Behavioral Learning Equilibria, Persistence Amplification & Monetary Policy

Cars Hommes[a], Mei Zhu[b*]

[a] CeNDEF, School of Economics, University of Amsterdam

and Tinbergen Institute, Netherlands

[b] Institute for Advanced Research & School of Economics, Shanghai University of Finance and Economics,

and the Key Laboratory of Mathematical Economics(SUFE), Ministry of Education, Shanghai 200433, China

April 13, 2016

Abstract

We generalize the concept of behavioral learning equilibrium (BLE) to a general high dimensional linear system and apply it to the standard New Keynesian model. Boundedly rational agents learn to use a simple AR(1) forecasting rule for each variable with parameters consistent with the observed sample mean and autocorrelation of past data. Agents do not fully recognize the more complex structure of the economy, but learn to use an optimal simple AR(1) rule. We find that BLE exists, under general stationarity conditions, typically with near unit root autocorrelation parameters. BLE thus exhibits a novel feature, persistence amplification: the persistence in inflation and output gap is much higher than the persistence in exogenous fundamental driving factors. In a boundedly rational world, coordination of individual expectations on an aggregate outcome described by our simple, parsimonious BLE seems more likely. We also consider monetary policy under BLE for different Taylor interest rate rules and study whether inflation and/or ouput gap targeting can stabilize coordination on near unit root BLE.

JEL classification: E32; E52; D83; D84

Keywords: Bounded rationality; Behavioral learning equilibriium; Adaptive learning; New Keynesian macro-model; Inflation persistence

* *E-mail addresses:* C.H.Hommes@uva.nl, zhu.mei@mail.shufe.edu.cn

Acknowledgements. This paper has been presented at the Computation in Economics and Finance (CEF) conference, June 20-22, 2015, Taipei, Taiwan, the Workshop Expectations in Dynamic Macroeconomic Models, August 13-15 2015, University of Oregon, USA and the Workshop on Agent-Based and DSGE Macroeconomic Modeling: Bridging the Gap, November 20, 2015, Surrey, UK. Stimulating discussions and comments from Klaus Adam, Bill Branch, Jim Bullard, George Evans, Paul Levine, Domenico Massaro, Bruce McGough, Bruce Preston and Mike Woodford are gratefully acknowledged. The research leading to these results has received funding from the European Communitys Seventh Framework Programme (FP7/2007-2013) under grant agreement "Integrated Macro-Financial Modeling for Robust Policy Design (MACFINROBODS)", grant no. 612796. Mei Zhu also acknowledges financial support from NSFC funding Studies on equilibria of heterogeneous agents models under adaptive learning (grant no. 11401365), NSFC (71541018) and China Scholarship Council (file no. 201506485009).

1 Introduction

Rational Expectations Equilibrium (REE) requires that economic agents' subjective probability distributions coincide with the objective distribution that is determined, in part, by their subjective beliefs. There is a vast literature that studies the drawbacks of REE. Some of these drawbacks include the fact that REE requires an unrealistic degree of computational power and perfect information on the part of agents. Alternatively, the adaptive learning literature (see, e.g., Evans and Honkapohja (2001, 2013) and Bullard (2006) for extensive surveys and references) replaces rational expectations with beliefs that come from an econometric forecasting model with parameters updated using observed time series. A large part of this literature involves studying under which conditions learning will converge to the rational expectations equilibrium. When the perceived law of motion (PLM) of agents is correctly specified, convergence of adaptive learning to an REE can occur. However, in general the PLM will be misspecified. As shown in White (1994), an economic model or a probability model is only a more or less crude approximation to whatever might be the "true" relationships among the observed data and consequently it is necessary to view economic and/or probability models as misspecified to some greater or lesser degree. Whenever agents have *misspecified* PLMs a reasonable learning process may settle down to some sort of misspecification equilibrium. In the literature, different types of misspecification equilibria have been proposed, e.g. Restricted Perceptions Equilibrium (RPE) where the forecasting model is underparameterized (Sargent, 1991; Evans and Honkapohja, 2001; Adam, 2003; Branch and Evans, 2010) and Stochastic Consistent Expectations Equilibrium (SCEE) (Hommes and Sorger, 1998; Hommes et al., 2013), where agents learn the optimal parameters of a simple, parsimonious AR(1) rule.[1]

A SCEE is a very natural misspecification equilibrium, where agents in the economy do not know the actual law of motion or even recognize all relevant explanatory variables, but rather prefer a parsimonious forecasting model. The economy is too complex to fully understand and therefore, as a first-order approximation, agents forecast the state of the economy by simple autoregressive models (e.g. Fuster et al., 2010). In the simplest model applying this idea, agents run an univariate AR(1) regression to generate out-of-sample forecasts of the state of the economy. Hommes and Zhu (2014) provide the first-order SCEE with an *intuitive behavioral* interpretation and refer to them as a *Behavioral Learn-*

[1] Branch (2006) provides a stimulating survey discussing the connection between these types of misspecification equilibria.

ing Equilibrium (BLE). Although it is possible for some agents to use more sophisticated models, one may argue that these practices are neither straightforward nor widespread. A simple, parsimonious BLE seems a more plausible outcome of the coordination process of individual expectations in large complex socio-economic systems (Grandmont, 1998).

Hommes and Zhu (2014) formalize the concept of BLE in the simplest class of models one can think of: a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. Agents do not recognize, however, that the economy is driven by an exogenous AR(1) process y_t, but simply forecast the state of the economy x_t using an univariate AR(1) rule. The parameters of the AR(1) forecasting rule are not free, but fixed (or learned over time) according to the observed sample average and first-order sample autocorrelation. Within this simple, but general, class of models Hommes and Zhu (2014) fully characterize the existence and multiplicity of BLE and provide stability conditions under a simple adaptive learning scheme –Sample Autocorrelation Learning (SAC-learning). Although this class of models is simple, it contains two important standard applications: an asset pricing model driven by autocorrelated dividends and the New Keynesian Philips curve with inflation driven by autocorrelated output gap (or marginal costs). As shown in Fuhrer (2009), however, the skeleton model of the New Keynesian Philips curve with AR(1) driving variable leaves implicit the determination of real output and the role of monetary policy in influencing output and inflation.

In this paper we extend the BLE concept to a general n-dimensional linear stochastic framework. As an application we consider the standard two-dimensional dynamic stochastic general equilibrium (DSGE) model-the New Keynesian model–and study the role of monetary policy under BLE. Agents are boundedly rational. They do not know the exact form of the actual law of motion because of cognitive limitations or simply prefer a parsimonious prediction rule. Agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. The same consistency requirements are imposed upon BLE to pin down the parameters of the forecasting model: for each endogenous variable observed sample averages and first-order sample autocorrelations match the corresponding parameters of the forecasting rule.

Numerous empirical studies show that overly parsimonious models with little parameter uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007; Enders, 2010). In a similar vein (but without analytical results) Slobodyan and Wouters (2012) study a New Keynesian DSGE model with agents using an AR(2) forecasting rule. Chung and Xiao (2014) and Xiao and Xu (2014) study learning and predictions with an AR(1) or VAR(1) model in a two dimensional New Keynesian model with limited information and show, based on simulations, that the simple AR(1) model is more likely to prevail in reality when they make predictions. Laboratory experiments in the NK framework also show that simple forecasting rules such as AR(1) describe individual forecasting behavior surprisingly well (Assenza et al., 2014; Pfajfar and Zakelj, 2016).

The main contributions of our paper are fourfold: (1) existence proofs of BLE in a general linear framework, (2) stability conditions of BLE under sample autocorrelation (SAC-) learning, (3) persistence and volatility amplification, and (4) monetary policy stabilization analysis under BLE. Many models of learning lead to excess volatility, that is, volatility under learning is typically much higher than under REE. Our BLE model exhibits another novel feature, *persistence amplification*: the persistence of inflation and output gap under BLE is significantly higher than under REE. In fact, even when autocorrelations of the exogenous shocks to fundamentals are small, inflation and output gap along BLE are near unit root processes. Monetary policy through inflation and output targeting may have strong effects upon this persistence and volatility amplification.

Related literature

The issue of persistence has been of great interest to macroeconomists and policymakers. A number of models of frictions have been proposed to replicate persistence, such as habit formation in consumption, indexation to lagged inflation in price-setting, rule-of-thumb behavior, or various adjustment costs (Phelps, 1968; Taylor, 1980; Fuhrer and Moore, 1992, 1995; Christiano et al., 2005; Smets and Wouters, 2003, 2005; Boivin and Giannoni, 2006; Giannoni and Woodford, 2003). These papers essentially improve the empirical fit by adding lags in the model equations. Estimating these rich models with frictions under the assumption of rational expectations one typically finds that substantial degrees of habit persistence and inflation indexation are supported by the data. Those additional sources of persistence appear, therefore, necessary to match the inertia of macroeconomic variable. These estimations also typically involve highly persistent structural shocks. Our BLE model is applied to a frictionless New Keynesian framework, but nevertheless exhibits strong. Learning causes persistence amplification: small autocorrelations of exogenous shocks are strongly amplified as agents learn to coordinate on a simple AR(1) forecasting rule with near unit root parameters consistent with observed sample average and sample autocorrelations. The high persistence of inflation and output thus arises from a self-fulfilling mistake (Grandmont, 1998).

Our BLE concept fits with the literature employing adaptive learning to analyze the evolution of U.S. inflation and monetary policy. Adaptive learning can help in understanding some particular historical episodes, such as high inflation in the 1980s, which are often harder to explain under rational expectations. For example, Orphanides and Williams (2003) consider a form of imperfect knowledge in which economic agents rely on adaptive learning to form expectations. This form of learning represents a relatively modest deviation from rational expectations that nests it as a limiting case. They find that policies that would be efficient under rational expectations can perform poorly when knowledge is imperfect. Milani (2007) also assumes that agents form expectations through adaptive learning using correctly specified economic models and updating the parameters through constant-gain learning (CGL) based on historical data. He shows empirically that when learning replaces rational expectations, the estimated degrees of habits and indexation drop closer to zero, suggesting that persistence arises in the model economy mainly from expectations and learning. Fuhrer (2009) provides a good survey on inflation persistence. He examines a number of empirical measures of reduced form persistence including the first-order autocorrelation and the autocorrelation function of the inflation series. He also investigates the sources of persistence, including learning of agents in a rational- expectation setting.

Our behavioral learning equilibrium concept is closely related to the Exuberance Equilibria (EE) in Bullard et al. (2008), where agents' perceived law of motion is misspecified. However, because of difficulty of computation, in Bullard et al. (2008) there are only numerical results on the exuberance equilibria, while here we analytically show the existence of BLE, its stability under learning and the persistence amplification in a general linear framework with application to the New Keynesian model. Another related misspecification equilibrium is Limited Information Learning Equilibrium (LILE) defined in Chung and Xiao (2014), which is defined by the least-squares projection of variables on the past information of the actual law of motion equal to that in the perceived law of motion. Different from the LILE, our general Behavioral Learning Equilibrium is defined by the conditions that sample means and first-order autocorrelations of each variable of the actual law of motion are consistent with those corresponding to the perceived law of motion. We further study the effects of monetary policy under the more plausible BLE. The concept of natural expectations in Fuster et al. (2010) and Fuster et al. (2011,

2012) is another misspecification concept, where agents use simple, misspecified models, e.g., linear autoregressive models. Natural expectations, however, do not pin down the parameters of the forecasting model through consistency requirements as for a restricted perceptions equilibrium nor do they allow the agents to learn an optimal misspecified model through empirical observations. Cho and Kasa (2015) study model validation in an environment where agents are aware of misspecification and try to detect it through adaptive learning. In our BLE misspecification is self-fulfilling and the outcome of the SAC-learning process.

The paper is organized as follows. Section 2 introduces the main concepts of BLE and Section 3 generalizes the existence and stability of BLE to a general n-dimensional linear system. Section 4 applies BLE to the two-dimensional New Keynesian model and studies existence of BLE, their stability under learning and the persistence amplification. Section 5 studies whether monetary policy can mitigate persistence and volatility amplification for different specifications of the Taylor rules. Finally, Section 6 concludes.

2 Main concepts

In Hommes and Zhu (2014), we introduced BLE in the simplest setting, a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. In this paper we generalize BLE to n-dimensional (linear) stochastic models driven by exogenous linear stochastic AR(1) processes of multiple shocks.

Let the law of motion of an economic system be given by the stochastic difference equation

$$\boldsymbol{x}_t \;=\; \boldsymbol{F}(\boldsymbol{x}^e_{t+1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.1}$$

where $\boldsymbol{x}_t$ is an $n \times 1$ vector of endogenous variables denoted by $[x_{1t}, x_{2t}, \cdots, x_{nt}]'$ and $\boldsymbol{x}^e_{t+1}$ is the expected value of $\boldsymbol{x}$ at date $t+1$. This denotation highlights that expectations may not be rational. Here $\boldsymbol{F}$ is a continuous n-dimensional vector function, $\boldsymbol{u}_t$ is a vector of exogenous stationary variables and $\boldsymbol{v}_t$ is a vector of white noise disturbances.

Agents are boundedly rational and do not know the exact form of the actual law of motion (2.1). They only use a simple, parsimonious forecasting model where agents' perceived law of motion (PLM) is a simple univariate AR(1) process for each variable to be forecasted. As shown in Enders (2010, p.84-85), coefficient uncertainty increases as the model becomes more complex, and hence it could be that an estimated AR(1) model forecasts a real ARMA(2,1) process better than an estimated ARMA(2,1) model. Numerous empirical studies also show that overly parsimonious models with little parameters uncertainty can provide better forecasts than models consistent with the actual data-generating complex process (e.g. Nelson, 1972; Stock and Watson, 2007; Clark and West, 2007). Thus agents' perceived law of motion (PLM) is assumed to be the simplest VAR model with minimum parameters, i.e. a VAR(1) process

$$\boldsymbol{x}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}) + \boldsymbol{\delta}_t, \tag{2.2}$$

where $\boldsymbol{\alpha}$ is a vector denoted by $[\alpha_1, \alpha_2, \cdots, \alpha_n]'$, $\boldsymbol{\beta}$ is a diagonal matrix[2] denoted by $\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & \beta_n \end{bmatrix}$ with $\beta_i \in (-1, 1)$ and $\{\boldsymbol{\delta}_t\}$ is a white noise process; $\boldsymbol{\alpha}$ is the unconditional mean of $\boldsymbol{x}_t$ and β_i is the first-order correlation coefficient of variable x_i. Given the perceived law of motion (2.2), the 2-period ahead forecasting rule for $\boldsymbol{x}_{t+1}$ that minimizes the mean-squared forecasting error is

$$\boldsymbol{x}^e_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\beta^2}(\boldsymbol{x_{t-1}} - \boldsymbol{\alpha}). \tag{2.3}$$

Combining the expectations (2.3) and the law of motion of the economy (2.1), we obtain the implied actual law of motion (ALM)

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}),\ \boldsymbol{u}_t,\ \boldsymbol{v}_t). \tag{2.4}$$

In the case that the ALM (2.4) is stationary, suppose the variance-covariance matrix $\boldsymbol{\Gamma}(0) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']$ and the first order covariance matrix $\boldsymbol{\Gamma}(1) := E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{x}_{t+1} - \overline{\boldsymbol{x}})']$, where $\overline{\boldsymbol{x}}$ is the mean of $\boldsymbol{x}_t$. Let $\boldsymbol{\Omega}$ be the diagonal matrix in which the ith diagonal element is the variance of the ith process, that is $\boldsymbol{\Omega} = \mathrm{diag}[\gamma_{11}(0), \gamma_{22}(0), \cdots, \gamma_{nn}(0)]$, where $\gamma_{ii}(0)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(0)$. Let $\boldsymbol{E}$ be the diagonal matrix in which the ith diagonal element is the first-order autocovariance of the ith process, that is $\boldsymbol{E} = \mathrm{diag}[\gamma_{11}(1), \gamma_{22}(1), \cdots, \gamma_{nn}(1)]$, where $\gamma_{ii}(1)$ is the ith diagonal entry of $\boldsymbol{\Gamma}(1)$. Let $\boldsymbol{G}$ denote the diagonal matrix in which the ith diagonal element is the first-order autocorrelation coefficient of the ith process $x_{i,t}$. Hence

$$\boldsymbol{G} = \boldsymbol{E}\boldsymbol{\Omega}^{-1}. \tag{2.5}$$

[2] Chung and Xiao (2014) also argue using simulations that the simple AR(1) model is more likely to prevail in reality because of limited information restrictions when they model predictions in a two dimensional New Keynesian model.

Behavioral Learning Equilibrium (BLE)

Extending Hommes and Zhu (2014), the concept of BLE is generalized as follows.

Definition 2.1 *A vector $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where μ is a probability measure, $\boldsymbol{\alpha}$ is a vector and $\boldsymbol{\beta}$ is a diagonal matrix with $\beta_i \in (-1, 1)$ $(i = 1, 2, \cdots, n)$, is called a behavioral learning equilibrium (BLE) if the following three conditions are satisfied:*

S1 The probability measure μ is a nondegenerate invariant measure for the stochastic difference equation (2.4);

S2 The stationary stochastic process defined by (2.4) with the invariant measure μ has unconditional mean $\boldsymbol{\alpha}$, that is, the unconditional mean of x_i is α_i, $(i = 1, 2, \cdots, n)$;

S3 Each element x_i for the stationary stochastic process of $\boldsymbol{x}$ defined by (2.4) with the invariant measure μ has unconditional first-order autocorrelation coefficient β_i, $(i = 1, 2, \cdots, n)$, that is, $\boldsymbol{G} = \boldsymbol{\beta}$.

That is to say, a BLE is characterized by two natural observable consistency requirements: the unconditional means and the unconditional first-order autocorrelation coefficients generated by the actual (unknown) stochastic process (2.4) coincide with the corresponding statistics for the perceived linear VAR(1) process (2.2), as given by the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This means that in a BLE agents correctly perceive the two simplest and most important statistics: the mean and first-order autocorrelation (i.e., persistence) of each relevant variable of the economy, without fully understanding its structure and recognizing all explanatory variables and cross correlations. Along a BLE the two parameters of each linear forecasting rule are pinned down by simple and observable statistics. Hence, agents do not fully understand the linear structure of the stochastic economy, e.g. they do not take the cross-correlation of state variables into account, but rather use a parsimonious univariate AR(1) forecasting rule for each state variable. A simple BLE may be a plausible outcome of the coordination process of expectations of a large population. Laboratory experiments within the New Keynesian framework also provide empirical evidence of the use of simple univariate AR(1) forecasting rules to forecast inflation and output gap (e.g. Adam, 2007; Pfajfar and Zakelj, 2016; Assenza et al., 2014).

Sample autocorrelation learning

In the above definition of BLE, agents' beliefs are described by the linear forecasting rule (2.3) with fixed parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. However, the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are usually unknown. In the adaptive learning literature, it is common to assume that agents behave like econometricians using time series observations to estimate the parameters as additional observations become available. Following Hommes and Sorger (1998), we assume that agents use sample autocorrelation learning (SAC-learning) to learn the parameters α_i and β_i, $i = 1, 2, \cdots, n$. That is, for any finite set of observations $\{x_{i,0}, x_{i,1}, \cdots, x_{i,t}\}$, the sample average is given by

$$\alpha_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} x_{i,k}, \tag{2.6}$$

and the first-order sample autocorrelation coefficient is given by

$$\beta_{i,t} = \frac{\sum_{k=0}^{t-1} (x_{i,k} - \alpha_{i,t})(x_{i,k+1} - \alpha_{i,t})}{\sum_{k=0}^{t} (x_{i,k} - \alpha_{i,t})^2}. \tag{2.7}$$

Hence $\alpha_{i,t}$ and $\beta_{i,t}$ are updated over time as new information arrives. It is easy to check that, independently of the choice of the initial values $(x_{i,0}, \alpha_{i,0}, \beta_{i,0})$, it always holds that $\beta_{i,1} = -\frac{1}{2}$, and that the first-order sample autocorrelation $\beta_{i,t} \in [-1, 1]$ for all $t \geq 1$.

As shown in Hommes and Zhu (2014), define

$$R_{i,t} = \frac{1}{t+1} \sum_{k=0}^{t} (x_{i,k} - \alpha_{i,t})^2,$$

then the SAC-learning is equivalent to the following recursive dynamical system

$$\begin{cases} \alpha_{i,t} = \alpha_{i,t-1} + \dfrac{1}{t+1}(x_{i,t} - \alpha_{i,t-1}), \\ \beta_{i,t} = \beta_{i,t-1} + \dfrac{1}{t+1} R_{i,t}^{-1} \Big[(x_{i,t} - \alpha_{i,t-1}) \big(x_{i,t-1} + \dfrac{x_{i,0}}{t+1} - \dfrac{t^2+3t+1}{(t+1)^2} \alpha_{i,t-1} - \dfrac{1}{(t+1)^2} x_{i,t} \big) \\ \qquad - \dfrac{t}{t+1} \beta_{i,t-1} (x_{i,t} - \alpha_{i,t-1})^2 \Big], \\ R_{i,t} = R_{i,t-1} + \dfrac{1}{t+1} \Big[\dfrac{t}{t+1} (x_{i,t} - \alpha_{i,t-1})^2 - R_{i,t-1} \Big]. \end{cases} \tag{2.8}$$

The actual law of motion under SAC-learning is therefore given by

$$\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2 (\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1}),\ \boldsymbol{u}_t,\ \boldsymbol{v}_t), \tag{2.9}$$

with $\alpha_{i,t}, \beta_{i,t}$ as in (2.8).

In Hommes and Zhu (2014), F is a one-dimensional linear function. In this paper $\boldsymbol{F}$ may be a general n-dimensional linear vector function.

3 Main results in a n-dimensional linear framework

Assume that a reduced form model is a n-dimensional linear stochastic process $\boldsymbol{x}_t$, driven by an exogenous VAR(1) process $\boldsymbol{u}_t$. More precisely, the actual law of motion of the economy is given by[3]

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{F}(\boldsymbol{x}^e_{t+1},\ \boldsymbol{u}_t,\ \boldsymbol{v}_t) = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{x}^e_{t+1} + \boldsymbol{b}_2\boldsymbol{u}_t + \boldsymbol{v}_t, && (3.1)\\ \boldsymbol{u}_t &= \boldsymbol{a} + \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, && (3.2)\end{aligned}$$

where $\boldsymbol{x}_t$ is an $n\times 1$ vector of endogenous variables, $\boldsymbol{b}_0$ and $\boldsymbol{a}$ are $n\times 1$ vectors of constants, $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{\rho}$ are $n\times n$ matrices of coefficients, $\boldsymbol{u}_t$ is an $n\times 1$ vector of exogenous variables which is assumed to follow a stationary VAR(1) as shown in (3.2), and $\boldsymbol{v}_t$ is an $n\times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_v}$. That is, here all of the eigenvalues of $\boldsymbol{\rho}$ are assumed to be inside the unit circle. In order to study BLE and REE more conveniently, we also assume all of the eigenvalues of $\boldsymbol{b}_1$ lie inside the unit circle[4]. In addition, $\boldsymbol{\varepsilon}_t$ is assumed to be an $n\times 1$ vector of i.i.d. stochastic disturbance terms with mean zero and finite absolute moments, with variance-covariance matrix $\boldsymbol{\Sigma_\varepsilon}$ and is independent of $\boldsymbol{v}_t$.

3.1 Rational expectations equilibrium

Under the assumption that agents are rational, assume the perceived law of motion (PLM) corresponding to the minimum state variable REE of the model

$$\boldsymbol{x}^*_t = \boldsymbol{\xi} + \boldsymbol{\eta}\boldsymbol{u}_t + \boldsymbol{v}_t. \tag{3.3}$$

Assuming that shocks $\boldsymbol{u}_t$ are observable when forecasting $\boldsymbol{x}_{t+1}$ the one-step ahead forecast is

$$E_t\boldsymbol{x}^*_{t+1} = \boldsymbol{\xi} + \boldsymbol{\eta}\boldsymbol{a} + \boldsymbol{\eta}\boldsymbol{\rho}\boldsymbol{u}_t, \tag{3.4}$$

and the corresponding actual law of motion is

$$\boldsymbol{x}^*_t = \boldsymbol{b}_0 + \boldsymbol{b}_1(\boldsymbol{\xi} + \boldsymbol{\eta}\boldsymbol{a} + \boldsymbol{\eta}\boldsymbol{\rho}\boldsymbol{u}_t) + \boldsymbol{b}_2\boldsymbol{u}_t + \boldsymbol{v}_t. \tag{3.5}$$

[3]As shown in Section 4 for the alternative case with lagged Taylor rule, our results on BLE still hold for the more general model including the term of lagged $\boldsymbol{x}_{t-1}$ in the RHS of Eq. (3.1).

[4]In the case when $\boldsymbol{b}_1$ has an eigenvalue outside the unit circle a typical time series will be exploding under naive expectations $\boldsymbol{x}^e_{t+1} = \boldsymbol{x}_{t-1}$. BLE under an AR(1) rule and SAC-learning then typically become non-stationary and exploding.

The rational expectations equilibrium (REE) is the fixed point of

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{\xi} + \boldsymbol{b}_1\boldsymbol{\eta}\boldsymbol{a}, & (3.6) \\ \boldsymbol{\eta} &= \boldsymbol{b}_1\boldsymbol{\eta}\boldsymbol{\rho} + \boldsymbol{b}_2. & (3.7) \end{aligned}$$

A straightforward computation (see Appendix A) shows that the mean of the REE $\overline{\boldsymbol{x}^*}$ satisfies

$$\overline{\boldsymbol{x}^*} = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_2(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}], \tag{3.8}$$

where $\boldsymbol{I}$ denotes a comfortable identity matrix throughout the paper.

In the special case $\boldsymbol{\rho} = \rho\boldsymbol{I}$ [5], the rational expectation equilibrium $\boldsymbol{x}_t^*$ satisfies

$$\boldsymbol{x}_t^* = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}\boldsymbol{b}_0 + (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}\boldsymbol{b}_1(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2\boldsymbol{a} + (\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2\boldsymbol{u}_t + \boldsymbol{v}_t. \tag{3.9}$$

Thus its unconditional mean is

$$\overline{\boldsymbol{x}^*} = E(\boldsymbol{x}_t^*) = (1 - \rho)^{-1}(\boldsymbol{I} - \boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0(1 - \rho) + \boldsymbol{b}_2\boldsymbol{a}]. \tag{3.10}$$

Its variance-covariance matrix is

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*} = E[(\boldsymbol{x}_t^* - \overline{\boldsymbol{x}^*})(\boldsymbol{x}_t^* - \overline{\boldsymbol{x}^*})'] = (1 - \rho^2)^{-1}(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2]' + \boldsymbol{\Sigma}_{\boldsymbol{v}}. \tag{3.11}$$

Furthermore, the first-order autocovariance is,

$$\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}} = E[(\boldsymbol{x}_t^* - \overline{\boldsymbol{x}^*})(\boldsymbol{x}_{t-1}^* - \overline{\boldsymbol{x}^*})'] = \rho(1 - \rho^2)^{-1}(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}[(\boldsymbol{I} - \rho\boldsymbol{b}_1)^{-1}\boldsymbol{b}_2]'. \tag{3.12}$$

The first-order autocorrelation of the i-element x_i^* of $\boldsymbol{x}^*$ is the i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*\boldsymbol{x}^*_{-1}}$ divided by the corresponding i-th diagonal element of matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}^*}$.

Note that in the special case $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \boldsymbol{0}$ and the first-order autocorrelation of the i-element u_i of $\boldsymbol{u}$ is equal to ρ, the persistence of the i-th variable x_i^* in the REE coincides exactly with the persistence of the exogenous driving force $u_{i,t}$. That is, in this case the persistence in the REE only inherits from the exogenous driving force.

3.2 Existence of BLE

Now assume that agents are boundedly rational and do not believe or do not recognize that the economy is driven by an exogenous VAR(1) process $\boldsymbol{u}_t$, but use a simple univariate

[5] Note that $\boldsymbol{\rho}$ is a matrix while ρ is a scalar number throughout the paper.

linear rule to forecast the state $\boldsymbol{x}_t$ of the economy. Given that agents' perceived law of motion is a special VAR(1) process as shown in(2.2), the actual law of motion becomes

$$\boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha})] + \boldsymbol{b}_2\boldsymbol{u}_t + \boldsymbol{v}_t, \tag{3.13}$$

with $\boldsymbol{u}_t$ given in (3.2). If all the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ for each $\beta_i \in [-1,1](i = 1,2,\cdots,n)$ lie inside the unit circle, then the system of $\boldsymbol{x}_t$ is stationary and hence its mean $\overline{\boldsymbol{x}}$ and first-order autocorrelation $\boldsymbol{G}$ exist.

The mean of $\boldsymbol{x}_t$ in (3.13) is computed as

$$\overline{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^2)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{\alpha} - \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\alpha} + \boldsymbol{b}_2(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}]. \tag{3.14}$$

Imposing the first consistency requirement of a BLE on the mean, i.e. $\overline{\boldsymbol{x}} = \boldsymbol{\alpha}$, and solving for $\boldsymbol{\alpha}$ yields

$$\boldsymbol{\alpha}^* = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_2\boldsymbol{(I - \rho)^{-1}}\boldsymbol{a}]. \tag{3.15}$$

Comparing with (3.8), we conclude that in a BLE the unconditional mean $\boldsymbol{\alpha}^*$ coincides with the REE mean. That is to say, in a BLE the state of the economy $\boldsymbol{x}_t$ fluctuates on average around its RE fundamental value $\boldsymbol{x}^*$.

Consider the second consistency requirement of a BLE on the first-order autocorrelation coefficient matrix $\boldsymbol{\beta}$ of the PLM. The second consistency requirement yields

$$\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{\beta}. \tag{3.16}$$

Recall from Section 2, both $\boldsymbol{G}$ and $\boldsymbol{\beta}$ are diagonal matrices. For convenience let G_i denote the i-th diagonal element of the matrix $\boldsymbol{G}$ in (2.5). Under the assumption that all of the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ for each $\beta_i \in [-1,1](i = 1,2,\cdots,n)$ lie inside the unit circle, from the theory of stationary linear time series, $G_i(\beta_1,\beta_2,\cdots,\beta_n) \in [-1,1]$ and is a smooth function with respect to $(\beta_1,\beta_2,\cdots,\beta_n)$ and other model parameters, see Appendix B[6]. Based on Brouwer's fixed-point theorem for $(G_1,G_2,\cdots,G_n)$, there exists $\boldsymbol{\beta}^* = (\beta_1^*,\beta_2^2,\cdots,\beta_n^*)$ with each $\beta_i^* \in [-1,1]$, such that $\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{\beta}$. We conclude:

Proposition 1 *If all the eigenvalues of $\boldsymbol{\rho}$ and $\boldsymbol{b}_1\boldsymbol{\beta}^2$ for each $\beta_i \in [-1,1]$ are inside the*

[6]For example, refer to the expression (3.9) in Hommes and Zhu (2014) for the special 1-dimensional case $n = 1$. In Section 4 we consider the two-dimensional New Keynesian model and will compute the (complicated) expressions of $G_1(\beta_1,\beta_2)$ and $G_2(\beta_1,\beta_2)$.

unit circle[7]*, there exists at least one behavioral learning equilibrium (BLE)* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ *for the economic system (3.13) with* $\boldsymbol{\alpha}^* = (\boldsymbol{I} - \boldsymbol{b}_1)^{-1}[\boldsymbol{b}_0 + \boldsymbol{b}_2(\boldsymbol{I} - \boldsymbol{\rho})^{-1}\boldsymbol{a}] = \overline{\boldsymbol{x}}^*$.

3.3 Stability under SAC-learning

In this subsection we study the stability of BLE under SAC-learning. The ALM of the economy under SAC-learning is given by

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{b}_0 + \boldsymbol{b}_1[\boldsymbol{\alpha}_{t-1} + \boldsymbol{\beta}_{t-1}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha}_{t-1})] + \boldsymbol{b}_2\boldsymbol{u}_t + \boldsymbol{v}_t, \\ \boldsymbol{u}_t = \boldsymbol{a} + \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \tag{3.17}$$

with $\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t$ updated based upon realized sample average and sample autocorrelation as in (2.8). Appendix C shows that the E-stability principle applies and that the stability under SAC-learning is determined by the associated ordinary differential equation (ODE)[8]

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha} = (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^2)^{-1}[\boldsymbol{b}_0 + b_1\boldsymbol{\alpha} - b_1\boldsymbol{\beta}^2\boldsymbol{\alpha} + b_2(\boldsymbol{I} - \boldsymbol{\rho})^{-1}a] - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G}(\boldsymbol{\beta}) - \boldsymbol{\beta}, \end{cases} \tag{3.18}$$

where $\overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the mean given by (3.14) and $\boldsymbol{G}(\boldsymbol{\beta})$ is the diagonal first-order autocorrelation matrix. A BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ corresponds to a fixed point of the ODE (3.18). Moreover, a BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is locally stable under SAC-learning, if it is a stable fixed point of the ODE (3.18). Therefore, we have the following property of SAC-learning stability.

Proposition 2 *A BLE* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ *is locally stable under SAC-learning if*

(i) all the eigenvalues of $(\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I})$ *have negative real parts*[9]*, and*

(ii) all the eigenvalues of $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*)$ *have real parts less than 1, where* $\boldsymbol{DG_\beta}$ *is a Jacobian matrix with the* (i,j)*-th entry* $\frac{\partial G_i}{\partial \beta_j}$.

Proof. See Appendix C.

Recall from Subsection 3.2 that $G_i(\beta_1, \beta_2, \cdots, \beta_n) \in (-1, 1)$ so that at least one BLE exists. If the BLE is unique, the BLE may be (locally) stable under SAC-learning. In the next section, we study BLE in a two-dimensional New Keynesian model.

[7] The Schur-Cohn criterion theorem provides necessary and sufficient conditions for all eigenvalues to lie inside the unit circle, see Elaydi (1999). For specific models, one may find sufficient conditions to guarantee all eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ for each $\beta_i \in [-1, 1]$ are inside the unit circle, as shown below for the NK model.

[8] See Evans and Honkapohja (2001) for discussion and a mathematical treatment of E-stability.

[9] The Routh-Hurwitz criterion theorem provides sufficient and necessary conditions for all the n eigenvalues having negative real parts, see Brock and Malliaris (1989).

4 Application: a New Keynesian model

4.1 A baseline model

Now apply these results within the framework of a standard New Keynesian model along the lines of Woodford (2003) and Galí (2008). Consider a simple version, linearized around the zero inflation steady state, given by

$$\begin{cases} y_t = y^e_{t+1} - \varphi(i_t - \pi^e_{t+1}) + u_{y,t}, \\ \pi_t = \lambda \pi^e_{t+1} + \gamma y_t + u_{\pi,t}, \end{cases} \tag{4.1}$$

where y_t is the aggregate output gap, π_t is the inflation rate, y^e_{t+1} and π^e_{t+1} are expected output gap and expected inflation. Following Bullard and Mitra (2002) and Bullard et al. (2008) we study the NK-model (4.1) with adaptive learning. The terms $u_{y,t}, u_{\pi,t}$ are stochastic shocks and are assumed to follow AR(1) processes

$$u_{y,t} = \rho_1 u_{y,t-1} + \varepsilon_{1,t}, \tag{4.2}$$

$$u_{\pi,t} = \rho_2 u_{\pi,t-1} + \varepsilon_{2,t}, \tag{4.3}$$

where $\rho_i \in [0,1)$ and $\{\varepsilon_{i,t}\}$ $(i = 1,2)$ are two uncorrelated i.i.d. stochastic processes with zero mean and finite absolute moments with corresponding variances σ_i^2.

The first equation in (4.1) is an IS curve that describes the demand side of the economy. In an economy of rational or boundedly rational agents, it is a linear approximation to a representative agent's Euler equation. The parameter $\varphi > 0$ is related to the elasticity of intertemporal substitution in consumption of a representative household. The second equation in (4.1) is the New Keynesian Phillips curve which describes the aggregate supply relation. This is obtained by averaging each firm's pricing decisions, while the parameter γ is related to the degree of price stickiness in the economy and the parameter $\lambda \in [0,1)$ is the discount factor of a representative household.

We supplement the equations in (4.1) with a policy rule, which represents the behavior of the monetary authority in setting the interest rate. In this section we assume a Taylor-type policy rule setting the nominal interest rate

$$i_t = \phi_\pi \pi_t + \phi_y y_t, \tag{4.4}$$

where i_t is the deviation of the nominal interest rate from the value that is consistent with inflation at target and output at potential and the parameters ϕ_π, ϕ_y, measuring the response of i_t to the deviation of inflation and output from long run steady states, are assumed to be nonnegative. The policy rule (4.4) is a *contemporaneous* Taylor rule, responding to current inflation and output, π_t, y_t. In the next sections we will also discuss *lagged* and *forward-looking* Taylor rules.

Substituting the Taylor-type policy rule in equation (4.4) into the equations in (4.1) and writing the model in matrix form gives

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{x}^e_{t+1} + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \end{cases} \tag{4.5}$$

where $\boldsymbol{x}_t = [y_t, \pi_t]'$, $\boldsymbol{u}_t = [u_{y,t}, u_{\pi,t}]'$, $\boldsymbol{\varepsilon}_t = [\varepsilon_{1,t}, \varepsilon_{2,t}]'$, $\boldsymbol{B} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\begin{bmatrix} 1 & \varphi(1-\lambda\phi_\pi) \\ \gamma & \gamma\varphi + \lambda(1+\varphi\phi_y) \end{bmatrix}$,
$\boldsymbol{C} = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\begin{bmatrix} 1 & -\varphi\phi_\pi \\ \gamma & 1+\varphi\phi_y \end{bmatrix}$, $\boldsymbol{\rho} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}$.

Before turning to BLE, we consider rational expectations equilibrium first.

4.1.1 Theoretical results

Comparing the NK model (4.5) with the general framework (3.1), we note that $\boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{b}_0 = \mathbf{0}$. The rational expectation equilibrium (REE) fixed point in (3.6-3.7) then simplifies to

$$(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{\xi} = \mathbf{0} \tag{4.6}$$

$$\boldsymbol{\eta} = \mathbf{B}\boldsymbol{\eta}\boldsymbol{\rho} + \mathbf{C}. \tag{4.7}$$

Bullard and Mitra (2002) show that the REE is unique (determinate) if and only if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$. The REE is then the stable stationary process with mean

$$\overline{\mathbf{x}^*} = \mathbf{0}. \tag{4.8}$$

In the symmetric case $\rho_i = \rho$ for $i = 1, 2, \cdots, n$, the REE $\mathbf{x}^*_t$ satisfies

$$\mathbf{x}^*_t = (\mathbf{I} - \rho\mathbf{B})^{-\mathbf{1}}\mathbf{C}\mathbf{u_t}. \tag{4.9}$$

Thus its covariance is

$$\mathbf{\Sigma_{x^*}} = \mathbf{E}(\mathbf{x^*_t} - \overline{\mathbf{x}^*})(\mathbf{x^*_t} - \overline{\mathbf{x}^*})' = (1-\rho^2)^{-1}(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}\mathbf{\Sigma}_{\boldsymbol{\varepsilon}}[(\mathbf{I} - \rho\mathbf{B})^{-1}\mathbf{C}]'. \tag{4.10}$$

Furthermore, the first-order autocorrelation of the i-element x_i of $\mathbf{x}$ is equal to ρ. That is, in this case the persistence of the REE coincides exactly with the persistence of the exogenous driving force $\mathbf{u}_t$ and the first-order autocorrelations of output gap and inflation are the same, i.e. symmetric, equal to the autocorrelation in the driving force. Inflation and output gap only inherit the persistence of the shocks.

Behavioral learning equilibria

Bullard and Mitra (2002) study adaptive learning in this NK setting. They consider a PLM which coincides with the minimum state variable solution (MSV) of the form

$$\boldsymbol{x}_t = \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{E}}\boldsymbol{x}^e_{t+1} + \widetilde{\boldsymbol{F}}\boldsymbol{u}_t, \tag{4.11}$$

where $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{F}}$ are conformable matrices. We will consider learning with misspecification. As in the general setup in Section 3, we assume that agents are boundedly rational and use simple univariate linear rules to forecast the output gap y_t and inflation π_t of the economy. We therefore deviate from Bullard and Mitra (2002) in two important ways: (i) our agents can not observe or do not use the exogenous shocks $\boldsymbol{u}_t$, and (ii) agents do not fully understand the linear stochastic structure and do not take into account the cross-correlation between inflation and output. Rather our agents learn a simple AR(1) univariate forecasting rule for each variable as shown in (2.2). This AR(1) rule however indirectly, in a boundedly rational way, takes exogenous shocks and cross-correlations of endogenous variables into account as agents learn the two parameters of each AR(1) rule consistent with the observable sample averages and first-order autocorrelations. The use of simple AR(1) rules is supported by evidence from the learning-to-forecast laboratory experiments in the NK framework in Adam (2007), Assenza et al. (2014) and Pfajfar and Zakelj (2016). The actual law of motion (4.5) becomes

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{B}[\boldsymbol{\alpha} + \boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \boldsymbol{\alpha})] + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho} u_{t-1} + \boldsymbol{\varepsilon}_t. \end{cases} \tag{4.12}$$

For the actual law of motion (ALM) (4.12), the REE determinacy condition $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$ implies the ALM is stationary, see Appendix D. Thus the means and first-order autocorrelations are

$$\begin{aligned} \overline{\boldsymbol{x}} &= (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B}\boldsymbol{\alpha} - \boldsymbol{B}\boldsymbol{\beta}^2\boldsymbol{\alpha}), \\ \boldsymbol{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \begin{bmatrix} G_1(\beta_1, \beta_2) & 0 \\ 0 & G_2(\beta_1, \beta_2) \end{bmatrix} = \begin{bmatrix} \mathrm{corr}(y_t, y_{t-1}) & 0 \\ 0 & \mathrm{corr}(\pi_t, \pi_{t-1})) \end{bmatrix}. \end{aligned}$$

In order to obtain analytic expressions for $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ we focus on the symmetric case with $\rho_1 = \rho_2 = \rho$. The first-order autocorrelations of output gap and inflation are obtained through complicated calculations (see Appendix E)[10]:

$$G_1(\beta_1,\beta_2) = \frac{\widetilde{f}_1}{\widetilde{g}_1} \tag{4.13}$$

$$G_2(\beta_1,\beta_2) = \frac{\widetilde{f}_2}{\widetilde{g}_2} \tag{4.14}$$

where

$$\begin{aligned}
\widetilde{f}_1 &= \sigma_1^2\Big\{(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\\
&\quad \rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\lambda\beta_2^2\rho\lambda_1\lambda_2]\Big\}+\sigma_2^2\Big\{(\varphi\phi_\pi(\rho+\lambda_1+\lambda_2)-\varphi\beta_2^2))\\
&\quad [\varphi\phi_\pi-\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\phi_\pi\rho\lambda_1\lambda_2]\\
&\quad [\varphi\phi_\pi(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\beta_2^2\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_1 &= \sigma_1^2\Big\{[(1+\lambda^2\beta_2^4)-2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)+(1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&\quad -\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\}\\
&\quad +\sigma_2^2\Big\{[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)-2\varphi\phi_\pi\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)+((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\\
&\quad -\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)\\
&\quad +((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{f}_2 &= \sigma_1^2\Big\{\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}+\sigma_2^2\Big\{[(1+\varphi\phi_y)(\rho+\lambda_1+\lambda_2)-\beta_1^2]\cdot\\
&\quad [(1+\varphi\phi_y)-\beta_1^2(\rho+\lambda_1+\lambda_2)]+[\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(1+\varphi\phi_y)\rho\lambda_1\lambda_2]\cdot\\
&\quad [(1+\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\beta_1^2\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_2 &= \sigma_1^2\Big\{\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big\}\\
&\quad +\sigma_2^2\Big\{[((1+\varphi\phi_y)^2+\beta_1^4)-2(1+\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[((1+\varphi\phi_y)^2+\beta_1^4)(\rho+\lambda_1+\lambda_2)-2(1+\varphi\phi_y)\beta_1^2\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+((1+\varphi\phi_y)^2+\beta_1^4)\rho\lambda_1\lambda_2]\Big\},
\end{aligned}$$

$$\begin{aligned}
\lambda_1+\lambda_2 &= \frac{\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\\
\lambda_1\lambda_2 &= \frac{\lambda\beta_1^2\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}.
\end{aligned}$$

[10]Numerical computations of the first-order autocorrelation coefficients of output gap and inflation are easily obtained from simulated time series generated by the system (4.12), and confirm the complicated expressions (4.13-4.14).

From these expressions, it is easy to see $G_1(\beta_1, \beta_2)$ and $G_2(\beta_1, \beta_2)$ are analytic functions with respect to β_1 and β_2, independent of $\boldsymbol{\alpha}$.

The actual law of motion (4.5) depends on eight parameters φ, λ, γ, ϕ_y, ϕ_π, ρ, σ_1^2 and σ_2^2. Only the ratio σ_1^2/σ_2^2 of noise terms matters for the persistence $G_i(\beta_1, \beta_2)$ in (4.13) and (4.14). Hence, the existence of BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ depends on seven structural parameters φ, λ, γ, ρ, ϕ_y, ϕ_π and σ_1^2/σ_2^2 of the NK-model.

Using Proposition 1 and Proposition 2 we have the following properties for the New Keynesian model:

Corollary 1 *Under the contemporaneous interest rate rule, if* $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, *then there exists at least one BLE* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$, *where* $\boldsymbol{\alpha}^* = \mathbf{0} = \overline{\boldsymbol{x}^*}$.

Corollary 2 *Under the contemporaneous interest rate rule and the condition* $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, *a BLE* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ *is locally stable under SAC-learning if all eigenvalues of* $\boldsymbol{DG}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*) = \left(\frac{\partial G_i}{\partial \beta_j}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ *have real parts less than 1.*

Proof. See Appendix F.

It is useful to discuss the special case in which shocks are not persistent, that is, $\rho = 0$ (no autocorrelation in the shocks). It is easy to see that

$$G_1(0,0)\big|_{\rho=0} = 0, \qquad G_2(0,0)\big|_{\rho=0} = 0.$$

That is to say $(\mathbf{0}, \mathbf{0})$ is a BLE for $\rho = 0$. Hence, when there is no persistence in the exogenous shocks, the BLE coincides with the rational expectation equilibrium.

It is also useful to discuss the non-stationary case, that is, when the coefficient matrix $\boldsymbol{B}$ for expectations $\boldsymbol{x}_{t+1}^e$ in (4.5) has at least one eigenvalue outside the unit circle. In that case, SAC-learning of an AR(1) rule typically leads to explosive dynamics with $\alpha_t \to \pm\infty$ and $\beta_t \to 1$. In the non-stationary case, learning of BLE thus typically leads to explosive time paths of inflation and output.

4.1.2 Persistence amplification

We illustrate these results by some typical numerical calculations for empirically plausible parameter values. We calibrate the parameters in order to match the stylized facts of autocorrelation functions of output gap and inflation. As in the Clarida et al. (1999) calibration we fix $\varphi = 1, \lambda = 0.99$. We fix $\gamma = 0.04$, which lies between the calibrations $\gamma = 0.3$ in Clarida et al. (1999) and $\gamma = 0.024$ in Woodford (2003). For the exogenous shocks, we set the ratio of shocks $\frac{\sigma_2}{\sigma_1} = 0.5$, which is within the possible range suggested in Fuhrer (2006). We consider the symmetric case $\rho_1 = \rho_2 = \rho = 0.5$, with weak persistence in the shocks. The baseline parameters on the policy response to inflation deviation and output gap follow a broad literature, $\phi_\pi = 1.5$, $\phi_y = 0.5$, see for example Fuhrer (2006, 2009). The numerical results shown below are robust across a range of plausible parameter values.

Figure 1 illustrates the existence of a unique BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$[11]. In order to obtain (β_1^*, β_2^*), we numerically compute the corresponding fixed point $\beta_2^*(\beta_1)$ satisfying $G_2(\beta_1, \beta_2^*) = \beta_2^*$ for each β_1 and the corresponding fixed point $\beta_1^*(\beta_2)$ satisfying $G_1(\beta_1^*, \beta_2) = \beta_1^*$ for each β_2 as illustrated in Figure 1. Hence their intersection point (β_1^*, β_2^*) satisfies $G_1(\beta_1^*, \beta_2^*) = \beta_1^*$ and $G_2(\beta_1^*, \beta_2^*) = \beta_2^*$. A striking feature of the BLE is that the first-order autocorrelation coefficients of output gap and inflation $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ are substantially higher than those at the REE, that is, persistence is much higher than the persistence $\rho(= 0.5)$ of the exogenous shocks. We refer to this phenomenon as *persistence amplification*. Agents fail to recognize the complete linear structure of the economy, but rather learn to coordinate on a simple AR(1) rule consistent with simple observable statistics, the mean and the first-order autocorrelation. As a result of this *self-fulfilling mistake*, shocks to the economy are strongly amplified.

To illustrate the persistence amplification more clearly, Figure 2 shows the autocorrelation functions of output gap and inflation at the BLE and the REE. Compared with REE, both the autocorrelation functions (ACF) of output gap and inflation at the BLE decay considerably slower. The autocorrelation functions of output gap and inflation are similar to empirical data. Along the BLE, the first-order autocorrelation coefficients of output gap is about 0.9, while after about 5 periods its values decay to about 0.5, which is consistent with empirical work, see for example Fuhrer (2006, 2009). Furthermore, the autocorrelation function of inflation with very high persistence is also close to empirical work for inflation data. That is, for plausible parameters the BLE are capable of reproducing a data-consistent degree of inertia.

Figure 3 illustrates how these results depend on the persistence ρ of the exogenous shocks. The figure shows the BLE, i.e. the first-order autocorrelations β_1^* of output gap and β_2^* of inflation, as a function of the parameter ρ. This figure clearly shows the *persistence amplification* along BLE, with much higher ACF than under RE, for all values

[11] Note that $(\alpha_1^*, \alpha_2^*) = (0, 0)$.

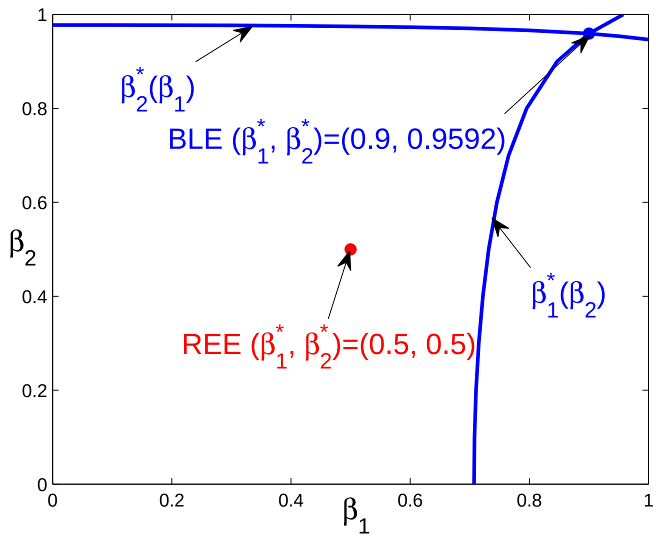
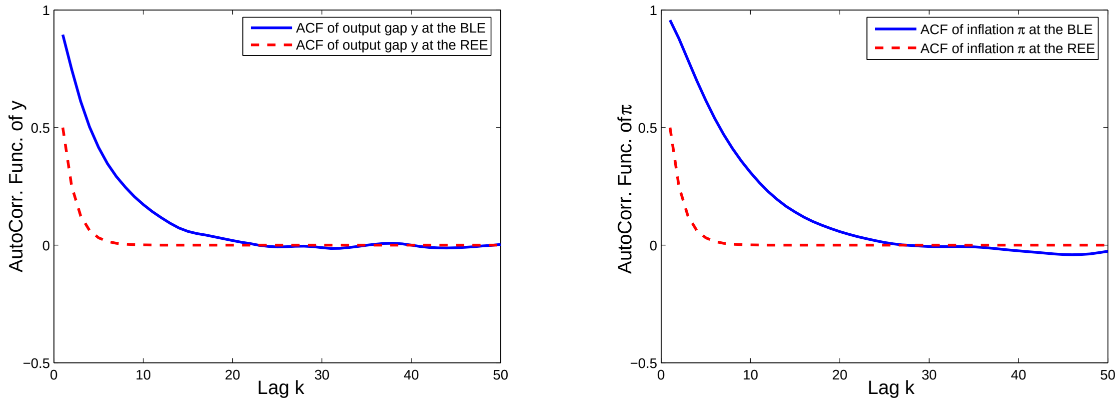
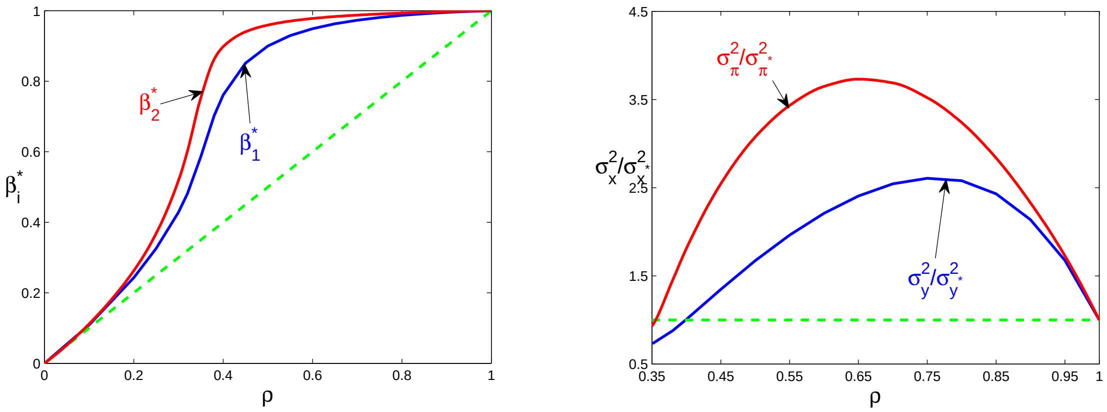


Figure 1: A unique BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ obtained as the intersection point of the fixed point curves $\beta_2^*(\beta_1)$ and $\beta_1^*(\beta_2)$. The BLE exhibits strong persistence amplification compared to REE (red dot, with $\rho = 0.5$). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.



(a) (b)

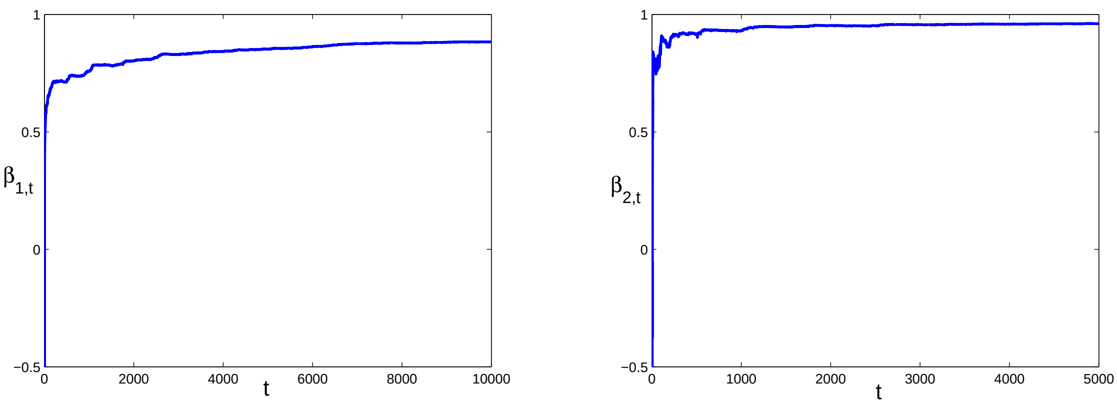
Figure 2: Autocorrelation functions of output gap y and inflation π for contemporaneous Taylor rule at the BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$ (blue plot) and at the REE $(\beta_1^*, \beta_2^*) = (0.5, 0.5)$ (red plot, dotted line). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.



(a)

(b)

Figure 3: BLE (β_1^*, β_2^*) as a function of the persistence ρ of the exogenous shocks, for the contemporaneous Taylor rule. (a) $\beta_i^*(i = 1, 2)$ with respect to ρ; (b) the ratio of variances $(\sigma_y^2/\sigma_{y^*}^2,\ \sigma_\pi^2/\sigma_{\pi^*}^2)$ of the BLE (β_1^*, β_2^*) w.r.t. the REE. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.



(a) First autocorrelation of y_t

(b) First autocorrelation of π_t

Figure 4: Time series of $\beta_{1,t}$ and $\beta_{2,t}$ under SAC-learning converging to the BLE $(\beta_1^*, \beta_2^*) = (0.9, 0.9592)$. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

of $0 < \rho < 1$. Especially for $\rho \geq 0.5$ we have $\beta_1^* \geq 0.9$ and $\beta_2^* > 0.95$, implying that output gap and inflation have significantly higher persistence than the exogenous driving forces. Moreover, Figure 3 also indicates an asymmetry, namely that the persistence of inflation is always larger than the persistence of output gap, which is consistent with empirical data. Figure 3 (right plot) also illustrates the *volatility amplification* under BLE compared to REE. For output gap the ratio of variances $\sigma_y^2/\sigma_{y^*}^2$ reaches a peak of about 2.5 for $\rho \approx 0.75$, while for inlfation the ratio of variances $\sigma_\pi^2/\sigma_{\pi^*}^2$ reaches its peak of about 3.5 for $\rho \approx 0.65$.

Finally, Figure 4 shows that the BLE is stable under SAC-learning, with $\beta_1 \to \beta_1^* \approx 0.9$ and $\beta_2 \to \beta_2^* \approx 0.9592$. In fact, based on our calculation, the two eigenvalues of the Jacobian matrix $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*)$ are $0.5012 \pm 0.7348i$ (the real parts less than 1), which shows the stability of the BLE under SAC-learning based on our theoretical results.

5 Monetary policy

BLE are characterized by persistence amplification, with much higher persistence in output and inflation than in the exogenous driving forces. This leaves an important role for monetary policy to stabilize inflation and output. In this section we study the effects of monetary policies on the persistence of inflation and output gap in the New Keynesian model for the same benchmark parameter values as before. Figure 5 shows how the BLE depend on the policy parameters ϕ_π and ϕ_y, measuring how strongly the interest rate responds to deviations of inflation and output from target.

Figure 5a shows that the first-order autocorrelation of output gap β_1^* becomes smaller as policy-makers respond more aggressively to output deviations (i.e. the curve shifts down as ϕ_y increases), while Figure 5b suggests that the first-order autocorrelation of inflation β_2^* becomes smaller as policy-makers more aggressively respond to inflation deviations (i.e. the curves are decreasing in ϕ_π). This illustrates the *direct* stabilizing effects of monetary policy: persistence of output (inflation) decreases in response to more aggressive output (inflation) targeting. This is in line e.g. with the results in Fuhrer (2009): when policy is more aggressive to adjust inflation to targets, the inflation tends to fluctuate more frequently and thus becomes less persistent. Furthermore, we find that if $\phi_y = 0$, that is, policy makers only care about inflation, the effects of adjustment are relatively large (the decrease of β_2^* is relatively large for $\phi_y = 0$ compared to $\phi_y = 0.5$; Figure 5b). A similar result holds for the direct effect on output. If $\phi_\pi = 1$, that is, policy makers

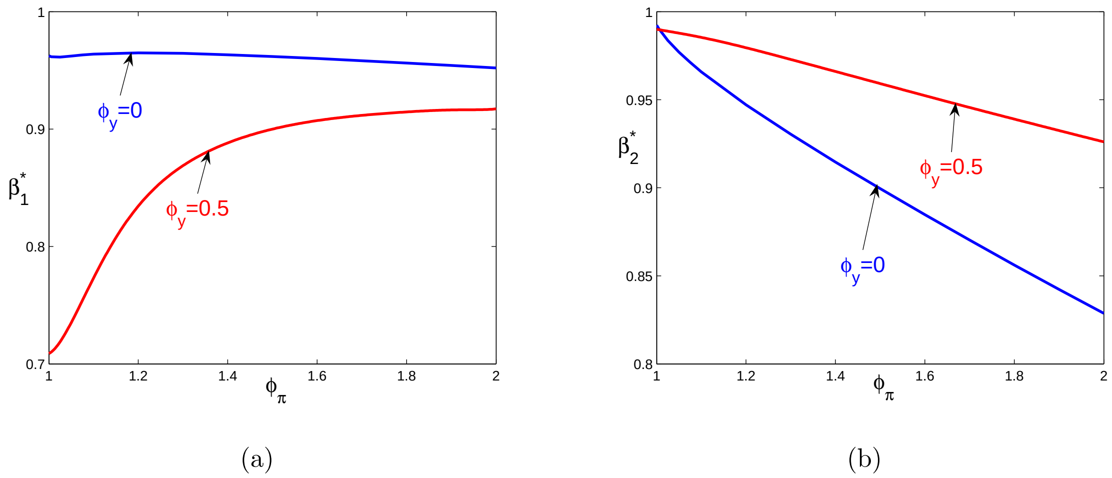


Figure 5: Effects of monetary policy with contemporaneous Taylor rule on BLE persistence of output β_1^* (a) and persistence of inflation β_2^* (b). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

mainly care about output, not inflation, the effects of increasing ϕ_y are relatively large; for $\phi_\pi = 1$ the decline of β_1^* is relatively large when ϕ_y increases compared to $\phi_\pi = 2$ (Figure 5a).

We provide some further intuitive explanations of these effects. Since monetary authorities are usually more concerned about inflation, we first discuss some intuitive explanations of the effects of policy parameters on the persistence of inflation. Using the model (4.1), (4.4) and (4.12), inflation dynamics is governed by

$$\pi_t = \frac{\beta_2^2[\gamma\varphi + \lambda(1+\varphi\phi_y)]}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\pi_{t-1} + \frac{\beta_1^2\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}y_{t-1} + \frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}u_{y,t} + \frac{1+\varphi\phi_y}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}u_{\pi,t}. \tag{5.1}$$

This equation is similar to the New Keynesian Philips curve with inflation driven by an exogenous AR(1) process for output gap (or marginal costs) as in Hommes & Zhu (2014), where the persistence of inflation increases as one of the coefficients (except for the white noise term) increases. But different from that case, here in (5.1) the persistence of the driving process y_t is not exogenous and constant, but rather output gap y_t is endogenous and its persistence varies with policy parameters. In fact there exists intricate interaction between y and π and all the four coefficients might change simultaneously, but with different directions as one policy parameter varies. The overall effect is a balance of various effects with different magnitude, as discussed below.

It is easy to see that all four coefficients in (5.1) are positive and decrease as ϕ_π

	$\frac{\beta_2^2[\gamma\varphi+\lambda(1+\varphi\phi_y)]}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\beta_1^2\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1+\varphi\phi_y}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
ϕ_π	$\frac{-\beta_2^2\gamma\varphi[\gamma\varphi+\lambda(1+\varphi\phi_y)]}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\beta_1^2\gamma^2\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\gamma^2\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\gamma\varphi(1+\varphi\phi_y)}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$
ϕ_y	$\frac{\beta_2^2\gamma\varphi^2(\lambda\phi_\pi-1)}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\beta_1^2\gamma\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\gamma\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{\gamma\varphi^2\phi_\pi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$

Table 1: The derivatives of the coefficients in (5.1) with respect to ϕ_π and ϕ_y and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

	$\frac{\gamma\varphi+\lambda(1+\varphi\phi_y)}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1+\varphi\phi_y}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
$\phi_y = 0$	$[0.9904, \searrow, 0.9537]$	$[0.0385, \searrow, 0.0370]$	$[0.0385, \searrow, 0.0370]$	$[0.9615, \searrow, 0.9259]$
$\phi_y = 0.5$	$[0.9903, \searrow, 0.9652]$	$[0.0260, \searrow, 0.0253]$	$[0.0260, \searrow, 0.0253]$	$[0.9740, \searrow, 0.9494]$

Table 2: The changing ranges of the coefficients of π in (5.1) with ϕ_π increasing from 1 to 2 and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

increases, as summarized in Table 1. To simplify the discussion below, we fix $\beta_1^2 = \beta_2^2 = 1$ at their upperbounds. Table 2 summarizes the ranges of the decrease of the four coefficients for the benchmark parameters $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$, as ϕ_π increases from 1 to 2. The dominating coefficient is the coefficient of π_{t-1}, which decreases from 0.9904 to 0.9537 (for $\phi_y = 0$), and all other coefficients are relatively small. This explains why the persistence of inflation decreases as ϕ_π increases from 1 to 2, as in Figure 5b. Furthermore, the changing range of the coefficient of π_{t-1} for $\phi_y = 0$ is larger than that for $\phi_y = 0.5$ as ϕ_π grows from 1 to 2.

In addition, from Table 3 it can be seen that for $\phi_\pi = 1$, or ϕ_π very close to one, the coefficient of π_{t-1} is also very close to one and decreases from 0.9904 to 0.9903 for $\phi_y \in [0, 0.5]$, which plays a dominant role in determining the persistence of π. This may explain why in Figure 5b for ϕ_π very close to one, the persistence of π for $\phi_y = 0$ is larger than for $\phi_y = 0.5$ while for large ϕ_π (e.g. $\phi_\pi = 2$) the persistence of π for $\phi_y = 0$ is smaller than for $\phi_y = 0.5$.

Similar intuition may be provided for the persistence of output gap. Using (4.1), (4.4) and (4.12), output dynamics becomes

$$y_t = \frac{\beta_1^2}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y} y_{t-1} + \frac{\beta_2^2\varphi(1 - \lambda\phi_\pi)}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y}\pi_{t-1} + \frac{1}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y} u_{y,t} - \frac{\varphi\phi_\pi}{1 + \gamma\varphi\phi_\pi + \varphi\phi_y} u_{\pi,t}. \tag{5.2}$$

The following Table 4 indicates the derivatives of the coefficients in (4.12) with respect to ϕ_y and ϕ_π. For given $\phi_y(= 0 \text{ or } 0.5)$, all coefficients decrease within some ranges as shown

	$\frac{\gamma\varphi+\lambda(1+\varphi\phi_y)}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\gamma}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1+\varphi\phi_y}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
$\phi_\pi = 1$	$[0.9904, \searrow, 0.9903]$	$[0.0385, \searrow, 0.0260]$	$[0.0385, \searrow, 0.0260]$	$[0.9615, \nearrow, 0.9740]$
$\phi_\pi = 2$	$[0.9537, \nearrow, 0.9652]$	$[0.0370, \searrow, 0.0253]$	$[0.0370, \searrow, 0.0253]$	$[0.9259, \nearrow, 0.9494]$

Table 3: The changing ranges of the coefficients of π in (5.1) with ϕ_y increasing from 0 to 0.5 and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

	$\frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\beta_2^2\varphi(1-\lambda\phi_\pi)}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{-\varphi\phi_\pi}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
ϕ_y	$\frac{-\beta_1^2\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\beta_2^2\varphi^2(1-\lambda\phi_\pi)}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{\varphi^2\phi_\pi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$
ϕ_π	$\frac{-\beta_1^2\gamma\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\beta_2^2\varphi[\lambda(1+\varphi\phi_y)+\gamma\varphi]}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\gamma\varphi}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$	$\frac{-\varphi(1+\varphi\phi_y)}{(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2}$

Table 4: The derivatives of the coefficients in (5.2) with respect to ϕ_y and ϕ_π and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

in Tables 4 and 5. However, different from the coefficients of π, here the coefficients of π_{t-1} and $u_{\pi,t}$ may become negative as ϕ_π grows from 1 to 2, as shown in Table 5. As ϕ_π grows from 1 to 2, in fact, the absolute value of the coefficient of π_{t-1} decreases first and then increase and the absolute value of the coefficient of $u_{\pi,t}$ increases[12]. Hence here the persistence of y is determined by complex interactions. It is possible that in some cases (e.g. $\phi_y = 0$) the persistence of y changes little (basically slightly decreasing for increasing ϕ_π), while in some other cases (e.g. $\phi_y = 0.5$) the persistence of y increases much as ϕ_π grows from 1 to 2, as shown in Figure 5a. Although the effect of ϕ_π on the persistence of y is complex, the direct effect of ϕ_y is rather clear. From Tables 5 and 6 it is easy to see that the absolute values of the four coefficient decreases as ϕ_y grows from 0 to 0.5 for given ϕ_π. For $\phi_\pi = 1$, not only all the absolute values of the four coefficients decreases as ϕ_y grows from 0 to 0.5, but also the persistence of π_{t-1} decreases, as shown in the above paragraph. Hence for $\phi_\pi = 1$ and other given parameters, the persistence of y decreases much from about 0.96 to 0.71, as shown in Figure 5a. However, for $\phi_\pi = 2$ although all absolute values of the four coefficients decreases as ϕ_y grows from 0 to 0.5, the persistence of the driving process of π_{t-1} increases, as shown in the above paragraph. In view of this and the changes of the coefficients, the persistence of y for $\phi_\pi = 2$ decreases less than that for $\phi_\pi = 1$, consistent with Figure 5a.

[12]Note that in the model of Hommes & Zhu (2014), from the expression of first-order autocorrelation coefficient of Eq.(4.16) on p.795, it can be seen that the first-order autocorrelation coefficient in fact depends on the absolute value of the coefficient.

	$\frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\varphi(1-\lambda\phi_\pi)}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{-\varphi\phi_\pi}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
$\phi_y = 0$	$[0.9615, \searrow, 0.9259]$	$[0.0096, \searrow, -0.9074]$	$[0.9615, \searrow, 0.9259]$	$[-0.9615, \searrow, -1.8519]$
$\phi_y = 0.5$	$[0.6494, \searrow, 0.6329]$	$[0.0065, \searrow, -0.6203]$	$[0.6494, \searrow, 0.6329]$	$[-0.6494, \searrow, -1.2658]$

Table 5: The changing ranges of the coefficients of y in (5.2) with ϕ_π increasing from 1 to 2 and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

	$\frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{\varphi(1-\lambda\phi_\pi)}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$	$\frac{-\varphi\phi_\pi}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}$
$\phi_\pi = 1$	$[0.9615, \searrow, 0.6494]$	$[0.0096, \searrow, 0.0065]$	$[0.9615, \searrow, 0.6494]$	$[-0.9615, \nearrow, -0.6494]$
$\phi_\pi = 2$	$[0.9259, \searrow, 0.6329]$	$[-0.9074, \nearrow, -0.6203]$	$[0.9259, \searrow, 0.6329]$	$[-1.8519, \searrow, -1.2658]$

Table 6: The changing ranges of the coefficients of y in (5.2) with ϕ_y increasing from 0 to 0.5 and given $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5, \beta_1^2 = \beta_2^2 = 1$.

Therefore, for the contemporaneous Taylor rule monetary policy can mitigate persistence amplification in inflation (output), when the policy maker is more aggressive to adjust inflation to its target (output to potential output). While the direct effects of monetary policy upon the persistence of inflation or output are clear, the indirect effects are more subtle. Stabilizing both inflation and output requires careful balancing of the direct and indirect effects and depends upon the relative importance of inflation versus output stabilization. In the following we will check if there exist the similar effects for different Taylor-type interest rules

5.1 Alternative specifications for setting interest rates

The monetary policy Taylor rule (4.4) sets the interest rate in response to contemporaneous output gap y_t and inflation π_t. This contemporaneous rule assumes that the monetary authority observes current output gap and inflation. Here we consider two alternative specifications for setting interest rates, widely used in the literature, and perhaps under more realistic informational assumptions of either forward looking expectations (e.g. through survey data) or observing lagged variables.

5.1.1 A forward-looking monetary policy rule

As shown in Bullard and Mitra (2002) and Bullard et al. (2008), another Taylor-type interest rate rule is to assume that the monetary authorities set their interest rate instrument in response to the forecasts of output and inflation deviations. This leads to the forward expectations specification for the interest rate equation, where (4.4) is replaced with

$$i_t = \phi_\pi \hat{E}_t \pi_{t+1} + \phi_y \hat{E}_t y_{t+1}. \tag{5.3}$$

Thus the system (4.5) becomes

$$\begin{cases} \mathbf{x}_t = \mathbf{B}\hat{E}_t \mathbf{x}_{t+1} + \mathbf{C}\mathbf{u_t}, \\ \mathbf{u_t} = \boldsymbol{\rho}\mathbf{u_{t-1}} + \boldsymbol{\varepsilon}_\mathbf{t}, \end{cases} \tag{5.4}$$

where $\mathbf{B} = \begin{bmatrix} 1-\varphi\phi_y & \varphi(1-\phi_\pi) \\ \gamma(1-\varphi\phi_y) & \gamma\varphi(1-\phi_\pi)+\lambda \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$.

Similar to the above contemporaneous data interest rate rules, if the actual law of motion with the PLM in (2.2) is stationary, the first-order autocorrelations (4.13) and (4.14) become[13]

$$G_1(\beta_1, \beta_2) = \frac{\widetilde{f}_1}{\widetilde{g}_1} \tag{5.5}$$

$$G_2(\beta_1, \beta_2) = \frac{\widetilde{f}_2}{\widetilde{g}_2} \tag{5.6}$$

where

$$\begin{aligned}
\widetilde{f}_1 &= \sigma_1^2\Big\{(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)] + [\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \\
&\quad \rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) - \lambda\beta_2^2\rho\lambda_1\lambda_2]\Big\} + \sigma_2^2\Big\{(\varphi(1-\phi_\pi)\beta_2^2)^2[(\rho+\lambda_1+\lambda_2) \\
&\quad -\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}, \\
\widetilde{g}_1 &= \sigma_1^2\Big\{[(1+\lambda^2\beta_2^4) - 2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2) + (1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] \\
&\quad -\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2) - 2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) + (1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\} \\
&\quad +\sigma_2^2\Big\{(\varphi(1-\phi_\pi)\beta_2^2)^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2) - (\rho\lambda_1\lambda_2)^2]\Big\},
\end{aligned}$$

[13] As in the baseline model above, the first -order sample autocorrelations of output gap and inflation computed based on the time series generated by the model are consistent with the complicated expression for G_1 and G_2 in Eqs. (5.5-5.6).

$$
\begin{aligned}
\widetilde{f}_2 &= \sigma_1^2\Big\{\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}+\sigma_2^2\Big\{[(\rho+\lambda_1+\lambda_2)-(1-\varphi\phi_y)\beta_1^2]\cdot\\
&\quad [1-(1-\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)]+[(1-\varphi\phi_y)\beta_1^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\rho\lambda_1\lambda_2]\cdot\\
&\quad [(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(1-\varphi\phi_y)\beta_1^2\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_2 &= \sigma_1^2\Big\{\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big\}\\
&\quad +\sigma_2^2\Big\{[1+((1-\varphi\phi_y)\beta_1^2)^2-2(1-\varphi\phi_y)\beta_1^2(\rho+\lambda_1+\lambda_2)+(1+((1-\varphi\phi_y)\beta_1^2)^2)\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[(1+((1-\varphi\phi_y)\beta_1^2)^2)(\rho+\lambda_1+\lambda_2)-2(1-\varphi\phi_y)\beta_1^2\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+((1-\varphi\phi_y)\beta_1^2)^2)\rho\lambda_1\lambda_2]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1+\lambda_2 &= (1-\varphi\phi_y)\beta_1^2+(\gamma\varphi(1-\phi_\pi)+\lambda)\beta_2^2,\\
\lambda_1\lambda_2 &= \lambda(1-\varphi\phi_y)\beta_1^2\beta_2^2.
\end{aligned}
$$

In this case we have some similar results on the existence and stability as in the baseline model. Since the coefficients matrices are different, the corresponding sufficient conditions change.

Corollary 3 *Under the forward looking interest rate rule, if* $\phi_y < \frac{1}{\varphi}$ *and* $1 < \phi_\pi < 1+\frac{\lambda}{\gamma\varphi}$*, then there exists at least one BLE* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$*, where* $\boldsymbol{\alpha}^* = \mathbf{0} = \overline{\mathbf{x}^*}$*. Furthermore, the BLE* $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ *is locally stable under SAC-learning if all the eigenvalues of* $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*) = \left(\frac{\partial G_i}{\partial \beta_j}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ *have real parts less than 1.*

Proof. See Appendix G.

For the same benchmark parameter values as before, the system has a BLE $(\beta_1^*, \beta_2^*) = (0.8326, 0.9605)$, with output and inflation much more persistent than the REE benchmark (with $\rho = 0.5$), as illustrated in Figure 6. Figure 7a illustrates how these results depend on the parameter ρ, the persistence of the exogenous shocks. As before, for the forward looking Taylor rule, the system also displays *persistence amplification*, with the persistence of inflation and output gap along BLE much higher than the persistence ρ of the exogenous shocks. Similarly, Figure 7b illustrates the excess volatility of inflation and output compared to the REE benchmark.

We also find similar results concerning the effects of monetary policy. Figure 8 illustrates how the BLE depend upon the the monetary policy parameters ϕ_π and ϕ_y under the forward-looking Taylor-type interest rule. The direct effects are similar as before: an increase of ϕ_y strongly reduces the persistence of output gap (i.e. the curve in Figure 8a shifts down as ϕ_y increases from 0 to 0.5), while an increase of ϕ_π strongly reduces the

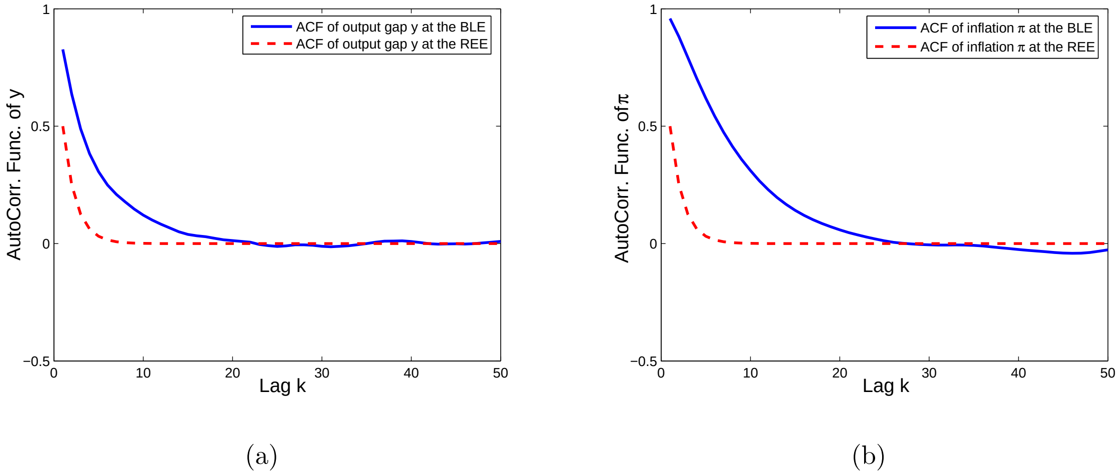


Figure 6: Autocorrelation functions of output gap y and inflation π with forward looking Taylor rule at the BLE $(\beta_1^*, \beta_2^*) = (0.8326, 0.9605)$. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \sigma_2/\sigma_1 = 0.5$.

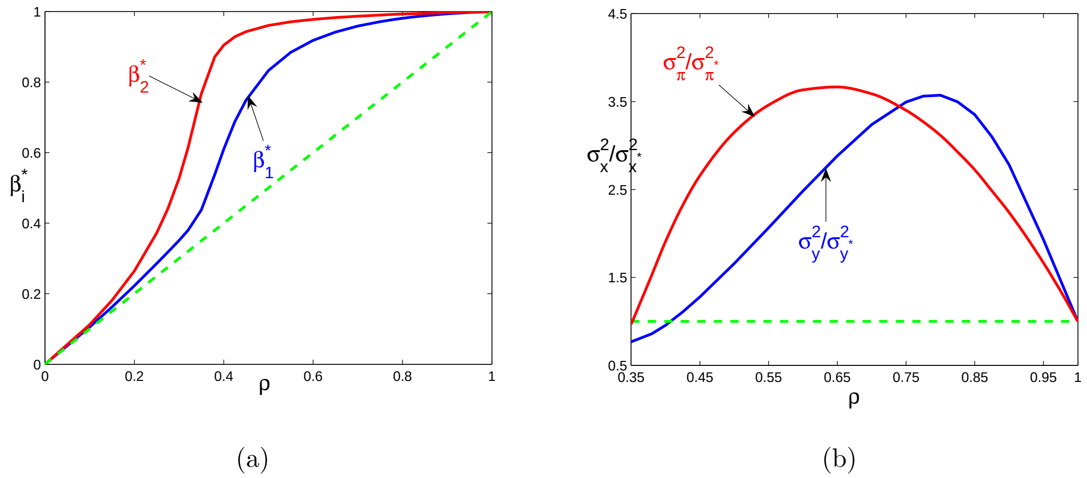


Figure 7: BLE (β_1^*, β_2^*) as a function of the persistence ρ of the exogenous shocks with forward-looking Taylor rule a) $\beta_i^*(i = 1, 2)$ with respect to ρ; b) the ratio of variances $(\sigma_y^2/\sigma_{y^*}^2,\ \sigma_\pi^2/\sigma_{\pi^*}^2)$ with respect to ρ at the corresponding BLE (β_1^*, β_2^*). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

	$(1-\varphi\phi_y)\beta_1^2$	$\varphi(1-\phi_\pi)\beta_2^2$	1	0
ϕ_y	$-\varphi\beta_1^2$	0	0	0
ϕ_π	0	$-\varphi\beta_2^2$	0	0

Table 7: The derivatives of the coefficients in (5.7) with respect to ϕ_y and ϕ_π for forward-looking Taylor rule.

persistence of inflation (i.e. the curves in Figure 8b are decreasing as ϕ_π increases from 1 to 2). The cross effects are smaller and ambiguous. As ϕ_π increases, the persistence in output may either decrease somewhat (for $\phi_y = 0$) or may increase (for $\phi_y = 0.5$ in Figure 8a); as ϕ_y increases, the persistence in inflation may either decrease somewhat (for $\phi_\pi = 1$) or may increase (for $\phi_\pi = 2$ in Figure 8b).

These graphical illustrations may be explained intuitively by looking at the output-inflation dynamics for the forward-looking Taylor rule, given by

$$
\begin{aligned}
y_t &= (1-\varphi\phi_y)\beta_1^2 y_{t-1} + \varphi(1-\phi_\pi)\beta_2^2\pi_{t-1} + u_{y,t}, &(5.7)\\
\pi_t &= (\gamma\varphi(1-\phi_\pi)+\lambda)\beta_2^2\pi_{t-1} + \gamma(1-\varphi\phi_y)\beta_1^2 y_{t-1} + \gamma u_{y,t} + u_{\pi,t}. &(5.8)
\end{aligned}
$$

Similar to the contemporaneous case, consider $\beta_1 = \beta_2 = 1$ and $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$. Tables 7 and 8 show the partial derivatives of y and π w.r.t. the parameters ϕ_y and ϕ_π. From Table 7 it is easy to see that for the equation (5.7) of y_t, the coefficient of y_{t-1} decreases from 1 to 0.5 as ϕ_y grows from 0 to 0.5 and the coefficient of π_{t-1} equals 0 if $\phi_\pi = 1$ and -1 if $\phi_\pi = 2$. Also at $\phi_\pi = 2$, the persistence increases a little as ϕ_y grows from 0 to 0.5. Therefore as ϕ_y grows, the persistence of y decreases and the decreasing range becomes smaller as ϕ_π grows from 1 to 2, as shown in Figure 8a. Furthermore, for the equation (5.8) of π_t, the coefficient of π_{t-1} decreases from 0.99 to 0.95 and the coefficient of y_{t-1} equals 0.04 if $\phi_y = 0$ and 0.02 if $\phi_y = 0.5$. Thus as ϕ_π grows from 1 to 2, the persistence of π decreases. Note that for $\phi_y = 0$ the persistence of y decreases and for $\phi_y = 0.5$ the persistence of y increases as ϕ_π grows as shown in Figure 8a. Hence the decreasing range for $\phi_y = 0.5$ is smaller than $\phi_y = 0$, as shown in Figure 8b.

	$(\gamma\varphi(1-\phi_\pi)+\lambda)\beta_2^2$	$\gamma(1-\varphi\phi_y)\beta_1^2$	γ	1
ϕ_π	$-\gamma\varphi\beta_2^2$	0	0	0
ϕ_y	0	$-\gamma\varphi\beta_1^2$	0	0

Table 8: The derivatives of the coefficients in (5.8) with respect to ϕ_π and ϕ_y for forward-looking Taylor rule.

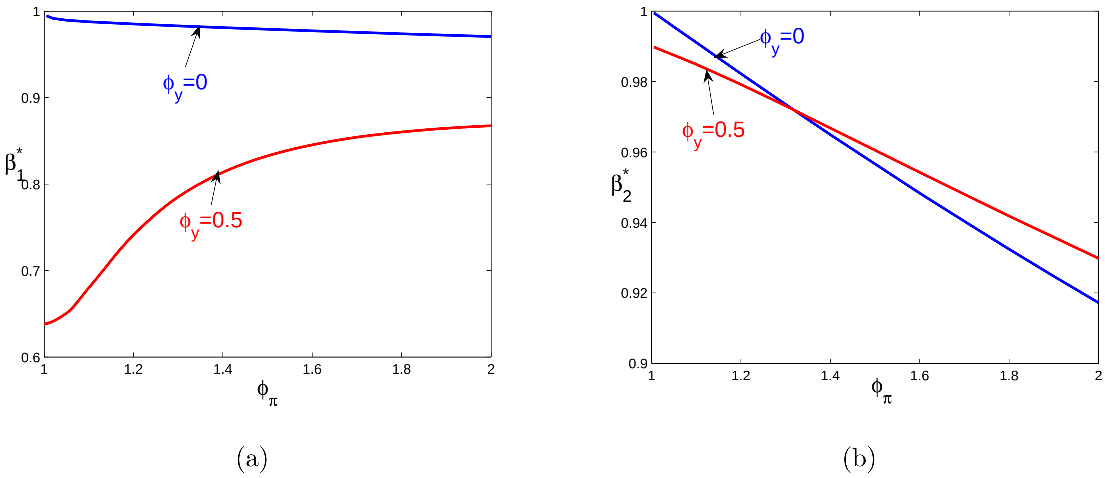


Figure 8: Effects of monetary policy with forward looking Taylor rule on BLE persistence of output β_1^* (a) and persistence of inflation β_2^* (b). Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \sigma_2/\sigma_1 = 0.5$.

5.1.2 A lagged monetary policy rule

As argued e.g. in Bullard and Mitra (2002), it may be viewed as realistic for central bank practice to posit that the monetary authorities must react to last quarter's observations on inflation and the output gap as contemporaneous values are not known yet. This leads to the lagged data specification for the interest rate equation, where (4.4) is replaced by

$$i_t = \phi_\pi \pi_{t-1} + \phi_y y_{t-1}. \tag{5.9}$$

Due to the lagged monetary policy rule the system (4.5) includes an additional lagged term x_{t-1} and becomes

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{A}\boldsymbol{x}_{t-1} + \boldsymbol{B}\boldsymbol{x}^e_{t+1} + \boldsymbol{C}\boldsymbol{u}_t, \\ \boldsymbol{u}_t = \boldsymbol{\rho}\boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t, \end{cases} \tag{5.10}$$

where $\boldsymbol{A} = \begin{bmatrix} -\varphi\phi_y & -\varphi\phi_\pi \\ -\gamma\varphi\phi_y & -\gamma\varphi\phi_\pi \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & \varphi \\ \gamma & \gamma\varphi + \lambda \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$.

Similar to the above two interest rate rules, if the actual law of motion with the PLM in (2.2) is stationary, the first-order autocorrelations (4.13) and (4.14) become

$$G_1(\beta_1, \beta_2) = \frac{\widetilde{f}_1}{\widetilde{g}_1} \tag{5.11}$$

$$G_2(\beta_1, \beta_2) = \frac{\widetilde{f}_2}{\widetilde{g}_2} \tag{5.12}$$

where

$$\begin{aligned}
\widetilde{f}_1 &= \sigma_1^2\Big\{(\rho + \lambda_1 + \lambda_2 - \lambda\beta_2^2)[1 - \lambda\beta_2^2(\rho + \lambda_1 + \lambda_2)] + [\lambda\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \\
&\quad \rho\lambda_1\lambda_2][(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) - \lambda\beta_2^2\rho\lambda_1\lambda_2]\Big\} + \sigma_2^2\Big\{(\varphi\phi_\pi - \varphi\beta_2^2)^2[(\rho + \lambda_1 + \lambda_2) \\
&\quad -\rho\lambda_1\lambda_2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2)]\Big\}, \\
\widetilde{g}_1 &= \sigma_1^2\Big\{[(1 + \lambda^2\beta_2^4) - 2\lambda\beta_2^2(\rho + \lambda_1 + \lambda_2) + (1 + \lambda^2\beta_2^4)(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2)] \\
&\quad -\rho\lambda_1\lambda_2[(1 + \lambda^2\beta_2^4)(\rho + \lambda_1 + \lambda_2) - 2\lambda\beta_2^2(\rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2) + (1 + \lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big\} \\
&\quad +\sigma_2^2\Big\{(\varphi\phi_\pi - \varphi\beta_2^2)^2[1 + \rho\lambda_1 + \rho\lambda_2 + \lambda_1\lambda_2 - \rho\lambda_1\lambda_2(\rho + \lambda_1 + \lambda_2) - (\rho\lambda_1\lambda_2)^2]\Big\},
\end{aligned}$$

$$
\begin{aligned}
\widetilde{f}_2 &= \sigma_1^2\Big\{\gamma^2[(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]\Big\}+\sigma_2^2\Big\{[(\rho+\lambda_1+\lambda_2)-(\beta_1^2-\varphi\phi_y)]\cdot\\
&\quad [1-(\beta_1^2-\varphi\phi_y)(\rho+\lambda_1+\lambda_2)]+[(\beta_1^2-\varphi\phi_y)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\rho\lambda_1\lambda_2]\cdot\\
&\quad [(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-(\beta_1^2-\varphi\phi_y)\rho\lambda_1\lambda_2]\Big\},\\
\widetilde{g}_2 &= \sigma_1^2\Big\{\gamma^2[1+\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2-\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-(\rho\lambda_1\lambda_2)^2]\Big\}\\
&\quad +\sigma_2^2\Big\{[(1+(\beta_1^2-\varphi\phi_y)^2)-2(\beta_1^2-\varphi\phi_y)(\rho+\lambda_1+\lambda_2)+(1+(\beta_1^2-\varphi\phi_y)^2)\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)]-\rho\lambda_1\lambda_2[(1+(\beta_1^2-\varphi\phi_y)^2)(\rho+\lambda_1+\lambda_2)-2(\beta_1^2-\varphi\phi_y)\cdot\\
&\quad (\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+(\beta_1^2-\varphi\phi_y)^2)\rho\lambda_1\lambda_2]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1+\lambda_2 &= \beta_1^2-\varphi\phi_y-\gamma\varphi\phi_\pi+(\gamma\varphi+\lambda)\beta_2^2,\\
\lambda_1\lambda_2 &= \lambda(\beta_1^2-\varphi\phi_y)\beta_2^2.
\end{aligned}
$$

Note that, because of the additional lagged term x_{t-1}, the rational expectations equilibrium is different compared to the system without this lagged term $\boldsymbol{x}_{t-1}$ in the previous two cases. The proofs concerning existence and stability of BLE, however, are straightforward extensions of the proofs without the additional term $\boldsymbol{x}_{t-1}$[14]. In this case we have similar results on the existence and stability on BLE as in the baseline model.

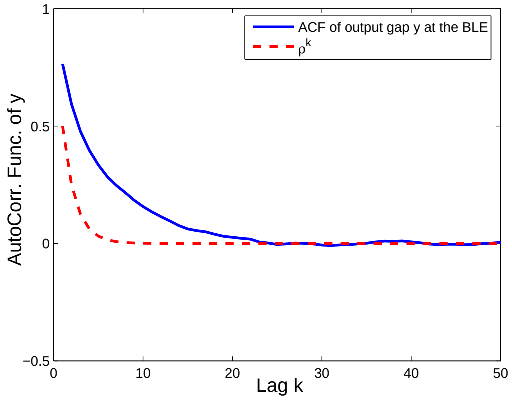
Corollary 4 *Under the lagged interest rate rule, if $\phi_y < \frac{1}{\varphi}$ and $1 < \phi_\pi < \frac{1-\varphi\phi_y}{\gamma\varphi}$, then there exists at least one BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$. Furthermore, the BLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is locally stable under SAC-learning if all the eigenvalues of $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*) = \left(\frac{\partial G_i}{\partial \beta_j}\right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ have real parts less than 1.*

Proof. See Appendix H.

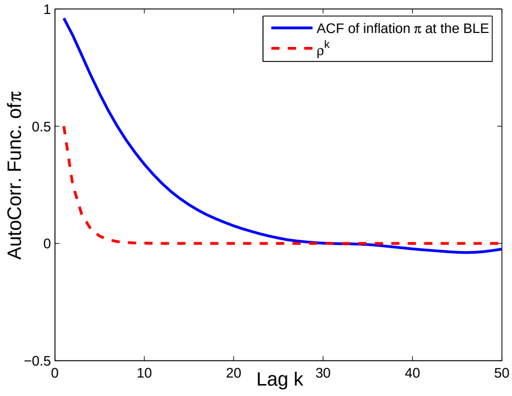
Figure 9 illustrates the ACF of output and inflation along the BLE and shows that they are much more persistent as the autocorrelations in the exogenous shocks. Figure 10 illustrates that the model with a lagged Taylor rule also exhibits *persistence amplification.*

Figure 11 suggests similar results for monetary policy with the lagged Taylor-type interest rule. There are strong direct effects, with the persistence of output (inflation) declining when ϕ_y (ϕ_π) increases. The indirect effects are again ambiguous.

[14] In particular, the system (5.10) can be rewritten in the form of Eq. (E.1) in appendix E, when BLE is considered. The difference from the baseline case lies in different coefficient matrices $\mathbf{z}_t = \boldsymbol{(A+B\beta^2)}\mathbf{z}_{t-1} + \boldsymbol{C\varepsilon}_t + \boldsymbol{C\rho I\varepsilon}_{t-1} + \cdots$ with matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as in the system (5.10). Then following the same idea of the proof, the first-order autocorrelations for the lagged monetary policy rule in (5.11) and (5.12) are obtained. Similar proofs of propositions 1 and 2, concerning existence and stability of BLE, can then be obtained.



(a)



(b)

Figure 9: Autocorrelation functions of output gap y and inflation π with lagged Taylor rule at the BLE $(\beta_1^*, \beta_2^*) = (0.7746, 0.9628)$. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \phi_\pi = 1.5, \phi_y = 0.5, \sigma_2/\sigma_1 = 0.5$.

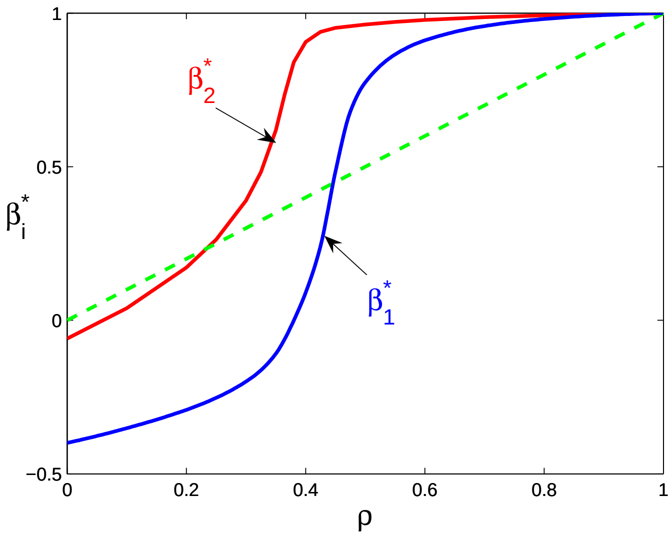


Figure 10: Effects of ρ with lagged Taylor rule, i.e. $\beta_i^*(i = 1, 2)$ with respect to ρ. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \phi_\pi = 1.5, \phi_y = 0.5, \frac{\sigma_2}{\sigma_1} = 0.5$.

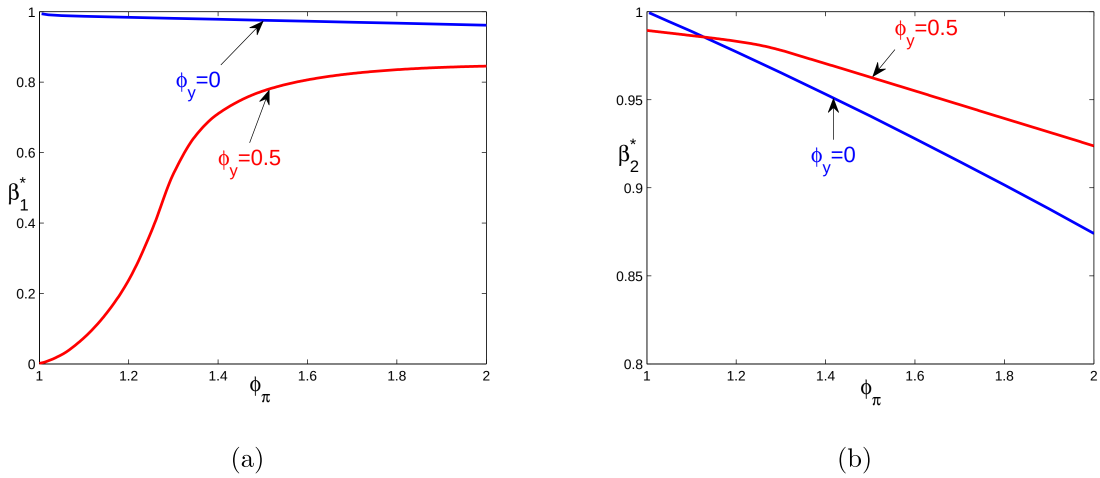


Figure 11: Effects of monetary policy with lagged Taylor rule. Parameters are: $\lambda = 0.99, \varphi = 1, \gamma = 0.04, \rho = 0.5, \sigma_2/\sigma_1 = 0.5$.

	$-\varphi\phi_y + \beta_1^2$	$-\varphi\phi_\pi + \varphi\beta_2^2$	1	0
ϕ_y	$-\varphi$	0	0	0
ϕ_π	0	$-\varphi$	0	0

Table 9: The derivatives of the coefficients in (5.13) with respect to ϕ_y and ϕ_π for forward-looking Taylor rule.

For the lagged monetary policy rule, output-inflation dynamics is given by

$$y_t = (-\varphi\phi_y + \beta_1^2)y_{t-1} + (-\varphi\phi_\pi + \varphi\beta_2^2)\pi_{t-1} + u_{y,t}, \tag{5.13}$$

$$\pi_t = (-\gamma\varphi\phi_\pi + (\gamma\varphi + \lambda)\beta_2^2)\pi_{t-1} + (-\gamma\varphi\phi_y + \gamma\beta_1^2)y_{t-1} + \gamma u_{y,t} + u_{\pi,t}. \tag{5.14}$$

From Tables 9 and 10, it can be seen that the monetary policy parameters ϕ_y and ϕ_π have similar effects as in the forward-looking case.

	$-\gamma\varphi\phi_\pi + (\gamma\varphi + \lambda)\beta_2^2$	$-\gamma\varphi\phi_y + \gamma\beta_1^2$	γ	1
ϕ_π	$-\gamma\varphi$	0	0	0
ϕ_y	0	$-\gamma\varphi$	0	0

Table 10: The derivatives of the coefficients in (5.14) with respect to ϕ_π and ϕ_y for forward-looking Taylor rule.

6 Concluding remarks

We have generalized the behavioral learning equilibrium concept to a general n-dimensional linear framework and applied it to the two-dimensional New Keynesian model. Boundedly rational agents use univariate AR(1) forecasting rules for all endogenous variables. Along a BLE the two parameters of each rule are pinned down by two observable statistics: the unconditional mean and the first-order autocorrelations. Hence, to a first-order approximation the simple linear forecasting rule is consistent with observed market realizations. Agents gradually update the two coefficients –sample mean and first-order autocorrelation– of their linear rule through sample autocorrelation learning. In the long run, agents thus learn to coordinate on the best univariate linear forecasting rule, without fully recognizing the more complex structure of the economy. In higher-dimensional system BLE exist under fairly general conditions and we provide simple stability conditions under learning. Coordination on a simple, parsimonious BLE is self-fulfilling and seems a plausible outcome of the coordination process of individual expectations in large complex socio-economic systems.

A striking feature of BLE is the strong *persistence amplification*: the persistence of output and inflation along a BLE is much higher, often near unit root, than the persistence in the exogenous shocks driving the economy. This leaves an important role for monetary policy with the goal of stabilizing inflation and output. We study monetary policies with a Taylor interest rate rule. There are strong direct effects: more aggressive inflation (output) targeting weakens the persistence in inflation (output). Indirect effects may be destabilizing however: more aggressive inflation (output) targeting may lead to more persistent output (inflation). To stabilize both inflation and output, monetary policy must therefore carefully balance between inflation and output targeting. More generally, to check the robustness of policy analysis under RE future work should study policy under more plausible behavioural learning equilibria.

Appendix

A Mean of the rational expectations equilibrium

Using (3.1-3.2) and (3.6-3.7) the mean satisfies

$$\begin{aligned}
\overline{\mathbf{x}^*} &= \boldsymbol{\xi} + \boldsymbol{\eta}\overline{\boldsymbol{u}} \\
&= (\mathbf{I}-\mathbf{b_1})^{-1}\mathbf{b_0} + (\mathbf{I}-\mathbf{b_1})^{-1}\mathbf{b_1}\boldsymbol{\eta}\mathbf{a} + \boldsymbol{\eta}\boldsymbol{(I-\rho)^{-1}a} \\
&= (\mathbf{I}-\mathbf{b_1})^{-1}\mathbf{b_0} + (\mathbf{I}-\mathbf{b_1})^{-1}[\mathbf{b_1}\boldsymbol{\eta}(\mathbf{I}-\boldsymbol{\rho}) + (\mathbf{I}-\mathbf{b_1})\boldsymbol{\eta}](\mathbf{I}-\boldsymbol{\rho})^{-1}\mathbf{a} \\
&= (\mathbf{I}-\mathbf{b_1})^{-1}[\mathbf{b_0} + \mathbf{b_2}(\mathbf{I}-\boldsymbol{\rho})^{-1}\mathbf{a}].
\end{aligned}$$

B Autocorrelation in the n-dimensional case

We rewrite model (3.13) as

$$\begin{cases} \boldsymbol{x}_t - \overline{\boldsymbol{x}} = \boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}) + \boldsymbol{b}_2(\boldsymbol{u}_t - \overline{\boldsymbol{u}}) + \boldsymbol{v}_t, \\ \boldsymbol{u}_t - \overline{\boldsymbol{u}} = \boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}) + \boldsymbol{\varepsilon}_t. \end{cases} \tag{B.1}$$

That is,

$$\begin{cases} \boldsymbol{x}_t - \overline{\boldsymbol{x}} = \boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}) + \boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}}) + \boldsymbol{b}_2\boldsymbol{\varepsilon}_t + \boldsymbol{v}_t, \\ \boldsymbol{u}_t - \overline{\boldsymbol{u}} = \boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}) + \boldsymbol{\varepsilon}_t. \end{cases} \tag{B.2}$$

$$\begin{aligned}
\boldsymbol{\Gamma}(-1) &= E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= E\Big[\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})' + \boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})' + \boldsymbol{b}_2\boldsymbol{\varepsilon}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})' + \boldsymbol{v}_t(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'\Big] \\
&= \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\Gamma}(0) + \boldsymbol{b}_2\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})'] \\
&= \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\Gamma}(0) + \boldsymbol{b}_2\boldsymbol{\rho}E[(\boldsymbol{u}_{t}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})'].
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\boldsymbol{\Gamma}(0) &= E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})'] \\
&= E\Big[\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})' + \boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})' + \boldsymbol{b}_2\boldsymbol{\varepsilon}_t(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})' + \boldsymbol{v}_t(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})'\Big] \\
&= \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\Gamma}(1) + \boldsymbol{b}_2\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})'] + \boldsymbol{b}_2E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] + E[\boldsymbol{v}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] \\
&= \boldsymbol{b}_1\boldsymbol{\beta}^2\boldsymbol{\Gamma}(1) + \boldsymbol{b}_2\boldsymbol{\rho}E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_{t}-\overline{\boldsymbol{x}})'] + \boldsymbol{b}_2\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_2' + \boldsymbol{\Sigma}_{\boldsymbol{v}}.
\end{aligned} \tag{B.4}$$

Note that $E[\boldsymbol{\varepsilon}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{\varepsilon}_t(\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}))' + \boldsymbol{\varepsilon}_t(\boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}))' + \boldsymbol{\varepsilon}_t(\boldsymbol{b}_2\boldsymbol{\varepsilon}_t)' + \boldsymbol{\varepsilon}_t(\boldsymbol{v}_t)'\Big] = \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_2'$ and $E[\boldsymbol{v}_t(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'] = E\Big[\boldsymbol{v}_t(\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}}))' + \boldsymbol{v}_t(\boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}}))' + \boldsymbol{v}_t(\boldsymbol{b}_2\boldsymbol{\varepsilon}_t)' + \boldsymbol{v}_t(\boldsymbol{v}_t)'\Big] = \boldsymbol{\Sigma}_{\boldsymbol{v}}$.

Based on (B.3), (B.4) and $\mathbf{\Gamma}(-1) = \mathbf{\Gamma}(1)'$,

$$\begin{aligned}\mathbf{\Gamma}(0) &= \boldsymbol{b}_1\boldsymbol{\beta}^2\mathbf{\Gamma}(0)(\boldsymbol{b}_1\boldsymbol{\beta}^2)' + \boldsymbol{b}_1\boldsymbol{\beta}^2 E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'](\boldsymbol{b}_2\boldsymbol{\rho})' \\ &\quad + \boldsymbol{b}_2\boldsymbol{\rho} E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})'] + \boldsymbol{b}_2\mathbf{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_2' + \mathbf{\Sigma}_{\boldsymbol{v}}.\end{aligned}$$

In order to obtain the expression of $\mathbf{\Gamma}(0)$, we use column stacks of matrices. Suppose $vec(\boldsymbol{K})$ is the vectorization of a matrix $\boldsymbol{K}$ and $\otimes$ is the Kronecker product[15]. Under the assumption that all the eigenvalues of $\boldsymbol{b}_1\boldsymbol{\beta}^2$ are inside the unit circle, based on the property of Kronecker product[16], it is easy to see all the eigenvalues of $(\boldsymbol{b}_1\boldsymbol{\beta}^2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)$ lie inside the unit circle and hence $[\boldsymbol{I} - (\boldsymbol{b}_1\boldsymbol{\beta}^2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}$ exist. Therefore,

$$\begin{aligned}vec(\mathbf{\Gamma}(0)) &= [\boldsymbol{I} - (\boldsymbol{b}_1\boldsymbol{\beta}^2) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}[((\boldsymbol{b}_2\boldsymbol{\rho}) \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2))vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) \\ &\quad + (\boldsymbol{I} \otimes (\boldsymbol{b}_2\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']) + vec(\boldsymbol{b}_2\mathbf{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{b}_2' + \mathbf{\Sigma}_{\boldsymbol{v}})]. \qquad \text{(B.5)}\end{aligned}$$

Thus in order to obtain $\mathbf{\Gamma}(1)$ and $\mathbf{\Gamma}(0)$, we need calculate $E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']$ and $E[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}})']$.

$$\begin{aligned}&E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'] \\ =\ & E\Big[\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{b}_2\boldsymbol{\varepsilon}_t(\boldsymbol{u}_t - \overline{\boldsymbol{u}})' + \boldsymbol{v}_t(\boldsymbol{u}_t - \overline{\boldsymbol{u}})'\Big] \\ =\ & E\Big[\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] + \boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] \\ & + \boldsymbol{b}_2\boldsymbol{\varepsilon}_t[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t'] + \boldsymbol{v}_t[(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})'\boldsymbol{\rho}' + \boldsymbol{\varepsilon}_t']\Big] \\ =\ & \boldsymbol{b}_1\boldsymbol{\beta}^2 E[(\boldsymbol{x}_{t-1} - \overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1} - \overline{\boldsymbol{u}})']\boldsymbol{\rho}' + \boldsymbol{b}_2\boldsymbol{\rho} E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']\boldsymbol{\rho}' + \boldsymbol{b}_2\mathbf{\Sigma}_{\varepsilon}.\end{aligned}$$

Correspondingly,

$$\begin{aligned}&vec(E[(\boldsymbol{x}_t - \overline{\boldsymbol{x}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) \\ =\ & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}[vec(\boldsymbol{b}_2\boldsymbol{\rho} E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']\boldsymbol{\rho}') + vec(\boldsymbol{b}_2\mathbf{\Sigma}_{\varepsilon})] \\ =\ & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}[(\boldsymbol{\rho} \otimes (\boldsymbol{b}_2\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t - \overline{\boldsymbol{u}})(\boldsymbol{u}_t - \overline{\boldsymbol{u}})']) + (\boldsymbol{I} \otimes \boldsymbol{b}_2)vec\mathbf{\Sigma}_{\varepsilon})] \\ =\ & [\boldsymbol{I} - \boldsymbol{\rho} \otimes (\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}[(\boldsymbol{\rho} \otimes (\boldsymbol{b}_2\boldsymbol{\rho}))[\boldsymbol{I} - \boldsymbol{\rho} \otimes \boldsymbol{\rho}]^{-1} + (\boldsymbol{I} \otimes \boldsymbol{b}_2)]vec(\mathbf{\Sigma}_{\varepsilon}). \qquad \text{(B.6)}\end{aligned}$$

[15] One property of column stacks is that the column stack of a product of three matrices is $vec(ABC) = (C' \otimes A)vec(B)$. For more details on this and related properties, see Magnus and Neudecker(1988, Chapter 2) and Evans and Honkapohja (2001, Section 5.7).

[16] The eigenvalues of $\widehat{A} \otimes \widehat{B}$ are the mn numbers $\lambda_r\mu_s, r = 1, 2, \cdots, m, s = 1, 2, \cdots, n$ where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of $m \times m$ matrix $\widehat{A}$ and $\mu_1, \cdots, \mu_n$ are the eigenvalues of $n \times n$ matrix $\widehat{B}$; see Lancaster and Tismenetsky (1985).

Furthermore,

$$
\begin{aligned}
& E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'] \\
= & E\Big[\boldsymbol{b}_1\boldsymbol{\beta}^2(\boldsymbol{x}_{t-1}-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{b}_2\boldsymbol{\rho}(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{b}_2\boldsymbol{\varepsilon}_t(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'+\boldsymbol{v}_t(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'\Big] \\
= & \boldsymbol{b}_1\boldsymbol{\beta}^2E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']+\boldsymbol{b}_2\boldsymbol{\rho}E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'].
\end{aligned}
$$

Thus based on (B.6),

$$
\begin{aligned}
& vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})']) \\
= & (\boldsymbol{I}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2))vec(E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})'])+(\boldsymbol{I}\otimes(\boldsymbol{b}_2\boldsymbol{\rho}))vec(E[(\boldsymbol{u}_t-\overline{\boldsymbol{u}})(\boldsymbol{u}_t-\overline{\boldsymbol{u}})']) \\
= & (\boldsymbol{I}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2))[\boldsymbol{I}-\boldsymbol{\rho}\otimes(\boldsymbol{b}_1\boldsymbol{\beta}^2)]^{-1}[(\boldsymbol{\rho}\otimes(\boldsymbol{b}_2\boldsymbol{\rho}))[\boldsymbol{I}-\boldsymbol{\rho}\otimes\boldsymbol{\rho}]^{-1}+(\boldsymbol{I}\otimes\boldsymbol{b}_2)]vec(\boldsymbol{\Sigma}_\varepsilon) \\
& +(\boldsymbol{I}\otimes(\boldsymbol{b}_2\boldsymbol{\rho}))[\boldsymbol{I}-\boldsymbol{\rho}\otimes\boldsymbol{\rho}]^{-1}vec(\boldsymbol{\Sigma}_\varepsilon). \qquad \text{(B.7)}
\end{aligned}
$$

Therefore based on (B.7), the expression of matrix $E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'$ can be obtained. Then by transposing the matrix $E[(\boldsymbol{x}_t-\overline{\boldsymbol{x}})(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})'$, we can obtain $vec(E[(\boldsymbol{u}_{t-1}-\overline{\boldsymbol{u}})(\boldsymbol{x}_t-\overline{\boldsymbol{x}})'])$. Furthermore, combining this with (B.6), we obtain the variance-covariance matrix $\boldsymbol{\Gamma}(0)$ from (B.5) and further $\boldsymbol{\Gamma}(1)$ from (B.3). Based on the properties of matrices operations, it is easy to see that the entries of matrices $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ are smooth functions with respect to $(\beta_1,\beta_2,\cdots,\beta_n)$ and the other related parameters. Thus the first-order autocorrelation coefficients of the nontrivial stochastic stationary system (3.13) are continuous functions with respect to $(\beta_1,\beta_2,\cdots,\beta_n)$ and the other related parameters.

C Proof of Proposition 2 (stability under SAC-learning)

Set $\gamma_t=(1+t)^{-1}$. For the state dynamics equations in (3.17) and (2.8)[17], since all functions are smooth, the SAC-learning rule satisfies the conditions (A.1-A.3) of Section 6.2.1 in Evans and Honkapohja (2001, p.124).

In order to check the conditions (B.1-B.2) of Section 6.2.1 in Evans and Honkapohja (2001, p.125), we rewrite the system in matrix form by

$$
\boldsymbol{X}_t=\widetilde{\boldsymbol{A}}(\boldsymbol{\theta}_{t-1})\boldsymbol{X}_{t-1}+\widetilde{\boldsymbol{B}}(\boldsymbol{\theta}_{t-1})\boldsymbol{W}_t,
$$

[17]For convenience of theoretical analysis, one can set $\mathbf{S_{t-1}}=\mathbf{R_t}$.

where $\boldsymbol{\theta}_t' = (\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t, \boldsymbol{R}_t), \boldsymbol{X}_t' = (1, \boldsymbol{x}_t', \boldsymbol{x}_{t-1}', \boldsymbol{u}_t')$ and $\boldsymbol{W}_t' = (1, \boldsymbol{v}_t', \boldsymbol{\varepsilon}_t')$,

$$\widetilde{\boldsymbol{A}}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boldsymbol{b_0} + \boldsymbol{b_1}(\boldsymbol{I} - \boldsymbol{\beta}^2)\boldsymbol{\alpha} + \boldsymbol{b_2 a} & \boldsymbol{b_1}\boldsymbol{\beta}^2 & \boldsymbol{0} & \boldsymbol{b_2}\boldsymbol{\rho} \\ \boldsymbol{0} & \mathbf{I} & \boldsymbol{0} & \boldsymbol{0} \\ \mathbf{a} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\rho} \end{pmatrix},$$

$$\widetilde{\boldsymbol{B}}(\boldsymbol{\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{b_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Based on the properties of eigenvalues, see e.g. Evans and Honkapohja (2001, p.117), all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ include 0 (multiple $n+1$), the eigenvalues of $\boldsymbol{\rho}$ and $\boldsymbol{b_1}\boldsymbol{\beta}^2$. Thus based on the assumptions, all the eigenvalues of $\widetilde{\boldsymbol{A}}(\boldsymbol{\theta})$ lie inside the unit circle. Moreover, it is easy to see all the other conditions for Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001) are also satisfied.

Since $\boldsymbol{x}_t$ is stationary, then the limits

$$\sigma_i^2 := \lim_{t\to\infty} E(x_{i,t} - \alpha_i)^2, \qquad \sigma^2_{x_i x_{i,-1}} := \lim_{t\to\infty} E(x_{i,t} - \alpha_i)(x_{i,t-1} - \alpha_i)$$

exist and are finite. Hence according to Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001, p.126), the associated ODE is

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{R}^{-1}[\boldsymbol{E} - \boldsymbol{\beta}\boldsymbol{\Omega}] = \boldsymbol{R}^{-1}\boldsymbol{\Omega}[\boldsymbol{E}\boldsymbol{\Omega}^{-1} - \boldsymbol{\beta}], \\ \dfrac{d\boldsymbol{R}}{d\tau} = \boldsymbol{\Omega} - \boldsymbol{R}, \end{cases} \tag{C.1}$$

where $\boldsymbol{R}$ is a diagonal matrix with the i-th diagonal entry R_i and $\boldsymbol{\Omega}$, $\boldsymbol{E}$ are also diagonal matrices as defined in Section 2. As shown in Evans and Honkapohja (2001), a BLE corresponds to a fixed point of the following ODE (C.2).

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{d\tau} = \overline{\boldsymbol{x}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) - \boldsymbol{\alpha}, \\ \dfrac{d\boldsymbol{\beta}}{d\tau} = \boldsymbol{G} - \boldsymbol{\beta}. \end{cases} \tag{C.2}$$

Note that $\boldsymbol{\beta}$ and $\boldsymbol{G}$ are both diagonal matrices. The Jacobian matrix of C.2 is, in fact, equivalent to

$$\begin{pmatrix} (\boldsymbol{I} - \boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1 - \boldsymbol{I}) & \boldsymbol{\varrho} \\ \boldsymbol{0} & \boldsymbol{D}\boldsymbol{G}_{\boldsymbol{\beta}}(\boldsymbol{\beta}^*) - \boldsymbol{I} \end{pmatrix},$$

where $\boldsymbol{DG_\beta}$ is a Jacobian matrix with the (i,j)-th entry $\frac{\partial G_i}{\partial \beta_j}$ and the form of matrix $\boldsymbol{\varrho}$ is omitted since it is not needed in the proof. Therefore, if all the eigenvalues of $(\boldsymbol{I}-\boldsymbol{b}_1\boldsymbol{\beta}^{*2})^{-1}(\boldsymbol{b}_1-\boldsymbol{I})$ have negative real parts, and all the eigenvalues of $\boldsymbol{DG_\beta}(\boldsymbol{\beta}^*)$ have real parts less than 1, the SAC-learning $(\boldsymbol{\alpha}_t,\boldsymbol{\beta}_t)$ converges to the BLE $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ as time t tends to ∞.

D Eigenvalues of matrix $\boldsymbol{B\beta}^2$

The characteristic polynomial of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ is given by $h(\nu)=\nu^2+c_1\nu+c_2$, where

$$c_1=-\frac{\beta_1^2+[\gamma\varphi+\lambda(1+\varphi\phi_y)]\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\quad c_2=\frac{\lambda\beta_1^2\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}.$$

Both of the eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ are inside the unit circle if and only if both of the following conditions hold (see Elaydi, 1999):

$$h(1)>0,\quad h(-1)>0,\quad |h(0)|<1.$$

It is easy to see $h(-1)>0, |h(0)|<1$ for any $\beta_i\in[-1,1]$. Note that

$$\begin{aligned} h(1) &= \frac{(1-\beta_1^2)(1-\lambda\beta_2^2)+\gamma\varphi\phi_\pi+\varphi\phi_y-(\gamma\varphi+\lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}, \\ &\geq \frac{\varphi[\gamma(\phi_\pi-1)+(1-\lambda)\phi_y]}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}. \end{aligned}$$

Thus if $\gamma(\phi_\pi-1)+(1-\lambda)\phi_y>0$, then $h(1)>0$. Therefore, both eigenvalues of $\mathbf{B}\boldsymbol{\beta}^{\mathbf{2}}$ lie inside the unit circle.

E First-order autocorrelation coefficients of output gap and inflation

Now we calculate $\boldsymbol{G}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Define $\boldsymbol{z}_t=\boldsymbol{x}_t-\boldsymbol{E}x_t$. Then in order to obtain $\boldsymbol{G}(\boldsymbol{\alpha},\boldsymbol{\beta})$, we first calculate $\boldsymbol{E}(z_tz_{t-1}')$ and $\boldsymbol{E}(z_tz_t')$. Rewrite model (4.12) into

$$\boldsymbol{z}_t=\boldsymbol{B\beta^2}\boldsymbol{z}_{t-1}+\boldsymbol{C\varepsilon}_t+\boldsymbol{C\rho I\varepsilon}_{t-1}+\cdots. \tag{E.1}$$

Thus

$$
\begin{aligned}
\boldsymbol{z}_t &= \boldsymbol{B\beta^2 z_{t-1}} + \boldsymbol{C\varepsilon_t} + \boldsymbol{C\rho I\varepsilon_{t-1}} + \cdots \\
&= (\boldsymbol{B\beta^2})^2\boldsymbol{z_{t-2}} + \boldsymbol{B\beta^2}(\boldsymbol{C\varepsilon_{t-1}} + \boldsymbol{C\rho I\varepsilon_{t-2}} + \cdots) + \boldsymbol{C\varepsilon_t} + \boldsymbol{C\rho I\varepsilon_{t-1}} + \cdots \\
&= \boldsymbol{C\varepsilon_t} + [\boldsymbol{C\rho I} + \boldsymbol{B\beta^2 C}]\boldsymbol{\varepsilon_{t-1}} + [\boldsymbol{C\rho^2 I} + \boldsymbol{B\beta^2 C\rho I} + (\boldsymbol{B\beta^2})^2\boldsymbol{C}]\boldsymbol{\varepsilon_{t-2}} + \cdots \\
&\quad + [\boldsymbol{C\rho^n I} + \boldsymbol{B\beta^2 C\rho^{n-1} I} + \cdots + (\boldsymbol{B\beta^2})^{n-1}\boldsymbol{C\rho I} + (\boldsymbol{B\beta^2})^n\boldsymbol{C}]\boldsymbol{\varepsilon_{t-n}} + \cdots \\
&= \boldsymbol{C\varepsilon_t} + \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}[\boldsymbol{\rho^2 I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^2\boldsymbol{C}]\boldsymbol{\varepsilon_{t-1}} \\
&\quad + \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}[\boldsymbol{\rho^3 I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^3\boldsymbol{C}]\boldsymbol{\varepsilon_{t-2}} + \cdots \\
&\quad + \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}[\boldsymbol{\rho^{n+1} I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C}]\boldsymbol{\varepsilon_{t-n}} + \cdots.
\end{aligned}
$$

Note ρ is a scalar number and $\boldsymbol{I}$ is a 2×2 identity matrix. Based on i.i.d. assumption of $\boldsymbol{\varepsilon}_t$,

$$
\begin{aligned}
\boldsymbol{E}\boldsymbol{z}_t\boldsymbol{z}_t' &= \boldsymbol{E}\{\boldsymbol{C\varepsilon_t} + \cdots + \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}[\boldsymbol{\rho^{n+1} I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C}]\boldsymbol{\varepsilon_{t-n}} + \cdots\}\cdot \\
&\quad \{\boldsymbol{\varepsilon_t' C'} + \cdots + \boldsymbol{\varepsilon_{t-n}'}[\boldsymbol{\rho^{n+1} I} - (\boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C})'][\boldsymbol{\rho I} - (\boldsymbol{C^{-1}B\beta^2 C})']^{-1}\boldsymbol{C'} + \cdots\} \\
&= \boldsymbol{C\Sigma C'} + \cdots + \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}[\boldsymbol{\rho^{n+1} I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C}]\boldsymbol{\Sigma} \\
&\quad [\boldsymbol{\rho^{n+1} I} - (\boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C})'][\boldsymbol{\rho I} - (\boldsymbol{C^{-1}B\beta^2 C})']^{-1}\boldsymbol{C'} + \cdots \\
&= \boldsymbol{C}[\boldsymbol{\rho I} - \boldsymbol{C^{-1}B\beta^2 C}]^{-1}\sum_{n=0}^{\infty}[\boldsymbol{\rho^{n+1} I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C}]\boldsymbol{\Sigma}[\boldsymbol{\rho^{n+1} I} - (\boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C})'] \cdot \\
&\quad [\boldsymbol{\rho I} - (\boldsymbol{C^{-1}B\beta^2 C})']^{-1}\boldsymbol{C'},
\end{aligned}
$$

where $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$.

In the following we try to obtain the expression of the matrix $\boldsymbol{E}\boldsymbol{z}_t\boldsymbol{z}_t'$ and hence we first calculate the matrix $\sum_{n=0}^{\infty}[\boldsymbol{\rho^{n+1} I} - \boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C}]\boldsymbol{\Sigma}[\boldsymbol{\rho^{n+1} I} - (\boldsymbol{C^{-1}}(\boldsymbol{B\beta^2})^{n+1}\boldsymbol{C})']$. Note that

$$
\boldsymbol{B\beta}^2 = \frac{1}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\begin{bmatrix} \beta_1^2 & \varphi(1-\lambda\phi_\pi)\beta_2^2 \\ \gamma\beta_1^2 & (\gamma\varphi+\lambda(1+\varphi\phi_y))\beta_2^2 \end{bmatrix}.
$$

$\boldsymbol{B\beta}^2$ has two eigenvalues[18]

$$
\begin{aligned}
\lambda_1 &= \frac{[\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]+\sqrt{[\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2-4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}, \\
\lambda_2 &= \frac{[\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]-\sqrt{[\beta_1^2+(\gamma\varphi+\lambda+\lambda\varphi\phi_y)\beta_2^2]^2-4\lambda\beta_1^2\beta_2^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}}{2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)}.
\end{aligned}
$$

[18] In the special case $\lambda_1 = \lambda_2$, although $\boldsymbol{B\beta}^2$ is not diagonalizable, the expressions of first-order autocorrelations (4.13) and (4.14) still hold based on the Jordan normal form of matrix $\boldsymbol{B\beta}^2$. Without loss of generality, in the following we assume $\lambda_1 \neq \lambda_2$.

Their corresponding eigenvectors are

$$\begin{aligned} P_1 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]', \\ P_2 &= \Big[\frac{\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y},\ \lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big]'. \end{aligned}$$

Let $\boldsymbol{P} = [P_1, P_2]$. Then

$$\boldsymbol{C^{-1}B\beta^2C} = \boldsymbol{C^{-1}P}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}(C^{-1}\boldsymbol{P})^{-1},$$

where

$$\begin{aligned} &\boldsymbol{C^{-1}P} \\ = &\begin{bmatrix} \frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\Big(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) & \frac{(1+\varphi\phi_y)\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \varphi\phi_\pi\Big(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) \\ \frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \Big(\lambda_1 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) & \frac{-\gamma\varphi(1-\lambda\phi_\pi)\beta_2^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y} + \Big(\lambda_2 - \frac{\beta_1^2}{1+\gamma\varphi\phi_\pi+\varphi\phi_y}\Big) \end{bmatrix} \\ =: &\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}. \end{aligned}$$

Correspondingly

$$(C^{-1}\boldsymbol{P})^{-1} = \frac{1}{d_1d_4 - d_2d_3}\begin{bmatrix} d_4 & -d_2 \\ -d_3 & d_1 \end{bmatrix},$$

where

$$d_1d_4 - d_2d_3 = \det(\boldsymbol{C}^{-1}P) = \varphi(1-\lambda\phi_\pi)\beta_2^2(\lambda_2 - \lambda_1).$$

Hence

$$\begin{aligned} \boldsymbol{C^{-1}(B\beta^2)^{n+1}C} &= \boldsymbol{C^{-1}P}\begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & \lambda_2^{n+1} \end{bmatrix}\boldsymbol{(C^{-1}P)^{-1}} \\ &= \frac{1}{d_1d_4 - d_2d_3}\begin{bmatrix} d_1d_4\lambda_1^{n+1} - d_2d_3\lambda_2^{n+1} & d_1d_2(\lambda_2^{n+1} - \lambda_1^{n+1}) \\ d_3d_4(\lambda_1^{n+1} - \lambda_2^{n+1}) & d_1d_4\lambda_2^{n+1} - d_2d_3\lambda_1^{n+1} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} &\boldsymbol{\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C} = \\ &\frac{1}{d_1d_4 - d_2d_3}\begin{bmatrix} d_1d_4(\rho^{n+1} - \lambda_1^{n+1}) - d_2d_3(\rho^{n+1} - \lambda_2^{n+1}) & -d_1d_2(\lambda_2^{n+1} - \lambda_1^{n+1}) \\ -d_3d_4(\lambda_1^{n+1} - \lambda_2^{n+1}) & d_1d_4(\rho^{n+1} - \lambda_2^{n+1}) - d_2d_3(\rho^{n+1} - \lambda_1^{n+1}) \end{bmatrix}. \end{aligned}$$

Therefore

$$[\boldsymbol{\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)'}] = \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} s_1(n+1) & s_2(n+1) \\ s_2(n+1) & s_3(n+1) \end{bmatrix},$$

where

$$\begin{aligned}
s_1(n+1) &= \sigma_1^2[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]^2 + \sigma_2^2[d_1d_2(\lambda_2^{n+1}-\lambda_1^{n+1})]^2, \\
s_2(n+1) &= \sigma_1^2 d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})] + \\
&\quad \sigma_2^2 d_1d_2(\lambda_1^{n+1}-\lambda_2^{n+1})[d_1d_4(\rho^{n+1}-\lambda_2^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_1^{n+1})], \\
s_3(n+1) &= \sigma_1^2[d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})]^2 + \sigma_2^2[d_1d_4(\rho^{n+1}-\lambda_2^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_1^{n+1})]^2.
\end{aligned}$$

Thus in order to obtain $\boldsymbol{\sum_{n=0}^{\infty}[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)']}$, in the following we first try to calculate the values of $\sum_{n=0}^{\infty} s_i(n+1), (i=1,2,3)$.

Since

$$\begin{aligned}
&\sum_{n=0}^{\infty}[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]^2, \\
=\ &\sum_{n=0}^{\infty}[(d_1d_4-d_2d_3)^2\rho^{2(n+1)} - 2d_1d_4(d_1d_4-d_2d_3)(\rho\lambda_1)^{n+1} + (d_1d_4)^2\lambda_1^{2(n+1)} \\
&+2d_2d_3(d_1d_4-d_2d_3)(\rho\lambda_2)^{n+1} - 2d_1d_2d_3d_4(\lambda_1\lambda_2)^{n+1} + (d_2d_3)^2\lambda_2^{2(n+1)}] \\
=\ &(d_1d_4-d_2d_3)^2\Big(\frac{1}{1-\rho^2}-1\Big) - 2d_1d_4(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_1}-1\Big) + (d_1d_4)^2\Big(\frac{1}{1-\lambda_1^2}-1\Big) \\
&+2d_2d_3(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_2}-1\Big) - 2d_1d_2d_3d_4\Big(\frac{1}{1-\lambda_1\lambda_2}-1\Big) + (d_2d_3)^2\Big(\frac{1}{1-\lambda_2^2}-1\Big) \\
=\ &(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2} - 2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1} + (d_1d_4)^2\frac{1}{1-\lambda_1^2} \\
&+2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2} - 2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{1}{1-\lambda_2^2},
\end{aligned}$$

similarly,

$$\begin{aligned}
&\sum_{n=0}^{\infty} d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})[d_1d_4(\rho^{n+1}-\lambda_1^{n+1}) - d_2d_3(\rho^{n+1}-\lambda_2^{n+1})] \\
=\ &d_3d_4\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_2}-\frac{1}{1-\rho\lambda_1}\Big) + \frac{d_1d_4}{1-\lambda_1^2} - \frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2} + \frac{d_2d_3}{1-\lambda_2^2}\Big\},
\end{aligned}$$

$$\sum_{n=0}^{\infty}[d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})]^2 = (d_3d_4)^2\Big[\frac{1}{1-\lambda_2^2} - \frac{2}{1-\lambda_1\lambda_2} + \frac{1}{1-\lambda_1^2}\Big],$$

then

$$
\begin{aligned}
\sum_{n=0}^{\infty} s_1(n+1) &= \sigma_1^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2} - 2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1} + (d_1d_4)^2\frac{1}{1-\lambda_1^2} \\
&\quad +2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2} - 2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{1}{1-\lambda_2^2}\Big] \\
&\quad +\sigma_2^2\Big[(d_1d_2)^2\Big(\frac{1}{1-\lambda_2^2} - \frac{2}{1-\lambda_1\lambda_2} + \frac{1}{1-\lambda_1^2}\Big)\Big] =: s_1^* \qquad \text{(E.2)} \\
\sum_{n=0}^{\infty} s_2(n+1) &= \sigma_1^2\Big[d_3d_4\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_2} - \frac{1}{1-\rho\lambda_1}\Big) + \frac{d_1d_4}{1-\lambda_1^2} - \frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2} + \frac{d_2d_3}{1-\lambda_2^2}\Big\}\Big] \\
&\quad +\sigma_2^2\Big[d_1d_2\Big\{(d_1d_4-d_2d_3)\Big(\frac{1}{1-\rho\lambda_1} - \frac{1}{1-\rho\lambda_2}\Big) + \frac{d_1d_4}{1-\lambda_2^2} - \frac{d_1d_4+d_2d_3}{1-\lambda_1\lambda_2} + \frac{d_2d_3}{1-\lambda_1^2}\Big\}\Big] \\
&=: s_2^*, \qquad \text{(E.3)} \\
\sum_{n=0}^{\infty} s_3(n+1) &= \sigma_1^2\Big[(d_3d_4)^2\Big(\frac{1}{1-\lambda_2^2} - \frac{2}{1-\lambda_1\lambda_2} + \frac{1}{1-\lambda_1^2}\Big)\Big] \\
&\quad +\sigma_2^2\Big[(d_1d_4-d_2d_3)^2\frac{1}{1-\rho^2} - 2d_1d_4(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_2} + (d_1d_4)^2\frac{1}{1-\lambda_2^2} \\
&\quad +2d_2d_3(d_1d_4-d_2d_3)\frac{1}{1-\rho\lambda_1} - 2d_1d_2d_3d_4\frac{1}{1-\lambda_1\lambda_2} + (d_2d_3)^2\frac{1}{1-\lambda_1^2}\Big] =: s_3^*. \qquad \text{(E.4)}
\end{aligned}
$$

Therefore it is natural to have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^{n+1}I - (C^{-1}(B\beta^2)^{n+1}C)'] \\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} \sum_{n=0}^{\infty} s_1(n+1) & \sum_{n=0}^{\infty} s_2(n+1) \\ \sum_{n=0}^{\infty} s_2(n+1) & \sum_{n=0}^{\infty} s_3(n+1) \end{bmatrix} \\
&= \frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} s_1^* & s_2^* \\ s_2^* & s_3^* \end{bmatrix}.
\end{aligned}
$$

where $s_i^*(i=1,2,3)$ is shown in (E.2)-(E.4). Based on this, thus we can further obtain the expression of $\boldsymbol{E} z_t \boldsymbol{z}_t'$.

Since

$$
\begin{aligned}
&\rho I - C^{-1}(B\beta^2)C \\
&= \frac{1}{d_1d_4-d_2d_3}\begin{bmatrix} d_1d_4(\rho-\lambda_1) - d_2d_3(\rho-\lambda_2) & -d_1d_2(\lambda_2-\lambda_1) \\ -d_3d_4(\lambda_1-\lambda_2) & d_1d_4(\rho-\lambda_2) - d_2d_3(\rho-\lambda_1) \end{bmatrix},
\end{aligned}
$$

then

$$
[\rho I - C^{-1}(B\beta^2)C]^{-1} = \frac{1}{\widetilde{m}}\begin{bmatrix} d_1d_4(\rho-\lambda_2) - d_2d_3(\rho-\lambda_1) & d_1d_2(\lambda_2-\lambda_1) \\ d_3d_4(\lambda_1-\lambda_2) & d_1d_4(\rho-\lambda_1) - d_2d_3(\rho-\lambda_2) \end{bmatrix},
$$

where $\widetilde{m} = (d_1d_4 - d_2d_3)(\rho - \lambda_1)(\rho - \lambda_2)$. Furthermore,

$$\boldsymbol{C}[\rho \boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)\boldsymbol{C}]^{-1} = \frac{1}{\widetilde{m}(1 + \gamma\varphi\phi_\pi + \varphi\phi_y)} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix},$$

where

$$\begin{aligned} k_1 &= d_1d_4(\rho - \lambda_2) - d_2d_3(\rho - \lambda_1) - \varphi\phi_\pi d_3d_4(\lambda_1 - \lambda_2), && \text{(E.5)} \\ k_2 &= d_1d_2(\lambda_2 - \lambda_1) - \varphi\phi_\pi[d_1d_4(\rho - \lambda_1) - d_2d_3(\rho - \lambda_2)], && \text{(E.6)} \\ k_3 &= \gamma[d_1d_4(\rho - \lambda_2) - d_2d_3(\rho - \lambda_1)] + (1 + \varphi\phi_y)d_3d_4(\lambda_1 - \lambda_2), && \text{(E.7)} \\ k_4 &= \gamma d_1d_2(\lambda_2 - \lambda_1) + (1 + \varphi\phi_y)[d_1d_4(\rho - \lambda_1) - d_2d_3(\rho - \lambda_2)]. && \text{(E.8)} \end{aligned}$$

Therefore,

$$\begin{aligned} & \boldsymbol{E}z_t\boldsymbol{z}_t' \\ = \; & \boldsymbol{C}[\rho \boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)\boldsymbol{C}]^{-1}\Big\{\sum_{n=0}^{\infty}[\rho^{n+1}\boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^{n+1}\boldsymbol{C}]\Sigma[\rho^{n+1}\boldsymbol{I} - (\boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^{n+1}\boldsymbol{C})']\Big\} \cdot \\ & [\rho \boldsymbol{I} - (\boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)\boldsymbol{C})']^{-1}\boldsymbol{C}' \\ = \; & \widetilde{k} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} s_1^* & s_2^* \\ s_2^* & s_3^* \end{bmatrix} \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix} \\ = \; & \widetilde{k} \begin{bmatrix} k_1^2s_1^* + 2k_1k_2s_2^* + k_2^2s_3^* & k_1k_3s_1^* + (k_1k_4 + k_2k_3)s_2^* + k_2k_4s_3^* \\ k_1k_3s_1^* + (k_1k_4 + k_2k_3)s_2^* + k_2k_4s_3^* & k_3^2s_1^* + 2k_3k_4s_2^* + k_4^2s_3^* \end{bmatrix}, \end{aligned}$$

where $\widetilde{k} = \frac{1}{\widetilde{m}^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(d_1d_4-d_2d_3)^2}$, s_i^* is given in (E.2)-(E.4) and k_i is given in (E.5)-(E.8).

With the expression of covariance matrix $\boldsymbol{E}z_t\boldsymbol{z}_t'$, in order to obtain the expression of first-order autocorrelation coefficient of output and inflation, we need to further calculate the first-order autocovariance $\boldsymbol{E}z_t\boldsymbol{z}_{t-1}'$.

Similar to the calculation of $\boldsymbol{E}z_t\boldsymbol{z}_t'$, we have

$$\begin{aligned} \boldsymbol{E}z_t\boldsymbol{z}_{t-1}' &= \boldsymbol{E}\{\boldsymbol{C}\boldsymbol{\varepsilon_t} + \cdots + \boldsymbol{C}[\rho \boldsymbol{I} - \boldsymbol{C}^{-1}\boldsymbol{B}\beta^2\boldsymbol{C}]^{-1}[\rho^{n+1}\boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^{n+1}\boldsymbol{C}]\boldsymbol{\varepsilon_{t-n}} + \cdots\} \cdot \\ & \quad \{\boldsymbol{\varepsilon_t'}\boldsymbol{C}' + \cdots + \boldsymbol{\varepsilon_{t-n}'}[\rho^n \boldsymbol{I} - (\boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^n\boldsymbol{C})'][\rho \boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}\beta^2\boldsymbol{C})']^{-1}\boldsymbol{C}' + \cdots\} \\ &= \boldsymbol{C}[\rho \boldsymbol{I} - \boldsymbol{C}^{-1}\boldsymbol{B}\beta^2\boldsymbol{C}]^{-1}\sum_{n=1}^{\infty}[\rho^{n+1}\boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^{n+1}\boldsymbol{C}]\Sigma[\rho^n \boldsymbol{I} - (\boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^n\boldsymbol{C})'] \cdot \\ & \quad [\rho \boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}\beta^2\boldsymbol{C})']^{-1}\boldsymbol{C}', \end{aligned}$$

and

$$[\rho^{n+1}\boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^{n+1}\boldsymbol{C}]\Sigma[\rho^n \boldsymbol{I} - (\boldsymbol{C}^{-1}(\boldsymbol{B}\beta^2)^n\boldsymbol{C})'] = \frac{1}{(d_1d_4 - d_2d_3)^2} \cdot \begin{bmatrix} w_1(n) & w_2(n) \\ w_3(n) & w_4(n) \end{bmatrix},$$

where

$$
\begin{aligned}
w_1(n) &= \sigma_1^2\big\{[d_1d_4(\rho^{n+1}-\lambda_1^{n+1})-d_2d_3(\rho^{n+1}-\lambda_2^{n+1})][d_1d_4(\rho^n-\lambda_1^n)-d_2d_3(\rho^n-\lambda_2^n)]\big\}\\
&\quad+\sigma_2^2(d_1d_2)^2(\lambda_2^{n+1}-\lambda_1^{n+1})(\lambda_2^n-\lambda_1^n),\\
w_2(n) &= \sigma_1^2\big\{d_3d_4(\lambda_2^n-\lambda_1^n)[d_1d_4(\rho^{n+1}-\lambda_1^{n+1})-d_2d_3(\rho^{n+1}-\lambda_2^{n+1})]\big\}\\
&\quad+\sigma_2^2\big\{d_1d_2(\lambda_1^{n+1}-\lambda_2^{n+1})[d_1d_4(\rho^n-\lambda_2^n)-d_2d_3(\rho^n-\lambda_1^n)]\big\},\\
w_3(n) &= \sigma_1^2\big\{d_3d_4(\lambda_2^{n+1}-\lambda_1^{n+1})[d_1d_4(\rho^n-\lambda_1^n)-d_2d_3(\rho^n-\lambda_2^n)]\big\}\\
&\quad+\sigma_2^2\big\{d_1d_2(\lambda_1^n-\lambda_2^n)[d_1d_4(\rho^{n+1}-\lambda_2^{n+1})-d_2d_3(\rho^{n+1}-\lambda_1^{n+1})]\big\},\\
w_4(n) &= \sigma_1^2\big\{(d_3d_4)^2(\lambda_2^{n+1}-\lambda_1^{n+1})(\lambda_2^n-\lambda_1^n)\big\}\\
&\quad\sigma_2^2\big\{[d_1d_4(\rho^{n+1}-\lambda_2^{n+1})-d_2d_3(\rho^{n+1}-\lambda_1^{n+1})][d_1d_4(\rho^n-\lambda_2^n)-d_2d_3(\rho^n-\lambda_1^n)]\big\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{n=1}^{\infty} w_1(n) &= \sigma_1^2\Big\{(d_1d_4-d_2d_3)^2\frac{\rho}{1-\rho^2}-d_1d_4(d_1d_4-d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1}+(d_1d_4)^2\frac{\lambda_1}{1-\lambda_1^2}\\
&\quad+d_2d_3(d_1d_4-d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2}-d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{\lambda_2}{1-\lambda_2^2}\Big\}\\
&\quad+\sigma_2^2(d_1d_2)^2\Big[\frac{\lambda_2}{1-\lambda_2^2}-\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2}+\frac{\lambda_1}{1-\lambda_1^2}\Big]=:w_1^*, &\text{(E.9)}\\
\sum_{n=1}^{\infty} w_2(n) &= \sigma_1^2d_3d_4\Big\{(d_1d_4-d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_2}-\frac{\rho}{1-\rho\lambda_1}\Big]+\frac{d_1d_4\lambda_1}{1-\lambda_1^2}-\frac{d_1d_4\lambda_1+d_2d_3\lambda_2}{1-\lambda_1\lambda_2}+\frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\}\\
&\quad+\sigma_2^2d_1d_2\Big\{(d_1d_4-d_2d_3)\Big[\frac{\lambda_1}{1-\rho\lambda_1}-\frac{\lambda_2}{1-\rho\lambda_2}\Big]+\frac{d_2d_3\lambda_1}{1-\lambda_1^2}-\frac{d_1d_4\lambda_1+d_2d_3\lambda_2}{1-\lambda_1\lambda_2}+\frac{d_1d_4\lambda_2}{1-\lambda_2^2}\Big\}\\
&=: w_2^*, &\text{(E.10)}\\
\sum_{n=1}^{\infty} w_3(n) &= \sigma_1^2d_3d_4\Big\{(d_1d_4-d_2d_3)\Big[\frac{\lambda_2}{1-\rho\lambda_2}-\frac{\lambda_1}{1-\rho\lambda_1}\Big]+\frac{d_1d_4\lambda_1}{1-\lambda_1^2}-\frac{d_1d_4\lambda_2+d_2d_3\lambda_1}{1-\lambda_1\lambda_2}+\frac{d_2d_3\lambda_2}{1-\lambda_2^2}\Big\}\\
&\quad+\sigma_2^2d_1d_2\Big\{(d_1d_4-d_2d_3)\Big[\frac{\rho}{1-\rho\lambda_1}-\frac{\rho}{1-\rho\lambda_2}\Big]+\frac{d_1d_4\lambda_2}{1-\lambda_2^2}-\frac{d_1d_4\lambda_2+d_2d_3\lambda_1}{1-\lambda_1\lambda_2}+\frac{d_2d_3\lambda_1}{1-\lambda_1^2}\Big\}\\
&=: w_3^*, &\text{(E.11)}\\
\sum_{n=1}^{\infty} w_4(n) &= \sigma_1^2(d_3d_4)^2\Big[\frac{\lambda_2}{1-\lambda_2^2}-\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2}+\frac{\lambda_1}{1-\lambda_1^2}\Big]+\sigma_2^2\Big\{(d_1d_4-d_2d_3)^2\frac{\rho}{1-\rho^2}\\
&\quad-d_1d_4(d_1d_4-d_2d_3)\frac{\rho+\lambda_2}{1-\rho\lambda_2}+(d_1d_4)^2\frac{\lambda_2}{1-\lambda_2^2}+d_2d_3(d_1d_4-d_2d_3)\frac{\rho+\lambda_1}{1-\rho\lambda_1}\\
&\quad-d_1d_2d_3d_4\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2}+(d_2d_3)^2\frac{\lambda_1}{1-\lambda_1^2}\Big\}=:w_4^*, &\text{(E.12)}
\end{aligned}
$$

then

$$
\sum_{n=1}^{\infty}[\rho^{n+1}I-C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^nI-(C^{-1}(B\beta^2)^nC)']=\frac{1}{(d_1d_4-d_2d_3)^2}\begin{bmatrix} w_1^* & w_2^*\\ w_3^* & w_4^*\end{bmatrix}.
$$

Therefore,

$$\begin{aligned}
& \boldsymbol{E} z_t \boldsymbol{z}_{t-1}' \\
= \ & C[\rho I - C^{-1}(B\beta^2)C]^{-1}\Big\{\sum_{n=1}^{\infty}[\rho^{n+1}I - C^{-1}(B\beta^2)^{n+1}C]\Sigma[\rho^n I - (C^{-1}(B\beta^2)^n C)']\Big\} \\
= \ & \widetilde{k}\begin{bmatrix} k_1^2 w_1^* + k_1 k_2(w_2^* + w_3^*) + k_2^2 w_4^* & k_1 k_3 w_1^* + k_1 k_4 w_2^* + k_2 k_3 w_3^* + k_2 k_4 w_4^* \\ k_1 k_3 w_1^* + k_2 k_3 w_2^* + k_1 k_4 w_3^* + k_2 k_4 w_4^* & k_3^2 w_1^* + k_3 k_4(w_2^* + w_3^*) + k_4^2 w_4^* \end{bmatrix},
\end{aligned}$$

where $\widetilde{k} = \frac{1}{\widetilde{m}^2(1+\gamma\varphi\phi_\pi+\varphi\phi_y)^2(d_1d_4-d_2d_3)^2}$, w_i^* is given in (E.9)-(E.11) and k_i is given in (E.5)-(E.8).

Therefore based on the expressions of the first-order autocovariance $\boldsymbol{E} z_t \boldsymbol{z}_{t-1}'$ and covariance $\boldsymbol{E} z_t \boldsymbol{z}_t'$, we can obtain the first-order autocorrelation coefficients $G_1(\beta_1,\beta_2)$ and $G_2(\beta_1,\beta_2)$ of output gap and inflation, i.e.,

$$\begin{aligned}
G_1(\beta_1,\beta_2) &= \frac{k_1^2 w_1^* + k_1 k_2(w_2^* + w_3^*) + k_2^2 w_4^*}{k_1^2 s_1^* + 2k_1 k_2 s_2^* + k_2^2 s_3^*}, \\
G_2(\beta_1,\beta_2) &= \frac{k_3^2 w_1^* + k_3 k_4(w_2^* + w_3^*) + k_4^2 w_4^*}{k_3^2 s_1^* + 2k_3 k_4 s_2^* + k_4^2 s_3^*},
\end{aligned}$$

where s_i^* is given in (E.2)-(E.4), k_i is given in (E.5)-(E.8) and w_i^* is given in (E.9)-(E.11).

From the expressions of s_i^* and w_i^*, it can be seen that $G_1(\beta_1,\beta_2)$ and $G_2(\beta_1,\beta_2)$ are extremely complicated. Hence in the following we try to simplify the expressions of $G_1(\beta_1,\beta_2)$ and $G_2(\beta_1,\beta_2)$ so that the separate expressions of λ_1 and λ_2 are not needed (i.e. G_1 and G_2 just depend on $\lambda_1+\lambda_2$ and $\lambda_1\cdot\lambda_2$). We first simplify the expression of the first-order autocorrelation $G_1(\beta_1,\beta_2)$ of output gap.

$$
\begin{aligned}
& k_1^2 w_1^* + k_1 k_2 (w_2^* + w_3^*) + k_2^2 w_4^* \\
= \; & \sigma_1^2 \Big[k_1^2 (d_1 d_4 - d_2 d_3)^2 \frac{\rho}{1-\rho^2} - k_1 (d_1 d_4 - d_2 d_3)(k_1 d_1 d_4 + k_2 d_3 d_4) \frac{\rho + \lambda_1}{1 - \rho\lambda_1} \\
& + (k_1 d_1 d_4 + k_2 d_3 d_4)^2 \frac{\lambda_1}{1-\lambda_1^2} + k_1 (d_1 d_4 - d_2 d_3)(k_1 d_2 d_3 + k_2 d_3 d_4) \frac{\rho + \lambda_2}{1 - \rho\lambda_2} \\
& + (k_1 d_2 d_3 + k_2 d_3 d_4)^2 \frac{\lambda_2}{1-\lambda_2^2} - (k_1 d_1 d_4 + k_2 d_3 d_4)(k_1 d_2 d_3 + k_2 d_3 d_4) \frac{\lambda_1 + \lambda_2}{1 - \lambda_1 \lambda_2} \Big] \\
& + \sigma_2^2 \Big[k_2^2 (d_1 d_4 - d_2 d_3)^2 \frac{\rho}{1-\rho^2} - k_2 (d_1 d_4 - d_2 d_3)(k_2 d_1 d_4 + k_1 d_1 d_2) \frac{\rho + \lambda_2}{1 - \rho\lambda_2} \\
& + (k_2 d_1 d_4 + k_1 d_1 d_2)^2 \frac{\lambda_2}{1-\lambda_2^2} + k_2 (d_1 d_4 - d_2 d_3)(k_2 d_2 d_3 + k_1 d_1 d_2) \frac{\rho + \lambda_1}{1 - \rho\lambda_1} \\
& + (k_2 d_2 d_3 + k_1 d_1 d_2)^2 \frac{\lambda_1}{1-\lambda_1^2} - (k_2 d_1 d_4 + k_1 d_1 d_2)(k_2 d_2 d_3 + k_1 d_1 d_2) \frac{\lambda_1 + \lambda_2}{1 - \lambda_1 \lambda_2} \Big] \\
= \; & \frac{1}{(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \cdot \\
& \Big\{ \sigma_1^2 \Big[k_1 (d_1 d_4 - d_2 d_3) \rho (1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2) \big[k_1 (d_1 d_4 - d_2 d_3)(1-\rho\lambda_1)(1-\rho\lambda_2) \\
& - (k_1 d_1 d_4 + k_2 d_3 d_4)(1-\rho^2)(1-\rho\lambda_2) + (k_1 d_2 d_3 + k_2 d_3 d_4)(1-\rho^2)(1-\rho\lambda_1) \big] \\
& - (k_1 d_1 d_4 + k_2 d_3 d_4) \lambda_1 (1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2) \big[k_1 (d_1 d_4 - d_2 d_3)(1-\lambda_1^2)(1-\lambda_1\lambda_2) \\
& - (k_1 d_1 d_4 + k_2 d_3 d_4)(1-\rho\lambda_1)(1-\lambda_1\lambda_2) + (k_1 d_2 d_3 + k_2 d_3 d_4)(1-\rho\lambda_1)(1-\lambda_1^2) \big] \\
& + (k_1 d_2 d_3 + k_2 d_3 d_4) \lambda_2 (1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2) \big[k_1 (d_1 d_4 - d_2 d_3)(1-\lambda_2^2)(1-\lambda_1\lambda_2) \\
& - (k_1 d_1 d_4 + k_2 d_3 d_4)(1-\rho\lambda_2)(1-\lambda_2^2) + (k_1 d_2 d_3 + k_2 d_3 d_4)(1-\rho\lambda_2)(1-\lambda_1\lambda_2) \big] \Big] \\
& + \sigma_2^2 \Big[k_2 (d_1 d_4 - d_2 d_3) \rho (1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2) \big[k_2 (d_1 d_4 - d_2 d_3)(1-\rho\lambda_1)(1-\rho\lambda_2) \\
& - (k_2 d_1 d_4 + k_1 d_1 d_2)(1-\rho^2)(1-\rho\lambda_1) + (k_2 d_2 d_3 + k_1 d_1 d_2)(1-\rho^2)(1-\rho\lambda_2) \big] \\
& - (k_2 d_1 d_4 + k_1 d_1 d_2) \lambda_2 (1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2) \big[k_2 (d_1 d_4 - d_2 d_3)(1-\lambda_2^2)(1-\lambda_1\lambda_2) \\
& - (k_2 d_1 d_4 + k_1 d_1 d_2)(1-\rho\lambda_2)(1-\lambda_1\lambda_2) + (k_2 d_2 d_3 + k_1 d_1 d_2)(1-\rho\lambda_2)(1-\lambda_2^2) \big] \\
& + (k_2 d_2 d_3 + k_1 d_1 d_2) \lambda_1 (1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2) \big[k_2 (d_1 d_4 - d_2 d_3)(1-\lambda_1^2)(1-\lambda_1\lambda_2) \\
& - (k_2 d_1 d_4 + k_1 d_1 d_2)(1-\rho\lambda_1)(1-\lambda_1^2) + (k_2 d_2 d_3 + k_1 d_1 d_2)(1-\rho\lambda_1)(1-\lambda_1\lambda_2) \big] \Big] \Big\}.
\end{aligned}
$$

Since

$$\begin{aligned}
& k_1(d_1d_4-d_2d_3)(1-\rho\lambda_1)(1-\rho\lambda_2)-(k_1d_1d_4+k_2d_3d_4)(1-\rho^2)(1-\rho\lambda_2) \\
& +(k_1d_2d_3+k_2d_3d_4)(1-\rho^2)(1-\rho\lambda_1) \\
= \; & k_1(d_1d_4-d_2d_3)\rho^2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_1+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_2 \\
& +k_1(d_1d_4-d_2d_3)\rho^2\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho^3\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho^3\lambda_1 \\
= \; & \rho\big[k_1(d_1d_4-d_2d_3)\rho-(k_1d_1d_4+k_2d_3d_4)\lambda_1+(k_1d_2d_3+k_2d_3d_4)\lambda_2\big] \\
& +\rho^2\big[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1\big],
\end{aligned}$$

$$\begin{aligned}
& k_1(d_1d_4-d_2d_3)(1-\lambda_1^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)(1-\rho\lambda_1)(1-\lambda_1\lambda_2) \\
& +(k_1d_2d_3+k_2d_3d_4)(1-\rho\lambda_1)(1-\lambda_1^2) \\
= \; & \lambda_1\big[k_1(d_1d_4-d_2d_3)\rho-(k_1d_1d_4+k_2d_3d_4)\lambda_1+(k_1d_2d_3+k_2d_3d_4)\lambda_2\big] \\
& +\lambda_1^2\big[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1\big],
\end{aligned}$$

$$\begin{aligned}
& k_1(d_1d_4-d_2d_3)(1-\lambda_2^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)(1-\rho\lambda_2)(1-\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)(1-\rho\lambda_2)(1-\lambda_1\lambda_2) \\
= \; & \lambda_2\big[k_1(d_1d_4-d_2d_3)\rho-(k_1d_1d_4+k_2d_3d_4)\lambda_1+(k_1d_2d_3+k_2d_3d_4)\lambda_2\big] \\
& +\lambda_2^2\big[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1\big],
\end{aligned}$$

then

$$\begin{aligned}
& k_1^2w_1^*+k_1k_2(w_2^*+w_3^*)+k_2^2w_4^* \\
= \; & \frac{1}{(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}\cdot \\
& \Big\{\sigma_1^2\Big[\big[k_1(d_1d_4-d_2d_3)\rho-(k_1d_1d_4+k_2d_3d_4)\lambda_1+(k_1d_2d_3+k_2d_3d_4)\lambda_2\big]\big[k_1(d_1d_4-d_2d_3)\rho^2(1-\lambda_1^2) \\
& (1-\lambda_2^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2)+(k_1d_2d_3+k_2d_3d_4)\lambda_2^2 \\
& (1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)\big]+\big[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4) \\
& \rho\lambda_1\big]\big[k_1(d_1d_4-d_2d_3)\rho^3(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3(1-\rho^2) \\
& (1-\rho\lambda_2)(1-\lambda_2^2)+(k_1d_2d_3+k_2d_3d_4)\lambda_2^3(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)\big]\Big]+ \\
& \sigma_2^2\Big[\big[k_2(d_1d_4-d_2d_3)\rho-(k_2d_1d_4+k_1d_1d_2)\lambda_2+(k_2d_2d_3+k_1d_1d_2)\lambda_1\big]\big[k_2(d_1d_4-d_2d_3)\rho^2(1-\lambda_2^2) \\
& (1-\lambda_1^2)(1-\lambda_1\lambda_2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)+(k_2d_2d_3+k_1d_1d_2)\lambda_1^2 \\
& (1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2)\big]+\big[k_2(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_2d_1d_4+k_1d_1d_2)\rho\lambda_1+(k_2d_2d_3+k_1d_1d_2) \\
& \rho\lambda_2\big]\big[k_2(d_1d_4-d_2d_3)\rho^3(1-\lambda_2^2)(1-\lambda_1^2)(1-\lambda_1\lambda_2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2^3(1-\rho^2) \\
& (1-\rho\lambda_1)(1-\lambda_1^2)+(k_2d_2d_3+k_1d_1d_2)\lambda_1^3(1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2)\big]\Big]\Big\}.
\end{aligned}$$

Note that

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^2(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^2(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2) \\
= \;& [k_1(d_1d_4-d_2d_3)\rho^2-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2+(k_1d_2d_3+k_2d_3d_4)\lambda_2^2]-[k_1(d_1d_4-d_2d_3)\rho^2 \\
& (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2)+(k_1d_2d_3+k_2d_3d_4)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2)] \\
& +\rho\lambda_1\lambda_2[k_1(d_1d_4-d_2d_3)\rho(\lambda_1^2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1(\rho^2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2(\rho^2+\lambda_1^2)]-\rho^2\lambda_1^2\lambda_2^2[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2 \\
& +(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1],
\end{aligned}
$$

and

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^3(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3(1-\rho^2)(1-\rho\lambda_2)(1-\lambda_2^3) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^3(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2) \\
= \;& [k_1(d_1d_4-d_2d_3)\rho^3-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3+(k_1d_2d_3+k_2d_3d_4)\lambda_2^3]-[k_1(d_1d_4-d_2d_3)\rho^3 \\
& (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3(\rho^2+\rho\lambda_2+\lambda_2^2)+(k_1d_2d_3+k_2d_3d_4)\lambda_2^3(\rho^2+\rho\lambda_1+\lambda_1^2)] \\
& +\rho\lambda_1\lambda_2[k_1(d_1d_4-d_2d_3)\rho^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2)].
\end{aligned}
$$

Because $k_1 = d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1)-\varphi\phi_\pi d_3d_4(\lambda_1-\lambda_2), k_2 = d_1d_2(\lambda_2-\lambda_1)-\varphi\phi_\pi[d_1d_4(\rho-\lambda_1)-d_2d_3(\rho-\lambda_2)]$, we can obtain

$$
\begin{aligned}
k_1d_1d_4+k_2d_3d_4 &= (d_1^2d_4+\varphi\phi_\pi d_2d_3^2-d_1d_2d_3-\varphi\phi_\pi d_1d_3d_4)d_4(\rho-\lambda_2) \\
&= (d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2), \\
k_1d_2d_3+k_2d_3d_4 &= (d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1), \\
k_2d_1d_4+k_1d_1d_2 &= (d_1d_4-d_2d_3)(d_1d_2-\varphi\phi_\pi d_1d_4)(\rho-\lambda_1), \\
k_2d_2d_3+k_1d_1d_2 &= (d_1d_4-d_2d_3)(d_1d_2-\varphi\phi_\pi d_2d_3)(\rho-\lambda_2).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho-(k_1d_1d_4+k_2d_3d_4)\lambda_1+(k_1d_2d_3+k_2d_3d_4)\lambda_2 \\
= \; & (d_1d_4-d_2d_3)[d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1)-\varphi\phi_\pi d_3d_4(\lambda_1-\lambda_2)]\rho-(d_1d_4-d_2d_3)\cdot \\
& (d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\lambda_1+(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\lambda_2 \\
= \; & (d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\rho-(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\rho \\
& -(d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\lambda_1+(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\lambda_2 \\
= \; & (d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)(\rho-\lambda_1)-(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)(\rho-\lambda_2) \\
= \; & (d_1d_4-d_2d_3)^2(\rho-\lambda_2)(\rho-\lambda_1) \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1),
\end{aligned}
$$

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1 \\
= \; & (d_1d_4-d_2d_3)[d_1d_4(\rho-\lambda_2)-d_2d_3(\rho-\lambda_1)-\varphi\phi_\pi d_3d_4(\lambda_1-\lambda_2)]\lambda_1\lambda_2-(d_1d_4-d_2d_3) \\
& (d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\rho\lambda_2+(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\rho\lambda_1 \\
= \; & (d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\lambda_1\lambda_2-(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\lambda_1\lambda_2 \\
& -(d_1d_4-d_2d_3)(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)\rho\lambda_2+(d_1d_4-d_2d_3)(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)\rho\lambda_1 \\
= \; & -(d_1d_4-d_2d_3)[(d_1d_4-\varphi\phi_\pi d_3d_4)(\rho-\lambda_2)(\rho-\lambda_1)\lambda_2-(d_2d_3-\varphi\phi_\pi d_3d_4)(\rho-\lambda_1)(\rho-\lambda_2)\lambda_1] \\
= \; & (d_1d_4-d_2d_3)(\rho-\lambda_2)(\rho-\lambda_1)[-(d_1d_4-\varphi\phi_\pi d_3d_4)\lambda_2+(d_2d_3-\varphi\phi_\pi d_3d_4)\lambda_1] \\
= \; & (d_1d_4-d_2d_3)(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi(1-\lambda\phi_\pi)\beta_2^2d_4\lambda_2+\varphi(1-\lambda\phi_\pi)\beta_2^2d_3\lambda_1] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\lambda\beta_2^2],
\end{aligned}
$$

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^2-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2+(k_1d_2d_3+k_2d_3d_4)\lambda_2^2 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\rho+\lambda_1+\lambda_2-\lambda\beta_2^2],
\end{aligned}
$$

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^3-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3+(k_1d_2d_3+k_2d_3d_4)\lambda_2^3 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\rho^2+\rho\lambda_1+\rho\lambda_2+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2-\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2) \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& k_1(d_1d_4-d_2d_3)\rho^3(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^3(\rho^2+\rho\lambda_2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^3(\rho^2+\rho\lambda_1+\lambda_1^2) \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\rho^2\lambda_1^2+\rho^2\lambda_1\lambda_2+\rho^2\lambda_2^2+\rho\lambda_1^2\lambda_2+\rho\lambda_1\lambda_2^2+\lambda_1^2\lambda_2^2],
\end{aligned}
$$

$$
\begin{aligned}
& \rho\lambda_1\lambda_2[k_1(d_1d_4-d_2d_3)\rho(\lambda_1^2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1(\rho^2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2(\rho^2+\lambda_1^2)] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\lambda\beta_2^2\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)-\rho\lambda_1\lambda_2(\lambda_1\lambda_2+\rho\lambda_1+\rho\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& \rho\lambda_1\lambda_2[k_1(d_1d_4-d_2d_3)\rho^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_1d_1d_4+k_2d_3d_4)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2) \\
& +(k_1d_2d_3+k_2d_3d_4)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2)] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\lambda\beta_2^2\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& (\rho\lambda_1\lambda_2)^2[k_1(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_1d_1d_4+k_2d_3d_4)\rho\lambda_2+(k_1d_2d_3+k_2d_3d_4)\rho\lambda_1] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\lambda\beta_2^2(\rho\lambda_1\lambda_2)^2],
\end{aligned}
$$

and

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\rho-(k_2d_1d_4+k_1d_1d_2)\lambda_2+(k_2d_2d_3+k_1d_1d_2)\lambda_1 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)(-\varphi\phi_\pi),
\end{aligned}
$$

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_2d_1d_4+k_1d_1d_2)\rho\lambda_1+(k_2d_2d_3+k_1d_1d_2)\rho\lambda_2 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\varphi\beta_2^2],
\end{aligned}
$$

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\rho^2-(k_2d_1d_4+k_1d_1d_2)\lambda_2^2+(k_2d_2d_3+k_1d_1d_2)\lambda_1^2 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi\phi_\pi(\rho+\lambda_1+\lambda_2)+\varphi\beta_2^2],
\end{aligned}
$$

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\rho^3-(k_2d_1d_4+k_1d_1d_2)\lambda_2^3+(k_2d_2d_3+k_1d_1d_2)\lambda_1^3 \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi\phi_\pi(\rho^2+\rho\lambda_1+\rho\lambda_2+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2) \\
& +\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\rho^2(\lambda_2^2+\lambda_1\lambda_2+\lambda_1^2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2) \\
& +(k_2d_2d_3+k_1d_1d_2)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2) \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& k_2(d_1d_4-d_2d_3)\rho^3(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2^3(\rho^2+\rho\lambda_1+\lambda_1^2) \\
& +(k_2d_2d_3+k_1d_1d_2)\lambda_1^3(\rho^2+\rho\lambda_2+\lambda_2^2) \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\rho^2\lambda_1^2+\rho^2\lambda_1\lambda_2+\rho^2\lambda_2^2+\rho\lambda_1^2\lambda_2 \\
& +\rho\lambda_1\lambda_2^2+\lambda_1^2\lambda_2^2](-\varphi\phi_\pi),
\end{aligned}
$$

$$
\begin{aligned}
& \rho\lambda_1\lambda_2[k_2(d_1d_4-d_2d_3)\rho(\lambda_1^2+\lambda_2^2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2(\rho^2+\lambda_1^2) \\
& +(k_2d_2d_3+k_1d_1d_2)\lambda_1(\rho^2+\lambda_2^2)] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi\beta_2^2\rho\lambda_1\lambda_2(\rho+\lambda_1+\lambda_2)+ \\
& \varphi\phi_\pi\rho\lambda_1\lambda_2(\lambda_1\lambda_2+\rho\lambda_1+\rho\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& \rho\lambda_1\lambda_2[k_2(d_1d_4-d_2d_3)\rho^2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-(k_2d_1d_4+k_1d_1d_2)\lambda_2^2(\rho^2+\rho\lambda_1+\lambda_1^2) \\
& +(k_2d_2d_3+k_1d_1d_2)\lambda_1^2(\rho^2+\rho\lambda_2+\lambda_2^2)] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[-\varphi\beta_2^2\rho\lambda_1\lambda_2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)],
\end{aligned}
$$

$$
\begin{aligned}
& (\rho\lambda_1\lambda_2)^2[k_2(d_1d_4-d_2d_3)\lambda_1\lambda_2-(k_2d_1d_4+k_1d_1d_2)\rho\lambda_1+(k_2d_2d_3+k_1d_1d_2)\rho\lambda_2] \\
= \; & (\varphi(1-\lambda\phi_\pi)\beta_2^2)^2(\lambda_2-\lambda_1)^2(\rho-\lambda_2)(\rho-\lambda_1)[\varphi\beta_2^2(\rho\lambda_1\lambda_2)^2].
\end{aligned}
$$

Hence

$$
\begin{aligned}
& k_1^2w_1^*+k_1k_2(w_2^*+w_3^*)+k_2^2w_4^* \\
= \; & \frac{(\varphi(1-\lambda\phi_\pi)\beta_2^2)^4(\lambda_2-\lambda_1)^4(\rho-\lambda_2)^2(\rho-\lambda_1)^2}{(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}\Big\{\sigma_1^2\big[(\rho+\lambda_1+\lambda_2-\lambda\beta_2^2)[1-\lambda\beta_2^2 \\
& (\rho+\lambda_1+\lambda_2)]+[\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\rho\lambda_1\lambda_2][(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\lambda\beta_2^2\rho\lambda_1\lambda_2]\big]+\sigma_2^2\cdot \\
& \big[(\varphi\phi_\pi(\rho+\lambda_1+\lambda_2)-\varphi\beta_2^2))[\varphi\phi_\pi-\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)]+[\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) \\
& -\varphi\phi_\pi\rho\lambda_1\lambda_2][\varphi\phi_\pi(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)-\varphi\beta_2^2\rho\lambda_1\lambda_2]\big]\Big\}.
\end{aligned}
$$

Following the similar procedures, we can obtain

$$
\begin{aligned}
& k_1^2s_1^*+2k_1k_2s_2^*+k_2^2s_3^* \\
= \; & \frac{(\varphi(1-\lambda\phi_\pi)\beta_2^2)^4(\lambda_2-\lambda_1)^4(\rho-\lambda_2)^2(\rho-\lambda_1)^2}{(1-\rho^2)(1-\rho\lambda_1)(1-\lambda_1^2)(1-\rho\lambda_2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)} \\
& \Big\{\sigma_1^2\Big[[(1+\lambda^2\beta_2^4)-2\lambda\beta_2^2(\rho+\lambda_1+\lambda_2)+(1+\lambda^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[(1+\lambda^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\lambda\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)+(1+\lambda^2\beta_2^4)\rho\lambda_1\lambda_2]\Big] \\
& +\sigma_2^2\Big[[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)-2\varphi\phi_\pi\varphi\beta_2^2(\rho+\lambda_1+\lambda_2)+((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2)] \\
& -\rho\lambda_1\lambda_2[((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)(\rho+\lambda_1+\lambda_2)-2\varphi\phi_\pi\varphi\beta_2^2(\rho\lambda_1+\rho\lambda_2+\lambda_1\lambda_2) \\
& +((\varphi\phi_\pi)^2+\varphi^2\beta_2^4)\rho\lambda_1\lambda_2]\Big]\Big\}.
\end{aligned}
$$

Therefor, from $G_1(\beta_1,\beta_2) = \frac{k_1^2 w_1^* + k_1 k_2 (w_2^* + w_3^*) + k_2^2 w_4^*}{k_1^2 s_1^* + 2k_1 k_2 s_2^* + k_2^2 s_3^*}$, we can obtain the expression of $G_1(\beta_1,\beta_2)$ as shown in (4.13). Similarly we can obtain the expression of $G_2(\beta_1,\beta_2)$ as shown in (4.14).

F Stability for the contemporaneous Taylor rule

Based on Proposition 2, we only need to show that both of the eigenvalues of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ have negative real parts if $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$.

The characteristic polynomial of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ is given by $h(\nu) = \nu^2 - c_1\nu + c_2$, where c_1 is the trace and c_2 is the determinant of matrix $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$. Direct calculation shows that

$$c_1 = \frac{-(1-\lambda)(1-\beta_1^2) - 2\varphi(\gamma\phi_\pi + \phi_y) + \varphi(\gamma + \lambda\phi_y)(1+\beta_2^2)}{\triangle\ (1+\gamma\varphi\phi_\pi + \varphi\phi_y)}, \tag{F.1}$$

$$c_2 = \frac{\varphi[\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y]}{\triangle\ (1+\gamma\varphi\phi_\pi + \varphi\phi_y)}, \tag{F.2}$$

where $\triangle = \frac{(1-\beta_1^2)(1-\lambda\beta_2^2) + \gamma\varphi\phi_\pi + \varphi\phi_y - (\gamma\varphi + \lambda\varphi\phi_y)\beta_2^2}{1+\gamma\varphi\phi_\pi + \varphi\phi_y}$.

Both of the eigenvalues of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ have negative real parts if and only if $c_1 < 0$ and $c_2 > 0$ (these conditions are obtained by applying the *Routh-Hurwitz criterion theorem*; see Brock and Malliaris, 1989). If $\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y > 0$, from Appendix D it is easy to see $\triangle > 0$. Furthermore,

$$c_1 \le \frac{-2\varphi[(\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y]}{\triangle\ (1+\gamma\varphi\phi_\pi + \varphi\phi_y)} < 0, \quad c_2 > 0.$$

G Proof of Corollary 3

Under the forward looking expectations interest rate rule, the characteristic polynomial of $\boldsymbol{B}\boldsymbol{\beta}^2$ is given by $h(\nu) = \nu^2 + c_1\nu + c_2$, where

$$c_1 = -[(1-\varphi\phi_y)\beta_1^2 + (\gamma\varphi(1-\phi_\pi) + \lambda)\beta_2^2], \quad c_2 = \lambda(1-\varphi\phi_y)\beta_1^2\beta_2^2.$$

Similarly, both of the eigenvalues of $\boldsymbol{B}\boldsymbol{\beta}^2$ are inside the unit circle if and only if the following conditions hold (see Elaydi, 1999): $h(1) > 0, \quad h(-1) > 0, \quad |h(0)| < 1$. Based on our assumption $\phi_y < \frac{1}{\varphi}$ and $1 < \phi_\pi < 1 + \frac{\lambda}{\gamma\varphi}$, it is easy to see $c_1 \le 0$ and $0 \le h(0) < 1$, and hence $h(-1) > 0$ for any $\beta_i \in [-1,1]$. Furthermore

$$h(1) = (1-\beta_1^2)(1-\lambda\beta_2^2) + \gamma\varphi(\phi_\pi - 1)\beta_2^2 + \varphi\phi_y\beta_1^2(1-\lambda\beta_2^2) > 0. \tag{G.1}$$

In addition, for the stability of BLE, we need to show both eigenvalues of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ have negative real parts if $\phi_y < \frac{1}{\varphi}$ and $1 < \phi_\pi < 1 + \frac{\lambda}{\gamma\varphi}$. The characteristic polynomial of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ is given by $h(\nu) = \nu^2 - c_1\nu + c_2$, where

$$
\begin{aligned}
c_1 &= \frac{1}{h(1)}[-(1-\lambda)(1-\beta_1^2) - \gamma\varphi(\phi_\pi - 1)(1+\beta_2^2) - \varphi\phi_y(1+\beta_1^2) + \lambda\varphi\phi_y(\beta_1^2+\beta_2^2)],\\
&\le \frac{1}{h(1)}[-\gamma\varphi(\phi_\pi - 1)(1+\beta_2^2) - \varphi\phi_y(1+\beta_1^2) + \lambda\varphi\phi_y(1+\beta_1^2)] < 0\\
c_2 &= \frac{\varphi}{h(1)}[\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y] > 0,
\end{aligned}
$$

where here $h(1)$ is given in (G.1). Therefore, based on the *Routh-Hurwitz criterion theorem* (see Brock and Malliaris (1989)), both eigenvalues of $(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{B} - \boldsymbol{I})$ have negative real parts. Then following the same ideas of Propositions 1 and 2, we obtain Corollary 3 on the existence and stability of BLE under the forward looking interest rate rule.

H Proof of Corollary 4

Under the lagged interest rate rule, the characteristic polynomial of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{\beta}^2$ is given by $h(\nu) = \nu^2 + c_1\nu + c_2$, where

$$c_1 = -[-\varphi\phi_y + \beta_1^2 - \gamma\varphi\phi_\pi + (\gamma\varphi + \lambda)\beta_2^2], \quad c_2 = (-\varphi\phi_y + \beta_1^2)\lambda\beta_2^2.$$

Based on our assumption $\phi_y < \frac{1}{\varphi}$ and $1 < \phi_\pi < \frac{1-\varphi\phi_y}{\gamma\varphi}$, it is easy to see $h(-1) = (1 - \varphi\phi_y - \gamma\varphi\phi_\pi) + \beta_1^2(1+\lambda\beta_2^2) + (\gamma\varphi + \lambda(1-\varphi\phi_y))\beta_2^2 > 0$ and $|h(0)| < 1$ for any $\beta_i \in [-1, 1]$. Furthermore

$$h(1) = (1-\beta_1^2)(1-\lambda\beta_2^2) + \varphi\phi_y(1-\lambda\beta_2^2) + \gamma\varphi(\phi_\pi - \beta_2^2) > 0. \tag{H.1}$$

In addition, for the stability of BLE, direct computations suggest that we need both eigenvalues of $(\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{A} + \boldsymbol{B} - \boldsymbol{I})$ have negative real parts if $\phi_y < \frac{1}{\varphi}$ and $1 < \phi_\pi < \frac{1-\varphi\phi_y}{\gamma\varphi}$. The characteristic polynomial of $(\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B}\boldsymbol{\beta}^2)^{-1}(\boldsymbol{A} + \boldsymbol{B} - \boldsymbol{I})$ is given by $h(\nu) = \nu^2 - c_1\nu + c_2$, where

$$
\begin{aligned}
c_1 &= \frac{1}{h(1)}[-(1-\lambda)(1-\beta_1^2) - (2 - \lambda(1+\beta_2^2))\varphi\phi_y - \gamma\varphi(2\phi_\pi - (1+\beta_2^2))] < 0,\\
c_2 &= \frac{\varphi}{h(1)}[\gamma(\phi_\pi - 1) + (1-\lambda)\phi_y] > 0,
\end{aligned}
$$

where here $h(1)$ is given in (H.1).

References

[1] Adam, K., 2003. Learning and Equilibrium Selection in a Monetary Overlapping Generations Model with Sticky Prices, *Review of Economic Studies* 70, 887-907.

[2] Adam, K., 2007. Experimental evidence on the persistence of output and inflation. Economic Journal 117, 603-636.

[3] Assenza, T., Heemeijer, P., Hommes, C. and Massaro, D., 2014. Individual Expectations and Aggregate Macro Behavior. CeNDEF Working Paper, University of Amsterdam.

[4] Brock, W.A., Malliaris, A.G., 1989. Differential Equations, Stability and Chaos in Dynamic Economics. North-Holland, Amterdam.

[5] Boivin, J., Giannoni, M., 2006. Has monetary policy become more effective? The Review of Economics and Statistics 88 (3), 445-462.

[6] Branch, W.A., 2006. Restricted perceptions equilibria and learning in macroeconomics, in: Colander, D. (Ed.), Post Walrasian Macroeconomics: Beyond the Dynamic Stochastic General Equilibrium Model. Cambridge University Press, New York, pp. 135-160.

[7] Branch, W.A., Evans, G.W., 2010. Asset return dynamics and learning. The Review of Financial Studies 23 (4), 1651-1680.

[8] Bullard, J., 2006. The learnability criterion and monetary policy. Federal Reserve Bank of St. Louis Review 88, 203-217.

[9] Bullard, J., Evans, G.W., Honkapohja, S., 2008. Monetary policy, judgment and near-rational exuberance. American Economic Review 98, 1163-1177.

[10] Bullard, J., Mitra, K., 2002. Learning about monetary policy rules. Journal of Monetary Economics 49, 1105-1129.

[11] Cho, I.-K. and Kasa, K. (2015), Learning and model validation, *Review of Economic Studies* 82, 45-82.

[12] Chung, H. and Xiao, W. (2014), Cognitive consistency, signal extraction and macroeconomic persistence, Working Paper Binghamton University.

[13] Clarida, R., Galí, J. and Gertler, M. (1999), The science of monetary policy: a New Keynesian perspective. *Journal of Economic Literature* 37, 1661-1707.

[14] Clark, T., West, K., 2007. Approximately normal tests for equal predictive accuracy in nested models. Journal of Econometrics 138, 291-311.

[15] Christiano, L.J., Eichenbaum, M., Evans, C.L., 2005. Nominal rigidities and the dynamic effects of a shock to monetary policy. Journal of Political Economy 113 (1).

[16] Elaydi, S.N., 1999. An Introduction to Difference Equations, 2nd edition. Springer, New York.

[17] Enders, W., 2010. Applied Econometric Time Series (3rd ed.). John Wiley & Sons, Inc., USA.

[18] Evans, G.W., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press, Princeton.

[19] Evans, G.W., Honkapohja, S., 2013. Learning as a rational foundation for macroeconomics and finance. In: Roman Frydman and Edmund S. Phelps (Eds.), Rethinking Expectations: The Way Forward for Macroeconomics (Chapter 2). Princeton University Press.

[20] Fuhrer, J., Moore, G., 1992. Monetary policy rules and the indicator properties of asset prices. Journal of Monetary Economics 29, 303-336.

[21] Fuhrer, J., Moore, G., 1995. Inflation persistence. Quarterly Journal of Economics 110(1): 127-159, February.

[22] Fuhrer, J.C., 2006. Intrinsic and inherited inflation persistence. International Journal of Central Banking 2, 49-86.

[23] Fuhrer, J.C., 2009. Inflation persistence. Federal Reserve Bank of Boston, working paper.

[24] Fuster, A., Hebert, B., Laibson, D., 2011. Natural expectations, macroeconomic Dynamics, and asset pricing. Forthcoming NBER Macroeconomics Annual 26.

[25] Fuster, A., Hebert, M., Laibson, D., 2012. Investment Dynamics with Natural Expectations. International Journal of Central Banking (Special Issue in Honor of Benj

[26] Fuster, A., Laibson, D. and Mendel, B., 2010. Natural expectations and macroeconomic fluctuations. Journal of Economic Perspectives 24, 67-84.

[27] Galí, J. (2008), Monetary policy, inflation, and the business cycle, An introduction to the New Keynesian framework, Princeton University Press, New Jersey.

[28] Giannoni, M., Woodford, M., 2003. Optimal inflation targeting rules. In: Bernanke, B.S., Woodford, M. (Eds.), Inflation Targeting. University of Chicago Press, Chicago.

[29] Grandmont, J.M., 1998. Expectation formation and stability in large socio-economic systems. Econometrica 66, 741-781.

[30] Hommes, C., Sorger, G., 1998. Consistent expectations equilibria. Macroeconomic Dynamics 2, 287-321.

[31] Hommes, C.H., Sorger, G., Wagener, F., 2013. Consistency of linear forecasts in a nonlinear stochastic economy. In: Bischi, G.I., Chiarella, C. and Sushko, I. (Eds.), *Global Analysis of Dynamic Models in Economics and Finance*, Springer-Verlag Berlin Heidelberg, pp. 229-287.

[32] Hommes, C., Zhu, M., 2014. Behavioral Learning Equilibria. Journal of Economic Theory 150, 778-814.

[33] Milani, F., 2007. Expectations, learning and macroeconomic persistence. Journal of Monetary Economics 54, 2065-2082.

[34] Lancaster, P., Tismenetsky, M., 1985. The Theory of Matrices (Second Edition with Applications). Academic Press, San Diego.

[35] Magnus, J., Neudecker, H., 1988. Matrix Differential calculus. Wiley, New York.

[36] Martelli, M., 1999. Introduction to Discrete Dynamical Systems and Chaos. Wiley, New York.

[37] Milani, F., 2007. Expectations, learning and macroeconomic persistence. Journal of Monetary Economics 54, 2065-2082.

[38] Nelson, C., 1972. The prediction performance of the FRB-MIT-PENN model of the US economy. American Economic Review 62, 902-917.

[39] Orphanides, A., Williams, J., 2003. Imperfect knowledge, inflation expectations and monetary policy. In: Bernanke, B., Woodford, M. (Eds.), Inflation Targeting. University of Chicago Press, Chicago.

[40] Phelps, E. S., 1968. Money-wage dynamics and labor-market equilibrium. Journal of Political Economy 76(4, Part 2), 678-711.

[41] Pfajfar, D., Žakeljz, B., 2016. Inflation expectations and monetary policy design: evidence from the laboratory. Forthcoming in Macroeconomic Dynamics.

[42] Sargent, T.J., 1991. Equilibrium with signal extraction from endogenous variables. Journal of Economic Dynamics & Control 15, 245-273.

[43] Slobodyan, S. and Wouters, R. (2012), Learning in a medium-scale DSGE model with expectations based on small forecasting models, *American Economic Journal: Macroeconomics* 4, 65-101.

[44] Smets, F., Wouters, R., 2003. Monetary policy in an estimated stochastic dynamic general equilibrium model of the euro area. Journal of the European Economic Association 1 (5), 1123C1175.

[45] Smets, F., Wouters, R., 2005. Comparing shocks and frictions in US and euro business cycles: a Bayesian DSGE approach. Journal of Applied Econometrics 20 (2), 161C183.

[46] Stock, J.H. and Watson, M.W., 2007. Why has inflation become harder to forecast? Journal of Money, Credit and Banking 39, 3-34.

[47] Taylor, J., 1980. Aggregate dynamics and staggered contracts. Journal of Political Economy 88, 1-23.

[48] White, H., 1994. Estimation, Inference and Specification Analysis. Cambridge University Press, Cambridge.

[49] Woodford, M., 2003. Interest and Prices. Princeton University Press, Princeton.

[50] Xiao, W. and Xu, J. (2014), Expectations and optimal monetary policy: a stability problem revisited, *Economics Letters* 124, 296-299.