Constrained Stochastic Cost Allocation*

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January 16, 2019

Abstract

This paper presents a model of a multi-divisional firm to share the joint yet uncertain and fixed cost of running a central operational unit. A firm aims at allocating these costs *ex ante*, subject to constraints imposed by the asymmetric and limited liabilities of the different divisions. We study solutions that are made up of a vector of *ex ante* payments which are allocated in absence of costs, and a remaining solution for the remaining variable component of the costs. Under a mild continuity condition we find different classes of *egalitarian* solutions. The class of egalitarian proportional solutions is characterized by dependency on the expected total cost instead of details of the distribution.In this class, it is the *constrained proportional rationing solution* which systematically minimizes the maximal transfer. A fundamentally different egalitarian solution is the *Stochastic Egalitarian constrained Equal Cost* solution. It is characterized using a local symmetry property which states that incremental costs should be distributed equally amongst those divisions with sufficient liability. This egalitarian solution has a smaller largest transfer than any egalitarian standard proportional solution. We conclude by showing how our results generalize when egalitarianism is replaced by a more general fairness property.

Keywords: stochastic cost allocation, egalitarian solution, rationing, constrained equal award rule, proportional rule.
JEL Classification: C79, D31, D81, M41.

1 Introduction

Consider a multi-divisional firm with a central service unit - to which all divisions have equal access. Running this shared facility is costly and the divisions are charged for the full and uncertain cost. We will focus on cost allocation, and study for *ex ante* contracts that the firm may use to share the *ex post* realized cost. The firm puts upper bounds on the liability of a division, just as long as the total of maximal liabilities is enough to cover the costs arising in the worst-case scenario. The maximal liabilities of the divisions may be the result from exogenous risk capital allocations within the firm, and are limiting the divisions' capacity to bear risk (see, e.g., Myers and Read, 2001).

We assume that it is up to a benevolent manager to allocate the risky cost, which is considered a social bad, amongst the divisions. We propagate an allocation of costs that reconciles the

*The authors thank Itzhak Gilboa, Hervé Moulin, Hans Schumacher, Juan D. Moreno-Ternero, and two anonymous reviewers for their comments on an earlier version of this paper.

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possible asymmetric way in which the divisions can be ultimately be held responsible for upon realization of the costs. In this paper we take egalitarianism as the fundamental principle that should govern the allocation. This means that ideally costs are shared equally by the divisions for any realization. However, this is not necessarily feasible in case liabilities are different and the realized costs are in this respect high enough. In such scenarios the divisions with high liabilities may *have to* contribute more than those with low liabilities. Despite the fact that the divisions cannot be treated equally in those cases, we will cherish the idea of cost allocations that are symmetric functions of the liability profiles.

We pose the question whether symmetric *ex ante* solutions exist according to which all agents face the same level of risk, even if this requires asymmetric solutions to realized instances of the constrained cost allocation problem. In order to make sense of intercomparison of the agents, we will assume that all have the same beliefs regarding the underlying probability distribution, and assume that agents are all try to minimize the expected cost. This makes the concept of egalitarianism useful and straightforward.

In non-trivial cases, where the liabilities are not high enough to be able to bear a fair share of $1/|N|$ of the realized cost, the solution will demand higher contributions from those agents with the higher liabilities. Egalitarian solutions see to it that in turn these agents will be compensated for by those with low liabilities in case of a low realized cost and that this compensation scheme via *transfers* is such that all agents are assured to be exposed, *ex ante*, to the same expected cost.

One particularly interesting class consists of solutions that define allocations for these *constrained cost allocation problems* by solving the natural dual of a *rationing problem* by a *rationing rule*. A rationing problem describes the situation in which we allocate a given amount (often referred to as estate) among a group of agents when the available amount is not enough to satisfy all their claims. A rationing rule calculates shares for agents such that 1) no agent gets more than his/her claim, and 2) all get a non-negative share.[1] With the realized cost as estate and the profile of liabilities as claims, each constrained cost allocation problem is the natural counterpart of a rationing problem. In fact, the constrained cost allocation problem generalizes the dual of a rationing problem to allow for a stochastic cost. Then using the mathematical equivalence between the two models, each rationing rule can be taken to define a cost allocation solution. We show that each rationing rule that is continuous in the "claims" component can be used to define an egalitarian solution. In particular, included are many solutions that are symmetric as function of the profile of liabilities, meaning that agents are regarded equal and possible asymmetries between the proposed allocations should be motivated by the differences in liability. We discuss two special subclasses of solutions therein, one that is generated by proportional rationing rules and the other that uses the constrained equal award rationing rule as basis.

Characteristic of the proportional solutions is that all satisfy an invariance property regarding the underlying distribution of the risk; as long as the expected cost is the same, we aim to have the same solution. We prove that egalitarian solutions with this *Invariance to expected total Cost-preserving Alternatives* (ICA) property are in fact proportional solutions. The egalitarian solution that we get by the constrained equal award rationing rule as generator is denoted as the Stochastic Egalitarian constrained Equal Cost (SEEC) solution. The solution is characterized as the unique egalitarian solution with the *Local Symmetry* (LS) property, according to which marginal increases of costs affect the agent's marginal contribution in the same way – for those agents whose liabilities are still not met.

Recall that our aim is to allocate equal cost shares whenever this is feasible. In case the liquidity is high enough, so that the *trivial solution* under the worst scenario realization of cost is

[1] For rationing problems in practice and rationing rules see, e.g., O'Neill (1982), Aumann and Maschler (1985), or the overviews of Moulin (2002), and Thomson (2003, 2015).

feasible, this completely determines the solution. Since then the solution should prescribe equal shares for all realizations of cost. We will refer to this property as *Triviality at Sufficient Liquidity* (TSL). The *constrained proportional solution* is introduced as a special egalitarian proportional solution with the TSL property. Additionally, we show that it systematically minimizes the maximal transfers within the class of egalitarian proportional solutions. More precisely, the vector of transfers generated by this solution Lorenz-dominates all others used by the egalitarian proportional solutions. Nevertheless, we show that SEEC also satisfies TSL and when liquidity is not high enough to assure a feasible equal share solution it uses a lower maximal transfer than does any egalitarian proportional solution. And thus the vector of transfers according to SEEC is not dominated by the vector of transfers corresponding to the constrained proportional solution. We discuss the egalitarian solution underpinned by the constrained equal loss rationing rule as an example of a solution that does not even satisfy TSL, despite the fact that this particular rationing rule on which the solution is constructed is credited egalitarian properties in the rationing framework.

Whereas egalitarian solutions play a central role in this paper, by no means this excludes other meaningful deviations from the egalitarian allocation. We finalize this paper by showing that much of the strict interpretation of egalitarianism may be relaxed, and that much of the reasoning throughout the paper still applies to find transfers such that costs are allocated in desired proportion of risks. Then this more general set-up bridges the gap between the theory of risk allocation and a social norm. It is the social norm that governs the risk allocation *ex ante*, subject to *ex post* feasibility defined by the individual liabilities. And those social norms exogenous to the model may require other fractions in which the expected total cost is shared *ex ante*. This concept of *fairness* is formalized by a given allocation of the expected total cost. Such a specific allocation may be the result of non-symmetric initially received contributions to bare the total stochastic cost. Our fairness constraint is akin to the financial fairness condition in risk-sharing (Pazdera et al., 2017). The concept of fairness was originally introduced for risk exchanges by Bühlmann and Jewell (1979). Bühlmann and Jewell (1979) and Pazdera et al. (2017) all consider the case of risk-sharing, where the agents have initial stochastic endowments to be shared, and the focus is on the induced incentives for agents to participate or not. In this paper we concentrate on the a fundamentally different problem of allocating the stochastic cost of a public good. Our general characterization of non-egalitarian solutions via financial fairness follows from a slight adaptation of our model formulation. In addition, the solutions inherit the very same structure of the pure egalitarianism that is elemental to the standard cost allocation rules like SEEC; basically, it only changes by a correction of the transfers at 0 cost.

Habis and Herings (2013) and Ertemel and Kumar (2018) study stochastic rationing problems as well, where besides the estate also the claims are considered stochastic. Like us, they both concentrate on *ex ante* solutions, but focus on notions of stability and enhancing cooperation. Kıbrıs and Kıbrıs (2013), Karagözoğlu (2014) and Boonen (2017) study an investment problem with an endogenous and stochastic estate, where bankruptcy problems are applied in case of default. Instead of investing with risk of a defaulting counterparty, we instead concentrate on a public good of which the cost needs to be allocated under liability constraints. In a problem to share the cost of a stochastic network, Hougaard and Moulin (2018) focus at determining cost shares *ex ante* such that these equal the expectation over the random realization of the network of the shares *ex post*. Xue (2018) studies egalitarian cost-sharing without liability constraints, but with uncertain claims to a divisible commodity.

Section 2 specifies the model, and Section 3 provides our construction of egalitarian solutions and transfers. The ordering of transfers is studied in Section 4. Sections 5 and 6 are devoted to the defining and characterizing of our solutions based on the proportional and constrained equal award rules. Section 7 compares these solutions based on a ranking of the transfers. Section 8 provides a generalization of the concept of egalitarianism to the case where we first exogenously allocate the expected total cost to the agents. Section 9 provides a remark in which we explain how our solution concept can be generalized to situations where egalitarian solutions do not exist, i.e., where some of the agents have too low liabilities to take a fair share of the risk. Finally, Section 10 concludes. All proofs are delegated to the appendix.

2 The constrained cost allocation model and solutions

In this section, we formalize the constrained cost allocation problem, which is summarized by a tuple (C, L, V). The stochastic cost C is shared among a fixed and finite set of *agents* N under liability constraints, that are summarized by a vector L. The preferences of the agents for stochastic allocations are denoted by V. The constrained cost allocation problem (C, L, V) is defined in Subsection 2.1. In Subsection 2.2, we define the concept of egalitarianism for cost allocation solutions.

2.1 The constrained cost allocation problem

Agents aim to share the cost of a risky project. The cost of the project, denoted by C, is a bounded, non-negative random variable on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the state space, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure. The class of all bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by $\mathcal{L}^\infty$.

We consider an ordered pair $(C, L) \in \mathcal{L}^\infty \times \mathbb{R}^N_{++}$, where C is the non-negative random cost that has to be allocated to the agents in $N = \{1, 2, \ldots, n\}$ under liquidity constraints. We assume that the allocation for an agent i is at most equal to its the liability $L_i > 0$. This liability of an agent is the maximal obligation to contribute. We assume that the range of C a subset of $[0, \bar{c}]$, where $\bar{c}$ is the supremum of all realizations of C. Moreover, (C, L) satisfies:

Admissibility: $L(N) := \sum_{i \in N} L_i \geq \bar{c}$.

Admissibility implies that whatever the realization of the project will turn out to be, the collective of agents can afford it. Below it will be convenient to think of L as an ordered vector, i.e.,

$$L_1 \leq \cdots \leq L_n.$$

Suppose that $X \in \mathcal{L}^\infty$ is considered as an agent's future cost, and $\mathbb{Q}^X$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}^X(0 \leq C \leq \bar{c}) = 1$, and if $(X, Y) \in (\mathcal{L}^\infty)^2$ is comonotonic,[2] then $\mathbb{Q}^X = \mathbb{Q}^Y$. Each agent wants to minimize $V(X)$, where V is such that

$$V(X) = \mathbb{E}_{\mathbb{Q}^X}[X], \text{ for all } X \in \mathcal{L}^\infty. \tag{1}$$

The set of all such preferences V is denoted $\mathcal{V}$. For instance, the function V can be a representation of the expectation or of *dual utility* (Yaari, 1987).[3] We denote $V(C)$ as the expected (total) cost.

[2] The ordered pair $(X, Y) \in (\mathcal{L}^\infty)^2$ is comonotonic if there exists a non-decreasing function h such that $(X, Y) \stackrel{d}{=} (X, h(X))$.

[3] If V is represented by dual utility, then

$$V(X) = \int_0^\infty g^V(1 - F_X(x))\, dx + \int_{-\infty}^0 [1 - g^V(1 - F_X(x))]\, dx, \text{ for all } X \in \mathcal{L}^\infty,$$

for a left-continuous and non-decreasing function $g^V : [0, 1] \to [0, 1]$ with $g^V(0) = 0$ and $g^V(1) = 1$, where F_X is the cumulative density function (CDF) of X. Then, $\mathbb{Q}^X(X \leq x) = 1 - g^V(1 - F_X(x))$ for all $x \in \mathbb{R}$.

We emphasise that we do not assume heterogeneous beliefs or heterogeneous preferences for the agents; the same $V \in \mathcal{V}$ is used by the collective of agents. In case of heterogeneous preferences we would need to compare interpersonal risk aversion, which can be considered problematic in several aspects. First of all, there is a complicating factor that there is no way to assure that agents will reveal their true preferences. For instance, Anthropelos and Karatzas (2017) show that agents would have an incentive to misrepresent their true risk aversion if it is self-reported. But even if we would know the true preferences, the problem of ambiguity involved in making such intercomparison is intrinsically hard to overcome, see Young (1990). This motivates the use of a "representative agent" model where the agents are assumed to have the same preferences over risky allocations.

We will call the ordered triple $(C, L, V) \in \mathcal{L}^\infty \times \mathbb{R}^N_{++} \times \mathcal{V}$ a *constrained cost allocation problem* if (C, L) is admissible and (C, L, V) satisfies the following condition:

V-sufficiency: $L \geq \frac{1}{n} V(C)\, \mathbf{e}$.[4]

This means that the liabilities are such that in principle each agent can take a fair share of the risky cost, as measured by V. The set of all such constrained cost allocation problems is denoted $\mathcal{P}$.

2.2 Solutions

For each realization $c \in [0, \overline{c}]$ of C, the set of feasible cost allocations is given by:

$$\mathcal{A}(c, L) := \left\{ x \in \mathbb{R}^N : x(N) = c, x \leq L \right\},$$

where inequalities of vectors are understood element-wise. This set $\mathcal{A}(c, L)$ is a finite dimensional bounded space, and the intersection of a finite number of closed half-spaces. Thus, it is a convex polytope, which is easily described as the convex hull of its extreme points $m^1, \ldots, m^n \in \mathbb{R}^n$ defined by

$$m^i_j := \begin{cases} L_j & \text{if } j \neq i, \\ c - L(N \backslash \{i\}) & \text{if } j = i. \end{cases}$$

In particular, this shows that the challenge is not to show existence of feasible cost allocations. Instead, we will focus on the notion of a solution, which selects cost allocation vectors for all possible realizations of the stochastic cost.

For each $(C, L, V) \in \mathcal{P}$, a cost allocation solution ψ maps every $c \in [0, \overline{c}]$ into $\mathcal{A}(c, L)$. So ψ defines a cost allocation for each realization of the total cost, and $\psi(C, L, V)(c) \in \mathcal{A}(c, L)$ for $c \in [0, \overline{c}]$ stands for the cost allocation solution for the realization c of random variable C. Note that this implies for all $(C, L, V) \in \mathcal{P}$ that

$$\psi\left(C, L, V\right)(c) \leq L \text{ and } \sum_{i \in N} \psi_i\left(C, L, V\right)(c) = c, \text{ for all } c \in [0, \overline{c}]. \tag{2}$$

Denote by $\mathcal{Z}$ the set of all cost allocation solutions (or, in short, solutions). At this point it is essential to realize that the only restriction on $\psi(C, L, V)(0)$ is that it is contained in $\mathcal{A}(0, L)$, and it is not necessarily equal to the zero vector. If non-zero, the cost allocation is negative for one or more agents and positive for one or more of the others. Negative cost allocations are possible in order to allow for compensations between agents. The standard literature on distributive justice

[4] Here, $\mathbf{e} \in \mathbb{R}^N$ is the vector with $\mathbf{e}_i = 1$ for all i. We will use $\mathbf{0}$ to denote the zero vector in $\mathbb{R}^N$.

is more restrictive in this sense, and allows for non-negative shares only.

Below we will discuss several classes of solutions, which all satisfy the following weak monotonicity property – which we think is compelling for any reasonable solution:

Comonotonicity: $c \mapsto \psi_i(C, L, V)(c)$ is non-decreasing on $[0, \bar{c}]$ for all $i \in N$.

Comonotonicity expresses the idea that if the aggregate cost C increases, no agent should benefit by paying less. So, it can be considered a weak property assuring that no agent benefits if the total cost increases.

Consider the set $\mathcal{Z}_0 \subset \mathcal{Z}$ of all comonotonic solutions ψ that are *zero-normalized* in the sense that $\psi(C, L, V)(0) = 0$ for all $(C, L, V) \in \mathcal{P}$. It is trivial but instructive to see that any solution ψ can be written as the sum of the solution at 0 cost and a zero-normalized solution $\psi_0 \in \mathcal{Z}_0$, as follows

$$\begin{aligned} \psi(C, L, V)(c) &= \psi(C, L, V)(0) + (\psi(C, L, V)(c) - \psi(C, L, V)(0)) \\ &= \psi(C, L, V)(0) + \psi_0(C, L, V)(c), \end{aligned}$$

for all $c \in [0, \bar{c}]$.

This paper focuses foremost on the class of solutions ψ that we will refer to as *standard*, i.e., those solutions for which $\psi_0 \in \mathcal{Z}_0$ can be written as

$$\psi_0(C, L, V)(c) = f(c, L, t, V(C)), \text{ for all } c \in [0, \bar{c}], \tag{3}$$

where $t = \psi(C, L, V)(0) \in \mathcal{A}(0, L)$ and $f(c, L, t, V(C)) \le L - t$ for all problems $(C, L, V) \in \mathcal{P}$. Let $\mathcal{Z}^{\text{S}} \subset \mathcal{Z}$ be the set of all standard solutions. Then solutions in $\mathcal{Z}^{\text{S}}$ capture the idea that asymmetries derived from the liabilities can be neutralized with respect to the preferences using a profile of *ex ante* transfers t; after realization of the costs the *ex post* cost allocation may still be influenced by V but only through t and $V(C)$. Note that the upper bound on f is meant to keep feasibility.

Example 1 The class of standard *proportional solutions* $\mathcal{Z}^{\text{SP}} \subset \mathcal{Z}^{\text{S}}$ consists of all solutions ψ^α given by

$$\psi^\alpha(C, L, V)(c) = t + \alpha(L, t, V(C)) \cdot c, \text{ for all } c \in [0, \bar{c}], \tag{4}$$

where $t \in \mathcal{A}(0, L)$ and $\alpha(L, t, V(C)) \in \mathbb{R}^N_+$ is a vector such that $\sum_{i \in N} \alpha_i(L, t, V(C)) = 1$ and $\alpha(L, t, V(C)) \le (L - t)/\bar{c}$. So, a standard proportional solutions conveys the idea that *within* each problem $(C, L, V) \in \mathcal{P}$ the costs are allocated to the agents in a fixed ratio. For the sake of exposition we will slightly abuse our notation and simplify $\alpha(L, t, V(C))$ to α, when no confusion arises. $\bigtriangledown$

The set of feasible cost allocations $\mathcal{A}(c, L)$ is a superset of the set of allocations in a rationing problem.[5] Therefore, on the one hand, we may take standard rationing rules to define cost

[5] Formally, a rationing problem is an ordered pair $(c, L) \in \mathcal{R} := \mathbb{R}_+ \times \mathbb{R}^N_+$ such that $\sum_{i \in N} L_i \ge c$. In the rationing framework L_i is interpreted as the justified claim of agent i on the amount of good c. Then, a rationing rule is a mapping $r : \mathcal{R} \to \mathbb{R}^N_+$ such that for each $(c, L) \in \mathcal{R}$ we have

$$\mathbf{0} \le r\,(c, L) \le L \text{ and } \sum_{i \in N} r_i\,(c, L) = c.$$

See, e.g., Moulin (2002) and Thomson (2003, 2015) for excellent overviews on the rich domain of rationing rules.

allocation solutions. On the other hand, each zero-normalized ψ_0 may be interpreted as the dual of a rationing rule by which for each realization of costs c the ordered pair (c, L) is assigned a cost allocation $\psi_0(C, L, V)(c) = \psi(C, L, V)(c) - \psi(C, L, V)(0) \leq L - t$, where $t = \psi(C, L, V)(0)$. It then can be argued that this dual rationing rule should take $L - t$ as the vector of liabilities, instead of L. Since ultimately, the liabilities in the model serve the goal of specifying the extent to which agents may be exposed to costs, and after having made the *ex ante* payment of t the room left for cost allocations is set at the vector $L - t$.
Let $\mathcal{Z}^{\text{R}} \subset \mathcal{Z}^{\text{S}}$ be the set of solutions ψ that agree with this idea so that it can be written as

$$\psi(C, L, V)(c) = t + \varphi^{V(C)}(c, L - t), \text{ for all } c \in [0, \bar{c}], \tag{5}$$

where $t = \psi(C, L, V)(0) \in \mathcal{A}(0, L)$ and $\varphi^{V(C)}$ is the mathematical equivalent of some rationing rule that may depend on $V(C)$.

Example 2 The class of *standard proportional rationing solutions* $\mathcal{Z}^{\text{SPR}} \subset \mathcal{Z}^{\text{SP}} \cap \mathcal{Z}^{\text{R}}$ consists of all solutions $\psi^\alpha \in \mathcal{Z}^{\text{SP}}$ based on the proportional rationing rule $r^p(c, L) := L/L(N) \cdot c$, so that[6]

$$\psi^\alpha(C, L, V)(c) = t + \alpha\, c = t + \frac{L - t}{L(N)}\, c, \text{ for all } c \in [0, \bar{c}],$$

where $t \in \mathcal{A}(0, L)$. Note that any such solution is feasible due to $\alpha = (L - t)/L(N) \leq (L - t)/\bar{c}$. There is only one standard proportional solution when $L(N) = \bar{c}$, which is the standard proportional rationing solution. Since in that situation, for fixed $t \in \mathcal{A}(0, L)$, the only α in (4) that we can chose is $\alpha = (L - t)/L(N)$. ▽

Example 3 Consider the *constrained equal award* rationing rule (see Aumann and Maschler, 1985) which is defined by $r_i^{\text{CEA}}(c, L) = \min\{\lambda, L_i\}$ for $i \in N$, where λ solves

$$\sum_{i \in N} \min\{\lambda, L_i\} = c,$$

and let φ^{EC} its mathematical equivalent in terms of cost allocation. Then the class of *constrained equal cost solutions* $\mathcal{Z}^{\text{CEC}} \subset \mathcal{Z}^{\text{R}}$ consists of solutions ψ such that

$$\psi(C, L, V)(c) = t + \varphi^{\text{EC}}(c, L - t), \text{ for all } c \in [0, \bar{c}],$$

where $t \in \mathcal{A}(0, L)$. ▽

Example 4 Consider the *constrained equal loss* rationing rule (see Aumann and Maschler, 1985) which is the dual of r^{CEA}, so that $r_i^{\text{CEL}}(c, L) = L - r^{\text{CEA}}(L(N) - c, L)$. Let φ^{CEL} its cost allocation mathematical equivalent. Then the class of *constrained equal remaining cost* solutions $\mathcal{Z}^{\text{CERC}} \subset \mathcal{Z}^{\text{R}}$ consists of solutions ψ that can be written as

$$\psi(C, L, V)(c) = t + \varphi^{\text{CEL}}(c, L - t), \text{ for all } c \in [0, \bar{c}],$$

where $t \in \mathcal{A}(0, L)$. ▽

[6]Here, dividing a vector by a positive scalar is understood element-wise.

3 Egalitarian solutions and transfers

The central issue addressed in this paper is the question how to deal with situations where individual liabilities are too skewed to be able to see to a pure egalitarian solution for all realized costs *ex post*, but that at least *ex ante* the risky asymmetric allocation is equally preferred by all the agents. The leading property is the following. We will call a solution *egalitarian* if the agents with homogeneous preferences in $\mathcal{V}$ all are subjected to the same level of risk, for all problems in $\mathcal{P}$. Formally:

Egalitarianism: ψ is called *egalitarian* if $V(\psi_i(C,L,V)) = V(\psi_j(C,L,V))$ for all $i,j \in N$ and all $(C,L,V) \in \mathcal{P}$.

This means that the advocated notion of egalitarianism agrees with the Dutta-Ray (1989) interpretation, ideally yielding Lorenz-dominant allocations with respect to the amounts of the cost to be rationed. We will see how the vectors of transfers $t = \psi(C,L,V)(0)$ play a key role in fine-tuning basic solutions to egalitarian solutions. And, particularly, we will need the possibility of negative cost allocations to accomplish this. Note, however, that comonotonicity implies that if $\psi(C,L,V)(0)$ is the zero vector, then allocations are all non-negative (and so in line with the standard literature). In Section 8, the concept of egalitarianism is generalized to the case where we asymmetrically allocate the expected total cost to the agents.

From (1) we get that if ψ satisfies comonotonicity, then for $i \in N$

$$V(\psi_i(C,L,V)) = \mathbb{E}_{\mathbb{Q}^{\psi_i(C,L,V)}}[\psi_i(C,L,V)] = \mathbb{E}_{\mathbb{Q}^C}[\psi_i(C,L,V)].$$

Therefore, any egalitarian and comonotonic solution satisfies

$$V(\psi_i(C,L,V)) = \tfrac{1}{n}V(C), i \in N. \tag{6}$$

This can be taken as additional argument to restrict the attention to V-sufficient constrained cost allocation problems. Note that admissibility and V-sufficiency are obviously necessary for an egalitarian and comonotonic solution to exist.

Remark 1 *If $V(C) = \bar{c}$, then C is deterministic $\mathbb{Q}^C$-almost surely. Then, finding the egalitarian solution, that is unique $\mathbb{Q}^C$-almost surely, is possible and trivial as a result of V-sufficiency. We assume in the sequel that $(C,L,V) \in \mathcal{P}$ is such that $V(C) < \bar{c}$.*

The subclasses of $\mathcal{Z}^{\mathrm{R}}$ introduced in Examples 2-4 are parametrized by the transfers t when the realized total cost is zero. The following shows that a rather mild condition on the rationing method underpinning a standard solution assures existence of the vector t so that the corresponding solution is egalitarian.

Theorem 1 *Consider $\psi \in \mathcal{Z}^{\mathrm{R}}$ and corresponding $\varphi^{V(C)}$ such that $\psi(C,L,V)(c) := t+\varphi^{V(C)}(c, L-t)$ for some $t \in \mathcal{A}(0,L)$ and all $(C,L,V) \in \mathcal{P}$. If $x \mapsto \varphi^{V(C)}(c,x)$ is continuous for all $c \in [0,\bar{c}]$, then $t = t(C,L,V) \in \mathcal{A}(0,L)$ may be chosen such that ψ is egalitarian.*

Theorem 1 provides a condition so that risk equalizing transfers do exist, but which may not be unique. However, for the previously discussed classes of egalitarian and proportional solutions there are unique vectors of transfers that make the solutions egalitarian, as we will show now.

Theorem 2 *The solution $\psi^\alpha \in \mathcal{Z}^{\mathrm{SP}}$ given by $\psi^\alpha(C,L,V)(c) = t + \alpha c$ for all $c \in [0,\bar{c}]$ and $t \in \mathcal{A}(0,L)$ is egalitarian if and only if*

$$t = V(C)(\tfrac{1}{n}\mathbf{e} - \alpha) \text{ and } \alpha \le \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

Corollary 1 *The unique egalitarian solution in* $\mathcal{Z}^{\text{SPR}}$ *is given by* $\psi^{\text{ESP}}(C,L,V) = t + \varphi^{\text{P}}(C, L-t)$ *for*

$$t = \frac{(\frac{1}{n}L(N)\,\mathbf{e} - L)V(C)}{L(N) - V(C)}.$$

Where for the proportional solutions the transfers can be explicitly found, this is not true for the egalitarian solutions based on either φ^{EC} or φ^{CEL}. Still, we have uniqueness of the transfers for the egalitarian solution based on φ^{EC} under a mild additional condition.

Theorem 3 *On the class* $(C,L,V) \in \mathcal{P}$ *such that* $L > \frac{1}{n}V(C)\,\mathbf{e}$*, there is a unique* $\psi^{\text{SEEC}} \in \mathcal{Z}^{\text{R}}$ *defined by* $\psi^{\text{SEEC}}(C,L,V) = t + \varphi^{\text{EC}}(C, L-t)$ *that is egalitarian.*

Remark 2 *If* $L_1 = \frac{1}{n}V(C)$*, then there may be multiple vectors* t *such that* $t + \varphi^{\text{EC}}(C, L-t)$ *is egalitarian. All such cost allocations are* $\mathbb{Q}^C$*-almost surely identical, and so we define* ψ^{SEEC} *as such allocation. In the above theorem we may exchange the condition* $L > \frac{1}{n}V(C)\,\mathbf{e}$ *with the condition that the underlying distribution of* C *has a positive density on* $[0, \bar{c}]$*.*

Example 5 In this example, we describe two solutions ψ that we will characterize in this paper. Let $V(C) = \mathbb{E}_{\mathbb{P}}[C]$, $N = \{1,2,3\}$, $C \sim Un(0,10)$, and $L = (2,3,8)$. So, $V(C) = 5$. It is easily verified that the problem (C,L,V) satisfies admissibility and V-sufficiency.

From Corollary 1, we compute a unique vector of transfers $t \approx (1.46, 0.83, -2.29)$ for the egalitarian solution $\psi^{\text{ESP}}(C,L,V)$. Moreover, we derive a unique vector of transfers $t \approx (0.51, -0.13, -0.38)$ for the egalitarian solution ψ^{SEEC}, defined in Theorem 3. The solutions $\psi^{\text{ESP}}(C,L,V)$ and $\psi^{\text{SEEC}}(C,L,V)$ are displayed in Figures 1 and 2. The solution $\psi^{\text{ESP}}(C,L,V)$ is linear, and the solution $\psi^{\text{SEEC}}(C,L,V)$ is piecewise linear such that marginal contributions due to cost increase are equally shared amongst the agents whose liabilities are not fully attained. We see that the transfers of $\psi^{\text{SEEC}}(C,L,V)$ are closer to $\mathbf{0}$ than the transfers of $\psi^{\text{ESP}}(C,L,V)$. This is the topic of that we study further in Section 7. $\triangledown$

4 Ranking of transfers

Basically, Theorem 1 says that problems arising in the *ex post* cost allocation process due to heterogeneity in liabilities can be repaired using fixed *ex ante* payments, so that the resulting solution is still egalitarian. The idea is that agents with the smaller liabilities are defaulting for higher levels of cost, and should therefore contribute more than the others when they can – at the lower cost levels. Below it is shown that this reverse ordering for transfers automatically follows for the fixed points resulting in Theorem 1, if only the standard solution uses an *order preserving* component $\varphi^{V(C)}$ as in (5):

Order preserving (OP): $\varphi_i^{V(C)}(\cdot, L) \leq \varphi_j^{V(C)}(\cdot, L)$ whenever $L_i \leq L_j$.[7]

This property is tantamount to the property discussed by Aumann and Maschler (1985) for rationing methods – or, here in the framework with costs rather than awards, for mappings $\varphi^{V(C)}$ that use a rationing rule r, i.e., $\varphi^{V(C)}(c, L-t) = r(c, L-t)$.

Theorem 4 *Consider* $\psi \in \mathcal{Z}^{\text{R}}$ *as in* (5) *with order preserving* $\varphi^{V(C)}$*, and assume that* $x \mapsto \varphi^{V(C)}(c,x)$ *is continuous for all* c*. Then for all* $(C,L,V) \in \mathcal{P}$ *we may choose a vector of transfers* $t = \psi(C,L,V)(0)$ *such that* ψ *is egalitarian and* $L_i \leq L_j \implies t_i \geq t_j$ *for all* $i,j \in N$*.*

[7]Note that if $\varphi^{V(C)}$ is order preserving, then it also satisfies the weaker and well-known property of *equal treatment*: $L_i = L_j \implies \varphi_i^{V(C)}(\cdot, L) = \varphi_j^{V(C)}(\cdot, L)$.

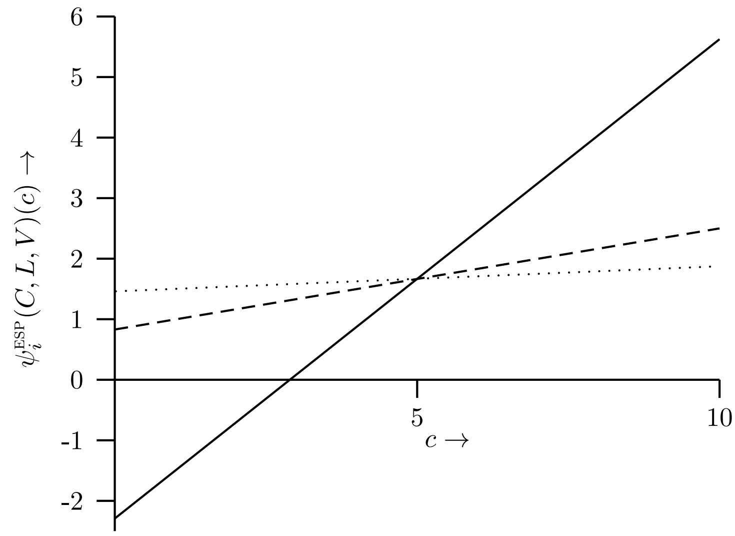

Figure 1: Graphical illustration of the egalitarian proportional solution $\psi^{\text{ESP}}(C, L, V)(c) = t + \varphi^P(c, L-t)$ corresponding to Example 5. The dotted line represents $\psi_1^{\text{ESP}}(C, L, V)(\cdot)$, the dashed line $\psi_2^{\text{ESP}}(C, L, V)(\cdot)$ and the solid line $\psi_3^{\text{ESP}}(C, L, V)(\cdot)$.

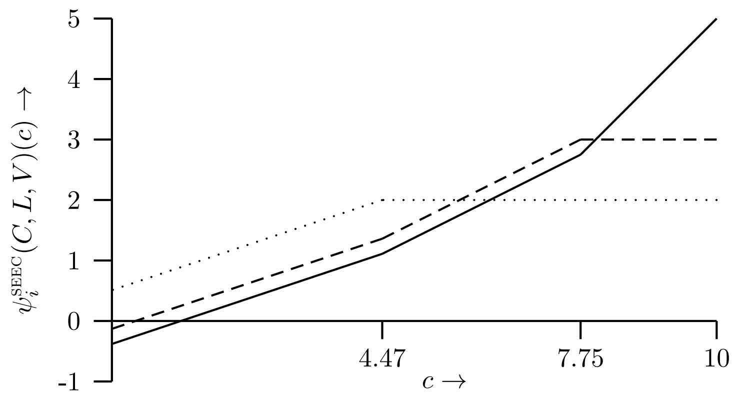

Figure 2: Graphical illustration of the egalitarian solution $\psi^{\text{SEEC}}(C, L, V)(c) = t + \varphi^{\text{EC}}(c, L - t)$ corresponding to Example 5. The dotted line represents $\psi_1^{\text{SEEC}}(C, L, V)(\cdot)$, the dashed line $\psi_2^{\text{SEEC}}(C, L, V)(\cdot)$ and the solid line $\psi_3^{\text{SEEC}}(C, L, V)(\cdot)$. Here, the "cut-off" points where the consecutive agents become tight are given by approximately $(4.47, 7.75, 10)$.

Note that $\varphi^{V(C)}(c, L) = \alpha c$ in the standard proportional solution preserve ordering if $\alpha_i \leq \alpha_j \Leftrightarrow L_i \leq L_j$. Moreover, both φ^{EC} and φ^{CEL} are order preserving.

5 Characterization of standard proportional solutions $\mathcal{Z}^{\mathbf{sp}}$

In this section, we characterize standard proportional solutions. The characterization is based on the following property:

Invariance to expected total Cost-preserving Alternatives (ICA): For all $(C, L, V) \in \mathcal{P}$, if $V^*(C) = V(C)$ then $\psi(C, L, V^*) = \psi(C, L, V)$.

The property ICA states that for all V such that $V(C)$ is the same, the corresponding solution is the same. This property implies that solutions depend on the distribution $\mathbb{Q}^C$ in V only via $V(C)$. Our next result shows that this property characterizes the class of standard proportional solutions.

Theorem 5 *An egalitarian $\psi \in \mathcal{Z}^{\text{sp}}$ satisfies ICA if and only if $\psi = \psi^\alpha$ with*

$$\psi(C, L, V)(0) = V(C)(\tfrac{1}{n}\, e - \alpha), \text{ and } \alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

6 Characterization of constrained egalitarian solutions $\mathcal{Z}^{\mathbf{cec}}$

We impose the following condition on solutions ψ:

Local Symmetry (LS): for all $(C, L, V) \in \mathcal{P}$, $i, j \in N$ and $c \in [0, \bar{c}]$, we have

$$\psi_i\,(C, L, V)\,(c) < L_i, \psi_j\,(C, L, V)\,(c) < L_j \implies \tfrac{\partial}{\partial c}\psi_i\,(C, L, V)\,(c) = \tfrac{\partial}{\partial c}\psi_j\,(C, L, V)\,(c).$$

A solution is locally symmetric if the marginal increases of the cost are accounted for by agents in a similar fashion – as long as the liability constraints are not met. The property LS states that, *ex post*, an additional unit in the cost is shared equally among the agents that can afford it. So, LS is a property that ensures *ex post* egalitarianism among marginal cost changes.[8]

Clearly, a solution $\psi \in \mathcal{Z}$ satisfying LS is comonotonic. In fact, we next show that ψ belongs to the class of constrained equal cost solutions.

Theorem 6 *Solution $\psi \in \mathcal{Z}$ has the property LS if and only if $\psi \in \mathcal{Z}^{\text{CEC}}$, i.e., we have*

$$\psi\,(C, L, V)\,(c) = t + \varphi^{\text{EC}}\,(c, L - t)\,, \text{ for all } c \in [0, \bar{c}],$$

for all $(C, L, V) \in \mathcal{P}$ and $t = \psi(C, L, V)(0) \in \mathcal{A}(0, L)$.

The fact that we may decompose any solution with the property LS in this fashion pins down our solution if we select transfers. The following is a direct consequence of Theorem 3, Remark 2 and Theorem 6.

Theorem 7 *There is a unique egalitarian $\psi \in \mathcal{Z}$ which satisfies LS, and that is $\psi = \psi^{\text{SEEC}}$.*

[8] Instead of considering marginal costs, it is easy to see that all results will still remain valid with a notion of local symmetry that takes care of non-infinitesimal increases of the cost. Rather than working with difference quotients, we prefer to stick to the present formulation for the sake of the exposition.

Remark 3 *In Koster and Boonen (2014), we provide an alternative characterization of* ψ^{SEEC} *based on properties stemming from the rationing literature. It relies on properties mimicking homonymous properties for rationing rules as there are the ideas of consistency, a secured lower bound, and the composition up property. These properties for rationing rules are translated to the stochastic context, where the transfers are already paid up-front. Consistency is a general idea that tells us how a solution behaves over problems with different agent sets (see, e.g., Moulin, 2000; Thomson, 2003). The secured lower bound provides a minimal allocation, and is introduced by Moreno-Ternero and Villar (2004) and populized by, e.g., Dominguez and Thomson (2006). The property composition up states that, given an increase of cost, new cost shares can be allocated from the information of the earlier cost shares alone. This property (and its dual) were introduced by Moulin (1987) and Young (1988). The characterization in Koster and Boonen (2014) relies then on the result of Yeh (2008) for rationing problems.*

7 Minimizing the use of transfers and sufficient liquidity

This paper focusses on solutions of allocating a stochastic cost when asymmetries between liabilities cause problems when at the outset the intention is to share costs equally. Under sufficient liquidity, however, we propose the trivial solution:

Triviality at Sufficient Liquidity (TSL): $L \geq \frac{1}{n}\bar{c}\,\mathbf{e} \implies \psi(C,L,V)(c) = \frac{1}{n}\,c\,\mathbf{e}$ for all $c \in [0,\bar{c}]$.

If every element of the liability vector L is large, then the property TSL requires that the allocated stochastic cost is always the same for all agents. In other words, if $\frac{1}{n}\,c\,\mathbf{e} \in \mathcal{A}(c,L)$ holds for all $c \in [0,\bar{c}]$, then $\psi(C,L,V)(c) = \frac{1}{n}\,c\,\mathbf{e}$. Notice that this condition is automatically satisfied by ψ^{SEEC}. However, TSL does not need to hold for standard proportional solutions, as one may show that TSL is not satisfied by the solutions in $\mathcal{Z}^{\text{SPR}}$.

Example 6 Consider the *constrained proportional solution* ψ^{CP} which is the egalitarian solution in $\mathcal{Z}^{\text{SP}}$ such that

$$\alpha = \frac{\varphi^{\text{EC}}(\bar{c},L) - \frac{1}{n}V(C)\mathbf{e}}{\bar{c} - V(C)}. \tag{7}$$

Note that α satisfies the condition in Theorem 5, so that egalitarianism implies

$$t = \psi^{\text{CP}}(C,L,V)(0) = V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha).$$

Obviously ψ^{CP} satisfies TSL, since $L \geq \frac{1}{n}\,\bar{c}\,\mathbf{e}$ implies $\varphi^{\text{EC}}(\bar{c},L) = \frac{1}{n}\,\bar{c}\,\mathbf{e}$, so that $\alpha = \frac{1}{n}\,\mathbf{e}$ and $t = \mathbf{0}$. The standard proportional rationing solution does not satisfy TSL. $\bigtriangledown$

For two transfer vectors $x, y \in \mathbb{R}$ we say that x Lorenz-dominates y if $\sum_{k=1}^{i} x_k \leq \sum_{k=1}^{i} y_k$ for all $i = 1,\ldots,n$. In that case, we will write $x \succeq_L y$. For any $t \in \mathcal{A}(0,L)$, it holds that $\mathbf{0} \succeq_L t$. So, this criterion aims to select transfers that are "closest" to the zero vector.

Theorem 8 *Consider an egalitarian* $\psi^{\alpha} \in \mathcal{Z}^{\text{SP}}$ *and* $(C,L,V) \in \mathcal{P}$. *Then if* $\alpha_i \leq \alpha_j \Leftrightarrow L_i \leq L_j$, *we have* $\psi^{\text{CP}}(C,L,V)(0) \succeq_L \psi^{\alpha}(C,L,V)(0)$ *for all* $(C,L,V) \in \mathcal{P}$.

This means that among the set of transfers generated by the egalitarian standard proportional solutions, the one corresponding to ψ^{CP} has the minimal largest transfer, and given the set of all minimizing this largest transfers it minimizes the second largest transfer and so on. However, compared to the standard proportional solution ψ^{CP}, the standard egalitarian solution ψ^{SEEC} calculates smaller largest transfers.

Theorem 9 *For all* $(C, L, V) \in \mathcal{P}$*, it holds* $\psi_1^{\text{SEEC}}(C, L, V)(0) \leq \psi_1^{\text{CP}}(C, L, V)(0)$.

Note that by Theorem 4 and the fact that φ^{EC} satisfies OP, the transfer of Agent 1 is the largest among the elements of the transfer vector $\psi^{\text{SEEC}}(C, L, V)(0)$. So, the intuition is that ψ^{SEEC} is better equipped for minimization of the largest transfer than ψ^{CP}, and thus than any egalitarian proportional solution due to Theorem 8. This is due to the local egalitarian feature in combination with a more extensive use of details of V. Recall that ψ^{CP} can only use $V(C)$ as characteristic of V, and specifies a linear function of the costs so that larger transfers are needed to assure that no agent is put at maximal liability before costs reach level $\bar{c}$. Whether this comparison is systematic in the sense that $\psi^{\text{SEEC}}(C, L, V)(0) \succeq_L \psi^{\text{CP}}(C, L, V)(0)$ will be left as an open problem.

Example 7 We return to the problem (C, L, V) of Example 5, but now we study a solution as in Example 4. Thus, we consider the constrained equal loss rationing rule, and the class of constrained equal remaining cost solutions $\mathcal{Z}^{\text{CERC}}$. We find a unique vector of transfers such that $\psi^{\text{SEERC}}(C, L, V)(c) = t + \varphi^{\text{CEL}}(c, L - t)$ is egalitarian. This unique vector of transfers is given by $t \approx (1.67, 1.67, -3.33)$. Then, $\psi_1^{\text{SEERC}}(C, L, V)(c) = \psi_2^{\text{SEERC}}(C, L, V)(c) = 1.67$ and $\psi_3^{\text{SEERC}}(C, L, V)(c) = -3.33 + c$ for all $c \in [0, 10]$. This solution is such that Agents 1 and 2 pay only their deterministic transfers, and do not bear any risk. All risk due to C is borne by Agent 3. This solution is displayed in Figure 3. Note that the transfers are not close to the zero vector $\mathbf{0}$. In general, the solution $\psi^{\text{SEERC}}(C, L, V)$ is piecewise linear.

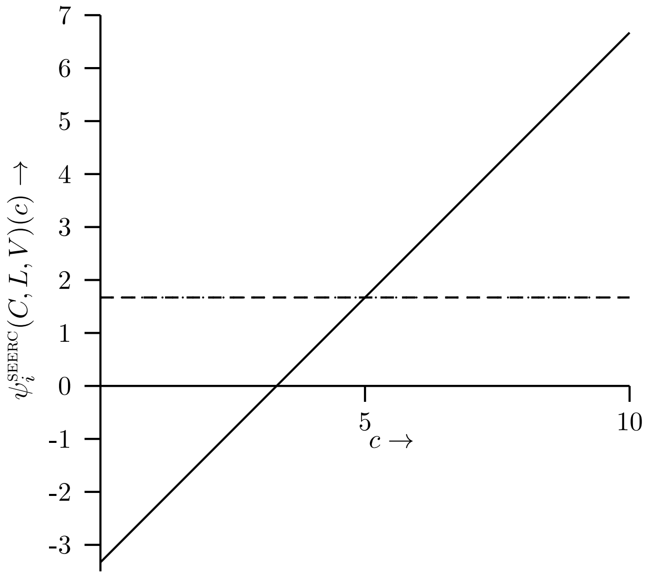

Figure 3: Graphical illustration of the egalitarian solution $\psi^{\text{SEERC}}(C, L, V)(c) = t + \varphi^{cel}(c, L - t)$ corresponding to Example 7. The dash-dotted line represents $\psi_1^{\text{SEERC}}(C, L, V)(\cdot) = \psi_2^{\text{SEERC}}(C, L, V)(\cdot)$, and the solid line $\psi_3^{\text{SEERC}}(C, L, V)(\cdot)$.

It is straightforward to check that the solution $\psi^{\text{SEERC}}(C, L, V)$ does not satisfy TSL. Moreover, note that the solution $\psi^{\text{SEEC}}(C, L, V)$ is be insensitive to choices of L_3, as long as $L_3 \geq 5$. The solution $\psi^{\text{SEERC}}(C, L, V)$ does not satisfy this property. In fact, it is very sensitive to choices of L_3 such that $5 \leq L_3 \leq 8$. For these reasons, we do not discuss this solution any further. $\triangledown$

8 Non-egalitarian solutions

In this section, we study an adaptation of the egalitarianism. We define the following property for a vector $a \in \mathbb{R}_+^N$ such that $a(N) = 1$ and $L \geq aV(C)$:

a-fairness: $V(\psi_i(C, L, V)) = a_i V(C)$ for all $i \in N$.

The property a-fairness is inspired by a desire to allocate the total C in a non-egalitarian manner. Here, the vector a is an exogenously given allocation to the agents as fraction of the expected total cost $V(C)$. For instance, a may be inspired by non-symmetric up-front contributions that the agents received to bear the stochastic cost C. Pazdera et al. (2017) study the related concept of financial fairness, where deterministic costs are allocated according to a given pricing measure. In our setting, V can be interpreted as price, where prices are measured by $V(X) = \mathbb{E}_{\mathbb{Q}}[X]$ for a given probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. Whereas Pazdera et al. (2017) study risk-sharing via Pareto-optimality with expected utilities, we want to characterize a specific solution via the property LS in the case where there are liability constraints.

Suppose we first allocate the amount $\delta_i := (a_i - \frac{1}{n})V(C)$ to every agent $i \in N$. The remaining constrained cost allocation problem is given by $\left(\hat{C}, \hat{L}, V\right) \in \mathcal{P}$, where

$$\hat{C} = C - \sum_{i \in N} \delta_i = C,$$
$$\hat{L} = L - \delta.$$

Then, a-fairness of the solution $\psi(C, L, V)$ is equivalent to V-egalitarianism of the solution $\hat{\psi}(\hat{C}, \hat{L}, V)$, where $\psi(C, L, V) = \delta V(C) + \hat{\psi}(\hat{C}, \hat{L}, V)$. Moreover, V-sufficiency of $(\hat{C}, \hat{L}, V)$ is equivalent to $L \geq aV(C)$ for the original problem (C, L, V). Hence, all results from Sections 3, 5, and 6 can be readily modified so that solutions are a-fair instead of V-egalitarian. For instance, Theorem 7 yields directly the following result.

Corollary 2 *On the class of (C, L, V) and $a \in \mathbb{R}_+^N$ such that $a(N) = 1$, (C, L) is admissible and $L > aV(C)$, there is a unique a-fair solution ψ that satisfies LS, and that is $\psi = \psi^a$ which is defined by*

$$\psi^a(C, L, V)(c) := t + \varphi^{\text{EC}}(c, L - t), \text{ for all } c \in [0, \bar{c}], \tag{8}$$

where $t \in \mathcal{A}(0, L)$ is a unique vector of transfers such that (8) is a-fair.

If there exists an $i \in N$ such that $L_i = a_i V(C)$, a similar remark as Remark 2 applies.

9 Generalization to liability profiles that are not V-sufficient

If the problem (C, L, V) is admissible but not V-sufficient, then it holds that

$$L_1 < \tfrac{1}{n} V(C).$$

In particular we have to conclude that egalitarian solutions do not exist. As the project might be beneficial for all parties, and all parties are needed to support the project, we consider an alternative solution. We want the solution to be as egalitarian as possible. For instance, we can lexicographic minimize the vector $V(\psi_i(C, L, V)), i \in N$. Then, we erase agent 1 that is not V-sufficient from the constrained cost allocation problem. Agent 1 pays the transfer $t = L_1$

regardless of the realization of C. Consider the reduced constrained cost allocation problem $\left(\tilde{C}, \tilde{L}, V\right)$, where

$$\tilde{C} = C - L_1,$$
$$\tilde{L} = L_{N\setminus 1}.$$

Clearly, this problem is again admissible, but is it V-sufficient? If not, i.e., if $L_2 < \frac{1}{n-1}V(C-L_1)$, remove Agent 2 in the same way for the new problem and continue. If the reduced problem becomes V-sufficient, then apply the solution to the reduced constrained cost allocation problem. Note that the cost $\tilde{C}$ may have negative realizations, but are bounded from below by $-L_1$. Then, it is easy to show that our results of this paper still hold true.

Hence, the idea is that where we are limited in our choices, we propose to adopt the idea of egalitarianism under participation constraints as in Dutta and Ray (1989).

10 Conclusion

This paper studies optimal cost allocation under liability constraints. The problem is a generalization of the dual of a rationing problem, where there is a stochastic estate. We aim to share the costs in an *ex ante* egalitarian way, which is defined in terms of an expectation. We show two conditions which are necessary and sufficient to have existence of egalitarian solutions. First, all cost levels can be accounted for. Second, the individual agent's liability equals at least a fair share of the expected total cost. Since there is typically not a unique egalitarian solution, we characterize specific egalitarian solutions by means of properties. The solutions that we propose are analogous to the proportional and constrained equal award rules within a rationing context. Within these classes of solutions, we introduce zero-sum transfers that are paid/received before the realization of the stochastic cost is known; this vector is the solution if the realized cost is zero. There is a unique vector of transfers to guarantee egalitarianism for a class of solutions that is intrinsically connected to continuous rationing methods.

The class of egalitarian proportional solutions is characterized by the property that the solutions may depend only on the information about the total risk. Within this class it is the constrained proportional rule that is consistent with the idea that when transfers are not necessary, the solution will do without these. A stronger result is that this solution determines a Lorenz-dominant vector of transfers within the class of proportional solutions. Then we considered another class of solutions, based on the egalitarian rationing solution known as the constrained equal award rule. Especially we characterize the SEEC solution which is the unique egalitarian solution in this class that can be considered locally egalitarian as well by using any possibility to share marginal costs equally amongst the agents. In particular, this solution uses transfers such that the largest transfer is smaller than the largest transfer used with the constrained proportional solution.

Even if egalitarianism is not possible, we provide a method that adopts the idea of egalitarianism under participation constraints of Dutta and Ray (1989). Finally, we generalize our results with the concept of fairness, which is a condition that relaxes egalitarianism.

We conclude this section with a suggestion for further research. We would like to extend the study of this paper to the setting where the preferences are given by a maximization of expected utility. If individuals minimize expected costs, it can be shown that any cost allocation is Pareto-optimal. For expected utilities, however, a notion of Pareto-optimality is relevant. For strictly concave utility functions, we do not expect multiple cost allocations that are both egalitarian and Pareto-optimal, and so a further characterizations based on the properties Invariance to expected total Cost-preserving Alternatives (ICA) or Local Symmetry (LS) are redundant.

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A Proofs

Proof of Theorem 1: First of all, we will show existence of a vector t with the desired property, by application of egalitarianism:

$$\begin{aligned} V(\psi(C,L,V)) = \tfrac{1}{n}V(C)\,\mathbf{e} \quad &\Leftrightarrow \quad t + V(\varphi^{V(C)}(C,L-t)) = \tfrac{1}{n}V(C)\,\mathbf{e} \\ &\Leftrightarrow \quad t = \tfrac{1}{n}V(C)\mathbf{e} - V(\varphi^{V(C)}(C,L-t)) \\ &\Leftrightarrow \quad L - t = L - \tfrac{1}{n}V(C)\,\mathbf{e} + V(\varphi^{V(C)}(C,L-t)) \end{aligned}$$

Define $F:\Delta\to\Delta$ by $F(x) = L - \frac{1}{n}V(C)\,\mathbf{e} + V(\varphi^{V(C)}(x,C))$, where $\Delta := \{y \in \mathbb{R}^N_+ : y(N) = L(N), y \geq L - \frac{1}{n}V(C)\mathbf{e}\}$ is compact and convex. Note that $V(\psi(C,L,V)) = \frac{1}{n}V(C)\,\mathbf{e}$ if and only if $F(L-t) = L-t$, so that for the given combination (C,L,V) we find a desired corresponding vector of transfers through fixed points of F. The function F is continuous on Δ by continuity of φ. Then the existence of a fixed point x^* of F follows by Brouwer's Fixed Point Theorem. So the vector of transfers we are looking for is given by $t^* = L - x^*$. Notice that $L - t^* \geq 0$, since we have

$$t^* = L - x^* \leq L - (L - \tfrac{1}{n}V(C)\,\mathbf{e}) = \tfrac{1}{n}V(C)\,\mathbf{e},$$

and $(C,L,V) \in \mathcal{P}$ satisfies V-sufficiency. Note that indeed t^* depends on (C,L,V). □

Proof of Theorem 2: The vector of transfers t can explicitly be found using egalitarianism, since

$$V(t+\alpha C) = \tfrac{1}{n}V(C)\,\mathbf{e} \quad \Leftrightarrow \quad t + \alpha V(C) = \tfrac{1}{n}V(C) \Leftrightarrow t = V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha)$$

Since $\alpha \geq \mathbf{0}, \alpha(N) = 1$ and V-sufficiency, it holds that $t \in \mathcal{A}(0,L)$. Only we need to have that $\alpha\, c \leq L - t$ for all $c \in [0,\bar{c}]$, so that a sufficient and necessary condition on α is that

$$\alpha\,\bar{c} \leq L - V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha) \Leftrightarrow \alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

□

Proof of Theorem 3: By Theorem 1, it suffices to show that the fixed point of the mapping F is unique. Recall $x = L - t$, where $t = \psi^{\text{SEEC}}(C,L,V)(0)$. Suppose that there are two fixed points, $\tilde{x}$ and $\bar{x}$, so that $\tilde{x} \neq \bar{x}$. Take the smallest index i such that $\tilde{x}_i \neq \bar{x}_i$. Since $\tilde{x}(N) = \bar{x}(N)$, we have without loss of generality that $\tilde{x}_i < \bar{x}_i$ for $i < n$. Since $\tilde{x}$ and $\bar{x}$ are ordered as a result of Theorem 4, we have, with $\tilde{x}_i^+ = \sum_{j<i}\tilde{x}_j + (n-i+1)\tilde{x}_i$,

$$\varphi_i^{\text{EC}}(c,\tilde{x}) = \begin{cases} \varphi_i^{\text{EC}}(c,\bar{x}) & \text{if } c \leq \tilde{x}_i^+, \\ \tilde{x}_i & \text{if } c > \tilde{x}_i^+, \end{cases}$$

and therefore

$$\varphi_i^{\text{EC}}(c,\bar{x}) - \varphi_i^{\text{EC}}(c,\tilde{x}) = \begin{cases} 0 & \text{if } c \leq \tilde{x}_i^+, \\ \min\left\{\frac{c - \tilde{x}_i^+}{n-i+1}, \bar{x}_i - \tilde{x}_i\right\} & \text{if } c > \tilde{x}_i^+. \end{cases}$$

Thus, $\mathbb{Q}^C(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x}) < \bar{x}_i - \tilde{x}_i) \geq \mathbb{Q}^C(C \leq \tilde{x}_i^+)$. Moreover, $\mathbb{Q}^C(C \leq \tilde{x}_i^+) = 0$ holds whenever $\mathbb{Q}^C(\varphi_i^{\text{EC}}(C,\bar{x}) = x_i) = 1$ or, equivalently, $\mathbb{Q}^C(\bar{t}_i + \varphi_i^{\text{EC}}(C,\bar{x}) = L_i) = 1$. So, $\mathbb{Q}^C(C \leq \tilde{x}_i^+) = 0$ whenever $V(\bar{t}_i + \varphi_i^{\text{EC}}(C,\bar{x})) = L_i$. Since $L > \frac{1}{n}V(C)\mathbf{e}$, this is not egalitarian. Thus, $\mathbb{Q}^C(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x}) < \bar{x}_i - \tilde{x}_i) > 0$ and so

$$V(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x})) < \bar{x}_i - \tilde{x}_i = (L_i - \bar{t}_i) - (L_i - \tilde{t}_i) = \tilde{t}_i - \bar{t}_i.$$

But then this leads to contradiction with egalitarianism as

$$\begin{aligned} V(\bar{t}_i + \varphi_i^{\mathrm{EC}}(C,\bar{x})) &= \bar{t}_i + V(\varphi_i^{\mathrm{EC}}(C,\bar{x})) \\ &< \bar{t}_i + (\tilde{t}_i - \bar{t}_i) + V(\varphi_i^{\mathrm{EC}}(C,\tilde{x})) = \tilde{t}_i + V(\varphi_i^{\mathrm{EC}}(C,\tilde{x})). \end{aligned}$$

□

Proof of Theorem 4: Consider again the mapping F that we used in the proof of Theorem 1, but now restricted to $\Delta^* := \{y \in \mathbb{R}_+^N : y_1 \le y_2 \le \cdots \le y_n, y(N) = L(N), y \ge L - \frac{1}{n}V(C)\mathbf{e}\}$, which is compact and convex, and we have $F(\Delta^*) \subseteq \Delta^*$. Then – again by using Brouwer's Fixed Point Theorem – there is at least one fixed point for F, say x^*. If $L_i \le L_j$ (or $i < j$), then $(L - t^*)_i = x_i^* \le x_j^* = (L - t^*)_j$ so that by OP we have $\varphi_i^{V(C)}(c, L - t^*) \le \varphi_j^{V(C)}(c, L - t^*)$. Consequently

$$t_i^* = \tfrac{1}{n}V(C) - V(\varphi_i^{V(C)}(c, L - t^*)) \ge \tfrac{1}{n}V(C) - V(\varphi_j^{V(C)}(c, L - t^*)) = t_j^*.$$

□

We continue with an intermediary result that we need in the proofs of Theorems 5 and 6.

Lemma 1 *If $\psi \in \mathcal{Z}$ satisfies comonotonicity, then the mapping $c \to \psi_i(C,L,V)(c)$ is Lipschitz-continuous.*

Proof of Lemma 1: We will show that $\|\psi(C,L,V)(c_1) - \psi(C,L,V)(c_2)\| \le \sqrt{n}\,|c_1 - c_2|$ for all $c_1, c_2 \in [0,\bar{c}]$. First recall that for fixed L and $c_1, c_2 \in [0,\bar{c}]$, $c_1 < c_2$, we have $\psi(C,L,V)(c_1) \le \psi(C,L,V)(c_2)$. Moreover, since $\psi \in \mathcal{Z}$, it holds that $\sum_{j\in N} \psi_j(C,L,V)(c_1) = c_1$ and $\sum_{j\in N}\psi_j(C,L,V)(c_2) = c_2$, so that for any choice of $i \in N$ we have

$$\psi_i(C,L,V)(c_2) - \psi_i(C,L,V)(c_1) + \sum_{j\ne i}(\psi_j(C,L,V)(c_2) - \psi_j(C,L,V))(c_1) = c_2 - c_1.$$

Now, suppose $\psi_i(C,L,V)(c_2) - \psi_i(C,L,V)(c_1) > c_2 - c_1$. Then, $\sum_{j\ne i}(\psi_j(C,L,V)(c_2) - \psi_j(C,L,V)(c_1)) < 0$, and there must be $k \in N$ with $\psi_k(C,L,V)(c_2) - \psi_k(C,L,V)(c_1) < 0$, which contradicts with the assumption that ψ satisfies comonotonicity. So we have $\psi_i(C,L,V)(c_2) - \psi_i(C,L,V)(c_1) \le c_2 - c_1$ for all $i \in N$.

Then,

$$\begin{aligned} \|\psi(C,L,V)(c_1) - \psi(C,L,V)(c_2)\|^2 &= \sum_{i\in N}(\psi_i(C,L,V)(c_2) - \psi_i(C,L,V)(c_1))^2 \\ &\le \sum_{i\in N}(c_2 - c_1)^2 = n(c_2 - c_1)^2, \end{aligned}$$

which proves our claim by taking the square-root on both sides. □

Proof of Theorem 5: The "if" part is easy, as it basically follows from Theorem 2. So now we will turn to the "only if" part. Since ψ is standard, we may write $\psi(C,L,V)(c) = t + f(c,L,t,V(C))$ with $t = \psi(C,L,V)(0)$. Suppose that V has cumulative distribution function F, and that ψ is egalitarian. Without loss of generality we assume that $\mathbb{Q}^C(c < C \le c_2) > 0$ for all

$0 \leq c_1 < c_2 \leq \bar{c}$, i.e., F is strictly increasing on $[0, \bar{c}]$. Below we will construct V^* with $V^*(C) = V(C)$, so that in turn ICA implies $\psi(C, L, V) = \psi(C, L, V^*)$. Especially we have $t = \psi(C, L, V)(0) = \psi(C, L, V^*)(0)$ so that for all i:

$$V^*(f_i(C, L, t, V^*(C))) = \tfrac{1}{n} V^*(C) - t_i = \tfrac{1}{n} V(C) - t_i = V(f_i(C, L, t, V(C))). \tag{9}$$

We will now show that an agent should then pay a proportion of the realized cost, that is fixed for all realizations. Assume that not each agent pays a fixed proportion of the realized cost after transfers, say agent i. Then we may construct V^* such that (9) is not satisfied, and we have the desired contradiction. For the sake of the exposition we will write $f_i(c, L, t, V(C))$ as $z(c)$. Since $c \mapsto z(c)$ is non-decreasing, it is Lipschitz-continuous due to Lemma 1, and hence differentiable almost everywhere. Since Agent i does not pay a fixed share of the realized cost after transfers, there must be c_1 and c_2 such that $z'(c_1)$ exists, and

- $z'(c_1) > \dfrac{z(c_2) - z(c_1)}{c_2 - c_1}$, or
- $z'(c_1) < \dfrac{z(c_2) - z(c_1)}{c_2 - c_1}$.

We will show a proof for the second case only, as a proof for the first case follows the same lines. So, assume that for small $\epsilon > 0$ and $c \in (c_1, c_1 + \epsilon)$ we have

$$\frac{z(c) - z(c_1)}{c - c_1} < \frac{z(c_2) - z(c_1)}{c_2 - c_1}. \tag{10}$$

We will show that for those cases we may shift some probability mass from $\mathbb{Q}^C$ to $\mathbb{Q}^{*C}$, and obtain a new distribution under which the expected cost share for i increases whilst $V^*(C) = \mathbb{E}_{\mathbb{Q}^{*C}}[C] = \mathbb{E}_{\mathbb{Q}^C}[C] = V(C)$. Then this conflicts with the assumption of an egalitarian solution. Define $F(c) = \mathbb{Q}^C(C \leq c)$ and $F^*(c) = \mathbb{Q}^{*C}(C \leq c)$.
The distribution $\mathbb{Q}^{*C}$ is constructed by removing mass $\Delta_F(\epsilon) := F(c_1+\epsilon) - F(c_1)$ on the interval $[c_1, c_1 + \epsilon]$ and replace it by point masses on c_1 and c_2. In order to keep the expected total cost under the redistribution constant, determine $\delta(\epsilon) \in (0, 1)$ such that

$$\delta(\epsilon)\Delta_F(\epsilon)c_1 + (1 - \delta(\epsilon))\Delta_F(\epsilon)c_2 + \int_{c_1+\epsilon}^{c_2} c\, dF(c) = \int_{c_1}^{c_2} c\, dF(c). \tag{11}$$

Then the cumulative distribution function F^* for $\mathbb{Q}^{*C}$ is specified by

$$F^*(c) = \begin{cases} F(c) & c < c_1 \text{ or } c \geq c_2, \\ F(c_1) + \delta(\epsilon)\Delta_F(\epsilon) & c \in [c_1, c_1 + \epsilon], \\ F(c) - (1 - \delta(\epsilon))\Delta_F(\epsilon) & c \in (c_1 + \epsilon, c_2), \end{cases}$$

and let $V^* \in \mathcal{V}$ be defined by $V^*(X) = \mathbb{E}_{\mathbb{Q}^{*C}}[X]$ for any $X \in \mathcal{L}^\infty$ that is comonotonic with C. Then by (11), we obtain $V^*(C) = V(C)$ and

$$\begin{aligned} V(z(C)) &= \int_0^{\bar{c}} z(c)\, dF(c), \\ V^*(z(C)) &= \int_0^{c_1} z(c)\, dF(c) + \int_{c_2}^{\bar{c}} z(c)\, dF(c) \\ &\quad + \delta(\epsilon)\Delta_F(\epsilon)z(c_1) + \int_{c_1+\epsilon}^{c_2} z(c)\, dF(c) + (1 - \delta(\epsilon))\Delta_F(\epsilon)z(c_2). \end{aligned}$$

It holds

$$
\begin{aligned}
V^*(z(C)) - V(z(C)) &= -\delta(\epsilon)\Delta_F(\epsilon)(z(c_2)-z(c_1)) + \Delta_F(\epsilon)z(c_2) - \int_{c_1}^{c_1+\epsilon} z(c)\,dF(c) \\
&= -\delta(\epsilon)\Delta_F(\epsilon)(z(c_2)-z(c_1)) + \int_{c_1}^{c_1+\epsilon} (z(c_2)-z(c))\,dF(c).
\end{aligned}
$$

Note that from equation (11) we get

$$
\delta(\epsilon)\Delta_F(\epsilon) = \frac{1}{c_2-c_1}\int_{c_1}^{c_1+\epsilon}(c_2-c)\,dF(c)
$$

and thus

$$
\begin{aligned}
V^*(z(C)) - V(z(C)) &= \int_{c_1}^{c_1+\epsilon}(c_2-c)\,dF(c)\cdot\frac{z(c_2)-z(c_1)}{c_2-c_1} + \int_{c_1}^{c_1+\epsilon}(z(c_2)-z(c))\,dF(c) \\
&= \int_{c_1}^{c_1+\epsilon}((c_2-c_1)+(c_1-c))\,dF(c)\cdot\frac{z(c_2)-z(c_1)}{c_2-c_1} \\
&\quad + \int_{c_1}^{c_1+\epsilon}((z(c_2)-z(c_1))+(z(c_1)-z(c)))\,dF(c) \\
&= -\Delta_F(\epsilon)(c_2-c_1)\frac{z(c_2)-z(c_1)}{c_2-c_1} + \int_{c_1}^{c_1+\epsilon}(c-c_1)\,dF(c)\cdot\frac{z(c_2)-z(c_1)}{c_2-c_1} \\
&\quad + \Delta_F(\epsilon)(z(c_2)-z(c_1)) - \int_{c_1}^{c_1+\epsilon}(z(c)-z(c_1))\,dF(c) \\
&= \int_{c_1}^{c_1+\epsilon}(c-c_1)\cdot\frac{z(c_2)-z(c_1)}{c_2-c_1}\,dF(c) - \int_{c_1}^{c_1+\epsilon}(z(c)-z(c_1))\,dF(c) \\
&> \int_{c_1}^{c_1+\epsilon}(c-c_1)\cdot\frac{z(c)-z(c_1)}{c-c_1}\,dF(c) - \int_{c_1}^{c_1+\epsilon}(z-z(c_1))\,dF(c) = 0.
\end{aligned}
$$

This gives the desired contradiction with (9). □

Proof of Theorem 6: Let $(C, L, V) \in \mathcal{P}$. Since solution $\psi \in \mathcal{Z}$ is comonotonic, the mapping $c \mapsto \psi(C, L, V)(c)$ is Lipschitz-continuous due to Lemma 1 and is therefore absolutely continuous. This implies

$$
\psi(C,L,V)(c) - \psi(C,L,V)(0) = \int_0^c \frac{\partial}{\partial s}\psi(C,L,V)(s)ds \text{ for all } c \in [0,\bar{c}]. \tag{12}
$$

Fix $\psi(C,L,V)(0)$, and let $k : N \to N$ be a bijection such that

$$
k(i) \le k(j) \Leftrightarrow L_{k(i)} - \psi_{k(i)}(C,L,V)(0) \le L_{k(j)} - \psi_{k(j)}(C,L,V)(0).
$$

Define iteratively constants $c_1, c_2, \ldots, c_n \in [0, \bar{c}\,]$ by

$$
\begin{aligned}
c_1 &:= n\left(L_{k(1)} - \psi_{k(1)}(C,L,V)(0)\right) \\
\ell &\ge 2 : c_\ell := \sum_{r=1}^{\ell-1}\frac{c_r}{n-r+1} + (n-\ell)\left(L_{k(\ell)} - \psi_{k(\ell)}(C,L,V)(0)\right).
\end{aligned}
$$

By construction, it holds $c_1 \leq c_2 \leq \cdots \leq c_n$. Then, by LS, we can write (12) for $c \in [c_{\ell-1}, c_\ell]$ as

$$\psi_{k(i)}(C,L,V)(c) = \begin{cases} \psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{i} \frac{c_r}{n-r+1} & \text{if } k(i) < \ell, \\ \psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{\ell-1} \frac{c_r}{n-r+1} + \frac{c - c_{\ell-1}}{n-\ell+1} & \text{if } k(i) \geq \ell. \end{cases}$$

Now this may be rewritten for random cost C with realization $c \in [0, \bar{c}]$ as

$$\begin{aligned} \psi_{k(i)}(C,L,V)(c) &= \left(\psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{n} \frac{c_r \wedge c}{n-r+1}\right) \wedge L_{k(i)} \\ &= \psi_{k(i)}(C,L,V)(0) + \left(\sum_{r=1}^{n} \frac{c_r \wedge c}{n-r+1}\right) \wedge \left(L_{k(i)} - \psi_{k(i)}(C,L,V)(0)\right) \\ &= \psi_{k(i)}(C,L,V)(0) + \varphi^{\text{EC}}_{k(i)}(c, L - \psi(C,L,V)(0)). \end{aligned}$$

Hence, $\psi(C,L,V)(c) = t + \varphi^{\text{EC}}(c, L-t)$ for $t = \psi(C,L,V)(0)$. This completes the proof. □

Proof of Theorem 8: If $L \geq \frac{1}{n}\bar{c}\,\mathbf{e}$, then $\varphi^{\text{EC}}(\bar{c}, L) = \frac{1}{n}\bar{c}\,\mathbf{e}$ so that $\alpha = \frac{1}{n}\,\mathbf{e}$ and $\psi^{\text{CP}}(C,L,V)(0) = \mathbf{0}$. Clearly, in $\mathcal{A}(0,L)$ the Lorenz dominant element is $\mathbf{0}$. Now assume that not $L \geq \frac{1}{n}\bar{c}\mathbf{e}$. Then pick $i \in N$ so that

$$\sum_{k<i} L_k + (n-i+1)L_i \leq \bar{c} < \sum_{k \leq i} L_k + (n-i)L_{i+1},$$

which exists due to admissibility. Then

$$\varphi^{\text{EC}}_j(\bar{c}, L) = \begin{cases} L_j & \text{if } j \leq i, \\ \frac{1}{n-i}(\bar{c} - \sum_{k<i} L_k) & \text{if } j > i. \end{cases}$$

Since ψ^α is egalitarian, by Theorem 5 it must hold that

$$\alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}$$

which means that for $j \leq i$

$$\alpha_j \leq \frac{\psi^{\text{SP}}_j(\bar{c}, L) - \frac{1}{n}V(C)}{\bar{c} - V(C)}$$

so that

$$\psi^{\text{SP}}_j(C,L,V)(0) = V(C)\left(\frac{1}{n} - \frac{\varphi^{\text{SP}}_j(\bar{c}, L) - \frac{1}{n}V(C)}{\bar{c} - V(C)}\right) \leq V(C)\left(\frac{1}{n} - \alpha_j\right) = \psi^\alpha_j(C,L,V)(0),$$

and thus we have for all $k = 1, \ldots, i$ that

$$\sum_{j=1}^{k} \psi^{\text{SP}}_j(C,L,V)(0) \leq \sum_{j=1}^{k} \psi^\alpha_j(C,L,V)(0).$$

Now suppose that there exists an $\ell > i$ such that

$$\sum_{j=1}^{\ell} \psi_j^{\text{SP}}(C,L,V)(0) > \sum_{j=1}^{\ell} \psi_j^{\alpha}(C,L,V)(0).$$

Assume without loss of generality that ℓ is the smallest number with this property, so that $1 < \ell < n$. Since α is ordered in the same way as L, a consequence of Theorem 4 is that for $j \geq \ell$

$$\psi_j^{\alpha}(C,L,V)(0) \leq \psi_\ell^{\alpha}(C,L,V)(0) < \psi_\ell^{\text{SP}}(C,L,V)(0).$$

But then this leads to a contradiction, since

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \psi_j^{\alpha}(C,L,V)(0) = \sum_{j\leq\ell} \psi_j^{\alpha}(C,L,V)(0) + \sum_{j>\ell} \psi_j^{\alpha}(C,L,V)(0) \\ &< \sum_{j\leq\ell} \psi_j^{\text{SP}}(C,L,V)(0) + \sum_{j>\ell} \psi_\ell^{\text{SP}}(C,L,V)(0) = 0. \end{aligned}$$

This means that ℓ cannot exist, and that for all $k = 1, \ldots, n$ we have

$$\sum_{j=1}^{k} \psi_j^{\text{SP}}(C,L,V)(0) \leq \sum_{j=1}^{k} \psi_j^{\alpha}(C,L,V)(0),$$

which concludes the proof. □

Proof of Theorem 9: Let $(C,L,V) \in \mathcal{P}$. Suppose that $\psi_1^{\text{SEEC}}(C,L,V)(0) > \psi_1^{\text{CP}}(C,L,V)(0)$. If $L_1 \geq \frac{1}{n}\bar{c}$, then by TSL that $\psi_1^{\text{SEEC}}(C,L,V)(0) = \psi_1^{\text{CP}}(C,L,V)(0) = 0$, which is a contradiction. Assume $L_1 < \frac{1}{n}\bar{c}$. By LS, we have that $\frac{\partial}{\partial c}\psi_1^{\text{SEEC}}(C,L,V)(c) \geq \frac{1}{n}$ whenever $\psi_1^{\text{SEEC}}(C,L,V)(c) < L_1$. Moreover,

$$\begin{aligned} \frac{\partial}{\partial c}\psi_1^{\text{CP}}(C,L,V)(c) = \alpha_1 &= \frac{\varphi_1^{\text{EC}}(\bar{c},L) - \frac{1}{n}V(C)}{\bar{c} - V(C)} \\ &= \frac{L_1 - \frac{1}{n}V(C)}{\bar{c} - V(C)} < \frac{\frac{1}{n}\bar{c} - \frac{1}{n}V(C)}{\bar{c} - V(C)} = \frac{1}{n}, \end{aligned}$$

for all $c \in [0,\bar{c}]$, and $\psi_1^{\text{CP}}(C,L,V)(c) < L_1$ for all $c \in [0,\bar{c})$. Thus, $\psi_1^{\text{SEEC}}(C,L,V)(c) > \psi_1^{\text{CP}}(C,L,V)(c)$ for all $c \in [0,\bar{c})$. Thus, since $V(C) < \bar{c}$, we have $V(\psi_1^{\text{SEEC}}(C,L,V)) > V(\psi_1^{\text{CP}}(C,L,V))$, which is a contradiction with egalitarianism of ψ^{SEEC} and ψ^{CP}. □