\omega)\beta E_t\widehat{\pi}_{t+1} + kE_t \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t} \widehat{y}_s + (1-\omega)\beta E_t \sum_{s=t+1}^{\infty} (\omega\beta)^{s-t} \widehat{\pi}_{s+1}$$

then gives

$$\begin{aligned} \widehat{\pi}_t &= k\widehat{y}_t + (1-\omega)\beta E_t\widehat{\pi}_{t+1} + \omega\beta E_t\widehat{\pi}_{t+1} \\ &= k\widehat{y}_t + \beta E_t\widehat{\pi}_{t+1}. \end{aligned}$$

Applying the same procedure to the HE-IS equation (2.12) yields the standard New Keynesian aggregate demand equation.

The analysis performed in this section shows that, when the microfoundations underpinning the New Keynesian model are solved under non rational heterogeneous expectations, the aggregate dynamics depend on long horizon forecasts as can be seen in equations (2.12) and (2.15). From a behavioral perspective it may seem a little awkward that boundedly rational agents are basing their decisions on long horizon forecasts. However this is a direct consequence of the fact that we are only departing from the benchmark model by relaxing the rationality assumption in the way agents form expectations, but we are keeping the assumption that agents behave optimally given their subjective beliefs. Moreover, it may seem that boundedly rational agents in the model with long horizon forecasts have to collect more information than agents in standard RE models with only one-period ahead forecasts. However this is not the case. In fact rational agents have to do more: they have to satisfy their intertemporal budget constraint, current and future Euler equations and their subjective probabilities have to be the same as the objective probabilities distribution determined by these beliefs. Boundedly rational agents, instead, do not have a complete model of determination of aggregate variables. As noted in Preston (2005), neither the aggregate demand relation (2.12) nor the Phillips curve (2.15) can be simplified as in the rational expectations equilibrium analysis where, since expectations are taken with respect to the correct distribution of future endogenous variables, the law of iterated expectations holds at the aggregate level and therefore only one period ahead expectations matter for aggregate dynamics.

3 Monetary policy with heterogeneous expectations

In this section we use the model with heterogeneous expectations developed in Section 2 for monetary policy analysis. Contemporary policy discussions argued that a desirable interest rate rule has to involve feedback from endogenous variables such as inflation and/or real activity. Many authors considered simple interest rate rules with endogenous components of the form:

$$\widehat{R}_t = \phi_\pi \widehat{\pi}_t + \phi_y \widehat{y}_t \tag{3.1}$$

and analyzed determinacy properties of the rational expectation equilibrium. Desirable monetary rules should avoid indeterminacy (i.e. multiple bounded equilibria) and sunspot fluctuations as emphasized in the original analysis of Sargent and Wallace (1975). Under rational expectations a necessary and sufficient condition for the equilibrium to be determinate is given by the *Taylor principle*:

$$k(\phi_\pi - 1) + (1 - \beta)\phi_y > 0\,, \tag{3.2}$$

stating that the monetary authority should respond to inflation and real activity by adjusting the nominal interest rate with "sufficient strength".[6] Recent studies investigated the validity of such a policy recommendation when expectations depart from the rational benchmark. In the context of a New Keynesian monetary model Bullard and Mitra (2002) assume that agents do not initially have rational expectations, and that they instead form forecasts by using recursive least squares. Using the *E-stability*[7] criterion they show that an interest rate rule that satisfies the Taylor principle induces learnability of the RE equilibrium. Preston (2005) studies the learnability of the RE equilibrium in a New Keynesian setting with long horizon forecasts. He shows that under least squares learning dynamics the Taylor principle (3.2) is necessary and sufficient for E-stability. However, most models studying the validity of classical policy recommendation in contexts that depart from the RE assumption assume a representative agent who is learning about the economy.

The focus of this section is to analyze the dynamical consequences of a policy regime as in (3.1) when agents have *heterogeneous beliefs*. The framework developed

[6]See Woodford (2003) for a proof.

[7]For a detailed presentation of the E-stability concept and its relation with real time learning, see Evans and Honkapohja (2001)

in the first part of the paper can now be applied to evaluate the desirability of the monetary policy rule (3.1) in the presence of heterogeneous expectations. In particular we want to investigate whether the standard advices to policy makers lead the dynamics to converge to the RE equilibrium in a world with heterogeneous expectations.

3.1 Specification of expectations and evolutionary dynamics

In this section we characterize individuals' expectation schemes and describe the evolution of beliefs over time. There is ample empirical evidence documenting that private sector beliefs, when proxied by surveys, are characterized by heterogeneity. Carroll (2003) and Branch (2004) analyze the Michigan survey data on inflation expectations and find results pointing in the direction of an intermediate degree of rationality. Pfajfar and Santoro (2010) study the same survey and document the fact that agents in different percentiles of the survey seem to be associated with forecasting schemes characterized by different degrees of rationality. Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010), and Assenza, Heemeijer, Hommes, and Massaro (2011) find evidence for pervasive heterogeneity of beliefs in learning to forecast experiments with human subjects. In particular, Pfajfar and Zakelj (2010) find a significant proportion of rational agents (around 40%) in a monetary policy experiment set up in the New Keynesian framework.

Building on the empirical evidence mentioned above, we assume that a fraction of agents are perfectly rational and the remainder are boundedly rational, using heuristics to forecast macroeconomic variables. We thus have in mind an economy in which some agents face cognitive problems in understanding and processing information and hence eventually make mistakes in forecasting macroeconomic variables while other agents have rational expectations.

Before describing the full model with both perfectly rational and boundedly rational agents, we will characterize the dynamic feedback system in which macroeconomic variables and heterogeneous subjective expectations co-evolve over time along the lines of Brock, Hommes, and Wagener (2005) and Diks and van der Weide (2005). We assume that agents do not fully understand how macroeconomic variables are determined and hence have biased forecasts. One might think about an economy in which subject do not know the target of the monetary authority and have biased beliefs about it. Moreover, as a result of cognitive limitations, there are differences in the use of information and thus heterogeneity in individual forecasts. As argued in De Grauwe (2010), the assumption of a simple biased forecast can be viewed as a parsimonious representation of a world where agents do not know the underlying economic model and have a biased view about this model.

The point predictors used by the agents are represented in the belief space Θ and parameterized by the belief parameter θ. Each value $\theta_i \in \Theta$ represents a strategy that fully characterizes the behavior of individuals of type i. We will consider the simplest possible case of constant predictors.[8] Within this class of simple rules we allow for differences in the conclusions that agents draw when processing information, as well as biases and idiosyncracies. Therefore in every period we will have a distribution of point predictions.[9] This specification is quite general in the sense that the individual point predictions can be thought of as the outcome of any mental process or estimation technique. Alternatively one could think of agents selecting between various Bayesian forecasting models each differing by their very strong priors. From the modeler's point of view, the strategy $\theta_{i,t}$ used by agents i at time t is a random variables distributed according to the probability

[8]Anufriev, Assenza, Hommes, and Massaro (2012) use a similar modeling approach in a simple model with inflation expectations while De Grauwe (2010) assumes that agents use simple constant heuristics to forecast output gap in the context of a behavioral DSGE model. In the experiments of Assenza, Heemeijer, Hommes, and Massaro (2011) individuals frequently employ constant forecasting rules.

[9]Anufriev, Assenza, Hommes, and Massaro (2012) use the concept of *Large Type Limit* (LTL) developed by Brock, Hommes, and Wagener (2005) to analyze inflation dynamics when the number of strategies available to the agents tend to infinity. The concept of *Continuous Beliefs System* (CBS) developed by Diks and van der Weide (2002) and used in this paper is a generalization of the LTL concept (see Diks and van der Weide (2003) for a discussion).

density function $\psi_t(\theta)$. Given that agents have limited cognitive abilities, their forecasts will typically be biased.[10] In order to limit the wilderness of bounded rationality and avoid completely irrational behavior we have to introduce discipline in the selection of rules. We will achieve this discipline by subjecting the choice of heuristics to a fitness criterion, and by introducing a selection mechanism that allows agents to learn from their forecasting mistakes.

Predictors will, in fact, be evaluated according to an *evolutionary fitness* measure. Given the performance of predictors at time $t-1$, denoted by $U_{t-1}(\theta)$, we assume as in Diks and van der Weide (2005) that the distribution of beliefs evolves over time as a function of past performances according to the *continuous choice model*:

$$\psi_t(\theta) = \frac{\upsilon(\theta) e^{\delta U_{t-1}(\theta)}}{Z_{t-1}}, \tag{3.3}$$

where Z_{t-1} is a normalization factor independent of θ given by

$$Z_{t-1} = \int_{\Theta} \upsilon(\vartheta) e^{\delta U_{t-1}(\vartheta)} d\vartheta,$$

and $\upsilon(\theta)$ is a *opportunity function* that can put different weights on different parts of the beliefs space. The parameter δ refers to the *intensity of choice* and measures how sensitive agents are to differences in performances. Notice that (3.3), which can be rewritten as

$$\psi_t(\theta) \propto \upsilon(\theta) e^{\delta U_{t-1}(\theta)}, \tag{3.4}$$

is a rule for updating the distribution of beliefs as new information becomes available similar to a Bayesian updating rule. In fact, $\upsilon(\theta)$ plays a role similar to a prior, reflecting the a priori faith of individuals in parameters within certain regions of the parameter space, and $\psi_t(\theta)$ to a posterior. The fitness measure enters

[10]The point predictions of the heuristics can coincide with the perfect foresight point forecast because we did not make any restrictive assumption on the support of the distribution of beliefs.

the beliefs distribution through the term $e^{\delta U_{t-1}(\theta)}$, which plays a role similar to the likelihood in Bayesian statistics. In fact, a Bayesian updating rule in the usual form is recovered exactly when $\delta = 1$ and the performance measure $U_{t-1}(\theta)$ is the log-likelihood function of an econometric model, given the available observations. In order to simplify the analysis we will assume that the performance measure is given by the past squared forecast error[11]

$$U_{t-1}(\theta) = -(\theta - x_{t-1})^2, \tag{3.5}$$

for $x \in \{y, \pi, R\}$. Having observed and compared overall past performance, all agents subsequently adapt their beliefs. The distribution of beliefs at time t is thus given by means of the continuous choice model (3.3). Co-evolution of the distribution of beliefs with the observed aggregate variables thus emerges through the ongoing evaluation of predictors. As in Diks and van der Weide (2005) we assume a constant opportunity function $v(\theta) = 1$, meaning that agents assign the same initial weight to all parameter values. Since the utility function is a quadratic function in the belief parameter θ, it can be shown that the distribution of beliefs is normal and its evolution is characterized by the following expression for mean and variance respectively[12]

$$\mu_t = x_{t-1} \tag{3.6}$$

$$\sigma_t^2 = \frac{1}{2\delta}. \tag{3.7}$$

In the absence of dependence among agents, the law of large number applies, and it follows that the average belief will converge to μ_t, that is

$$\int_i \theta_{i,t} di = \mu_t \; . \tag{3.8}$$

[11]Branch (2004) finds empirical evidence for dynamic switching depending on the squared errors of the predictors in survey data on individuals' expectations, while Pfajfar and Zakelj (2010) and Hommes (2011) find empirical evidence for dynamic switching depending on the squared errors of the predictors in experimental data on individuals' expectations.

[12]see Diks and van der Weide (2005) for details.

Rational versus biased beliefs

We now turn to the description of the heterogeneous expectations model with rational and biased beliefs. We assume that a constant proportion n_{RE} of agents is rational, while the remaining $(1 - n_{RE})$ fraction of agents has boundedly rational beliefs.[13] We can thus rewrite the heterogeneous New Keynesian model described by the aggregate equations (2.12) and (2.15) as

$$\begin{aligned}
\widehat{y}_t &= n_{RE} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left((1-\beta)\widehat{y}_s - \frac{\beta}{\sigma}(\widehat{R}_s - \widehat{\pi}_{s+1}) \right) \\
&\quad + (1 - n_{RE}) \int_i \widetilde{E}_{i,t} \sum_{s=t}^{\infty} \beta^{s-t} \left((1-\beta)\widehat{y}_s - \frac{\beta}{\sigma}(\widehat{R}_s - \widehat{\pi}_{s+1}) \right) di \\
\widehat{\pi}_t &= n_{RE} E_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t} \left(k\widehat{y}_s + (1-\omega)\beta\widehat{\pi}_{s+1} \right) \\
&\quad + (1 - n_{RE}) \int_i \widetilde{E}_{i,t} \sum_{s=t}^{\infty} (\omega\beta)^{s-t} \left(k\widehat{y}_s + (1-\omega)\beta\widehat{\pi}_{s+1} \right) di,
\end{aligned} \tag{3.9}$$

where E_t denotes the rational expectation operator.

We follow Kreps (1998) and Sargent (1999) in assuming that agents solve an anticipated utility problem, i.e. when agents solve their optimization problem they hold their expectation operator fixed and assume that it remains fixed for all future periods. Using this assumption we can simplify boundedly rational individual forecasts over the long horizon as

$$\widetilde{E}_{i,t} \sum_{s=t}^{\infty} \beta^{s-t} \widehat{x}_s = \frac{1}{1-\beta} \theta_{i,x,t}, \tag{3.10}$$

where $\theta_{i,x,t}$ denotes the biased forecasts of agents i in period t for $x \in \{y, \pi, R\}$. Using the results in (3.6), (3.7) and (3.8) we have that the average forecast of boundedly rational biased agents is given by

$$\int_i \theta_{i,x,t} di = \mu_{x,t} = \widehat{x}_{t-1}, \tag{3.11}$$

[13]This assumption can be justified using a combination of continuous and discrete choice predictor selection (see Massaro (2012) for details).

for $x \in \{y, \pi, R\}$. Therefore, using the results (3.10) and (3.11), we can rewrite system (3.9) as

$$\begin{aligned}
\widehat{y}_t &= n_{RE} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left((1-\beta)\widehat{y}_s - \frac{\beta}{\sigma}(\widehat{R}_s - \widehat{\pi}_{s+1}) \right) + (1-n_{RE}) \mathrm{V}_y(\widehat{y}_{t-1}, \widehat{R}_t, \widehat{R}_{t-1}, \widehat{\pi}_{t-1}) \\
\widehat{\pi}_t &= n_{RE} E_t \sum_{s=t}^{\infty} (\omega\beta)^{s-t} \left(k\widehat{y}_s + (1-\omega)\beta\widehat{\pi}_{s+1} \right) + (1-n_{RE}) \mathrm{V}_\pi(\widehat{y}_{t-1}, \widehat{\pi}_{t-1}),
\end{aligned} \tag{3.12}$$

where $\mathrm{V}_x(.)$ denotes a function linear in its arguments, for $x \in \{y, \pi\}$, described in equations (B.2) and (B.3) in Appendix B.

Using the results derived before and closing the model with the interest rate rule (3.1) we can rewrite system (3.12) in the standard matrix form as[14]

$$\begin{bmatrix} E_t\widehat{y}_{t+1} \\ E_t\widehat{\pi}_{t+1} \\ \widehat{Y1}_{t+1} \\ \widehat{\Pi 1}_{t+1} \end{bmatrix} = \begin{bmatrix} \gamma_{yy} & \gamma_{y\pi} & \gamma_{yY1} & \gamma_{y\Pi 1} \\ \gamma_{\pi y} & \gamma_{\pi\pi} & \gamma_{\pi Y1} & \gamma_{\pi\Pi 1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{y}_t \\ \widehat{\pi}_t \\ \widehat{Y1}_t \\ \widehat{\Pi 1}_t \end{bmatrix}, \tag{3.13}$$

where $\widehat{Y1}_t = \widehat{y}_{t-1}$ and $\widehat{\Pi}_t = \widehat{\pi}_{t-1}$. We are now ready to investigate the validity of the standard monetary policy recommendations in the context of a New Keynesian model with a rich variety of forecasting rules, including the rational expectations predictor.

3.2 Numerical analysis of determinacy

Model (3.13) has the form of a rational expectations model with predetermined variables. Techniques for analyzing the determinacy properties of a linear model under rational expectations are well known (see, for example, Blanchard and Kahn (1980)). The determinacy properties of the model depend on the magnitude of the eigenvalues of the transition matrix in (3.13). In our model with heterogeneous agents we have two predetermined variables, so that determinacy obtains when

[14] Description of the coefficients is given in Appendix B.

two eigenvalues are inside the unit circle. Fewer eigenvalues inside the unit circle imply explosiveness and more imply indeterminacy. Given the large number of the model's parameters we will perform a numerical analysis of determinacy properties. The model's parameters could be estimated but such an estimation is beyond the scope of this simple monetary policy exercise. We will therefore use calibrated values for the structural parameters as in Woodford (2003), namely $\beta = 0.99$, $\sigma = 0.157$, $k = 0.024$ and $\omega = 0.66$, and treat the policy parameters ϕ_π and ϕ_y, together with the fraction of rational agents n_{RE}, as bifurcation parameters.[15] We will follow Branch and McGough (2009) and use $0 < \phi_\pi < 2$, $0 < \phi_y < 2$ as the benchmark policy space.

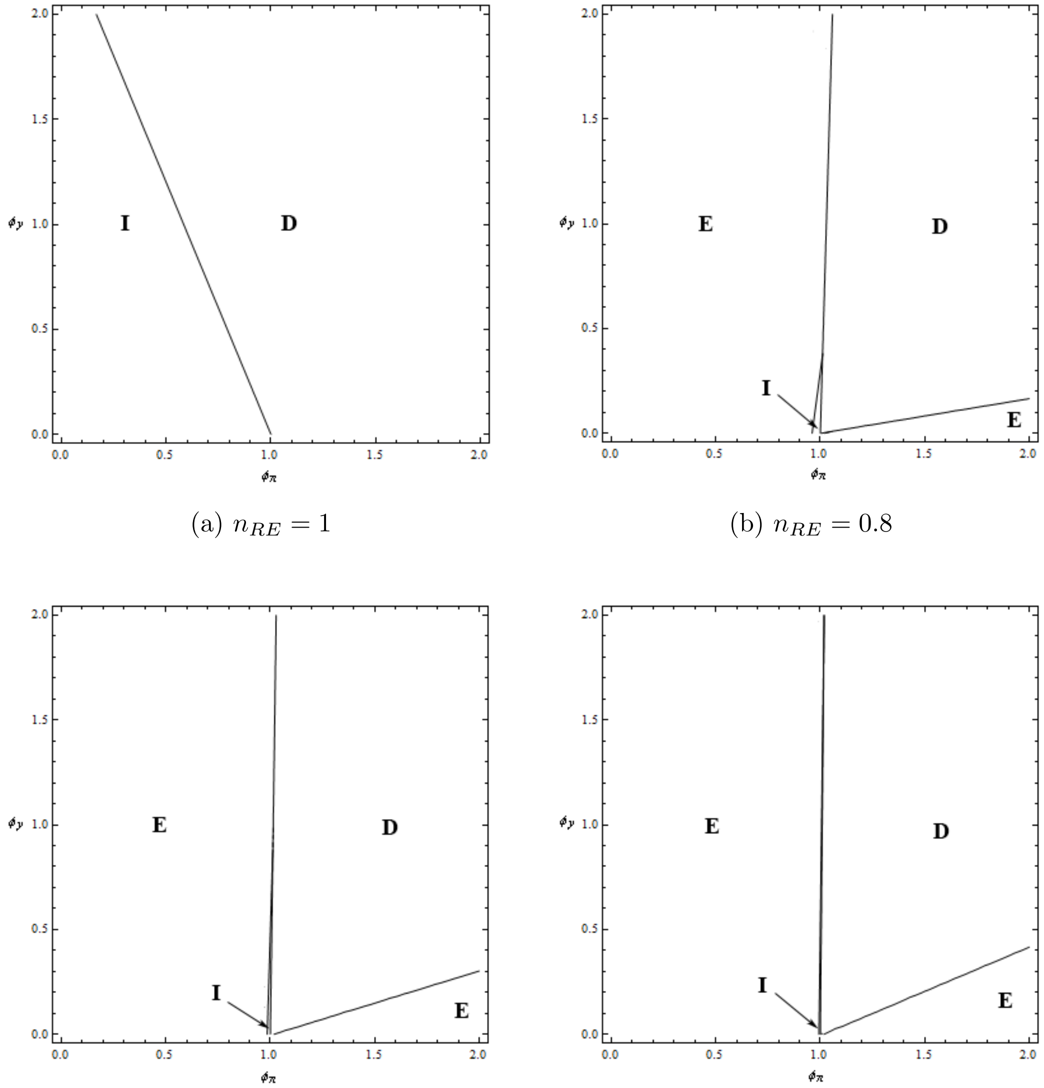
Even if studying the determinacy properties of the model in the 3-dimensional $(\phi_\pi, \phi_y, n_{RE})$ space does not present particular problems from the computational point of view, it is easier to visualize results in the 2-dimensional (ϕ_π, ϕ_y) space for a given fraction n_{RE} of rational agents.[16]

Fig. 1 shows how the determinacy properties of the model change when we allow for small departures from the representative rational agent benchmark. We present determinacy results for some empirically relevant cases. The choice of the fractions of rational agents in the economy reflects the findings of Gali and Gertler (1999) who estimated a reduced form NKPC with a fraction of forward looking rational firms and a fraction of firms with backward looking behavior and found a degree of rationality between 0.6 and 0.8, and the findings of Pfajfar and Zakelj (2010) who provided evidence of a degree of rationality around 0.4 in a monetary policy laboratory experiment.

Fig. 1a indicates the outcome under full rationality and it is consistent with the usual prescription of the RE literature: an interest rate rule satisfying the Taylor principle ensures determinacy. However, when we add a fraction of boundedly rational agents in the economy, the boundaries of the determinacy regions are

[15]The fundamental result of this section, namely that the presence of bounded rationality represents an important source of instability which may alter significantly the determinacy properties of the model, is robust across calibrations.

[16]See Massaro (2012) for the full bifurcation analysis in the 3-dimensional $(\phi_\pi, \phi_y, n_{RE})$ space.



(a) $n_{RE} = 1$

(b) $n_{RE} = 0.8$

(c) $n_{RE} = 0.6$

(d) $n_{RE} = 0.4$

Figure 1: Determinacy properties for different values of the fraction n_{RE}.
D denotes determinacy, **I** denotes indeterminacy, and **E** denotes explosiveness.
(a): RE benchmark, $n_{RE} = 1$, **(b):** Gali and Gertler (1999) (upper bound), $n_{RE} = 0.8$, **(c):** Gali and Gertler (1999) (lower bound), $n_{RE} = 0.6$, **(d):** Pfajfar and Zakelj (2010), $n_{RE} = 0.4$.

sensibly altered. We indeed observe in Fig. 1b that when a small portion of non-rational agents is added to the economy, the region corresponding to determinacy decreases in size, meaning that policy rules that obey the Taylor principle may not enforce determinacy.

It is important to notice that, in the presence of bounded rationality, the dynamics out of the determinacy region are quite different from the benchmark RE model. In fact, when boundedly rational agents are present in the economy, the model can show instability (i.e., explosive equilibria) instead of indeterminacy (i.e., multiple bounded equilibria). As the fraction of rational agents decreases further, the determinacy region keeps on shrinking as shown in Figs. 1c and 1d. Notice also that the indeterminacy region shrinks as n_{RE} decreases.

The intuition for such result can be found in the logic that produces unique stable dynamics in a New Keynesian model with homogeneous rational expectations. The stabilization mechanism after a shock in the New Keynesian model relies on unstable dynamics in the sense that, by obeying the Taylor principle, the monetary authority induces dynamics that will explode in any equilibrium but one. Ruling out explosive paths guarantees then uniqueness of output and inflation equilibrium paths.[17] However the presence of boundedly rational agents, whose expectations co-evolve with macroeconomic variables in a dynamic feedback system, introduces backward-looking components in the dynamics of the model. In the presence of backward-looking agents in the economy, parameters' regions that ensured unstable eigenvalues and thus determinate equilibrium in a completely forward-looking model, may now induce unstable dynamics.

Of course the results presented in Fig. 1 are not exhaustive of all possible determinacy scenarios, but they suffice to make clear the main point of our simple monetary policy exercise, namely that the presence of bounded rationality may alter significantly the determinacy properties of the model, and therefore that in a world with heterogeneous expectations an interest rate rule that obeys the Taylor principle does not necessarily guarantee a determinate equilibrium.

An important question at this point concerns the parameter values that can lead to determinacy for plausible degrees of heterogeneity. We will consider fractions of rational agents in the interval $n_{RE} \in [0.4, 0.8]$ as reasonable degrees of heterogeneity. In fact, the extremes of the considered interval correspond respectively

[17] See for example Cochrane (2010) for a discussion.

to the minimum and the maximum fraction of agents estimated in the empirical literature cited above. Given our parametrization, an example of a combination of policy coefficients leading to determinacy over the entire benchmark interval of heterogeneity degrees is $\phi_\pi = 1.5$ and $\phi_y = 0.5$. This set of coefficients is of a comparable order of magnitude of estimated Taylor rule coefficients in the econometric literature.[18]

The results of this section are of a crucial importance for the conduct of a sound monetary policy. In fact the rationale for recommendations advising to conduct policies within the determinacy region is based on the fact that determinacy reduces volatility of inflation and output. It is therefore very important to account for bounded rationality when designing monetary policy since policies constructed to achieve determinacy under homogeneous rational expectations may be destabilizing when expectations are heterogeneous.

4 Conclusion

Recent papers provided empirical evidence in favor of heterogeneity in individual expectations using survey data as well as experimental data. Building on this evidence, we derived a general micro-founded version of the New Keynesian framework for the analysis of monetary policy in the presence of heterogeneous expectations. We model individual behavior as being optimal given subjective expectations and derive a law of motion for output and inflation by explicitly aggregating individual decision rules. One advantage of our approach is that it is sufficiently general to consider a rich ecology of forecasting rules, ranging from simple heuristics to the very sophisticated rational expectation predictor. The RE benchmark is indeed a special case of our heterogeneous expectations New Keynesian model.

We designed an economy where some agents have rational expectations while others use simple heuristics to forecast macroeconomic variables. After having char-

[18]See, e.g., Taylor (1999), Judd and Rudebusch (1998), Clarida, Gali, and Gertler (2000) and Orphanides (2004).

acterized the dynamic feedback system in which aggregate variables and subjective expectations co-evolve over time, we performed a simple monetary policy exercise to illustrate the implications of expectations' heterogeneity on the determinacy properties of the model. Our central finding is that in a world with heterogeneous agents, the Taylor principle does not necessarily guarantee a unique equilibrium. Therefore policy makers should seriously take into account bounded rationality when designing monetary policies. In fact policy attempts to achieve determinacy under homogeneous rational expectations may destabilize the economy even when only a small fraction of boundedly rational agents is present in the economy.

This paper provides a theoretical DSGE framework for the analysis of monetary policy in the presence of heterogeneous beliefs. Future work should try to estimate the degree of rationality in the economy and investigate further the implications of heterogeneity in the way agents form their beliefs for the global dynamics of the economy and optimal monetary policy design.

Appendix

A Derivation of consumption rule (2.5)

Consider household i's first order conditions

$$\widehat{c}_{i,t} = \widetilde{E}_{i,t}\widehat{c}_{i,t+1} - \sigma^{-1}\left(\widehat{R}_t - \widetilde{E}_{i,t}\widehat{\pi}_{t+1}\right) \tag{A.1}$$

$$\widehat{h}_{i,t} = \gamma^{-1}\left(\widehat{w}_t - \sigma\widehat{c}_{i,t}\right) \tag{A.2}$$

and iterate forward the flow budget constraint to get the intertemporal budget constraint[19]

$$\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\widehat{c}_{i,s} = \widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\beta^{-1}\widetilde{b}_{i,t-1} + \left(1-\eta^{-1}\right)\left(\widehat{w}_s + \widehat{h}_{i,s}\right) + \eta^{-1}\widehat{d}_s\right). \tag{A.3}$$

By substituting (A.2) into (A.3) we get

$$\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\frac{\gamma + \left(1-\eta^{-1}\right)\sigma}{\gamma}\right)\widehat{c}_{i,s} = \widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\beta^{-1}\widetilde{b}_{i,t-1} + \frac{\left(1-\eta^{-1}\right)(1+\gamma)}{\gamma}\widehat{w}_s + \eta^{-1}\widehat{d}_s\right). \tag{A.4}$$

We can now iterate equation (A.1) to get

$$\widetilde{E}_{i,t}\widehat{c}_{i,s} = \widehat{c}_{i,t} + \sigma^{-1}\widetilde{E}_{i,t}\sum_{k=t}^{s}\left(\widehat{R}_k - \widehat{\pi}_{k+1}\right). \tag{A.5}$$

Substituting (A.5) in the LHS of (A.4) gives

$$\begin{aligned}&\left(\frac{\gamma + \left(1-\eta^{-1}\right)\sigma}{\gamma}\right)\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\widehat{c}_{i,t} + \sigma^{-1}\sum_{k=t}^{s}\left(\widehat{R}_k - \widehat{\pi}_{k+1}\right)\right)\\ =\ &\left(\frac{\gamma + \left(1-\eta^{-1}\right)\sigma}{\gamma}\right)\left(\frac{1}{1-\beta}\widehat{c}_{i,t} + \sigma^{-1}\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\sum_{k=t}^{s}\beta^{s-t}\left(\widehat{R}_k - \widehat{\pi}_{k+1}\right)\right).\end{aligned} \tag{A.6}$$

[19] In deriving (A.3) we used the fact that $b = 0$ from market clearing and we imposed the standard no-Ponzi game condition. In fact, even if expectations are not rational we assume that households are not allowed to believe that they can borrow (and consume) as much as they want and just pay off the interest payments by borrowing more. Optimality conditions then require $\lim_{s\to\infty}\widetilde{E}_{i,t}\beta^{s-t}\widetilde{b}_{i,s+1} = 0$.

Since the term

$$\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\sum_{k=t}^{s}\beta^{s-t}\left(\widehat{R}_k-\widehat{\pi}_{k+1}\right)=\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\frac{\beta}{1-\beta}\left(\widehat{R}_s-\widehat{\pi}_{s+1}\right),$$

we can rewrite (A.6) as

$$\left(\frac{\gamma+\left(1-\eta^{-1}\right)\sigma}{\gamma}\right)\left(\frac{1}{1-\beta}\widehat{c}_{i,t}+\frac{\beta}{\sigma}\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\frac{\beta^{s-t}}{1-\beta}\left(\widehat{R}_s-\widehat{\pi}_{s+1}\right)\right). \tag{A.7}$$

Substituting now (A.7) into (A.4) we finally get a consumption rule for agent i

$$\widehat{c}_{i,t}=\zeta_b\widetilde{b}_{i,t-1}+\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\zeta_w\widehat{w}_s+\zeta_d\widehat{d}_s\right)-\frac{\beta}{\sigma}\widetilde{E}_{i,t}\sum_{s=t}^{\infty}\beta^{s-t}\left(\widehat{R}_s-\widehat{\pi}_{s+1}\right), \tag{A.8}$$

where $\zeta_b\equiv\dfrac{\eta\gamma}{\beta\eta\gamma+\beta(\eta-1)\sigma}$, $\zeta_w\equiv\dfrac{(1-\beta)(\eta-1)(1+\gamma)}{\eta(\gamma+\sigma)-\sigma}$, and $\dfrac{\gamma-\beta\gamma}{\eta(\gamma+\sigma)-\sigma}$.

B Derivation of systems (3.12) and (3.13)

Start from system (3.9) which can be rewritten using results (3.10) and (3.11) as

$$\begin{aligned}
\widehat{y}_t &= n_{RE}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(1-\beta)\widehat{y}_s-\frac{\beta}{\sigma}(\widehat{R}_s-\widehat{\pi}_{s+1})\right)+\\
&\quad(1-n_{RE})\left(\widehat{y}_{t-1}-\frac{\beta}{\sigma}\widehat{R}_t-\frac{\beta^2}{(1-\beta)\sigma}\widehat{R}_{t-1}+\frac{\beta}{(1-\beta)\sigma}\widehat{\pi}_{t-1}\right)\\
\widehat{\pi}_t &= n_{RE}E_t\sum_{s=t}^{\infty}(\omega\beta)^{s-t}\left(k\widehat{y}_s+(1-\omega)\beta\widehat{\pi}_{s+1}\right)+(1-n_{RE})\left(\frac{k}{1-\omega\beta}\widehat{y}_{t-1}+\frac{(1-\omega)\beta}{1-\omega\beta}\widehat{\pi}_{t-1}\right).
\end{aligned} \tag{B.1}$$

Therefore we have that $\mathrm{V}_x(.)$, for $x\in\{y,\pi\}$, in (3.12) are defined as

$$\mathrm{V}_y(\widehat{y}_{t-1},\widehat{R}_t,\widehat{R}_{t-1},\widehat{\pi}_{t-1}) = \widehat{y}_{t-1}-\frac{\beta}{\sigma}\widehat{R}_t-\frac{\beta^2}{(1-\beta)\sigma}\widehat{R}_{t-1}+\frac{\beta}{(1-\beta)\sigma}\widehat{\pi}_{t-1} \tag{B.2}$$

$$\mathrm{V}_\pi(\widehat{y}_{t-1},\widehat{\pi}_{t-1}) = \frac{k}{1-\omega\beta}\widehat{y}_{t-1}+\frac{(1-\omega)\beta}{1-\omega\beta}\widehat{\pi}_{t-1} \tag{B.3}$$

As standard in the learning literature we assumed that the current interest rate is observed by non-rational agents while current output and inflation are not. The term $\widehat{R}_{t-1}$ shows up in the aggregate demand via the performance measure (3.5) in the heuristics selection process. This is due to the assumption that the selection of heuristics takes place at the beginning of period t, before observing $\widehat{R}_t$.

Closing the model with the interest rate rule $\widehat{R}_t = \phi_\pi \widehat{\pi}_t + \phi_y \widehat{y}_t$, after some algebraic manipulations we can rewrite the system (B.1) as

$$E_t\widehat{y}_{t+1} = q_{yy}\widehat{y}_t + q_{yE\pi}E_t\widehat{\pi}_{t+1} + q_{y\pi}\widehat{\pi}_t + q_{yY1}\widehat{y}_{t-1} + q_{y\Pi 1}\widehat{\pi}_{t-1} \tag{B.4}$$

$$E_t\widehat{\pi}_{t+1} = \gamma_{\pi y}\widehat{y}_t + \gamma_{\pi\pi}\widehat{\pi}_t + \gamma_{\pi Y1}\widehat{y}_{t-1} + \gamma_{\pi\Pi 1}\widehat{\pi}_{t-1} \tag{B.5}$$

with

$$\begin{aligned}
\Omega_y &= \left(1 + \frac{\beta(1-n_{RE})}{\sigma}\phi_y\right)^{-1}\beta^{-1} \\
q_{yy} &= \Omega_y\left(1 - n_{RE} + \beta - \left(\frac{\beta^3(1-n_{RE})}{(1-\beta)\sigma} - \frac{\beta}{\sigma}\right)\phi_y\right) \\
q_{yE\pi} &= \Omega_y\left(-\frac{n_{RE}\beta}{\sigma} - \frac{\beta^2(1-n_{RE})}{\sigma}\phi_\pi\right) \\
q_{y\pi} &= \Omega_y\left(\frac{\beta^2(1-n_{RE})}{(1-\beta)\sigma} - \left(\frac{\beta^3(1-n_{RE})}{(1-\beta)\sigma} - \frac{\beta}{\sigma}\right)\phi_\pi\right) \\
q_{yY1} &= \Omega_y\left(-(1-n_{RE})\left(1 - \frac{\beta^2}{(1-\beta)\sigma}\phi_y\right)\right) \\
q_{y\Pi 1} &= \Omega_y\left(-(1-n_{RE})\left(\frac{\beta}{(1-\beta)\sigma} - \frac{\beta^2}{(1-\beta)\sigma}\phi_\pi\right)\right) \\
\Omega_\pi &= (n_{RE}(1-\omega)\beta + \omega\beta)^{-1} \\
\gamma_{\pi y} &= \Omega_\pi\left(-n_{RE}k + \frac{\omega\beta(1-n_{RE})}{1-\omega\beta}\right) \\
\gamma_{\pi\pi} &= \Omega_\pi\left(1 + \frac{(1-\omega)\omega\beta^2(1-n_{RE})}{1-\omega\beta}\right) \\
\gamma_{\pi Y1} &= \Omega_\pi\left(-(1-n_{RE})\left(\frac{k}{1-\omega\beta}\right)\right) \\
\gamma_{\pi\Pi 1} &= \Omega_\pi\left(-(1-n_{RE})\frac{(1-\omega)\beta}{1-\omega\beta}\right)
\end{aligned}$$

Plugging (B.5) into (B.4) and rearranging terms we finally get

$$\begin{aligned}
E_t\widehat{y}_{t+1} &= \gamma_{yy}\widehat{y}_t + \gamma_{y\pi}\widehat{\pi}_t + \gamma_{yY1}\widehat{y}_{t-1} + \gamma_{y\Pi 1}\widehat{\pi}_{t-1} \\
E_t\widehat{\pi}_{t+1} &= \gamma_{\pi y}\widehat{y}_t + \gamma_{\pi\pi}\widehat{\pi}_t + \gamma_{\pi Y1}\widehat{y}_{t-1} + \gamma_{\pi\Pi 1}\widehat{\pi}_{t-1}
\end{aligned} \tag{B.6}$$

where

$$\begin{aligned}
\gamma_{yy} &= q_{yy} + q_{yE\pi}\gamma_{\pi y} \\
\gamma_{y\pi} &= q_{y\pi} + q_{yE\pi}\gamma_{\pi\pi} \\
\gamma_{yY1} &= q_{yY1} + q_{yE\pi}\gamma_{\pi Y1} \\
\gamma_{y\Pi 1} &= q_{y\Pi 1} + q_{yE\pi}\gamma_{\pi\Pi 1}.
\end{aligned}$$

System (B.6) can be rewritten in matrix form as in (3.13).

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