Intertemporal Greenwald-Stiglitz

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1 Introduction

In the early '90s Greenwald and Stiglitz (GS) proposed a theoretical framework to analyze the behaviour of firms in the presence of asymmetric information and bankruptcy costs (Greenwald and Stiglitz, 1993). In a nutshell, they show that when the firm maximizes expected profits net of bankruptcy costs, output (y) is increasing with net worth (a). For the sake of discussion, let's assume: $y_t = y(a_{t-1})$.

The level of net worth is determined, in turn, by a law of motion: $a_t = l(a_{t-1}, y_t)$. Substituting the first equation into the second one, one can track over time the sequence of net worth and production. This dynamic linkage is the source of the GS variant of the financial accelerator.[1]

One of the most frequent objections to this approach focuses on the following feature: Agents optimize period by period without taking into account the law of motion of net worth. In other words the firm is choosing y but not a, which is determined "mechanically" by the law of motion. The problem with GS is twofold: agents' maximization is not intertemporal and the capital structure of the firm is not chosen optimally.

In this paper we take this objection seriously and cast the original framework in an intertemporal setting in order to show the complications (but also the insights) that arise from intertemporal optimization and optimal capital structure decisions.

The paper is organized as follows. In section 2 we summarize the main tenets of a simplified GS framework that is cast in a period by period optimization setting (as in the 1993 paper). In section 3 we propose and discuss an intertemporal GS optimization setting. Section 3 concludes.

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[1] There are two other approaches in the financial accelerator literature. Bernanke-Gertler (1989, 1990) and Bernanke-Gertler-Gilchrist (1999) emphasize ex post asymmetric information and agency costs: Kiyotaki and Moore (1997) focus on collateral constraints.

2 Period by period optimization

The firm's (operating) profit is $\pi_t = (u_t - c)\, y_t$ where u_t is a idiosyncratic shock distributed as a uniform on $(0, 2)$, c is the (operating) average and marginal cost $(c < 1)$ and y_t is production. Hence expected operating profits are

$$E(\pi_t) = (1 - c)\, y_t \tag{1}$$

since $E(u_t) = 1$. Following GS, we assume that bankruptcy costs are increasing in the scale of production, i.e. $b(y_t)$ with $b_y(y_t) > 0$, and the probability of bankruptcy is decreasing with net worth, i.e. $\Phi(a_{t-1})$ with $\Phi_a(a_{t-1}) < 0$. The firm maximizes "profitability" V, i.e. expected operating profits net of expected bankruptcy costs. The maximization problem *in period* t is

$$\max_{y_t} V_t = (1 - c)\, y_t - b(y_t)\, \Phi(a_{t-1}) \tag{2}$$

Net worth is a *pre-determined variable* and will be considered given in the maximization problem.

The FOC of problem (2) is:

$$1 - c = b_y(y_t)\, \Phi(a_{t-1}) \tag{3}$$

i.e. the expected marginal operating profit $(1-c)$ must be equal to the expected marginal bankruptcy cost $b_y(y_t)\, \Phi(a_{t-1})$.[2]

From the FOC one gets optimal output:

$$y_t = b_y^{-1}\left(\frac{1 - c}{\Phi(a_{t-1})}\right) \tag{4}$$

where $b_y^{-1}(.)$ denotes the inverse function of $b_y(.)$.

The law of motion of the firm's net worth is:

$$a_t = \alpha + \rho a_{t-1} + (1 - c)\, y_t \tag{5}$$

Equations (4) and (5) define a system of non-linear difference equations in y_t, a_t. Substituting the first equation into the second one we get:

$$a_t = \alpha + \rho a_{t-1} + (1 - c)\, b_y^{-1}\left(\frac{1 - c}{\Phi(a_{t-1})}\right)$$

i.e. a non-linear difference equation in a_t.

[2] A different but equivalent interpretation goes as follows: the expected marginal revenue (1) must be equal to the expected marginal cost, consisting of the expected marginal operating cost (c) and the expected marginal bankruptcy cost $b_y(y_t)\, \Phi(a_{t-1})$.

2.1 Example 1

Suppose $b(y_t) = \frac{\gamma}{2} y_t^2$ and $\Phi(a_{t-1}) = \frac{\varphi}{a_{t-1}}$ where γ and φ are positive parameters, $\varphi \leq a_{t-1}$.[3] Then (4) becomes $y_t = \frac{1-c}{\varphi\gamma} a_{t-1}$ and (5) becomes $a_t = \alpha + \left[\rho + \frac{(1-c)^2}{\varphi\gamma}\right] a_{t-1}$. Both equations are linear. In this case,the steady state values of a and y are:

$$\begin{aligned} a_s &= \left[1 - \rho - \frac{1}{\varphi\gamma}(1-c)^2\right]^{-1} \alpha \\ y_s &= \frac{1-c}{\varphi\gamma}\left[1 - \rho - \frac{1}{\varphi\gamma}(1-c)^2\right]^{-1} \alpha \end{aligned}$$

In order to have a positive and stable steady state we assume $\frac{1}{\varphi\gamma}(1-c)^2 < 1 - \rho$.

We propose now a useful reformulation of the example. Let's define a new variable $x_t := \frac{y_t}{a_{t-1}}$ which we can conveniently interpret as proxy of *"leverage"*. If technology were linear in capital, for example, and there were a production lag ($y_t = \nu k_{t-1}$), then $x_t := \nu \frac{k_{t-1}}{a_{t-1}}$ where $\frac{k_{t-1}}{a_{t-1}}$ is indeed one definition of leverage.[4]

Remark 1 *In our example, period by period optimization leads to* $x_t = \frac{1-c}{\varphi\gamma} = \bar{x}$ *i.e. leverage is constant (in and out of the steady state). The law of motion of net worth becomes* $a_t = \alpha + [\rho + (1-c)\bar{x}] a_{t-1}$.*In the steady state* $a_s = \{1 - [\rho + (1-c)\bar{x}]\}^{-1} \alpha$ *and* $y_s = \bar{x} a_s$. *The steady state is stable if* $\bar{x} < (1-\rho)/(1-c)$ *i.e. if leverage is not "too high".*

3 Intertemporal optimization

In this section we assume that the firm has to decide optimally how much to produce and whether to pay dividends or not in each period of an infinite time horizon. The firm will take into account the law of motion of net worth:

$$a_{t+s} = \alpha + \rho a_{t+s-1} + (1-c) y_{t+s} - d_{t+s} \tag{6}$$

where $(1-c) y_{t+s}$ are profits expected in period t+s [5] and d_{t+s} are dividends. The intertemporal maximization problem consists in choosing a production plan $y_{t+s}, \quad s = 0, 1, 2...$ that maximizes the expected value *at t* of the discounted sum of operating profits net of bankruptcy costs over an infinite time horizon:

$$\sum_{s=0}^{\infty} \beta^s V_{t+s} = \sum_{s=0}^{\infty} \beta^s \left[(1-c) y_{t+s} - b(y_{t+s}) \Phi(a_{t+s-1})\right] \tag{7}$$

[3] This condition bounds the probability of bankruptcy to range between zero (when $a \to \infty$) and one $(a = \varphi)$. The parameter φ therefore, can be interpreted as a bankruptcy threshold.

[4] The balance sheet identity is $k = a + b$ where b represents debt. Hence $k/a = (b/a) + 1$.The ratios k/a and b/a are two possible definitions of leverage.

[5] Notice that expected profits depend only on the scale of activity, which is a control variable. This is due to the fact that in each period $t+s$, $s = 0, 1, 2$...the idiosyncratic shock u_{t+s} is drawn from a time invariant uniform distribution such that $E(u_{t+s}) = 1$.

subject to the following sequence of constraints:

$$d_{t+s} = \alpha + \rho a_{t+s-1} + (1-c)\, y_{t+s} - a_{t+s} \geq 0 \qquad s = 0, 1, 2... \tag{8}$$

Notice that (8) can be rewritten as follows:

$$a_{t+s} \leq \alpha + \rho a_{t+s-1} + (1-c)\, y_{t+s} \qquad s = 0, 1, 2.. \tag{9}$$

The constraint is binding if the firm does not distribute dividends, which means that it devotes all the profits to the accumulation of net worth. Therefore we can write the intertemporal optimization problem of the firm as max (7) subject to (9). The Lagrangian is

$$\begin{aligned} \mathcal{L}_t &= \sum_{s=0}^{\infty} \beta^s \left[(1-c)\, y_{t+s} - b\left(y_{t+s}\right) \Phi\left(a_{t+s-1}\right) \right] + \\ &\quad + \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} \left[\alpha + \rho a_{t+s-1} + (1-c)\, y_{t+s} - a_{t+s} \right] \end{aligned}$$

Here is a section of the Lagrangian:

$$\begin{aligned} &\mathcal{L}_t = (1-c)\left(y_t + \beta y_{t+1} + ...\right) - \left[b\left(y_t\right) \Phi\left(a_{t-1}\right) + \beta \Phi\left(a_t\right) b\left(y_{t+1}\right) + ... \right] + \\ &+ \lambda_t \left[\alpha + \rho a_{t-1} + (1-c)\, y_t - a_t \right] + \beta \lambda_{t+1} \left[\alpha + \rho a_t + (1-c)\, y_{t+1} - a_{t+1} \right] + ... \end{aligned}$$

The FOCs with respect to y_t and a_t are:

$$(1+\lambda_t)(1-c) = b_y\left(y_t\right) \Phi\left(a_{t-1}\right) \tag{10}$$

$$-\beta \Phi_a\left(a_t\right) b\left(y_{t+1}\right) = \lambda_t - \beta \rho \lambda_{t+1} \tag{11}$$

From (10) it is clear that $\lambda_t > 0$.The LHS of (11) is positive since $\Phi_a(a) < 0$ by assumption. Hence $\lambda_t > \beta \rho \lambda_{t+1}$. This means that the Lagrange multipliers cannot be zero and therefore the firm does not pay dividends. The constraints are binding.

Notice that, since $\lambda_t > 0$, in the present scenario the firm is producing

$$y_t = b_y^{-1} \left((1+\lambda_t) \frac{(1-c)}{\Phi\left(a_{t-1}\right)} \right) \tag{12}$$

i.e. more than the optimal quantity in the case of one period optimization (see equation (4)).

Remark 2 *In the intertemporal optimization problem, the firm does not pay dividends, i.e. it devotes expected profits only to the accumulation of net worth, and sets the optimal scale of activity at a level higher than the level that maximizes expected profits net of bankruptcy cost.*

From the fact that the constraints are binding, from (6) we derive (5). Moreover, substituting the equations for the multipliers derived from (10) into (11) one gets

$$-\beta b\left(y_{t+1}\right)\Phi_a\left(a_t\right)=\frac{1}{1-c}\left[b_y\left(y_t\right)\Phi\left(a_{t-1}\right)-\beta\rho b_y\left(y_{t+1}\right)\Phi\left(a_t\right)\right]-(1-\beta\rho) \tag{13}$$

Remark 3 *The dynamics of the economy under scrutiny (i.e. the evolution over time of y_t and of a_t will be determined jointly by equations (5) and (13). Notice that equation (5) is linear. Equation (13) is determined by the specific functional form for the functions $b(y)$ and $\Phi(a)$.*

3.0.1 Example 2

Suppose, as in the example above, that $b(y)=\frac{\gamma}{2}y^2$ and $\Phi(a)=\frac{\varphi}{a}$. In this case, after some algebra and rearrangement, from (13) we obtain the following non linear first order difference equation in the state variable "leverage":

$$\frac{(1-c)\beta}{2}x_{t+1}^2+\beta\rho x_{t+1}+\frac{(1-\beta\rho)(1-c)}{\varphi\gamma}=x_t \tag{14}$$

where $x_{t+1}:=\dfrac{y_{t+1}}{a_t}$. The steady state ratio can be determined solving for x the following quadratic equation: $\frac{(1-c)\beta}{2}x^2-(1-\beta\rho)x+\frac{(1-\beta\rho)(1-c)}{\varphi\gamma}=0$. The roots of this equation are

$$x_s=\frac{1-\beta\rho}{\beta(1-c)}\pm\frac{\sqrt{1-\beta\rho}}{\beta(1-c)}\sqrt{1-\beta\rho-2\frac{(1-c)^2\beta}{\varphi\gamma}}$$

We assume that the discriminant is positive $\left(1-\beta\rho>2\frac{(1-c)^2\beta}{\varphi\gamma}\right)$. The phase plot is depicted in figure (1).

It is easy to see that the greater root is a stable steady state, while the smaller root is unstable.Once the steady state ratio is determined, we can retrieve the steady state net worth from (5). In fact in the steady state: $a(1-\rho)=\alpha+(1-c)x_s a$ which implies

$$\begin{aligned}a_s&=\frac{\alpha}{1-\rho-(1-c)x_s}\\ y_s&=x_s a_s=\frac{\alpha x_s}{1-\rho-(1-c)x_s}\end{aligned}$$

4 Conclusions

In this paper we cast the GS financial accelerator framework – which was originally defined in a period by period optimization setting – in an intertemporal context. In order to get clearcut results, we simplify the original framework to

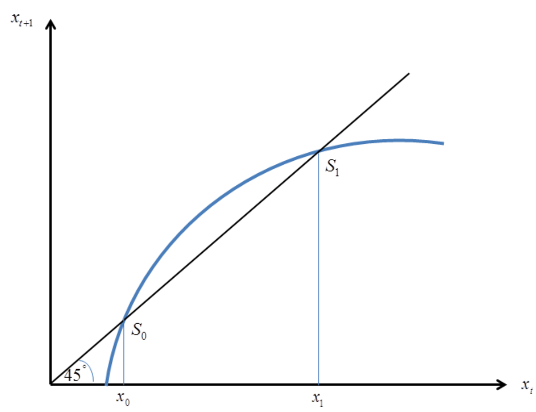

Figure 1: Phase plot of equation (14)

a great extent. Under specific restrictions on parameters and functional forms (cost of bankruptcy quadratic in output and bankruptcy probability decreasing and convex in net worth), in a period by period optimization, leverage is constant (in and out of the steady state) while in an intertemporal setting, leverage is changing over time and evolves according to a quadratic law of motion. This dynamic model, under appropriate restriction, has two steady states. The "low leverage" steady state is characterized by low net worth and low output and is unstable. The "high leverage" steady state, on the contrary, is characterized by high net worth and high output.

References

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