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Approximating Walrasian equilibria

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Abstract This paper demonstrates that the auctioneer can reliably estimate demand and supply schedules in a pure exchange economy by *assuming* that all agents have Cobb-Douglas utility functions. We prove global convergence to the Walrasian equilibrium for a set of CES exchange economies, in which each agent has a CES utility function in the range from a Leontief to a Cobb-Douglas utility function. In the unstable Scarf economies, prices spiral towards the Walrasian equilibrium in the direction found by Scarf. This heuristic illustrates a less-is-more effect, because it needs less information while achieving the same level of accuracy.

KEYWORDS: Walrasian equilibrium, computation, CES economy, heuristics, less-is-more.

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1 Introduction

The amount of information that is required for computing a Walrasian equilibrium is typically understood as identifying what an auctioneer needs to *know*. For instance, [8] proves that convergence to the Walrasian equilibrium from any starting point in any economy requires the knowledge of most elements of the Jacobian of the aggregate excess demand function.[1] We propose that the auctioneer can reliably *estimate* demand and supply schedules in an exchange economy by *assuming* that all agents have Cobb-Douglas utility functions. Here, "reliable" means implying global convergence to the Walrasian equilibrium. Demand at previously quoted prices suffices to identify the hypothetical preferences. This admits computation of the unique equilibrium prices of the associated Cobb-Douglas economy, which feed into the next iteration.

After deriving some auxiliary results (section 2), we will prove convergence in exchange economies in which traders have preferences that can be represented by CES utility functions ranging from Leontief to Cobb-Douglas functions (section 3). In section 4, we apply our approach to the examples proposed by Scarf [9]. In the so-called unstable Scarf economies, prices spiral towards the Walrasian equilibrium in the same direction as found by Herbert Scarf. Finally, section 5 offers some concluding thoughts.

[1] An algorithm that globally (from any starting point) and universally (in all economies) converges to the Walrasian equilibrium is called an effective price mechanism. [4] proves that the algorithm proposed by [11] is an example of an efficient price mechanism under standard conditions on utility functions and consumption sets.

2 Preliminaries

Consider an exchange economy, ξ, consisting of n agents and m commodities. Each agent i has preferences that can be represented by a continuous, quasi-concave, monotone utility function, $u_i : \mathbb{R}_+^m \to \mathbb{R}$. Traders also have non-negative endowments, $\mathbf{w}_i \in \mathbb{R}_+^m$. By assumption, prices, $\mathbf{p}$, are non-negative and add up to 1, $\mathbf{p} \in S^{m-1} = \left\{\mathbf{p} \in \mathbb{R}_+^m | \sum_j p_j = 1\right\}$.

After he has called prices $\mathbf{p}^k > 0$ and after each trader has responded with demand $\mathbf{x}_i\left(\mathbf{p}^k\right)$, the auctioneer *estimates* demand schedules by *assuming* that each trader has Cobb-Douglas preferences. In particular:

$$x_{ji}(\mathbf{p}|\mathbf{p}^k) = \frac{p_j^k x_{ji}(\mathbf{p}^k)}{\mathbf{p}^k \cdot \mathbf{w}_i} \frac{\mathbf{p} \cdot \mathbf{w}_i}{p_j}.$$

The unique equilibrium prices of the associated Cobb-Douglas economy feed into the next iteration.[2] Call this price process $\mathcal{P}$.

Let $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = \sum_i \mathbf{x}_i\left(\mathbf{p}|\mathbf{p}^k\right) - \sum_i \mathbf{w}_i$ be the aggregate excess function of the associated Cobb-Douglas economy. Define $\mathbf{p}^{k+1}$ as the vector of equilibrium prices in the associated Cobb-Douglas economy. Hence, for $\mathbf{p}^{k+1}$ we have $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) - \sum_i \mathbf{w}_i = \mathbf{0}$.

Lemma 1 *If* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$, $\mathbf{p}^{k+1} \neq \mathbf{p}^k$ *and if* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq 0$, *then* $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) > 0$.

Proof The Cobb-Douglas economy has an aggregate excess demand function that is characterized by gross substitution (GS); this implies WARP at the

[2] By assumption, trivial cases are excluded; e.g., an economy in which each trader exclusively prefers the commodity of which he is already the sole owner.

aggregate level, provided that prices are compared with the equilibrium price, i.e. with $\mathbf{p}^{k+1}$, (c.f. [7, 17.F.3]):

$$\left(\mathbf{p}^{k+1}-\mathbf{p}\right)\cdot\left(\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)-\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)\right)>0\Leftrightarrow$$
$$\left(\mathbf{p}^{k+1}-\mathbf{p}\right)\cdot\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)>0\Leftrightarrow$$
$$\mathbf{p}^{k+1}\cdot\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)>0.$$

Here, we used $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)=\mathbf{0}$ and Walras' Law.

Together with $\mathbf{z}\left(\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)=\mathbf{0}$, lemma 1 implies that the hyperplane $\mathbf{p}^{k+1}\cdot\mathbf{a}=0$ is tangential to $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)$ in $\mathbf{p}=\mathbf{p}^{k+1}$. The aggregate excess demand function $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)$ does not need to be convex, but for all mixtures $\lambda\mathbf{p}+(1-\lambda)\,\mathbf{p}^{k+1}$ with $0<\lambda<1$ we have

Corollary 1 *If* $\mathbf{p}^{k+1}=\mathcal{P}\left(\mathbf{p}^{k}\right)$, $\mathbf{p}\neq\mathbf{p}^{k+1}$, $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)\neq 0$ *and if* $0<\lambda<1$, *then*

$$\lambda\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)>\mathbf{z}\left(\lambda\mathbf{p}+(1-\lambda)\,\mathbf{p}^{k+1}|\mathbf{p}^{k}\right).$$

Proof Suppose the contrary, then there exists a λ^{*} such that $0<\lambda^{*}<1$ and

$$\lambda^{*}\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)\leq\mathbf{z}\left(\lambda^{*}\mathbf{p}+(1-\lambda^{*})\,\mathbf{p}^{k+1}|\mathbf{p}^{k}\right)\Rightarrow$$
$$\lambda^{*}(1-\lambda^{*})\,\mathbf{p}^{k+1}\cdot\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)\leq 0$$

This contradicts lemma 1. The implication is due to multiplying both sides by $\lambda^{*}\mathbf{p}+(1-\lambda^{*})\,\mathbf{p}^{k+1}$ and applying Walras' Law.

Next, we show that the intersection between the $\mathbf{q}\cdot\mathbf{a}=0$ hyperplane and the aggregate excess demand function $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^{k}\right)$ is unique.

Lemma 2 *If* $\mathbf{q} > \mathbf{0}$*,* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$ *and* $\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = 0$ *then* $\mathbf{q} = \mathbf{p}$*.*

Proof Let $\mathbf{p} > \mathbf{0}$, $\mathbf{q} > \mathbf{0}$ and $\mathbf{q} \neq \mathbf{p}$; furthermore suppose that $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$ and $\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = 0$. From the latter and from the fact that $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ satisfies GS we have $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)$. Multiplying both sides by $\mathbf{q}$ would give $0 \neq 0$; hence $\mathbf{q} = \mathbf{p}$.

The following lemma states that if $\mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right)$ lies below the $\mathbf{p} \cdot \mathbf{a} = 0$ hyperplane for some $\mathbf{p}$, then the intersection of $\mathbf{p} \cdot \mathbf{a} = 0$ and $\mathbf{z}\left(\cdot|\mathbf{p}^k\right)$ lies above the $\mathbf{q} \cdot \mathbf{a} = 0$ hyperplane.

Lemma 3 *If* $\mathbf{p} \cdot \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right) < 0$ *and* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$*, then* $\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) > 0$*.*

Proof Let $\mathbf{p}$ be a price such that $\mathbf{p} \cdot \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right) < 0$ and $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) \neq \mathbf{0}$. From lemma 1, it follows that $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right) > 0$; hence there exists a λ^* such that $0 < \lambda^* < 1$ and $\left(\lambda^* \mathbf{p} + (1-\lambda^*)\,\mathbf{p}^{k+1}\right) \cdot \mathbf{z}\left(\mathbf{q}|\mathbf{p}^k\right) = 0$. From lemma 2, we find that $\lambda^* \mathbf{p} + (1-\lambda^*)\,\mathbf{p}^{k+1} = \mathbf{q}$ and hence

$$\mathbf{q} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) = (1-\lambda^*)\,\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right) > 0.$$

Figure 1 depicts a slice of $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ in $\mathbb{R}^m$. Although $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ appears as a convex function in figure 1, we do not assume it to be convex.

Most of the analysis below refers to agents having Constant Elasticity of Substitution (CES) utility functions, ranging from Leontief to Cobb-Douglas utility functions. In that case, for each agent i we have

$$u_i(\mathbf{x}) = \left(\sum_j \alpha_{ji} x_{ji}^{\rho_i}\right)^{1/\rho_i}$$

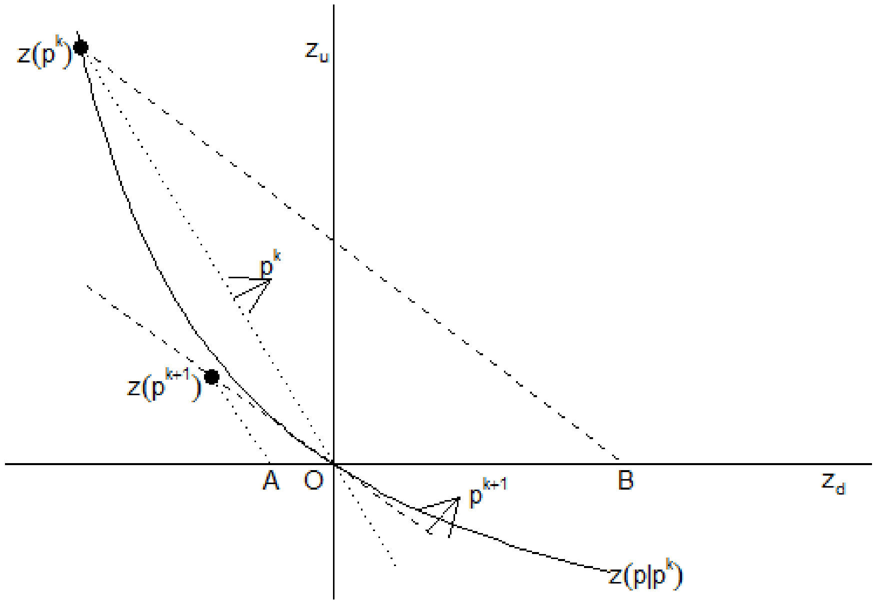

Fig. 1: *The function* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ *passes through the origin O at prices* $\mathbf{p}^{k+1}$*;* $\mathbf{p}^{k+1}\cdot\mathbf{q}=0$ *is a hyperplane that is tangential to* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$*. The* $\mathbf{p}^k\cdot\mathbf{q}=0$ *hyperplane intersects* $\mathbf{z}\left(\mathbf{p}|\mathbf{p}^k\right)$ *at* $\mathbf{z}\left(\mathbf{p}^k\right)$*, which is a graphical expression of Walras' Law. This is also why* $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ *lies somewhere on the* $\mathbf{p}^{k+1}\cdot\mathbf{q}=0$ *hyperplane. Commodities u and d have the largest increase and decrease in prices in going from iteration k to k+1. Conditions* $z_d\left(\mathbf{p}^k\right)<z_d\left(\mathbf{p}^{k+1}\right)\leq 0\leq z_u\left(\mathbf{p}^{k+1}\right)<z_u\left(\mathbf{p}^k\right)$ *are sufficient for having AO smaller than OB (triangles AO*$\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ *and OB*$\mathbf{z}\left(\mathbf{p}^k\right)$ *are congruent), i.e.* $\mathbf{p}^k\cdot\mathbf{z}\left(\mathbf{p}^{k+1}\right)+\mathbf{p}^{k+1}\cdot\mathbf{z}\left(\mathbf{p}^k\right)>0$*, implying that* $\mathbf{z}\left(\mathbf{p}\right)$ *satisfies WARP in moving from* $\mathbf{p}^k$ *to* $\mathbf{p}^{k+1}$*:* $\left(\mathbf{p}^{k+1}-\mathbf{p}^k\right)\cdot\left(\mathbf{z}\left(\mathbf{p}^{k+1}\right)-\mathbf{z}\left(\mathbf{p}^k\right)\right)<0$.

with $\rho_i<1$ and $\rho\neq 0$. If $\rho_i\to 0$, then the CES utility function converges to the Cobb-Douglas utility function. It will be convenient to define $\sigma_i=\frac{1}{1-\rho_i}$. If $\sigma_i\to 0$, the CES preferences approximate Leontief preferences. If $\sigma_i=1$ (i.e., if $\rho_i\to 0$), then CES preferences coincide with Cobb-Douglas preferences.

Below, we will consider CES preferences with $\sigma_i \leq 1$.[3] Given prices $\mathbf{p}$, trader i's optimal demand for commodity j can be written as:

$$x_{ji}(\mathbf{p}) = \left(\frac{\alpha_{ji}}{p_j}\right)^{\sigma_i} \frac{\mathbf{p} \cdot \mathbf{w}_i}{\sum_r \alpha_{ri}^{\sigma_i} p_r^{1-\sigma_i}}.$$

3 Price dynamics

Lemma 4 proves the one-to-one correspondence between equilibria of an exchange economy and its associated Cobb-Douglas economy.

Lemma 4 *Let $\xi = \{(u_i, \mathbf{w}_i)_{i=1}^n\}$ be an exchange economy with one or more Walrasian equilibria; furthermore, let $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$. If $\mathbf{p}^{k+1} = \mathbf{p}^k$ then $\mathbf{p}^k$ is a Walrasian equilibrium price vector of ξ. Conversely, each Walrasian equilibrium price vector $\mathbf{p}^*$ of ξ is also an equilibrium of the associated Cobb-Douglas economy.*

Proof If $\mathbf{p}^{k+1} = \mathbf{p}^k$ then $\sum_i \mathbf{w}_i = \sum_i \mathbf{x}_i(\mathbf{p}^{k+1}|\mathbf{p}^k) = \sum_i \mathbf{x}_i(\mathbf{p}^k|\mathbf{p}^k) = \sum_i \mathbf{x}_i(\mathbf{p}^k)$. Furthermore, $\sum_i x_i(\mathbf{p}^*|\mathbf{p}^*) = \sum_i x_i(\mathbf{p}^*) = \sum_i \mathbf{w}_i$.

After starting with strictly positive prices, the price adjustment process keeps generating strictly positive prices, implying that $\mathcal{P}$ cannot result in a boundary solution.

Lemma 5 *Let $\mathbf{p}^0 > 0$; if $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$ then $\mathbf{p}^{k+1} > 0$.*

[3] Due to GS, tâtonnement can be expected to do well for values $\sigma_i > 1$, but our argument requires $\sigma_i \leq 1$.

Proof By definition, prices $\mathbf{p}^{k+1}$ clear an associated Cobb-Douglas economy. Supposing $p_j^{k+1} = 0$, then demand for commodity j in the associated Cobb-Douglas economy would be infinite, which is inconsistent with an equilibrium; hence, $\forall j : p_j^{k+1} > 0$.

Lemma 6 *If all agents have CES preferences with* $0 \le \sigma_i \le 1$*,* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$ *and if* $\mathbf{p}^{k+1} \neq \mathbf{p}^k$*, then* $\mathbf{p}^k \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) \le 0$*, with equality applying if and only if all agents have Cobb-Douglas preferences.*

Proof Define $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ as:

$$\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right) = \frac{\mathbf{p}^{k+1} \cdot \mathbf{w}_i}{\mathbf{p}^{k+1} \cdot \mathbf{x}_i\left(\mathbf{p}^k\right)} \mathbf{x}_i\left(\mathbf{p}^k\right).$$

This point, $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$, is the demand at $\mathbf{p}^{k+1}$ if preferences are Leontief, i.e. if commodities are treated as pure complements (c.f. figure 2). In this case, $\mathbf{x}_i\left(\mathbf{p}^k\right)$ is the Leontief demand at $\mathbf{p}^k$, and $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ is scaled to lie on the $\mathbf{p}^{k+1}$ budget constraint. We want to show that for each trader i we have

$$\mathbf{p}^k \cdot \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) > \mathbf{p}^k \cdot \mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right). \tag{3.1}$$

The inequality can be obtained if there exists a hyperplane $\mathbf{p}^k \cdot \mathbf{q} = \theta$ that separates $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ and $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ in accordance with the inequality. We can determine this hyperplane by looking for hypothetical endowments $\tilde{\mathbf{w}}_i$ that (i) lie on the $\mathbf{p}^{k+1}$ budget constraint and (ii) that yield $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right) = \mathbf{x}_i\left(\mathbf{p}^k|\mathbf{p}^k, \tilde{\mathbf{w}}_i\right)$ and $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) = \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k, \tilde{\mathbf{w}}_i\right)$. For a graphical version of the argument, see figure 2 below.

Dropping index i, we can rewrite $x_{ji}^L\left(\mathbf{p}^{k+1}\right)$ as

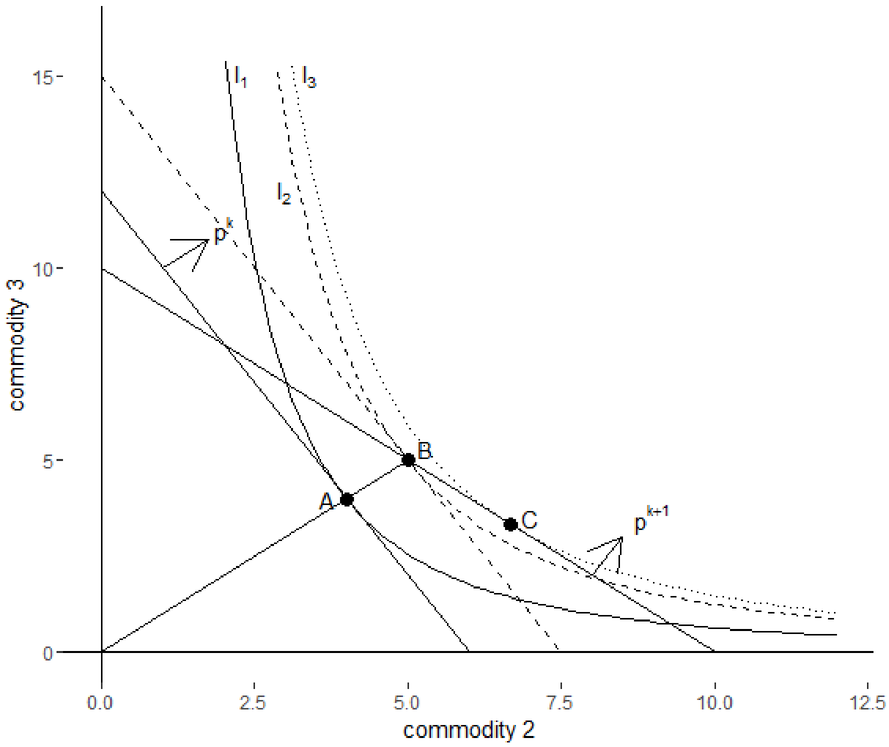

Fig. 2: *Graphical explanation of the proof of inequality 3.1. In response to prices* $\mathbf{p}^k$*, a trader demands A. The auctioneer constructs hypothetical Cobb-Douglas preferences that rationalize this choice, represented by the solid indifference curve* I_1*. The auctioneer expects that the trader will demand C at the new prices* $\mathbf{p}^{k+1}$*: The dotted indifference curve* I_3 *is tangential to the new budget constraint. Point B is the demand at prices* $\mathbf{p}^{k+1}$ *if the trader has Leontief preferences (instead of the unknown CES or the hypothetical Cobb-Douglas preferences). Inequality 3.1 expresses the fact that C is not affordable at prices* $\mathbf{p}^k$ *for another trader, with endowments B and the hypothetical Cobb-Douglas preferences. The point is proved by observing that this other trader prefers B at prices* $\mathbf{p}^k$ *and that he prefers C if prices are equal to* $\mathbf{p}^{k+1}$ *(i.e., while B is also affordable at* $\mathbf{p}^{k+1}$*). The implication is that C is not affordable at* $\mathbf{p}^k$ *(otherwise this other trader would have chosen C instead of B). Hence, the dashed budget constraint through B is part of a hyperplane that separates B and C.*

$$x_j^L\left(\mathbf{p}^{k+1}\right)=\frac{p_j^k x_j\left(\mathbf{p}^k\right)}{\mathbf{p}^k\cdot\mathbf{w}}\frac{\mathbf{p}^k\cdot\frac{\mathbf{p}^{k+1}\mathbf{w}}{\mathbf{p}^{k+1}\mathbf{x}(\mathbf{p}^k)}\mathbf{x}\left(\mathbf{p}^k\right)}{p_j^k}.$$

Furthermore, we also have

$$\begin{aligned}x_j\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)&=\frac{p_j^k x_j\left(\mathbf{p}^k\right)}{\mathbf{p}^k\cdot\mathbf{w}}\frac{\mathbf{p}^{k+1}\mathbf{w}}{p_j^{k+1}}\\&=\frac{p_j^k x_j\left(\mathbf{p}^k\right)}{\mathbf{p}^k\cdot\mathbf{w}}\frac{\mathbf{p}^{k+1}\cdot\frac{\mathbf{p}^{k+1}\mathbf{w}}{\mathbf{p}^{k+1}\mathbf{x}(\mathbf{p}^k)}\mathbf{x}\left(\mathbf{p}^k\right)}{p_j^{k+1}}.\end{aligned}$$

Therefore, we can argue that (i) an agent who is endowed with $\tilde{\mathbf{w}}=\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and has a demand function $\mathbf{x}\left(\mathbf{p}|\mathbf{p}^k,\tilde{\mathbf{w}}\right)$ will demand $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ if prices are equal to $\mathbf{p}^k$ and $\mathbf{p}^{k+1}$ respectively; (ii) if prices are equal to $\mathbf{p}^{k+1}$ then both $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ are affordable, though $\mathbf{x}\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ is preferred; (iii) if prices are equal to $\mathbf{p}^k$, then $\mathbf{x}^L\left(\mathbf{p}^{k+1}\right)$ is preferred to other affordable options; (iv) the individual demand function $\mathbf{x}_i\left(\mathbf{p}|\mathbf{p}^k,\tilde{\mathbf{w}}\right)$ satisfies WARP; and (v) hence, $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ is not affordable at prices $\mathbf{p}^k$. This demonstrates the existence of a suitable separating hyperplane, and hence, for all traders we have $\mathbf{p}^k\cdot\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)>\mathbf{p}^k\cdot\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$.

If preferences are CES with $0\le\sigma_i\le1$, then $\mathbf{x}_i\left(\mathbf{p}^{k+1}\right)$ lies between $\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)$ and $\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$ on the $\mathbf{p}^{k+1}$ budget constraint, i.e.,

$$\mathbf{x}_i\left(\mathbf{p}^{k+1}\right)=\left(1-\theta_i\right)\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)+\theta_i\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)$$

with $0\le\theta_i\le1$. If $0\le\theta_i<1$, then by virtue of inequality 3.1, we now have

$$\mathbf{p}^k\cdot\mathbf{x}_i\left(\mathbf{p}^{k+1}\right)=\left(1-\theta_i\right)\mathbf{p}^k\cdot\mathbf{x}_i^L\left(\mathbf{p}^{k+1}\right)+\theta_i\mathbf{p}^k\cdot\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right)<\mathbf{p}^k\cdot\mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right).$$

Summing over i yields:

$$\begin{aligned}
\mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}\right) < \mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}|\mathbf{p}^k\right) \Leftrightarrow \\
\mathbf{p}^k \cdot \sum_i \mathbf{x}_i\left(\mathbf{p}^{k+1}\right) < \mathbf{p}^k \cdot \sum_i \mathbf{w}_i \Leftrightarrow \\
\mathbf{p}^k \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0.
\end{aligned}$$

The second inequality is due to the fact that $\mathbf{p}^{k+1}$ is an equilibrium price vector of the associated Cobb-Douglas economy. If $\forall i : \theta_i = 1$, then the inequality becomes an equality.

The following lemma shows that constraints on $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ accumulate, which allows us to prove global convergence in proposition 1.

Lemma 7 *If* $\mathbf{p}^{k+1} = \mathcal{P}\left(\mathbf{p}^k\right)$, $\mathbf{p}^{k+1} \neq \mathbf{p}^k$, *and some traders do not have Cobb-Douglas preferences, then* $\forall r \leq k :\ \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$.

Proof For $r = k$, the result follows directly from lemma 6. Suppose $\mathbf{z}\left(\mathbf{p}^{k+1}\right) \neq \mathbf{0}$ and $\mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) \geq 0$; if $\mathbf{0} < \mathbf{p}^{k-1} \neq \mathbf{p}^{k+1}$, then according to lemma 2 $\mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) > 0$. Therefore, there exists a θ^*, $0 < \theta^* < 1$, for which

$$\left(\theta^* \mathbf{p}^{k-1} + (1-\theta^*)\,\mathbf{p}^k\right) \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) = 0.$$

But then, also due to lemma 2, $\theta^* \mathbf{p}^{k-1} + (1-\theta^*)\,\mathbf{p}^k = \mathbf{p}^{k+1}$. This implies for $\mathbf{z}\left(\mathbf{p}^k\right) \neq \mathbf{0}$

$$\begin{aligned}
\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^k\right) &= \\
\left(\theta^* \mathbf{p}^{k-1} + (1-\theta^*)\,\mathbf{p}^k\right) \cdot \mathbf{z}\left(\mathbf{p}^k\right) &= \\
\theta^* \mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^k\right) &< 0
\end{aligned}$$

which contradicts lemma 1. Hence, $\mathbf{p}^{k-1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$. This argument can be repeated to obtain $\mathbf{p}^{r} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ for other $r < k-1$.

Proposition 1 *Let $\xi = \{(u_i, \mathbf{w}_i)_{i=1}^{n}\}$ be a CES exchange economy with $\forall i :$ $0 \leq \sigma_i \leq 1$. Assume that the Walrasian equilibrium exists, and let $\mathbf{p}^*$ denote the Walrasian equilibrium prices. Price adjustment process $\mathcal{P}$ converges globally to $\mathbf{p}^*$.*

Proof We have $\forall r \leq k : \mathbf{p}^{r} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ from lemma 7. This implies $\mathbf{p}^{k+1} \neq \sum_{r=0}^{k} \theta^r \mathbf{p}^r$ for any $\{\theta^r\}_r$ with $0 < \theta^r < 1$ and $\sum_r \theta^r = 1$. To see this, suppose $\mathbf{p}^{k+1} = \sum_r \theta^r \mathbf{p}^r$ for appropriate $\{\theta^r\}_r$; then $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) = \sum_r \theta^r \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$, because, by assumption, each term $\mathbf{p}^{r} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$; however, this violates Walras' Law. Note that this also rules out the occurrence of cycles: $\mathbf{p}^k \neq \mathbf{p}^{k+j}$ for all k and j. Let

$$F(k) = \left\{ \mathbf{p} | \mathbf{p} = \sum_{r=0}^{k} \theta^r \mathbf{p}^r,\ 0 < \theta^r < 1,\ \sum_{r=0}^{k} \theta^r = 1 \right\} \subset S^{m-1}$$

be the set of "forbidden" subsequent prices. For as long as $\mathcal{P}$ has not yet converged, a new $\mathbf{p}^{k+1} \in S^{m-1} - F(k)$ will be selected. We have $F(k) \subseteq F(k+1)$, with equality applying only if $\mathbf{p}^{k+1} = \mathbf{p}^{k} = \mathbf{p}^*$. We will show that convergence of $F(k)$ implies that $\mathbf{p}^k$ converges.

To the contrary, assume that $F(k)$ converges and that prices do not converge; then, there exist $\{\theta^r\}_r$ with $0 < \theta^r < 1$ and $\sum_r \theta^r = 1$, such that $\mathbf{p}^{k+1} = \sum_{r=0}^{k} \theta^r \mathbf{p}^r + \boldsymbol{\delta}^{k+1}$ and where $||\boldsymbol{\delta}^{k+1}||$ can be made arbitrarily small. As a result we would have $\mathbf{p}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) = \sum_r \theta^r \mathbf{p}^r \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) + \boldsymbol{\delta}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right) < 0$ because the sum is strictly negative and $\boldsymbol{\delta}^{k+1} \cdot \mathbf{z}\left(\mathbf{p}^{k+1}\right)$ can be made arbitrarily small for suitable values of k, because $\mathbf{z}\left(\mathbf{p}^{k+1}\right)$ is bounded. This violates Walras' Law and hence prices must converge whenever $F(k)$ converges. Since $F(k) \subseteq S^{m-1}$ this set cannot increase beyond bounds; it must converge and so prices converge to a Walrasian equilibrium price vector $\mathbf{p}^*$.

4 Application to the Scarf economies

Herbert Scarf has demonstrated that tâtonnement may fail to converge to the Walrasian equilibrium. Scarf [9] provides three examples consisting of a small exchange economy with three traders and three commodities. The three examples differ in the initial allocation only. In the stable version, tâtonnement does converge to the Walrasian equilibrium. The other two examples are unstable; here, tâtonnement leads to prices orbiting around the Walrasian equilibrium values, in a clockwise (cw) or counter-clockwise (ccw) direction.

Figure 2 shows the convergence of $\mathcal{P}$ in Scarf's examples, as implemented by Anderson et al. [1]. Here, all three economies have the same unique equilibrium, with $\mathbf{p}^* = (1, 40, 20)$.

Convergence is fast: For instance, starting from $\mathbf{p}^1 = (1, 1, 1)$, it takes 15 iterations to obtain the equilibrium prices in two decimals in the stable case and 65 iterations in the unstable cases. In the unstable economies, the direction of the spiral is consistent with Scarf's predictions.

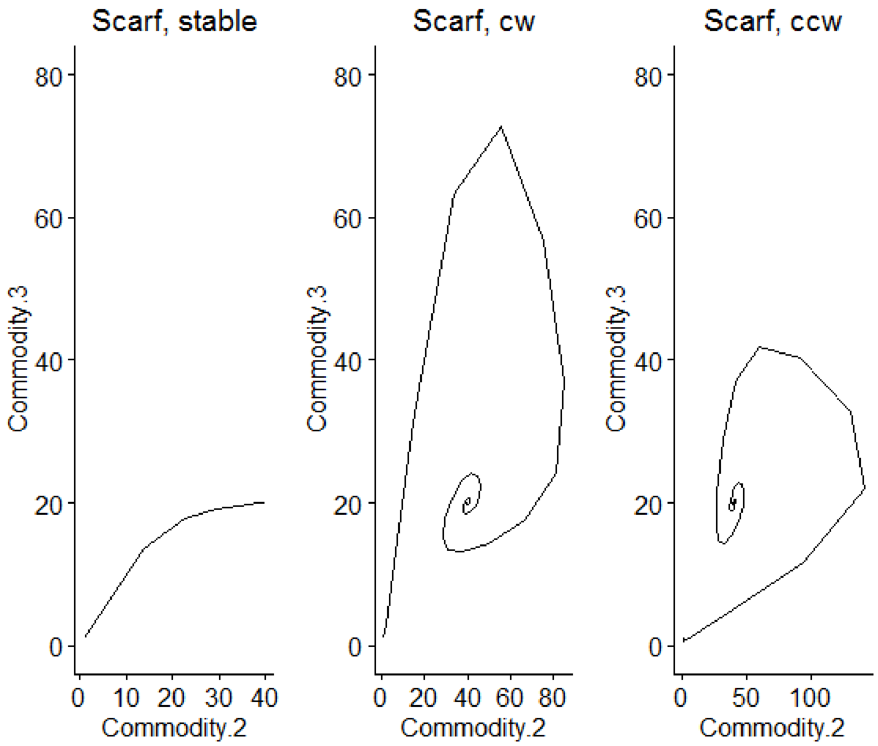

Fig. 3: *Convergence of price adjustment process $\mathcal{P}$ in the Scarf economies to the equilibrium prices,* $\left(p_2^*, p_3^*\right) = (40, 20)$. *Convergence depends on the initial allocation.*

5 Discussion

Cobb-Douglas approximation provides a new perspective on the information that is needed for proving global convergence. Instead of having to know the Jacobian of the aggregate excess demand function, the auctioneer can effectively *estimate* its behavior by *assuming* that demand is generated by Cobb-Douglas preferences. This illustrates a less-is-more effect, because within the specified class of CES-economies, the heuristics performs at least as good as alternative algorithms that require more information. Opposition to the accuracy - effort trade-off is a central theme in Gigerenzer and Pachur [3].

The fact that we can prove global convergence for economies in which substitution is at most equal to a Cobb-Douglas economy seems to be due to the price adjustment process $\mathcal{P}$'s correctly anticipating income effects. Therefore, there is no need to ascertain that substitution effects dominate income effects.

Our price process $\mathcal{P}$ also illustrates that it is possible to have prices at which the aggregate excess demand function $\mathbf{z}(\mathbf{p})$ satisfies WARP, even though $\mathbf{z}(\mathbf{p})$ does not satisfy WARP at all prices. The latter condition therefore appears to be unduly restrictive.

The algorithm also presents a new perspective on the Sonnenschein-Mantel-Debreu (SMD)-result (c.f. Sonnenschein [10], Mantel [6, 5], and Debreu [2]. This is often interpreted as saying that price dynamics can be as “bad” as desired. In a large class of CES exchange economies, however, there exists another, simple and well-behaved, alternative.

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