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Markov–Perfect Nash Equilibria in Models With a Single Capital Stock

Engelbert Dockner · Florian Wagener

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Abstract Many economic problems can be formulated as dynamic games in which strategically interacting agents choose actions that determine the current and future levels of a single capital stock. We study necessary as well as sufficient conditions that allow us to characterise Markov-perfect Nash equilibria for these games. These conditions can be translated into an auxiliary system of ordinary differential equations that helps us to explore stability, continuity and differentiability of these equilibria. The techniques are used to derive detailed properties of Markov-perfect Nash equilibria for several games including voluntary investment in a public capital stock, the inter-temporal consumption of a reproductive asset, and the pollution of a shallow lake.

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1 Introduction

Many economic problems can be formulated as dynamic games in which strategically interacting agents choose actions based on an inter-temporal objective that determine

E.J. Dockner
Institute for Finance, Banking and Insurance Department of Finance, Accounting and Statistics WU Vienna University of Economics and Business, Heiligenstädter Strasse 46-48, A – 1190, Austria,
E-mail: engelbert.dockner@wu.ac.at

F.O.O. Wagener (corresponding author)
CeNDEF, School of Economics, Universiteit van Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands, t + 31 20 525 4034, f + 31 20 525 4349
E-mail: f.o.o.wagener@uva.nl

the current and future levels of a single capital stock. When formulated in continuous time these games are differential games with a single state variable.[1]

In this paper we formulate a class of differential games in which n players either exploit or accumulate a single capital stock by choosing Markov strategies, where they select their current actions by fixing a policy function that relates the current state of the system (the single capital stock) to current actions. State dependent Markov (or feedback) strategies can be contrasted to strategies that are set as simple time paths at the beginning of the game with the need for every player to pre-commitment to the announced time profile throughout the entire duration of the game. From an economics modeling point of view pre-commitment is a very strong assumption for a dynamic game and largely unattractive.[2] On the contrary Markov equilibrium strategies exhibit several desirable properties such as subgame perfectness, in case they are derived using backward induction, and no commitment, allowing rival players to immediately react to unexpected changes in the state of the system.

Finding subgame perfect Markov Nash equilibrium strategies of a differential game, even if the game is of the linear-quadratic type, is a formidable analytical problem. For instance, to find a Markov-perfect Nash equilibrium in the general case of n players and m state variables requires to solve a system of n coupled nonlinear implicit m-dimensional partial differential equations (PDE). In case the underlying economic system can be described by a single state variable (a single capital stock) the system of PDE's collapses to a system of ordinary differential equations in the value functions that is much easier to deal with. Because of this tractability, the paper focuses on the least complex situation $m = 1$.

This system of ordinary differential equations in the value functions can be solved explicitly and the Markov-perfect Nash equilibrium (MPNE) can be derived analytically only for a restricted class of specific functional forms of the primitives of the model. Starting with the pioneering work of Case (1979), differential game theorists have modified this approach. Instead of working with the ordinary differential equations in the value functions, they derive a system of differential equations in shadow prices, that is, in the first derivatives of the value functions. Structurally this system is much simpler to work with, in particular when symmetric equilibrium strategies are analyzed. For the shadow price system reduces then to a single quasi-linear[3] differential equation, explicitly dependent on the state variable, which for specific functional forms of the state equation and the objective functionals can be solved explicitly.

Using the shadow price system approach, Tsutsui and Mino (1990) derived nonlinear Markov equilibria for a linear quadratic differential game. The same approach was used by Dockner and Long (1994) in a model of transboundary pollution control and by Wirl (1996) in a public goods investment problem.

[1] For a general introduction to the theory of differential games we refer the reader to Dockner et al. (2000).

[2] Pre-commitment strategies that are set as time functions only are also referred to as open-loop strategies and the corresponding dynamic game as an open-loop game.

[3] A differential equation is called quasi-linear if it is linear in the highest derivatives of the unknown function.

For a differential game with m state variables and n players, Rincón-Zapatero et al. (1998)[4] show that the approach introduced by Case (1979) can be made systematic when two assumptions are satisfied. The game must have an equal number of state and control variables and equilibria must be restricted to interior MPNE. Rincón-Zapatero et al. (1998) differentiate the Bellman equations to arrive at a system of quasi-linear partial differential equations in the shadow prices. Using the maximum condition they are able to eliminate shadow prices and arrive at what can be referred to as a generalised Euler equation system (GEES) that is a system of partial differential equations in the state and control variables. By solving an example they already point out that if the game is characterised by a single state variable the GEES reduces to a linear system of ordinary differential equations.[5]

The shadow price system and the GEES can be seen as two approaches to characterise MPNE, which are often mathematically equivalent. In this paper we extend these two approaches substantially for n-player differential games with a single state variable and an infinite horizon. The n-dimensional system of ordinary quasi-linear non-autonomous differential equations in shadow prices is used to derive an auxiliary $(n+1)$-dimensional system of ordinary autonomous differential equations whose solution trajectories trace out graphs of the equilibrium strategies. The auxiliary system opens up the opportunity to geometrically analyze and study Markov-perfect Nash equilibria for games with general functional forms. We introduce the concepts of local and global Markov equilibria and point out how the auxiliary system can be used to identify these two types of equilibria. In addition, the auxiliary system can be used to gain important insights into the continuity and differentiability properties of MPNE. Points where the Markov strategies are continuous but not differentiable can conveniently be described by singularities of the auxiliary system. Moreover, using the auxiliary system we are also able to find non-continuous Markov-perfect Nash strategy equilibria. The derivation of the auxiliary system and its use to characterise non-continuous and non-differentiable Markov-perfect Nash equilibria for differential games with a single state variable and general functional forms comprises, together with the analysis of several economic examples, the main contributions of this paper.

Dynamic games with a single capital stock can be applied in resource economics where n agents exploit a single renewable or exhaustible resource so as to maximise the present value of future consumption, see for example Levhari and Mirman (1980), Sundaram (1989), Benhabib and Radner (1992), Clemhout and Wan (1994), Dutta and Sundaram (1993), Dockner and Sorger (1996), Rincón-Zapatero et al. (1998) and Benchekroun (2003).

A game with n agents investing in a single public stock of capital also fits the class of differential games analyzed in this paper, see Fershtman and Nitzan (1991), Wirl (1996) and Rowat (2007). Dynamic public bads games arise in case of transboundary pollution where the emissions of countries accumulate a single stock of pollution that incurs costs for each player. For more details on these kinds of game problems see

[4] See also the paper by Rincón-Zapatero (2004) for differential games and Josa-Fombellida and Rincón-Zapatero (2007) for stochastic control problems.

[5] Kossioris et al. (2008) apply the shadow price system approach to an environmental economics problem.

van der Ploeg and de Zeeuw (1992), Dockner and Long (1994) and Dockner et al. (1996). Finally, environmental economists have recently started to explore equilibria in the shallow lake problem. This problem is structurally similar to the exploitation of a single renewable resource stock but with a non-concave production function. Recent papers dealing with the shallow lake problem include Brock and Starrett (2003), Mäler et al. (2003), Wagener (2003), Kossioris et al. (2008), Kiseleva and Wagener (2010), Kossioris et al. (2010).

The article is organised as follows. Section 2 presents the general theory to derive MPNE for the class of differential games with a single state variable and includes a linear quadratic example. Section 3 makes use of the auxiliary system to study non-continuity and non-differentiability of MPNE. Section 4 applies the approach presented to two distinct examples from resource and environmental economics and section 5 concludes.

2 General theory

In this section we derive the auxiliary system for general feedback Nash equilibria in a dynamic game with a single state variable. In this game, n players choose Markov strategies, $u_i(x)$, to maximise an inter-temporal objective function. The strategies determine the level of a single capital stock, x, that is governed by the state dynamics. For this game we characterise Markov perfect Nash equilibria that are either differentiable, or continuous, or have at most a finite number of jump points.

2.1 Definitions

We consider a game where n players can, at every point $t \geq 0$ in time, choose actions from a given action set. These actions determine the evolution of an underlying state variable $x(t)$ that takes values in a state space X; we shall call $x(t)$ the 'state of the game at time t'. The initial value of the state will be denoted by x_0:

$$x(0) = x_0 \in X.$$

We restrict attention to the case that X is a closed interval of the real line $\mathbb{R}$. For every $x \in X$, the set $U(x)$ of actions u available to one of the players is the closure of a convex open subset of $\mathbb{R}^q$. We do not require the sets $U(x)$ to be bounded. The union

$$U = \bigcup_{x \in X} \{x\} \times U(x) \subset X \times \mathbb{R}^q$$

is the *action space* of the player. We shall also assume that the action space is the closure of an open set.

Each player formulates his action choices in terms of a strategy, which specifies for each point in time which action to take. A *Markov strategy* is characterised by the requirement that the actions at each point in time are conditioned only on the state of the system. That is, a Markov strategy is a function $u : X' \to \mathbb{R}^q$, where X' is a subinterval of X with $x_0 \in X'$, such that if $x(t) \in X'$, the agent takes at time t the action $u(x(t))$. Necessarily we have that

$$u(x) \in U(x)$$

for all $x \in X'$; or, equivalently, the graph of u should be contained in the action space U. If $X' = X$, we call the Markov strategy *globally defined* or just *global*; if $X' \subset X$, it is said to be *locally defined* or *local*.

Let i be an index, running from 1 to n, which denotes the different players; player i's strategy is then a real-valued function u_i defined on the interval X'_i. Let

$$X' = \bigcap_{i=1}^{n} X'_i$$

be the common domain of definition of all the strategies, and introduce the *strategy vector* $\mathbf{u} : X' \to (\mathbb{R}^q)^n$, given as

$$\mathbf{u}(x) = (u_1(x), \cdots, u_n(x)) .$$

Note that the elements of the strategy vector are the individual strategies. Also introduce the vector

$$\mathbf{u}_{-i}(x) = (u_1(x), \cdots, u_{i-1}(x), u_{i+1}(x), \cdots, u_n(x))$$

of strategies other than the strategy of player i. Given a strategy vector $\mathbf{u}$, the state variable will evolve according to a state equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mathbf{u}(x)), \quad x(0) = x_0. \tag{1}$$

On the right hand side of equation (1), the time argument of the function x is suppressed for readability; this will be done throughout the article.

It will be assumed that the vector field f satisfies the consistency requirement that for all available actions it is 'inward pointing' on the boundary ∂X of the state space X. In the present context, this means the following. For $x \in \partial X$ let $\nu(x)$ be an outward pointing unit 'vector': that is, $\nu(x) = 1$ if x is the upper endpoint of X, and $\nu(x) = -1$ if x is the lower endpoint. Let

$$\mathbf{U}(x) = U_1(x) \times \cdots \times U_n(x) \subset (\mathbb{R}^q)^n$$

be the set of available actions at x, and let $\mathbf{U} = \cup_x \{x\} \times \mathbf{U}(x)$. Then f is *inward pointing (with respect to* $\mathbf{U}$*)*, if for $x \in \partial X$ the inequality

$$f(x, \mathbf{u}) \cdot \nu(x) \leq 0$$

holds for all $\mathbf{u} \in \mathbf{U}(x)$.

The pay-off of the players will depend on their strategies as well as on the state dynamics. In this article, we assume that the pay-off of the strategy choice u_i of player i, given that the other players play $\mathbf{u}_{-i}$, is of the general form

$$J_i(u_i, \mathbf{u}_{-i}) = \int_0^\infty L_i(x, \mathbf{u}(x)) \mathrm{e}^{-\rho t} \mathrm{d}t.$$

We shall assume the functions f and L_i to be *smooth*, which is meant to be 'infinitely often differentiable', but which could be read as 'as often differentiable as is necessary'. But even with smooth data, the set of strategies that are available to the players has to be restricted in order for the dynamics and the pay-offs to be at least well–defined. The specification of the available strategy set is an integral part of the specification of the game in question.

Attention will be restricted to Markov strategies that are piecewise continuously differentiable and bounded. That is, we assume that for every strategy u_i there are finitely many non-overlapping intervals that cover X_i' and which are such that u_i is continuously differentiable on the interior of each interval. Note that the strategies are not required to be continuous.

The right hand side of (1) is not necessarily Lipschitz-continuous, and the theorem of existence and uniqueness of solutions to differential equations does not apply at those points where Lipschitz-continuity fails to hold. By assumption, these points are a subset of the endpoints of the non-overlapping intervals covering X_i'; in particular there are only finitely many of them. We need to specify in what sense we interpret (1) at those points; this is done in Appendix A

We say that, given the strategies $\mathbf{u}_{-i}$ of the other players, a strategy u_i is *admissible* or *available to player* i, if it is a bounded piecewise continuously differentiable function on X' such that its graph is contained in U_i, and such that it satisfies the following consistency condition. Let $F(x) = f(x, u_i(x); \mathbf{u}_{-i}(x))$, fix a point $\hat{x}$, and denote by F_L and F_R respectively the left and right limit of $F(x)$ as x tends to $\hat{x}$. The condition requires that if $F_L > 0 > F_R$, then the value of $u_i(\hat{x})$ satisfies

$$F(\hat{x}) = f(\hat{x}, u_i(\hat{x}); \mathbf{u}_{-i}(\hat{x})) = 0. \tag{2}$$

Note that such a value always exists as a consequence of the intermediate value theorem and the convexity of $U_i(\hat{x})$. This mathematical condition may be interpreted as follows: by choosing the strategy u_i such that $F_L > 0$ and $F_R < 0$, player i intends to stabilise the system at $x = \hat{x}$. But then the choice of $u_i(\hat{x})$ should reflect that $\hat{x}$ is a stable steady state.

The space of strategies available to player i, given X and U_i, is denoted by $\mathscr{A}_i$. The spaces X and $\mathbf{U}$, together with the dynamics f and the instantaneous pay-offs L_i and the space of available strategies $\mathscr{A}_i$, for $i = 1, \ldots, n$, define a differential game $\mathscr{G}$.

We recall the definition of a Markov-perfect Nash equilibrium strategy of a game.

Definition 1 The strategy vector $\mathbf{u}^*$ is an *(global) Markov perfect Nash equilibrium* of $\mathscr{G}$, if

$$J_i(u_i^*, \mathbf{u}_{-i}^*) \geq \max_{u_i \in \mathscr{A}_i} J_i(u_i, \mathbf{u}_{-i}^*)$$

for each i; that is, if for each player his strategy is a best response to the strategies of the other players.

When investigating Markov equilibrium strategies for a differential game, the phenomenon is encountered that the Hamilton-Jacobi equation, which characterises these strategies, has typically many solutions that are not defined on the whole state space (see for instance Wirl 1996, Kossioris et al. 2008). In the context of the original game, these solutions are not admissible, as they do not specify the action of the players if the state leaves the domain of definition of one of the strategies. To address this, we introduce the concept of a local Markov perfect Nash equilibrium as follows.

Spaces X', U_i' and $\mathscr{A}_i'$ define a *restriction* $\mathscr{G}'$ of the game $\mathscr{G}$, if $X' \subset X$, $U_i' \subset U_i$, if f is inward pointing on $\partial X'$ with respect to $\mathbf{U}'$, and if $\mathscr{A}_i'$ is the set of strategies available to player i, given X' and U_i'.

Definition 2 The strategy vector $\mathbf{u}^*$ is a *local Markov perfect Nash equilibrium* of $\mathscr{G}$, if it is a global Markov perfect Nash equilibrium for a suitable restriction $\mathscr{G}'$ of $\mathscr{G}$.

In economic terms, a local Markov perfect Nash equilibrium might arise if the players can commit cooperatively on restricting their action spaces and then proceed to play non-cooperatively with the restricted action spaces. See Rowat (2007) for a discussion of this 'endogenising' of the state space. Alternatively, the restriction of the action spaces could be imposed by a regulating agency. We shall see that in several examples, local Markov perfect Nash equilibria may improve on the global equilibria.

As a consequence, we obtain the notion of a state that can be *stabilised*, or is *stabilisable*, by a local Markov-perfect Nash strategy.

Definition 3 A state $x_* \in X$ can be *stabilised by a local Markov-perfect Nash equilibrium strategy* $\mathbf{u}^*$, if there is a restriction $\mathscr{G}' = (X', \{U_i, \mathscr{A}_i'\})$ of the differential game such that

1. $\mathbf{u}^*$ is a Markov perfect Nash equilibrium for $\mathscr{G}'$;
2. X' contains x_* in its interior;
3. x^* is a stable steady state for the stock evolution dynamics

$$\dot{x} = f(x, \mathbf{u}^*(x)).$$

Stabilisable steady states are those long-term steady states which can be obtained in a non-cooperative differential game if the players play a (local) Nash equilibrium. One might want to call them *Nash attractors*.

2.2 The vector Hamilton-Jacobi equation.

Given an n-person differential game $\mathscr{G}$, the present value Pontryagin function[6] of player i reads as

$$P_i(x, p_i, u_i; \mathbf{u}_{-i}) = L_i(x, \mathbf{u}) + p_i f(x, \mathbf{u}).$$

If this function is maximised, with respect to u_i, at

$$u_i = v_i(x, p_i; \mathbf{u}_{-i}) \tag{3}$$

the Hamilton function is defined as

$$H_i(x, p_i; \mathbf{u}_{-i}) = P_i\left(x, p_i, v_i(x, p_i; \mathbf{u}_{-i}); \mathbf{u}_{-i}\right).$$

[6] Also called Hamilton, pre-Hamilton or unmaximised Hamilton function.

The Hamilton-Jacobi equation for the value function of player i reads then as

$$\rho V_i(x) = H_i(x, V_i'(x); \mathbf{u}_{-i}(x)).$$

Introduce the notations

$$\mathbf{v}(x, \mathbf{p}, \mathbf{u}) = (v_1(x, p_1; \mathbf{u}_{-1}), \cdots, v_n(x, p_n; \mathbf{u}_{-n}))$$

and

$$\frac{\partial \mathbf{v}}{\partial \mathbf{u}_-} = \begin{pmatrix} 0 & \frac{\partial v_1}{\partial u_2} & & & \frac{\partial v_1}{\partial u_n} \\ \frac{\partial v_2}{\partial u_1} & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \frac{\partial v_{n-1}}{\partial u_n} \\ \frac{\partial v_n}{\partial u_1} & & & \frac{\partial v_n}{\partial u_{n-1}} & 0 \end{pmatrix}.$$

In order to eliminate the functions $u_i(x)$ from the problem, the system of equations

$$F_i(x, \mathbf{u}) = u_i - v_i(x, p_i; \mathbf{u}_{-i}) = 0, \qquad i = 1, \cdots, n \tag{4}$$

has to be solved for the u_i; in vector notation, this system reads as

$$\mathbf{F}(x, \mathbf{u}) = \mathbf{u} - \mathbf{v}(x, \mathbf{p}, \mathbf{u}) = 0.$$

We assume that this system is solvable for $\mathbf{u}$, and that the solution

$$\mathbf{u} = \hat{\mathbf{u}}(x, \mathbf{p})$$

is a continuously differentiable function of x and $\mathbf{p}$. For instance a sufficient condition for the solvability of the system is that the matrix

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} = I - \frac{\partial \mathbf{v}}{\partial \mathbf{u}_-}$$

is invertible everywhere.

Consequently, it may be assumed that the *game* Hamilton functions G_i, $i = 1, \cdots, n$ of the players can be written as

$$G_i = G_i(x, V_1', \cdots, V_n') = H_i(x, V_i'; \hat{\mathbf{u}}_{-i}(x, \mathbf{V}'(x))),$$

and that we solve the following vector Hamilton-Jacobi equation

$$\rho \mathbf{V}(x) = \mathbf{G}(x, \mathbf{V}'(x)), \tag{5}$$

where $\mathbf{G} = (G_1, \cdots, G_n)$. Taking derivatives with respect to x and substituting

$$\mathbf{p}(x) = \mathbf{V}'(x)$$

yields

$$\rho \mathbf{p}(x) = \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{G}}{\partial \mathbf{p}} \mathbf{p}'(x). \tag{6}$$

Note that $\partial \mathbf{G}/\partial x$ is an n-dimensional vector, whereas $\partial \mathbf{G}/\partial \mathbf{p}$ is an $n \times n$ matrix. We obtain finally the equation

$$\frac{\partial \mathbf{G}}{\partial \mathbf{p}} \mathbf{p}'(x) = \rho \mathbf{p} - \frac{\partial \mathbf{G}}{\partial x}. \tag{7}$$

Equation (7) is referred to as the *shadow price system*. Due to the special structure of our class of games the shadow price system is a system of quasi-linear differential equations in $\mathbf{p}(x)$. As already pointed out, the shadow price system approach traces back to the analysis of Case (1979) who studied non-linear Markov equilibria for the sticky price model, which was also analysed in detail by Tsutsui and Mino (1990). The shadow price system approach has subsequently been applied by Dockner and Long (1994) and by Wirl (1996) to derive non-linear symmetric Markov-perfect Nash equilibria.

If the relation $\mathbf{u} = \hat{\mathbf{u}}(x, \mathbf{p})$ can be solved for $\mathbf{p}$, which is often possible if $\mathbf{u}$ is in the interior of the action space $\mathbf{U}$, then we can rewrite equation (7) in terms of the actions $\mathbf{u}$, which is often more convenient in applications; we do this regularly in the examples given further below. Actually Rincón-Zapatero et al. (1998) analyse a version of equation (7) expressed in controls rather than costates for state spaces of general dimension, and demonstrate its applicability by considering specific examples. However, in problems where control constraints become active, or more generally when the relation between controls and costates is not one-to-one, the analysis has to be done in terms of the shadow price vector $\mathbf{p}$.

In the important symmetric special case that all players are equal and play the same strategies, the vector $\mathbf{G}$ of game Hamilton functions is invariant under permutations of the p_i,

$$G_i(x, p_1, \cdots, p_n) = G_{\sigma(i)}(x, p_{\sigma(1)}, \cdots, p_{\sigma(n)}),$$

where σ is any permutation of n elements. If this is the case, the game is called *symmetric*.

Symmetric Markov-perfect Nash equilibria then correspond to value functions $\mathbf{V}(x) = (V(x), \cdots, V(x))$, which are sought as solutions of the scalar Hamilton-Jacobi equation

$$\rho V(x) = G(x, V'(x)), \tag{8}$$

where

$$G(x, p) = \frac{1}{n!} \sum G_{\sigma(i)}(x, p_{\sigma(i)}, \cdots, p_{\sigma(n)}), \tag{9}$$

where the sum is taken over all permutations of n elements. Of course, the sum is just equal to $G_1(x, p, \cdots, p)$.

2.3 Sufficiency.

In the following we shall be interested, amongst other things, in finding geometric criteria that characterise possible jump points of Nash equilibrium strategies. As these are usually connected to points of nondifferentiability of the value function, the question arises in what sense the Hamilton-Jacobi equation is satisfied in such points. Crandall and Lions (1983) have shown that the value function of an optimal control problem is usually the only *viscosity solution* of the Hamilton-Jacobi equation. It has since then been widely accepted that this is the 'right' solution concept.

We recall the notion of viscosity solutions of scalar Hamilton-Jacobi equations of the form

$$\rho V - H(x, V'(x)) = 0. \tag{10}$$

For this we give some preliminary definitions.

Definition 4 A vector p is a *subgradient* of a function V at a point $\hat{x}$, if for all x in a neighbourhood of $\hat{x}$ we have

$$V(x) \geq V(\hat{x}) + \langle p, x - \hat{x} \rangle. \tag{11}$$

Equivalently, this can be expressed by requiring $\hat{x}$ to be a local minimiser of the difference $V - v$, where

$$v(x) = V(\hat{x}) + \langle p, x - \hat{x} \rangle. \tag{12}$$

The *subdifferential* $D_- V$ of V at $\hat{x}$ is the set of all subgradients.

Supergradients and superdifferentials are defined analogously.

The following definition of viscosity solutions, though not the most general, suffices for our purposes. It is adapted from chapter II of Fleming and Soner (2006).

Definition 5 Let V be a continuous function on X.

1. The function V is a *viscosity supersolution* of (10), if for all subgradients $p \in D_- V(x)$ we have
$$H(x, p) - \rho V(x) \leq 0. \tag{13}$$
2. The function V is a *viscosity subsolution* of (10), if for all supergradients $p \in D_+ V(x)$ we have
$$H(x, p) - \rho V(x) \geq 0. \tag{14}$$
3. Finally, V is a *viscosity solution* of (10), if it is both a viscosity subsolution and a viscosity supersolution.

If V is differentiable at x, then $D_+ V(x) = D_- V(x) = \{V'(x)\}$, and equation (10) holds in the classical sense.

The following theorem says that solving equation (5) indeed gives us a Markov perfect Nash equilibrium.

Theorem 1 *Let* $\mathbf{V}$ *be a continuous and piecewise continuously differentiable vector-valued function, such that* V_i *is a viscosity solution of*

$$\rho V_i(x) = H_i(x, V_i'(x); \mathbf{u}^*_{-i}(x)). \tag{15}$$

Assume for every admissible trajectory x *that*

$$\lim_{t \to \infty} \mathbf{V}(x(t)) \mathrm{e}^{-\rho t} = 0. \tag{16}$$

Let $\mathbf{u}^* : X \to \mathbb{R}^n$ *be a vector-valued function that satisfies the following conditions.*

1. *The strategy vector* $\mathbf{u}^*$ *is admissible.*

2a. *If* $\mathbf{V'}$ *is differentiable at* x*, then*

$$\mathbf{u}^*(x) = \hat{\mathbf{u}}(x, \mathbf{V'}(x)).$$

2b. *If* $\mathbf{V'}$ *is not differentiable at* x*, then either*

$$\mathbf{u}^*(\hat{x}) = \lim_{x \uparrow \hat{x}} \hat{\mathbf{u}}(x, \mathbf{V'}(x))$$

or

$$\mathbf{u}^*(\hat{x}) = \lim_{x \downarrow \hat{x}} \hat{\mathbf{u}}(x, \mathbf{V'}(x)).$$

Then $\mathbf{u}^*$ *is a Markov-perfect Nash equilibrium of the differential game.*

For instance, condition (16) is satisfied if all admissible trajectories are bounded. The proof of theorem 1 is given in appendix B.

We shall call $\mathbf{V}$ a viscosity solution of the equation

$$\rho \mathbf{V} = \boldsymbol{G}(x, \mathbf{V'}),$$

if for each i the function V_i is a viscosity solution of the associated equation (15).

2.4 Auxiliary system.

Recall the definition of the *adjoint* matrix A^* of a given matrix A: it is the matrix whose (i,j)'th element is the cofactor of A that is obtained by deleting the j'th row and i'th column of A and taking the determinant of the remaining matrix. We have that $AA^* = (\det A)I$, where I is the identity matrix; hence $A^{-1} = (\det A)^{-1}A^*$ if $\det A \neq 0$. Multiplying an equation of the form

$$Ax = b$$

from the left with A^* yields

$$(\det A)x = A^* b.$$

Multiplying the shadow price system (7) from the left with the cofactor matrix

$$(\partial \mathbf{G} / \partial \mathbf{p})^*$$

yields therefore

$$\left(\det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p})\right) \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}x}(x) = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p})\right)^* \left(\rho \mathbf{p}(x) - \frac{\partial \mathbf{G}}{\partial x}(x, \mathbf{p})\right).$$

The *auxiliary system* to equation (5) is now defined as

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \left(\dfrac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p})\right)^* \left(\rho \mathbf{p} - \dfrac{\partial \mathbf{G}}{\partial x}(x, \mathbf{p})\right), \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = \det \dfrac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p}), \end{cases} \tag{17}$$

where $s \in \mathbb{R}$ is some real parameter that has no immediate economic significance. In fact, we have the following result.

Theorem 2 *Let the function $\mathbf{V}(x)$ be continuous and piecewise continuously differentiable, and let it be a viscosity solution of*

$$\rho\mathbf{V}(x) = \mathbf{G}(x, \mathbf{V}'(x)). \tag{18}$$

Set $\mathbf{p}(x) = \mathbf{V}'(x)$ whenever the derivative is defined. Assume that x_0 and $\mathbf{p}_0$ are such that $\mathbf{p}$ is defined and continuous at x_0, such that $\mathbf{p}_0 = \mathbf{p}(x_0)$, and such that

$$\det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}_0) \neq 0.$$

Then $\mathbf{p}$ is continuously differentiable in a neighbourhood of x_0, and its graph is traced out by the curve

$$s \mapsto (x(s), \mathbf{p}(s))$$

that satisfies (17) *with initial conditions $(x(0), \mathbf{p}(0)) = (x_0, \mathbf{p}_0)$.*

This theorem characterises solutions of the vector Hamilton-Jacobi equation (18) whenever they are differentiable, by relating the graph $x \mapsto (x, \mathbf{V}'(x))$ to solution curves of the auxiliary system (17). This relation is general, and in particular, it can be applied to find non-symmetric Markov perfect Nash equilibrium strategies.

When attention is restricted to symmetric Nash equilibria, the auxiliary system simplifies to

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}s} = \rho p - \dfrac{\partial G}{\partial x}(x, p), \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial G}{\partial p}(x, p). \end{cases} \tag{19}$$

Mathematically speaking, these equations are the characteristic equations of the Hamilton-Jacobi equation (5). However, in crucial contrast to the 'one-player' optimal control situation, the parameter s is different from the time parameter t. In a way, it is this fact that allows the occurrence of many Nash equilibrium strategies.

2.5 A linear-quadratic example

This subsection illustrates the theory by applying it to a standard economic problem, the analysis of private investment in a public capital stock.

This game was introduced by Fershtman and Nitzan (1991). They assumed that each agent derives quadratic utility from the consumption of the public capital stock. Investment in the stock is costly and results in quadratic adjustment costs. Fershtman and Nitzan solved both the open-loop game and the game with Markov strategies and found that the dynamic free rider problem is more severe when agents use linear Markov strategies. Wirl (1996) challenged these results and studied the identical linear quadratic game but solved it for non-linear Markov equilibria. He found that if the discount rate is small enough non-linear Markov strategies can support equilibrium outcomes that are close to the efficient provision of the public capital. Finally, Rowat (2007) derived explicit analytic expressions for the non-linear Markov equilibria.

There are n players; player i voluntarily invests in the nonnegative public capital stock x at a rate $u_i \geq 0$. The single public capital stock evolves according to

$$\dot{x} = \sum_{j=1}^{n} u_j - \delta x; \tag{20}$$

here $\delta > 0$ is the constant depreciation rate. Following Fershtman and Nitzan we assume that player i's utility functional is given by

$$J_i = \int_0^\infty \left(ax - \frac{b}{2}x^2 - \frac{1}{2}u_i^2 \right) \mathrm{e}^{-\rho t} \mathrm{d}t, \tag{21}$$

where $a, b > 0$ are positive parameters. Note that compared to the formulation of Wirl (1996), one parameter has been scaled away. We see from this formulation that both X and $U_i(x)$, for all i and all $x \in X$, are equal to the interval $[0, \infty)$.

The corresponding present value Pontryagin function becomes

$$P_i(x, p_i, u_i; \mathbf{u}_{-i}) = ax - \frac{b}{2}x^2 - \frac{1}{2}u_i^2 + p_i \left(\sum_{j=1}^{n} u_j - \delta x \right).$$

The function $u_i \mapsto P_i(x, p_i, u_i; \mathbf{u}_{-i})$ is maximised at

$$u_i = v_i(p_i) = \begin{cases} p_i & p_i \geq 0, \\ 0 & p_i < 0. \end{cases}$$

The present value Hamilton function H_i of player i reads as

$$H_i(x, p_i; \mathbf{u}_{-i}) = \begin{cases} ax - \dfrac{b}{2}x^2 - \dfrac{1}{2}p_i^2 + p_i \left(\displaystyle\sum_{j \neq i} u_j - \delta x \right), & \text{if } p_i \geq 0, \\ ax - \dfrac{b}{2}x^2 + p_i \left(\displaystyle\sum_{j \neq i} u_j - \delta x \right), & \text{otherwise.} \end{cases}$$

We now restrict our attention to the symmetric case, for which all players use the same strategy. The symmetric version of equation (5) reads as

$$\rho V = G(x, V') = \begin{cases} ax - \dfrac{b}{2}x^2 + \dfrac{2n-1}{2}(V')^2 - \delta x V', & \text{if } V' \geq 0, \\ ax - \dfrac{b}{2}x^2 - \delta x V', & \text{otherwise.} \end{cases} \tag{22}$$

Fershtman and Nitzan (1991) obtained a solution to this equation by the well-known method of substituting $V(x) = c_0 + c_1 x + c_2 x^2$ and comparing coefficients of x. Wirl (1996) pointed out that due to the fact that the Hamilton-Jacobi equation (22) has no initial conditions, there may be actually more solutions to this equation. He derived his conclusions from the shadow price system (7), which takes the form

$$\begin{cases} \Big((2n-1)p - \delta x \Big) p' &= (\rho + \delta)p - a + bx \quad \text{if } p \geq 0, \\ -\delta x p' &= (\rho + \delta)p - a + bx \quad \text{otherwise.} \end{cases} \tag{23}$$

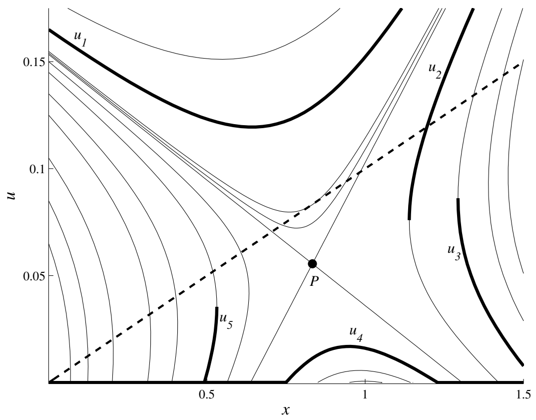

Figure 1 *Solutions of the auxiliary system (drawn curves) as well as the line of equilibria* $l_1 : \mathrm{d}x/\mathrm{d}t = 0$ *(thickly dashed line) and the isocline* $l_2 : \mathrm{d}x/\mathrm{d}s = 0$ *of the auxiliary system (thinly dashed line). Parameters:* $a = 0.1$, $b = 0.1$, $\delta = 0.2$, $\rho = 0.1$.

Note that while equation (22) was an implicit nonlinear first order ordinary differential equation in V, equation (23) is easily rewritten as an explicit equation in p with non-constant coefficients. Rowat (2007) derives an explicit solution for this equation by carefully considering the singularity locus $(2n-1)p - \delta x = 0$. We do not repeat his approach here but refer to his paper instead.

The auxiliary system associated to (23) takes for $p \geq 0$ the form

$$\frac{\mathrm{d}p}{\mathrm{d}s} = (\rho+\delta)p - a + bx, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = (2n-1)p - \delta x, \tag{24}$$

while for $p < 0$ it reads as

$$\frac{\mathrm{d}p}{\mathrm{d}s} = (\rho+\delta)p - a + bx, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = -\delta x. \tag{25}$$

Note that the derivatives are taken with respect to a parameter s which has no *a priori* economic interpretation; the point of the auxiliary system is that its solution trajectories

$$s \mapsto (x(s), p(s))$$

trace out graphs of solutions $p = p(x)$ of equation (23). This follows in the region $p \geq 0$ from the chain rule, which states that

$$\frac{\mathrm{d}p}{\mathrm{d}x}(x(s)) = \frac{\dfrac{\mathrm{d}p}{\mathrm{d}s}(s)}{\dfrac{\mathrm{d}x}{\mathrm{d}s}(s)} = \frac{(\rho+\delta)p - a + bx}{(2n-1)p - \delta x}. \tag{26}$$

This is the same expression as in equation (23). Some trajectories of the auxiliary system are shown in figure 1. There, solutions of the auxiliary system are represented by drawn curves. They can, locally and for $p > 0$, be interpreted as the graphs of possible symmetric feedback strategies $u(x) = p(x)$.

This system has a single steady state

$$P \ : \ (x_P, p_P) = \left(\frac{(2n-1)a}{(2n-1)b + \delta^2 + \delta\rho}, \frac{a\delta}{(2n-1)b + \delta^2 + \delta\rho} \right)$$

In figure 1 five generic strategies $u_1, \cdots, u_5$ are highlighted. We shall show that none of these can correspond to Markov perfect Nash equilibrium strategies.

First consider u_1 and u_2. Both have the unstable eigenspace of the equilibrium P as its asymptotic limit as $x \to \infty$. It is straightforward to show that the unstable eigenvector, corresponding to the unstable eigenvalue λ_u, is of the form $(1, v)$ with

$$v = \frac{\delta + \lambda_u}{2n-1} > 0$$

and $\lambda_u > \delta$ for all $\rho \geq 0$. It follows that $v > \delta/n$, and consequently that for each strategy, like u_1 and u_5, that tends asymptotically towards the unstable eigenspace of P there is a state $\bar{x} > 0$ such that $\mathrm{d}x/\mathrm{d}t > 0$ for all $x > \bar{x}$. But such a strategy cannot correspond to a Markov perfect Nash equilibrium, since for x sufficiently large the integrand of (21) is negative, and a strategy for which $u(x) = 0$ whenever $x > \bar{x}$ is a deviation with a better payoff.

Next we note that there is no interval $X' \subset X$ such that $f(x, u_3(x))$ is inward pointing into X'; therefore u_3 cannot be even a local Markov perfect Nash equilibrium strategy.

Finally we consider u_4 and u_5. Note that both are in the region that $\dot{x} = nu - \delta x < 0$, and both satisfy $u_j(x) = 0$ if x is sufficiently small. Hence $x(t)$ tends to 0 as $t \to \infty$. Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\delta x = \frac{\mathrm{d}x}{\mathrm{d}s}$$

if $u_j(x) = 0$, it follows that in this particular case $s = t$. Moreover, from the auxiliary system, it follows that

$$p(t) = C\mathrm{e}^{(\rho+\delta)t} + \mathsf{o}(\mathrm{e}^{(\rho+\delta)t}),$$

with $C < 0$. Consequently, the transversality condition

$$\lim_{t\to\infty} p(t)\mathrm{e}^{-\rho t} \geq 0 \tag{27}$$

is not satisfied for these strategies.

After all strategies are removed that cannot correspond to Markov Nash equilibria, we retain a single global Markov-perfect Nash equilibrium and a family of local Markov-perfect Nash equilibrium strategies; these are illustrated in figure 2.

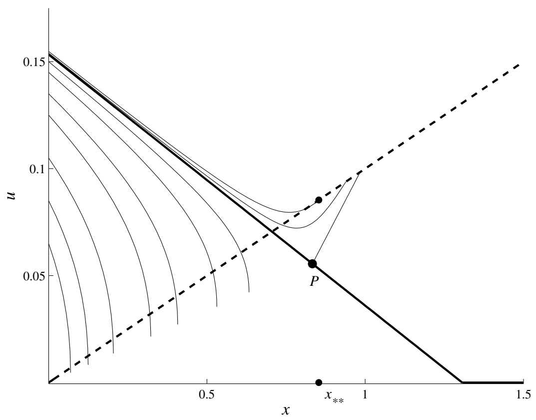

Figure 2 *Local (thin curves) and global (thick curve) Markov-perfect Nash equilibrium strategies, together with the lines l_1 and l_2. Also indicated is the supremum $\bar{x}_*$ of the state values that can be stabilised by a local Markov-perfect Nash equilibrium strategy. Parameters as in figure 1.*

Properties of equilibria. The feedback strategy that is formed by the upper two invariant manifolds of the steady state P of the auxiliary system is of the 'kink' or 'corner' type to be discussed below in subsection 3.1. Also the globally defined strategy, thickly drawn in figure 2, has a corner: it is located at the point where the invariant manifold of P intersects the horizontal axis. This corner is however of a different kind, as it represents a control constraint that becomes active.

Consider the line $l_1 = \{(x, p) \, : \, f(x, p) = 0\}$ of equilibria of the state dynamics (the broken thickly drawn line in the figure): the quantity $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive above l_1, and negative below. From the figure, it is readily apparent that points on l_1 close to the origin (lower left hand corner) are stable, while points on l_1 in the upper right hand corner are unstable. Hence there is a point on l_1 where equilibria change from stable to unstable; it is the unique point (x_{**}, p_{**}) where a solution curve of the auxiliary system touches the line l_1.

Let $p(x)$ determine a local Markov-perfect Nash equilibrium strategy. The stock then evolves according to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, p(x)) = np(x) - \delta x. \tag{28}$$

Let x_* be a steady state equilibrium of this equation; then $p_* = p(x_*) = (\delta/n)x_*$ and $(x_*, p_*) \in l_1$. This equilibrium is stable if

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, p(x)) \Big|_{x=x_*} = n\frac{\mathrm{d}p}{\mathrm{d}x}(x_*) - \delta < 0.$$

This stability condition holds, using (26), when

$$\frac{\mathrm{d}p}{\mathrm{d}x}(x_*) = \frac{(\rho+\delta)p_* - a + bx_*}{(2n-1)p_* - \delta x_*} = \frac{(\rho+\delta)\frac{\delta}{n}x_* - a + bx_*}{(2n-1)\frac{\delta}{n}x_* - \delta x_*} < \frac{\delta}{n}$$

is satisfied. This condition can be simplified to read as

$$x_* < \frac{a}{b + \frac{\delta\rho}{n} + \frac{\delta^2}{n^2}} = x_{**}.$$

In other words, the value x_{**} is the supremum of the stock values that can be stabilised by a local Markov-perfect Nash equilibrium strategy. In the present situation we have that for every $x_* < x_{**}$, there is an equilibrium strategy determined by p such that x_* is a stable steady state under the dynamics (28).

The maximal stream of utility derived from consuming the public good, that is, the maximum of $ax - \frac{1}{2}bx^2$, is obtained at $x_m = b/a$. Note that as the number n of players tends to infinity, the value x_{**}, and with it the region of stock values that can be stabilised, increases towards x_m. This is to be expected: as the adjustment costs are convex, it is better in terms of average costs per player that they are distributed over more players.

From figure 1 we can also draw conclusions about which strategies maximise the pay–off for the players, if the initial state $x_0 = x(0)$ of the system is given; we obtain from equation (22) that

$$\rho V = ax - \frac{b}{2}x^2 + \frac{2n-1}{2}p^2 - \delta xp = G(x,p). \tag{29}$$

For fixed x the value of G, and hence of V, increases for increasing p if $p > \delta x/(2n-1)$.

Consider first the case that $x_0 = 0$. Then

$$\rho V(0) = G(0,p) = \frac{2n-1}{2}p^2,$$

and we see that the highest payoff is attained if p is chosen as large as possible; from figure 2 we infer that this corresponds to the strategy that ends at the semi-stable steady state $x = x_{**}$.

In general, for fixed x, the function $p \mapsto G(x,p)$ is convex, taking its minimum at $p = \delta x/(2n-1)$. It follows that to maximise payoff for all players, the initial value of p has to be taken as large as is feasible for $x \leq x_P$. Beyond that point, the solutions with maximal p have to be compared with the globally defined strategy. For x sufficiently large, there is only a single candidate, which is necessarily optimal.

3 Structure of MPNE

In the preceding section we derived the auxiliary system from the shadow price system and documented its use to derive qualitative insights into (symmetric) Markov perfect Nash equilibria for infinite horizon games. In this section we will make use of the *auxiliary* system to gain insights into the general structure of Markov-perfect Nash equilibria. In particular, we demonstrate that non-differentiability of equilibrium strategies corresponds to singularities of the auxiliary system and the number and values of discontinuities of Markov-perfect Nash equilibrium strategies are related to solutions of the game Hamilton-Jacobi equation $\rho\mathbf{V}(x) = \mathbf{G}(x, \mathbf{V}'(x))$.

3.1 Corner points.

Let $\mathbf{p} : X \to \mathbb{R}^n$ be a given function. The graph of $\mathbf{p}$ is said to have a *corner point* or *corner* at $(x_0, \mathbf{p}(x_0))$, if there is a neighbourhood U of x_0 such that $\mathbf{p}$ is continuous on U, differentiable on $U\backslash\{x_0\}$, and such that the left and right limits of $\mathbf{p}'$ at x_0 exist and satisfy

$$\lim_{x\uparrow x_0} \mathbf{p}'(x) \neq \lim_{x\downarrow x_0} \mathbf{p}'(x).$$

Theorem 2 already answers the question of when a continuous equilibrium Markov shadow price vector $\mathbf{p}(x)$ may fail to be differentiable at certain (isolated) points x_0: it is necessary that

$$\det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}(x_0)) = 0$$

at such points. This is therefore a necessary criterion for the occurrence of corner points in Markov-perfect Nash equilibrium strategies. More generally, we have

Theorem 3 *Let the same assumptions about* $\mathbf{V}$ *as in theorem 2 hold. Assume that* $\mathbf{p} = \mathbf{V}'$ *has a corner at* $(x_0, \mathbf{p}(x_0))$. *Then necessarily the following equations hold:*

$$0 = \det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}(x_0)), \tag{30}$$

$$0 = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}(x_0))\right)^* \left(\rho \mathbf{p}(x_0) - \frac{\partial \mathbf{G}}{\partial x}(x_0, \mathbf{p}(x_0))\right). \tag{31}$$

The proof of this result is immediate. For the symmetric case, we have the following corollary.

Corollary 1 *Let the same assumptions as in theorem 2 hold with respect to the function* $\mathbf{V}$. *Assume moreover that the game is symmetric, and that* $\mathbf{V}$ *has the form*

$$\mathbf{V}(x) = (V(x), \cdots, V(x)).$$

If $p(x) = V'(x)$ *has a corner at* $(x_0, p(x_0))$, *then this point is a steady state of the auxiliary system* (19)*.*

Summing up: if $\mathbf{p}$ is continuous, we have the following two implications. If $\mathbf{p}$ has a corner at $(x_0, \mathbf{p}(x_0))$, then (30) is satisfied; obversely, if (30) is not satisfied at $(x_0, \mathbf{p}(x_0))$, then $\mathbf{p}$ is differentiable at x_0.

3.2 Jump points.

The function $\mathbf{p} : X \to \mathbb{R}^n$ is said to have an *isolated jump point* at x_0, or simply to *jump* at x_0, if there is a neighbourhood U of x_0 such that $\mathbf{p}$ is continuous on $U\backslash\{x_0\}$, and such that the left and right limits of $\mathbf{p}$ at x_0 exist and satisfy

$$\lim_{x\uparrow x_0} \mathbf{p}(x) \neq \lim_{x\downarrow x_0} \mathbf{p}(x).$$

Analogously to theorem 3 the following result gives a necessary condition for the costate function of an equilibrium strategy to have an isolated jump point.

Theorem 4 *Let the function* $\mathbf{V}(x)$ *be continuous, piecewise continuously differentiable, and let it solve the vector Hamilton-Jacobi equation* (5). *If* $\mathbf{p}(x) = \mathbf{V}'(x)$ *has an isolated jump discontinuity at* $x = x_0$*, then necessarily*

$$\lim_{x \uparrow x_0} \mathbf{G}(x, \mathbf{p}(x)) = \lim_{x \downarrow x_0} \mathbf{G}(x, \mathbf{p}(x)).$$

Proof This is a direct consequence of the vector Hamilton-Jacobi equation (5) together with the continuity of $\mathbf{V}$.

We make a couple of remarks concerning this theorem. First, we note that it is possible to give *a priory* conditions that ensure the continuity of $\mathbf{V}$: the relevant condition is that the system dynamics are locally controllable for every player.

If $\mathbf{V}$ is continuous, the possible values of $\mathbf{p}$ to jump starting from a point $(x_0, \mathbf{p}_0)$ are evidently the solutions of the system of equations

$$\mathbf{G}(x_0, \mathbf{p}) = \mathbf{G}(x_0, \mathbf{p}_0).$$

For the symmetric situation, we have the following result.

Theorem 5 *Let the game be symmetric, and let the function*

$$\mathbf{V}(x) = (V(x), \cdots, V(x))$$

be continuous, piecewise continuously differentiable, and let it be a viscosity solution of the vector Hamilton-Jacobi equation (5)*, or, equivalently, let* $V(x)$ *be a viscosity solution of the scalar Hamilton-Jacobi equation*

$$\rho V = G(x, V'(x)).$$

Assume that $G(x, p)$ *is strictly convex in* p*, and that* $p(x) = V'(x)$ *has a jump discontinuity at* $x = \hat{x}$*; that is, assume that the left and right limits* p_L *and* p_R *of* $p(x)$ *exist as* $x \to \hat{x}$*.*

Then

$$G(x, p_L) = G(x, p_R) \quad \textit{and} \quad p_L \leq p_R.$$

The equality follows from the left and right continuity of $G(x, V'(x))$ at $x = \hat{x}$. The inequality is a consequence of theorem 8 in the appendix.

4 Applications

The class of differential games introduced in the preceding sections is fairly general and allows us to study Markov equilibria for a variety of different examples. Here we apply the techniques of the auxiliary system to two alternative models that have been dealt with in the literature: (i) the exploitation of a reproductive asset (Benhabib and Radner 1992, Dockner and Sorger 1996) and (ii) the shallow lake problem (Mäler et al. 2003, Brock and Starrett 2003, Wagener 2003, Kossioris et al. 2008, Kiseleva and Wagener 2010).

4.1 Exploitation of reproductive assets.

Consider the problem where n agents strategically exploit a single reproductive asset, like fish or other species (see Dockner and Sorger 1996). The reproduction of the stock x occurs at rate $h(x)$, whereas player i extracts the stock at rate u_i. Hence, the state dynamics are given by

$$\dot{x} = h(x) - \sum_{i=1}^{n} u_i. \tag{32}$$

The instantaneous utility that agent i derives from the consumption of the stock is assumed to be of the constant elasticity type

$$L_i(u_i) = \frac{u_i^{1-\sigma}}{1-\sigma}$$

with $0 < \sigma < 1$, so that the utility functional of player i therefore is

$$J_i = \int_0^\infty \frac{u_i^{1-\sigma}}{1-\sigma} \mathrm{e}^{-\rho t} \mathrm{d}t.$$

The function P_i becomes

$$P_i = \frac{u_i^{1-\sigma}}{1-\sigma} + p_i \left(h(x) - \sum_{j=1}^{n} u_j \right).$$

From $\partial P_i / \partial u_i$ we obtain $p_i = u_i^{-\sigma}$ and $u_i = p_i^{-1/\sigma}$, and the game Hamilton functions read as

$$G_i = \frac{1}{1-\sigma} p_i^{(\sigma-1)/\sigma} + p_i \left(h(x) - \sum_{j=1}^{n} p_j^{-1/\sigma} \right).$$

In the symmetric case $p_1 = \cdots = p_n = p$, this simplifies to

$$G = \frac{1-n+n\sigma}{1-\sigma} p^{(\sigma-1)/\sigma} + ph(x),$$

and we obtain the auxiliary system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} &= \dfrac{n-1-n\sigma}{\sigma} p^{-1/\sigma} + h(x), \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} &= (\rho - h'(x))p. \end{cases}$$

Using the relation $u = p^{-1/\sigma}$, we find the form of the auxiliary system in state-control variables:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} &= \dfrac{n-1-n\sigma}{\sigma} u + h(x), \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} &= \dfrac{h'(x)-\rho}{\sigma} u. \end{cases} \tag{33}$$

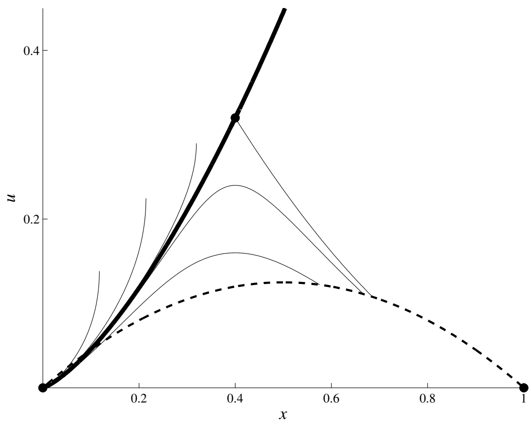

Figure 3 *local (thin) and global (thick) Markov perfect Nash equilibria, obtained from the auxiliary system* (33)*, together with the isocline* $\dot{x} = h(x) - nu = 0$ *(dashed) in the symmetric two player case of the fishery model with production function* $h(x) = x(1-x)$ *and parameters* $\rho = 0.2$ *and* $\sigma = 0.8$.

The situation $\sigma = (1 - 1/n)$ is special, as then the factor $(n - 1 - n\sigma)/\sigma$ vanishes and the system can be integrated, yielding

$$u(x) = Ch(x)^{n/(n-1)} \exp\left(-\frac{n\rho}{n-1}\int_{x_0}^{x} h(\xi)^{-1}\mathrm{d}\xi\right).$$

Compare equation (4) of Dockner and Sorger (1996).

Stability of steady states. As in the linear-quadratic example given in subsection 2.5, for a given symmetric Nash equilibrium strategy $u(x)$, the state dynamics are given as $f(x) = h(x) - nu(x)$. A state-control pair (x,u), with $u = u(x)$, corresponds to a steady state for these dynamics if $f(x) = h(x) - nu = 0$, that is, if $u = h(x)/n$. The state is locally attracting if $f'(x) < 0$. We compute, using the relation $u = h(x)/n$:

$$\begin{aligned} f'(x) &= h'(x) - nu'(x) = h'(x) - n\frac{\frac{\mathrm{d}u}{\mathrm{d}s}}{\frac{\mathrm{d}x}{\mathrm{d}s}} \\ &= h'(x) - n\frac{\frac{h'(x)-\rho}{\sigma}u}{\frac{n-1-n\sigma}{\sigma}u + h(x)} = \frac{n\rho - h'(x)}{n-1}. \end{aligned}$$

It follows that (x,u) corresponds to an attracting steady state if

$$\rho < \frac{h'(x)}{n},$$

and to an unstable state if the inequality sign is reversed. In particular, if $h'(x) < 0$, then $(x, h(x)/n)$ always corresponds to an unstable equilibrium for the state dynamics. Moreover, since the derivative $h'(x)$ is bounded from above, if

$$\rho > \max \frac{h'(x)}{n}$$

then the state dynamics does not have stable equilibria in the interior of the state space.

Let us finally consider the "semi-stable" state $\bar{x}$ that satisfies

$$\rho = \frac{h'(\bar{x})}{n};$$

this point is in the boundary of the set of all stabilisable states. Compare it to the optimal long term steady state $x_{\text{collusive}}$ of the collusive outcome, for which

$$\rho = h'(x_{\text{collusive}})$$

holds, the so-called "golden rule". The strategic behaviour in the semi-stable state $\bar{x}$ can be described as each player behaving as if he had a private fish stock available with reproduction rate $h(\bar{x})/n$.

Analysis of the auxiliary system. We shall assume that $\rho \leq h'(0)$. Then there is a unique $x_\rho \in [0, 1)$ such that $h'(x_\rho) = \rho$. The auxiliary system has then fixed points

$$(x, u) \in \left\{(0, 0), (1, 0), \left(x_\rho, \frac{\sigma}{1 - n(1 - \sigma)} h(x_\rho)\right)\right\}.$$

Note that the third equilibrium satisfies $u > 0$ only if $n < 1/(1 - \sigma)$.

Theorem 6 *Assume that x_1 is such that h is strictly decreasing for $x > x_1$. Then every n-player symmetric Markov perfect Nash equilibrium $x \mapsto u(x)$ satisfies*

$$u(x) \geq \frac{h(x)}{n}$$

for all $x \geq x_1$.

Proof Let $\bar{u}$ be an n-player symmetric Markov perfect Nash equilibrium, and assume that for $x_0 > x_1$ the inequality is violated, that is

$$\bar{u}(x_0) < \frac{h(x)}{n}. \tag{34}$$

If $x(0) = x_0$, this implies that $x(t) > x_0$ for all $t > 0$, and therefore, since all solutions of

$$\dot{x} = h(x) - n\bar{u}(x)$$

are increasing, and since h is strictly decreasing for $x > x_1$, that

$$\bar{u}(x(t)) \leq \frac{h(x(t))}{n} < \frac{h(x_0)}{n}$$

for all $t > 0$. Consequently

$$J(\bar{u}, \cdots, \bar{u}) < \bar{J}_0 \stackrel{\text{def}}{=} \rho^{-1} \frac{(h(x_0)/n)^{1-\sigma}}{1 - \sigma}.$$

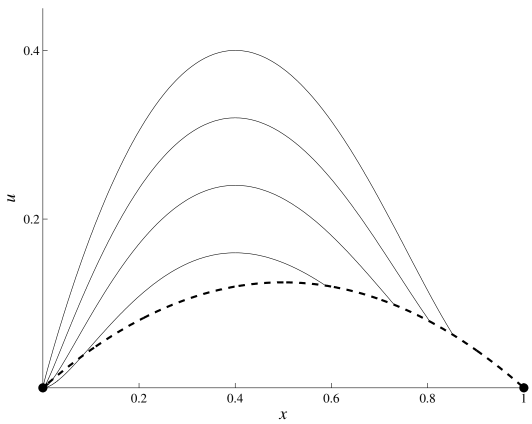

Figure 4 *Local Markov-perfect Nash equilibria in the symmetric two player case of the fishery model with production function $h(x) = x(1-x)$ and parameters $\rho = 0.2$ and $\sigma = 0.4$.*

Now assume that player 1 deviates by playing the constant strategy

$$u_1(x) = h(x_0) - (n-1)\bar{u}(x_0).$$

The system dynamics

$$\dot{x} = h(x) - \sum_{i=1}^{n} u_i(x) = h(x) - u_1(x) - (n-1)\bar{u}(x)$$

has then $x = x_0$ as steady state. From equation (34) it follows that

$$u_1(x_0) > h(x_0) - (n-1)\frac{h(x_0)}{n} = \frac{h(x_0)}{n} = \bar{u}(x_0);$$

hence

$$J_1(u_1, \bar{u}, \cdots, \bar{u}) > \bar{J}_0 > J_1(\bar{u}, \cdots, \bar{u}).$$

We finally obtain that $\bar{u}$ cannot be a Nash equilibrium strategy.

Using the theorem, we have plotted the symmetric Markov perfect Nash equilibria in figures 3 and 4. A characteristic feature of these strategy equilibria is that if the initial fish stock is higher than the semi-stable threshold value $\bar{x}$ introduced above, it cannot be stabilised. Moreover, for these non-stabilisable initial stocks, we see that as the initial stock is larger, the eventually reached steady state stock grows smaller.

Asymmetric strategies. Here the assumption is dropped that the players play symmetric strategies; for simplicity, we restrict to the two–player case $n = 2$ and assume that $\sigma = 1/2$ holds. Then

$$G_i(x, p) = \frac{2}{p_i} + p_i \left(h(x) - \frac{1}{p_1^2} - \frac{1}{p_2^2} \right).$$

The system (7) takes the form

$$\begin{pmatrix} h(x) - \dfrac{1}{p_1^2} - \dfrac{1}{p_2^2} & 2\dfrac{p_1}{p_2^3} \\ 2\dfrac{p_2}{p_1^3} & h(x) - \dfrac{1}{p_1^2} - \dfrac{1}{p_2^2} \end{pmatrix} \begin{pmatrix} \dfrac{\mathrm{d}p_1}{\mathrm{d}x} \\ \dfrac{\mathrm{d}p_2}{\mathrm{d}x} \end{pmatrix} = (\rho - h(x)) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Using the relation $p_i = u_i^{-1/2}$, we get

$$\begin{pmatrix} h - u_1 - u_2 & 2u_1 \\ 2u_2 & h - u_1 - u_2 \end{pmatrix} \begin{pmatrix} \dfrac{\mathrm{d}u_1}{\mathrm{d}x} \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}x} \end{pmatrix} = 2(h' - \rho) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

It is convenient to consider instead of u_1 and u_2 the quantities $v = u_1 + u_2$ and $w = u_1 - u_2$; the systems then takes the simpler form

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= 2\frac{h' - \rho}{\Delta}\left(hv - 2v^2 + w^2\right), \\ \frac{\mathrm{d}w}{\mathrm{d}x} &= 2\frac{h' - \rho}{\Delta}(h - v)w, \end{aligned}$$

with $\Delta(x) = h^2 - 2hv + w^2 = (h - v)^2 + w^2 - v^2$. The auxiliary system to this system of equations reads as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= \Delta = h^2 - 2hv + w^2, \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= 2(h' - \rho)\left(hv - 2v^2 + w^2\right), \\ \frac{\mathrm{d}w}{\mathrm{d}s} &= 2(h' - \rho)(h - v)w. \end{aligned}$$

Note that the plane $w = 0$, corresponding to the symmetric case $u_1 = u_2$, is invariant under the flow of the auxiliary system; in other words, that case is nested in the present one.

We will not give a full analysis of this system, leaving that to future work. However, we would like to point out one consequence of the equation $\dot{w} = 2(h' - \rho)(h - v)w$. Recall that $\dot{x} = h - v$; hence, if the system is on a time path for which the stock decreases, the factor $h - v < 0$, and the sign of $\dot{w}/w$ is the opposite of $h' - \rho$.

In the example above, the factor $h' - \rho$ is positive for small ρ and small x, and it follows that then the differences between strategies decay exponentially if the stock decreases towards an equilibrium close to $x = 0$. Conversely, if ρ sufficiently large, differences between strategies increase exponentially, which can be interpreted as a mad scramble to exploit the last remnants of the stock.

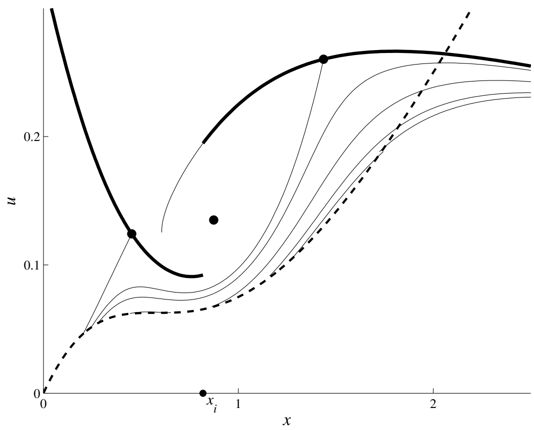

Figure 5 *Local (thin) and global (thick) Markov perfect Nash equilibrium strategies in the symmetric two player case of the lake game, for parameter values* $(b, c, \rho) = (0.65, 0.5, 0.03)$. *The global Nash equilibrium strategy is discontinuous at* x_i.

4.2 Shallow lake.

Consider the following environmental problem. There are n players (countries, communes, farmers) sharing the use of a lake. Each player has revenues from farming, for which artificial fertiliser is used. The use of fertiliser has two opposing effects: more fertiliser means better harvests and hence higher revenues from farming. On the other hand fertiliser is washed from the fields by rainfall and eventually accumulates a stock of phosphorus in the shallow lake. The higher the level of phosphorus the higher are the costs (for fresh water, decreased income from tourism) to the player. Since the level of the stock of phosphorus is the result of activities of all players sharing the lake, the resulting problem can best be described by a differential game. The shallow lake system has been investigated by Brock and Dechert (2008), Mäler et al. (2003), Wagener (2003), Kossioris et al. (2008), Kiseleva and Wagener (2010); we refer to these papers for background information. Particularly, Kossioris et al. (2008) found local Markov-perfect Nash equilbria of the 2-player game by analysing the shadow price equation expressed in controls numerically, but they failed to find the global equilibrium.

Let the stock variable x represent the amount of phosphorus in a shallow lake and let u_i be the amount of fertiliser used by farmer i. Assuming a concave technology to produce farming output and quadratic costs coming from the stock x, player i maximises intertemporal utility

$$J_i = \int_0^\infty (\log u_i - cx^2)\mathrm{e}^{-\rho t}\mathrm{d}t.$$

The level of phosphorus is assumed to evolve according to the following state equation:

$$\dot{x} = f(x, u) = \sum_{i=1}^{n} u_i - bx + \frac{x^2}{x^2+1}.$$

Here b is the constant rate of self-purification (sedimentation, outflow) and the non-linear term $x^2/(x^2+1)$ is the result of biological effects in the lake.

For this differential game the function P_i is given by

$$P_i = \log u_i - cx^2 + p_i \left(\sum_{j=1}^{n} u_j - bx + \frac{x^2}{x^2+1} \right).$$

Maximising over u_i yields that $u_i = -1/p_i$. Restricting again our attention to symmetric strategies, we find on setting $p_j = p$ for all $j = 1, \cdots, n$ that

$$G(x, p) = -\log(-p) - cx^2 - n + p \left(-bx + \frac{x^2}{x^2+1} \right).$$

The auxiliary system now reads as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial G}{\partial p} & = -\dfrac{1}{p} - bx + \dfrac{x^2}{x^2+1}, \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} = \rho p - \dfrac{\partial G}{\partial x} & = (\rho + b)p + 2cx - \dfrac{2px}{(x^2+1)^2}, \end{cases}$$

or, in terms of controls, as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = u - bx + \dfrac{x^2}{x^2+1}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} & = -(\rho + b)u + 2cu^2 x + \dfrac{2ux}{(x^2+1)^2}. \end{cases}$$

Solutions to the auxiliary system are given in figure 5. The most important feature of the solution set is that there is a globally defined non-continuous Markov-perfect Nash equilibrium strategy, indicated by a thick line in the figure. Indeed, it has been known for some time that the Hamilton-Jacobi equation of some economic optimal control problems may have jumps in the policy function, see Skiba (1978), and for the shallow lake model Mäler et al. (2003) and Wagener (2003). Since the game Hamilton-Jacobi equation for the case of two or more players is identical to that of the one player case, the same jump occurs. Note that the Nash strategies that are parametrised by parts of the stable and unstable manifolds of one of the saddle points of the auxiliary system are continuous, but not continuously differentiable everywhere.

Finally notice that the auxiliary system does not depend on the number of agents, and therefore coincides with the state–control system of the shallow lake optimal control problem. In practical terms, this means that figure 5 can be used to analyse the situation for any number of players. The only difference is in the symmetric time dynamics

$$\dot{x} = nu - bx + \frac{x^2}{1+x^2}.$$

Increasing the number of players n leads to a decrease of the isocline $\dot{x} = 0$. In particular, though this will not be demonstrated here, for large values of n no states in the low-pollution region can be stabilised by a locally defined feedback Nash equilibrium strategy.

5 Conclusions

In this article, a framework has been elaborated to find sufficient conditions as well as necessary conditions for Markov-perfect Nash equilibrium strategies in differential games with a single state variable. The Nash equilibria have been characterised as solutions of a system of explicit first order ordinary differential equations, usually nonlinear.

By analyzing a series of classical examples, we have shown that this characterization can be used to find both direct analytic information, by integration of the equations, and indirect qualitative information, by a geometric analysis of the solution curves of an auxiliary system in the phase space.

Additionally, we have addressed the issues of continuity and differentiability of Markov strategies in this class of differential games. In particular, in the shallow lake model, we have shown the existence of a non-continuous Markov-perfect Nash equilibrium. Our simple approach is capable enough to deliver interesting insights into a large class of capital accumulation games.

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We would like to thank Christophe Deissenberg, Colin Rowat and Aart de Zeeuw, as well as two anonymous referees for helpful remarks and comments. The work of Florian Wagener has been supported under the CeNDEF Pionier grant and a MaGW-VIDI grant, both from the Netherlands Organisation for Science (NWO).

A Evolution near non-Lipschitz points

For continuous one-dimensional vector fields $F : X \to \mathbb{R}$, where X is a closed interval of $\mathbb{R}$, Peano's theorem (Peano 1890) guarantees the existence of a positive constant $T > 0$, possibly infinite, and a differentiable function $x : [0, T] \to \mathbb{R}$ satisfying

$$\dot{x} = F(x) \tag{35}$$

for all $t \in [0, T]$, and such that $x(T) \in \partial X$.

At points $\hat{x}$ where the right hand side F of an ordinary differential equation has an isolated discontinuity, Peano's theorem does not apply. For our purposes, it is sufficient to have the existence of continuous functions $x(t)$ that satisfy (35) for all $t \in [0, \infty)\backslash N$, where N is a discrete set, that is, a set without limit points. For the purpose of this appendix, we shall call these *piecewise solutions* in analogy to piecewise differentiable functions. Piecewise solutions are a special case of Carathéodory solutions, which are absolutely continuous functions $x(t)$, satisfying (35) almost everywhere on $[0, \infty)$ (cf. Hajek 1979).

The theorem of this section gives a condition for one-dimensional vector fields with isolated jump discontinuities to have piecewise solutions.

Theorem 7 *Let $U \subset \mathbb{R}$ be an open interval including $\hat{x}$, and let F restricted to $U \backslash \{\hat{x}\}$ be continuous, non-zero and such that the right and left limits F_R and F_L of $F(x)$ exist as x tends to $\hat{x}$ from the right and from the left respectively. Assume that*

$$\text{if} \quad F_L \geq 0 \geq F_R, \quad \text{then} \quad F(\hat{x}) = 0.$$

Then for all $x_0 \in U$ there exists a piecewise solution of (35) *that satisfies $x(0) = x_0$ and that is defined for all t such that $x(t) \in U$.*

Proof In the proof, 'trajectory' will indicate a solution x of the differential equation, whose existence is guaranteed by Peano's theorem, that is, as long as $x(t) \neq \hat{x}$. A statement about a trajectory x that holds 'for all t' will always mean 'for all t such that $x(t) \in U$'.

There are a number of different situations. Firstly, if $F_L = F_R$, then F is continuous on U, and Peano's theorem yields the existence of a solution to the differential equation for all t.

Secondly, F_L and F_R may be both non-negative or both non-positive. Assume for definiteness that both are non-negative. Then a trajectory starting at $x_0 > \hat{x}$ does not decrease, never reaches $\hat{x}$, and yields a piecewise solution for all t, while a trajectory x_1 starting at $x_0 \leq \hat{x}$ reaches $\hat{x}$ at some time $T \geq 0$; if $T = \infty$, then x_1 is already a piecewise solution. Assume therefore that T is finite. Introduce

$$G(x) = \begin{cases} F(x) & x > \hat{x}, \\ F_R & x \leq \hat{x}. \end{cases} \tag{36}$$

Let then x_2 be a solution of $\dot{x} = G(x)$ with initial condition $x_2(T) = \hat{x}$, which exists as G is continuous. The trajectory that is equal to $x_1(t)$ for $0 \leq t < T$ and $x_2(t)$ for $t \geq T$ is a piecewise solution.

Thirdly, there is the possibility that $F_L > 0 > F_R$. By assumption we then have $F(\hat{x}) = 0$. A trajectory x_1 starting at $x_0 < \hat{x}$ will satisfy $\lim_{t \uparrow T} x_1(T) = \hat{x}$ for some finite time T. Concatenation with the constant trajectory $x_2(t) = \hat{x}$ for $t \geq T$ again yields a piecewise solution. The case $x_0 > \hat{x}$ is handled in the same manner.

Lastly, there is the situation that $F_L < 0 < F_R$. As above, a trajectory with initial value $x_0 > \hat{x}$ is increasing, hence defined for all t and yields a piecewise solution; likewise, trajectories starting at $x_0 < \hat{x}$ are decreasing and are also defined for all $t \geq 0$. If finally $x_0 = \hat{x}$, let G be defined as in (36), and let x be a solution of $\dot{x} = G(x)$ with $x(0) = x_0$. As $G(x) > 0$ for all x, $x(t)$ is increasing in t and satisfies therefore $G(x(t)) = F(x(t))$ for all $t > 0$. Hence it is a piecewise solution as well. ◻

B Proof of the sufficiency theorem

In this section, the proof of theorem 1 is given. Before starting, we make a general remark on supergradients of viscosity solutions $V : X \to \mathbb{R}$ to the Hamilton-Jacobi equation

$$\rho V = G(x, V'),$$

where $G : X \times \mathbb{R} \to \mathbb{R}$.

Theorem 8 *Let $G = G(x, p)$ be a continuous function that is strictly convex in p, let $\hat{x}$ be a point in X, let U be an open neighbourhood of $\hat{x}$ in X, and let V be a viscosity solution of the Hamilton-Jacobi equation* (10) *that is continuously differentiable on $U \backslash \{\hat{x}\}$. Then necessarily*

$$\lim_{x \uparrow \hat{x}} V'(x) \leq \lim_{x \downarrow \hat{x}} V'(x)$$

and

$$\lim_{x \uparrow \hat{x}} G_p(x, V'(x)) \leq \lim_{x \downarrow \hat{x}} G_p(x, V'(x)).$$

Corollary 2 *Let G and V be as in theorem 8. Consider the state evolution equation*

$$\dot{x} = F(x) = G_p(x, V'(x)), \tag{37}$$

defined for all x where V' is differentiable in x. If at a point $\hat{x}$ the left and right limits F_L and F_R of F exist, then $F_L \leq F_R$.

Remark 1 It follows from theorem 7 that under the conditions of theorem 8 the state evolution equation (37) has a piecewise solution.

Remark 2 Theorem 8 applies for instance to the global Markov-perfect Nash equilibrium of the shallow lake model, discussed in subsection 4.2.

Proof (Of theorem 8). Let

$$p_L = \lim_{x \uparrow \hat{x}} V'(x), \quad p_R = \lim_{x \downarrow \hat{x}} V'(x),$$

and assume by contradiction that

$$p_L > p_R.$$

Note that since V is C^1 on $U \backslash \{\hat{x}\}$, we have

$$\rho V(\hat{x}) = G(\hat{x}, p_L) = G(\hat{x}, p_R).$$

Moreover, since $p_L > p_R$

$$[p_R, p_L] \subset D_+ V(\hat{x}).$$

Since $G(\hat{x}, p)$ is strictly convex in p, it follows that for $\bar{p} \in (p_R, p_L)$, we have

$$G(\hat{x}, \bar{p}) < \rho V(\hat{x})$$

On the other hand, since V is a viscosity solution and $\bar{p} \in D_+ V(\hat{x})$, it follows that

$$G(\hat{x}, \bar{p}) \geq \rho V(\hat{x}).$$

This is impossible, hence $p_L \leq p_R$. □

We now give the proof of theorem 1.

Proof We have to show the following. If the strategy vector of the players other than player i equals $\mathbf{u}^*_{-i}$, then u^*_i is a best response of player i; in other words, $u_i(x) = u^*_i(x)$ solves the optimisation problem of player i.

The proof consists of two parts, and it rests on the verification that $V_i(x)$ is the value function of the optimisation problem of player i. Let $u_i(x)$ be any admissible strategy, set

$$\bar{\mathbf{u}}(x) = (u_i(x), \mathbf{u}^*_{-i}(x)),$$

and let $\bar{x}$ be any piecewise solution of

$$\dot{\bar{x}} = f(\bar{x}, \bar{\mathbf{u}}(\bar{x})), \qquad \bar{x}(0) = x_0, \tag{38}$$

whose existence is guaranteed by theorem 7. The first part of the proof shows that then

$$\int_0^\infty L_i(\bar{x}, \bar{\mathbf{u}}(x)) \mathrm{e}^{-\rho t} \mathrm{d}t \leq V_i(x_0); \tag{39}$$

that is, $V_i(x_0)$ is an upper bound for the payoff of player i.

Then for $u_i = u^*_i(x)$, and $x = x^*$ being any piecewise solution of

$$\dot{x} = f(x, \mathbf{u}^*(x)), \quad x(0) = x_0, \tag{40}$$

the second part of the proof demonstrates the opposite inequality

$$\int_0^\infty L_i(x^*, \mathbf{u}^*(x^*)) \mathrm{e}^{-\rho t} \mathrm{d}t \geq V_i(x_0). \tag{41}$$

Together, these inequalities show that u^*_i is a best response of player i.

Part one. As $\bar{\mathbf{u}}$ is piecewise differentiable, the set C of points where $f(x, \bar{\mathbf{u}}(x))$ fails to be differentiable is a set of isolated points.

Let D be the set of states at which $\mathbf{V}$ fails to be differentiable. By assumption this is a set of isolated points as well. Take $\varepsilon \in (0,1)$ arbitrarily. Because of condition (16) there is a constant $T > 1/\varepsilon > 0$ such that

$$\left| V_i(\bar{x}(T))\mathrm{e}^{-\rho T} \right| < \varepsilon. \tag{42}$$

Let $\Sigma \subset [0,T]$ be such that

$$\bar{x}(t) \in C \cup D$$

if and only if $t \in \Sigma$. As $\bar{x}$ is a piecewise solution, there are time points

$$0 \leq t_1 < t_2 < \cdots < t_k \leq T,$$

such that the set Σ is the union of the finite set $\Sigma_1 = \{t_1, \cdots, t_{k-1}\}$ and the interval $\Sigma_2 = [t_k, T]$, where it is understood that Σ_1 may be empty and Σ_2 may have zero length. Note that

$$f(\bar{x}(t), \bar{u}(\bar{x}(t))) = 0$$

if $t \in \Sigma_2$.

Also set

$$x_j = \bar{x}(t_j), \quad j = 1, \cdots, k.$$

From (42) it follows that

$$\begin{aligned} -V_i(x_0) &\leq \varepsilon + V_i(\bar{x}(T))\mathrm{e}^{-\rho T} - V_i(x_0) \\ &= \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{\mathrm{d}}{\mathrm{d}t} \left(V_i(\bar{x})\mathrm{e}^{-\rho t} \right) \mathrm{d}t + \int_{t_k}^{T} \frac{\mathrm{d}}{\mathrm{d}t} \left(V_i(x_k)\mathrm{e}^{-\rho t} \right) \mathrm{d}t. \end{aligned} \tag{43}$$

As $V_i(x)$ is differentiable in the open intervals (x_{j-1}, x_j) as a function of x and $\bar{x}$ is differentiable in (t_{j-1}, t_j) as a function of t, the differentiations can be performed to yield

$$-V_i(x_0) \leq \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(V_i'(\bar{x}) f\big(\bar{x}, \bar{\mathbf{u}}(\bar{x})\big) - \rho V_i(\bar{x}) \right) \mathrm{e}^{-\rho t} \mathrm{d}t \tag{44}$$

$$+ \int_{t_k}^{T} \left(p_i f\big(x_k, \bar{\mathbf{u}}(x_k)\big) - \rho V_i(x_k) \right) \mathrm{e}^{-\rho t} \mathrm{d}t. \tag{45}$$

Here, the constant $p_i \in D_- V_i(x_k)$ is an arbitrary subderivative of V_i at x_k; as x_k is a steady state, we have that the inserted term

$$p_i f(x_k, \bar{\mathbf{u}}(x_k)) = 0.$$

We compute

$$
\begin{aligned}
&\int_0^T L_i(\bar{x}, \bar{\mathbf{u}}) \mathrm{e}^{-\rho t} \mathrm{d}t - V_i(x_0) \\
&\qquad \le \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big(L_i(\bar{x}, \bar{\mathbf{u}}(\bar{x})) + V_i'(\bar{x}) f(\bar{x}, \bar{\mathbf{u}}) - \rho V_i(\bar{x}) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad\quad + \int_{t_k}^{T} \Big(L_i(x_k, \bar{\mathbf{u}}(x_k)) + p_i f(x_k, \bar{\mathbf{u}}(x_k)) - \rho V_i(x_k) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad = \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big(P_i\big(\bar{x}, V_i'(\bar{x}), u_i(\bar{x}); \mathbf{u}^*_{-i}(\bar{x})\big) - \rho V_i(\bar{x}) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad\quad + \int_{t_k}^{T} \Big(P_i(x_k, p_i, u_i(x_k); \mathbf{u}^*_{-i}(x_k)) - \rho V_i(x_k) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad \le \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big(H_i\big(\bar{x}, V_i'(\bar{x}); \mathbf{u}^*_{-i}(\bar{x})\big) - \rho V_i(\bar{x}) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad\quad + \int_{t_k}^{T} \Big(H_i(x_k, p_i; \mathbf{u}^*_{-i}(x_k)) - \rho V_i(x_k) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad \le \varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big(G_i\big(\bar{x}, \mathbf{V'}(\bar{x})\big) - \rho V_i(\bar{x}) \Big) \mathrm{e}^{-\rho t} \mathrm{d}t = \varepsilon.
\end{aligned}
$$

Note that for the second inequality, we used that $H_i(x, p_i) = \max_{u_i} P_i(x, p_i, u_i)$. In the last inequality, we used that V_i is a viscosity supersolution of the Hamilton-Jacobi equation (15). Letting $\varepsilon \to 0$ now yields inequality (39).

Part two. It remains to show the opposite inequality (41) if $u_i(x) = u_i^*(x)$ for all x, and $x = x^*$ solving (40). Let C denote the isolated set of states where $f(x, \mathbf{u}^*(x))$ fails to be differentiable; and let the set D be defined as above. Repeat the construction of T, the t_i, and the sets Σ_1 and Σ_2, but now with $\bar{x}$ replaced by x^*.

With a analogous reasoning as used to derive (44), we can show that

$$
\begin{aligned}
-V_i(x_0) \ge -\varepsilon &+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \big(V_i'(x^*) f\big(x^*, \mathbf{u}^*(x^*)\big) - \rho V_i(x^*) \big) \mathrm{e}^{-\rho t} \mathrm{d}t \\
&+ \int_{t_k}^{T} \big(p_i f\big(x_k, \mathbf{u}^*(x_k)\big) - \rho V_i(x_k) \big) \mathrm{e}^{-\rho t} \mathrm{d}t,
\end{aligned}
\tag{46}
$$

where $p_i \in D_- V_i(x_k)$ is any subderivative of V_i at x_k.

Again, if the interval Σ_2 is nontrivial, the point x_k is a steady state of equation (38), with $\bar{\mathbf{u}}$ replaced by $\mathbf{u}^*$. Introduce

$$F(x) = f(x, \mathbf{u}^*(x)).$$

By assumption, the strategy vector $\mathbf{u}^*$, and consequently the function F, is either left or right continuous at x_k – say it is left continuous, the other case being similar. Setting

$$p_{iL} = \lim_{x \uparrow x_k} V_i'(x),$$

it follows by continuity that

$$P_i(x^*, V_i'(x^*), u_i; \mathbf{u}^*_{-i}(x^*)) \to P_i(x^*, p_{iL}, u_i; \mathbf{u}^*_{-i}(x^*)) \quad \text{as} \quad x \uparrow x_k,$$

and hence that

$$u_i^*(x_k) = \text{arg max}_{u_i} P_i(x_k, p_{iL}, u_i; \mathbf{u}^*_{-i}(x_k)). \tag{47}$$

Inequality (46) implies that

$$
\begin{aligned}
&\int_0^T L_i(x^*, \mathbf{u}^*(x^*)) \mathrm{e}^{-\rho t} \mathrm{d}t - V_i(x_0) \\
&\geq -\varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \big(L_i(x^*, \mathbf{u}^*(x^*)) + V_i'(x^*) f\big(x^*, \mathbf{u}^*(x^*)\big) - \rho V_i(x^*) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad + \int_{t_k}^{T} \big(L_i(x_k, \mathbf{u}^*(x_k)) + p_{iL} f\big(x_k, \mathbf{u}^*(x_k)\big) - \rho V_i(x_k) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t \\
&= -\varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \big(P_i(x^*, V_i'(x^*), u_i^*(x^*); \mathbf{u}_{-i}^*(x^*)) - \rho V_i(x^*) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad + \int_{t_k}^{T} \big(P_i(x_k, p_{iL}, u_i^*(x^*); \mathbf{u}_{-i}^*(x_k)) - \rho V_i(x_k) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t,
\end{aligned}
$$

by definition of P_i, and

$$
\begin{aligned}
&= -\varepsilon + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \big(H_i(x^*, V_i'(x^*); \mathbf{u}_{-i}^*(x^*)) - \rho V_i(x^*) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t \\
&\qquad + \int_{t_k}^{T} \big(H_i(x_k, p_{iL}; \mathbf{u}_{-i}^*(x_k)) - \rho V_i(x_k) \big) \, \mathrm{e}^{-\rho t} \mathrm{d}t,
\end{aligned}
$$

by equation (47).

All the terms of the sum vanish, since

$$H_i(x, V_i'(x); \mathbf{u}_{-i}^*(x)) = G_i(x, \mathbf{V}'(x)) = \rho V_i(x)$$

whenever $x \in (x_{j-1}, x_j)$. The final integral vanishes as well, as by continuity

$$\rho V_i(x) = G_i(x, \mathbf{V}'(x)) \to H_i(x_k, p_{iL}; \mathbf{u}_{-i}^*(x_k)) \quad \text{as} \quad x \uparrow x_k.$$

Continuity of V_i then implies that

$$\rho V_i(x_k) = H_i(x_k, p_{iL}; \mathbf{u}_{-i}^*(x_k)).$$

Sending $\varepsilon \to 0$ demonstrates inequality (41). □

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