Differential games of public investment: Markovian best responses in the general case *

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Abstract

We define a differential game of public investment with a discontinuous Markovian strategy space. The best response correspondence for the game is well-behaved: a best response exists and uniquely maps almost all profiles of opponents' strategies back to the strategy space. Our chosen strategy space thus makes the differential game well-formed as a static game in Markovian strategies, resolving a long-standing open problem in the literature. We provide a user-friendly necessary and sufficient condition for constructing the best response. Our methods do not require specific functional forms. Our theory has general applications, including to problems of noncooperative control of stock pollutants, harvesting of natural resources, and joint investment problems.

1 Introduction

The dynamic public goods game is an important economic problem which shows up in different settings, including joint investment projects between firms, allocation of effort among members of a team, harvesting renewable resources under common access, and non-cooperative mitigation of climate change. As with infinite-horizon dynamic games in general, these games typically admit the possibility of multiple equilibria—even under Markovian strategies, which condition each player's investment flow on the current state of the accumulating capital stock only. A natural question, with both positive and normative implications, is to ask what the entire set of Markov-perfect Nash equilibria (MPE) is. Except for special cases, the extant literature has not been able to address this issue.

We develop methods which enable progress on this question. We leverage the tractability of a continuous-time framework, which allows us to focus on the local properties of value

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functions, without having to know the global properties.[1] We study a differential game,[2] making two contributions. First, we extend the space of Markovian strategies to include strategies that are discontinuous in the state variable, and show that this yields a well-defined best response correspondence, putting differential games on a conceptually sound foundation. Second, we give a concrete necessary and sufficient condition which can be used to construct a best response.

We restrict the players to use Markovian strategies, or policy rules, so that the control $a(t)$ is given as a function ϕ of the scalar state variable $y(t)$, that is $a(t) = \phi(y(t))$. The appropriate choice of strategy space—the set from which ϕ can be chosen—has long been an open question in differential games (Başar and Olsder, 1982; Fudenberg and Tirole, 1991; Dockner et al., 2000). To understand why, note that computing the payoffs in a differential game requires the determination of a trajectory which is a solution to the state dynamics $\dot{y}(t) = f(y(t), a(t))$. Under Markovian strategies, if the function $f(x, q)$ and the strategies $\phi(x)$ are Lipschitz-continuous, then a unique classical solution trajectory exists. However, for a large class of models the optimal response to Lipschitz-continuous dynamics feature indifference initial states, often called Skiba points, at which there are multiple optimal solutions. In such a situation, the best response cannot be described as a Lipschitz-continuous function of the state (Skiba, 1978; Wagener, 2003), and the best-response correspondence does not map the space of Lipschitz-continuous functions back to itself.

We therefore allow the players to use discontinuous Markovian strategies: player i's policy rule ϕ_i is selected from a space $\mathscr{S}_i$ of functions with a finite number of kinks or discontinuities. We cannot apply to the Picard-Lindelöf theorem on existence and uniqueness of solutions to differential equations. Hence, following Barles et al. (2013, 2014), we use a generalised solution concept to discontinuous dynamics, adapting the payoffs accordingly.

Our Theorem 1 shows that the differential game is well-formed: the optimal control chosen by player i to any profile of other players' strategies $\phi_{-i} \in \mathscr{S}_{-i} \backslash \mathscr{E}$ exists and can be described as a Markovian policy rule $\phi_i \in \mathscr{S}_i$. The exceptional set $\mathscr{E}$, for which the best response does not map to $\mathscr{S}_i$, is small: loosely speaking, it is shown to be a 'shy' set, that is, an infinite-dimensional analogue of a zero-measure set. We give a sufficient condition for identifying combinations of the other players' strategies that belong to $\mathscr{E}$.

The best-response correspondence thus maps $\mathscr{S}_{-i}$ into $\mathscr{S}_i$ so that, modulo exceptional cases, the static game of choosing Markovian policy rules has a complete payoff matrix, in that all strategy profiles induce a vector of payoffs; each player has a best response in $\mathscr{S}_i$; and each player can choose any strategy in $\mathscr{S}_i$ independently of the strategies chosen simultaneously by the other players.[3]

[1]These benefits mirror recent advances in continuous-time macroeconomics (Achdou et al., 2014, 2022; Brunnermeier and Sannikov, 2016), or in the literature on dynamic games and contracts (e.g. Sannikov, 2008; Faingold and Sannikov, 2011).

[2]Dockner et al. (2000) and Başar and Zaccour (2018) contain extensive overviews of differential games.

[3]The existing literature has often used an admissibility criterion on strategy profiles, which implies

Our Theorem 2 is practical, as it gives a user-friendly necessary and sufficient condition for a best response. While its proof is technical, this condition is easy to use in applied settings. We provide the result under general conditions, not requiring the use of particular functional forms.

In a companion paper, we use these results to analyse the fixed points of the best response correspondence, the Nash equilibria in Markovian strategies. In particular, we show how our methods can be used to construct the entire set of symmetric equilibria with finitely many discontinuities to the canonical problem of noncooperative mitigation of climate change (van der Ploeg and de Zeeuw, 1992; Dockner and Long, 1993) under a general functional specification. The main result of this companion paper is that the type of equilibrium most commonly discussed in the literature is Pareto-dominated by all other symmetric equilibria.[4] This raises questions about the importance of this equilibrium—a focal point in the literature—from both positive and normative perspectives. In the paper, we also construct the Pareto-dominant equilibria and obtain meaningful intuition for them. There is a large literature of other applications; we believe it is worthwhile to also take a second look at these, using our methods. Moreover, these methods can also shed light on asymmetric equilibria, something the existing literature has largely ignored (but see De Frutos and Martín-Herrán (2018)).

We use two primary tools in our analysis. The first is dynamic programming in the guise of the theory of viscosity solutions (Bardi and Capuzzo-Dolcetta, 2008). We apply viscosity theory to optimal control under discontinuous dynamics, building on the results by Barles et al. (2013, 2014), who consider exogenous discontinuities in dynamics. These methods allow us to construct the value function to a player's problem. We also rely on the theory of nonlinear dynamical systems to show that the best response is Markovian. Crucially, our ultimate goal is to understand equilibria, in which strategies—including any discontinuities—are endogenous. This means we cannot rule out complicated cases *a priori*, but have to deal with them in our analysis.

The present paper makes two contributions. First, we put the theory of Markov-perfect equilibria in differential games on a sound theoretical footing, as our specification makes the best-response correspondence well-behaved (at least when the state variable is a scalar). This issue has been an open problem for decades (Başar and Olsder, 1982; see also Fudenberg and Tirole 1991; Dockner et al. 2000). Our results demonstrate the usefulness of continuous-time methods in deriving novel and general results in the analysis of dynamic strategic interactions. Multiplicity of Markovian equilibria is also present in discrete-time dynamic games, but their analysis in general is typically quite difficult. Our results flow from the fact that, in continuous time, the value function can be analysed and constructed using local information only.

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that the set of allowed strategies depends on strategies chosen by other players. See e.g. Dockner et al. (2000).

[4]In a linear-quadratic framework, this would be the linear strategy obtained by the guess-and-verify method.

Second, our paper helps consolidate and clarify the long literature on multiple MPE in differential games, starting with Tsutsui and Mino (1990) and Dockner and Long (1993), and subsequently continued by e.g. Dockner and Sorger (1996), Sorger (1998), Rubio and Casino (2002), Rowat (2007), and Dockner and Wagener (2014).

The rest of the paper proceeds as follows. Section 2 sets up the basic model. Section 3 gives an example of the kind of situations our methods are designed to handle. Section 4 develops the conceptual framework of the paper. The main results, existence of the best response and its characterisation, are given in Section 5. Proofs are relegated to the Appendix.

2 Model

Time is continuous and runs to infinity: $t \in [0, \infty)$. The state space $\mathcal{X} = [x_{\min}, x_{\max}]$ is a compact interval in $\mathbb{R}$. There are N players, indexed by $i \in \{1, \dots, N\}$. The interior of a set S is denoted $\mathring{S}$; its boundary is $\partial S = S \backslash \mathring{S}$.

Player i has access to an *action variable* $q_i \in \mathcal{Q}_i \subset \mathbb{R}$ through an *action schedule* $a_i : [0, \infty) \to \mathcal{Q}_i$. We assume the *control set* $\mathcal{Q}_i = [q_{i,\ell}, q_{i,u}]$ to be nonempty, convex and compact,[5] and action schedules to be measurable functions. We collect actions and action schedules into vectors $q = (q_1, \dots, q_N)$ and $a(t) = (a_1(t), \dots, a_N(t))$. We use the notation $q_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N)$ and we write $q = (q_i, q_{-i})$. Similarly, we write $\mathcal{Q}_{-i} = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_{i-1} \times \mathcal{Q}_{i+1} \times \dots \times \mathcal{Q}_N$. A *Markovian strategy* for player i is a map $\phi_i : \mathcal{X} \to Q_i$.

The state evolution depends on the current state, but not on calendar time: given a vector of action schedules a, the differential equation governing the state evolution is

$$\dot{y}(t) = f(y(t), a(t)). \tag{1}$$

A function $y : [0, \infty) \to \mathcal{X}$ satisfying $y(0) = x$ and (1) almost everywhere is a *classical trajectory*, and (y, a) a *classical trajectory–action pair*: these notions will be extended below. We distinguish between state and action variables x and q, and state trajectories y and action schedules a.

A function defined on an open set is *real analytic* if for any point in the set it can be represented, in a nonempty neighbourhood of the point, as a convergent power series with real coefficients. In this article we say that a function $\psi(x)$ is *piecewise real analytic*, if it is real analytic at all points, excepting a finite number of discontinuities, and such that the function and its derivative have finite limits as x approaches a discontinuity.

[5]We could allow multivariate controls as in Dockner and Wagener (2014); our results would follow, given additional assumptions along the way. Our key insights are best conveyed without such complications.

Assumption 1. *The function $f(x,q)$ is continuous, real analytic in x and q, and satisfies $f_{q_i} > 0$ everywhere.*

The state variable is a public good, or public bad, in that the players' action variables reflect their contributions to investing in or disinvesting from it. In what follows, we focus on the latter case. The primitive of the payoffs is the flow felicity function:

Definition 1. *The* felicity *of player i when playing q_i at state x is $u_i(x, q_i)$.*

Assumption 2. *The felicity u_i is real analytic in x and q_i and satisfies $(u_i)_x < 0$ everywhere. Over time, felicity is exponentially discounted at a positive rate $\rho_i > 0$. For all x and q_{-i}, the set $\{(\eta_0, \eta_1) : \eta_0 \leq u_i(x, q_i),\ \eta_1 = f(x, q_i, q_{-i}),\ q_i \in \mathcal{Q}_i\}$ is convex.*

There is a unique maximiser $q_i = q_i^*(x, p, q_{-i})$ of $u_i(x, q_i) + pf(x, q_i, q_{-i})$ in Q_i. Moreover, there are real analytic functions $p_{i,\ell}(x, q_{-i})$ and $p_{i,u}(x, q_{-i})$, such that $q_i^* = q_{i,\ell}$ if $p \leq p_{i,\ell}$, $q_i^* = q_{i,u}$ if $p \geq p_{i,u}$, and q_i^* is a real analytic function of (x, p, q_{-i}) if $p_{i,\ell} < p < p_{i,u}$.

The assumption of piecewise real analyticity covers the vast majority of parametrised models in the literature, which are usually specified using polynomial, rational, algebraic or elementary transcendental functions. We could allow f or the u_i to have singularities in their domain of definition: but such singularities are as a rule only included for reasons of analytical convenience, which do not apply in our context.

To be able to work with a compact state space, we have to specify the boundary behaviour. If $f(x_{\min}, a(t)) < 0$ or $f(x_{\max}, a(t)) > 0$, the state leaves the state space, the system is stopped, and player i receives a boundary value payoff β_i.

Assumption 3. *The boundary payoffs satisfy $\beta_i(x_{\min}) \geq \max_{q_i} u_i(x_{\min}, q_i)/\rho_i$ as well as $\beta_i(x_{\max}) \leq \min_{q_i} u_i(x_{\max}, q_i)/\rho_i$.*

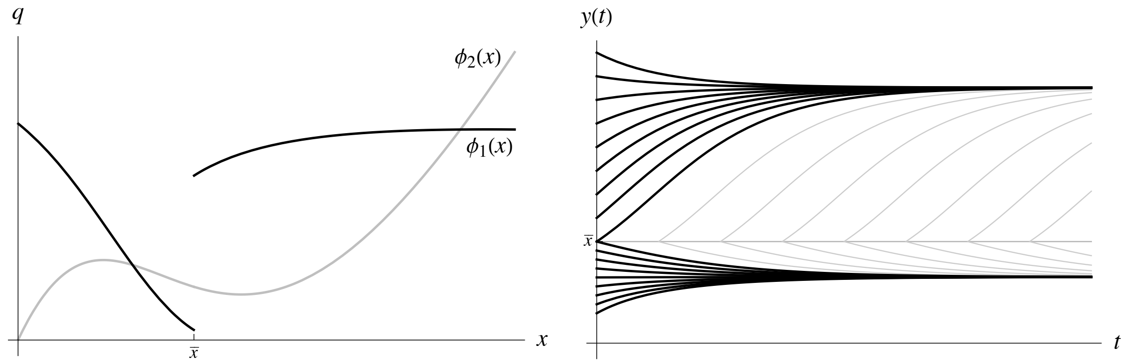
This assumption is used to derive that the state variable is a public bad: to see its necessity, note that if, for instance, the boundary payoff $\beta_i(x_{\max})$ is large, close to $x_{\max}$ the state might be a public good, as it allows the players to reach a high boundary payoff.

Let Θ denote the infimum of the set $\{t > 0 : y(t) \notin \mathcal{X}\}$ if that quantity is finite and ∞ otherwise, and introduce $\mathcal{T} = [0, \Theta]$. In the absence of discontinuities, the overall payoff is given by the sum of future discounted felicity, or

$$\int_0^\Theta \exp(-\rho_i t) u_i(y(t), a_i(t))\, \mathrm{d}t + \exp(-\rho_i \Theta)\beta_i(y(\Theta)). \tag{2}$$

For the payoffs to be consistent with the fundamentals of the model when strategies can be discontinuous, we will require a richer description of the payoffs for situations in which there is no classical solution to the dynamics given by equation (1). We thus postpone the full payoff specification until Section 4.3.

The basic set-up is one of dynamic public investment. Adjusting the signs of the partial derivatives of the felicity function, or modifying the dynamics, will allow the model to be interpreted, for instance, as one of joint investment into a common project with depreciating capital, or as a model of renewable resource exploitation.



(a) Discontinuous best response ϕ_1

(b) Many trajectories starting at $\bar{x}$

Figure 1:

3 Discontinuous strategies

We now discuss a simple example to motivate the contribution of this paper.

Consider the model of the previous section with $N = 2$, specifying $u_i(x, q_i) = \alpha_0 q_i - \alpha_1 q_i^2/2 - \gamma x^2/2$ and $f(x, q) = q_1 + q_2 - \delta x$. This is the canonical transboundary pollution model (van der Ploeg and de Zeeuw, 1992; Dockner and Long, 1993).

We are interested in Markovian strategies, which determine players' action schedules according to $a_i(t) = \phi_i(y(t))$ for policy rules ϕ_i. A Markov-perfect Nash equilibrium is a pair (ϕ_1, ϕ_2) such that each ϕ_i is a best response to ϕ_{-i}, starting from any initial state $x \in X$.[6]

Consider the situation that ϕ_1 is the best response to ϕ_2. To compute the payoffs given by equation (2), we need to determine the trajectory $y(t)$ induced by equation (1) and the pair (ϕ_1, ϕ_2); that is, the solution to

$$\dot{y}(t) = \phi_1(y(t)) + \phi_2(y(t)) - \delta y(t), \qquad y(0) = x \tag{3}$$

for any $x \in X$. By the Picard-Lindelöf theorem, a unique classical solution is guaranteed to exist if the right-hand side $f^{\phi}(x) = f(x, \phi_1(x), \phi_2(x))$ is Lipschitz continuous in x.

Lipschitz continuity of the best response ϕ_1 however fails to obtain in general. Take for instance $\phi_2(x) = \beta x - x^2/(1 + x^2)$, which is clearly Lipschitz-continuous. Then player 1 faces an optimisation problem with concave-convex dynamics, the solution of which is known, for an open set of parameters, to feature an "indifference" or "Skiba" point $\bar{x}$ (Wagener, 2003). This is a discontinuity of the optimal policy function $q_1 = \phi_1(x)$ of player 1, see Figure 1(a). The resulting dynamics have two locally stable steady states x_1^s and x_2^s: which is optimally reached depends on the initial state. For $x \leq \bar{x}$, the optimal trajectory satisfies $y(t) \to x_1^s$, and for $x \geq \bar{x}$, $y(t) \to x_2^s$. The Markovian best response does not give rise to a unique solution trajectory at the initial point $x = \bar{x}$. In other words, there exist perfectly standard optimisation problems such that the best response

[6]We will be more precise in defining the equilibrium in the next section.

to Lipschitz-continuous strategies does not exist in the space of Lipschitz-continuous functions, and such that there are for some initial states multiple solution trajectories.

In the following we allow the players to use discontinuous strategies. More precisely, the strategies can have a finite number of jump discontinuities. Everywhere else, they are real analytic, and the strategies and their first derivatives have finite one-sided limits when approaching a discontinuity. Both existence and uniqueness of solutions are now problematic.

Two new situations arise. First, the 'push-push' situation, when $f^{\phi}_{-} \equiv \lim_{x\uparrow\bar{x}} f^{\phi}(x) > 0$, $f^{\phi}_{+} \equiv \lim_{x\downarrow\bar{x}} f^{\phi}(x) < 0$ and $f^{\phi}(\bar{x}) \neq 0$. The natural solution in the neighbourhood of $\bar{x}$ is for the solution to reach $\bar{x}$ in finite time t_1 and remain there. However, there is no classical solution which satisfies the dynamics (3) for almost all $t > t_1$. This is for instance realised by the strategies $\phi_i(x) = \frac{1}{2}\delta(x+1)$ for $x \leq 1$ and $\phi_i(x) = \frac{1}{2}\delta(x-1)$ for $x > 1$, $i = 1, 2$.

The second new situation is 'pull-pull', for which $f^{\phi}_{-} < 0$ and $f^{\phi}_{+} > 0$. This arises in the situation of Figure 1(a). There are two classical solutions with initial state $\bar{x}$. As pull-pull is the mirror image of push-push, there is moreover a continuum of 'irregular' natural solutions, indexed by a parameter $t_1 \geq 0$, such that $y(t) = \bar{x}$ for $0 \leq t \leq t_1$ and $y(t) \neq \bar{x}$ for $t > t_1$. These are illustrated in Figure 1(b).

In what follows, we specify strategy spaces $\mathscr{S}_i$, extend the notion of a solution to the dynamics, adapting payoffs accordingly, so that best responses to opponents' strategies in $\mathscr{S}_{-i}$ almost always exist and belong to $\mathscr{S}_i$.

Başar and Olsder (1982, Section 5.3) note that "non-Lipschitz strategies cannot easily be put into a rigorous mathematical framework". Typically, the complications arising from the possibility of non-Lipschitz strategies are assumed away by either requiring strategies to be Lipschitz continuous, or with an admissibility requirement that strategy *profiles* which lead to pull-pull or push-push dynamics are not admitted. The latter approach implies that the strategies player i can choose depend on the strategies chosen by the other players. Our approach allows the space of admissible strategy profiles to simply be the product set of individual strategy spaces, as is standard in game theory, while allowing for non-Lipschitz strategies.

4 Markov-perfect Nash equilibrium

In this section, we describe the strategy spaces, set up an individual player's optimisation problem and define Markovian best responses. Finally, we present the equilibrium concept.

4.1 Markovian strategies

The players use Markovian strategies $\phi_i : \mathcal{X} \to \mathcal{Q}_i$, conditioning their actions on the current state variable only, in the following precise sense. To every strategy is associated an *adapted covering* $\mathscr{X}$ of $\mathcal{X}$ by a finite number of closed intervals that have non-empty and mutually disjunct interiors; restricted to one of these covering intervals, the function ϕ_i is real analytic on the interior of the interval. The restricted function and its derivative can be continuously extended to the covering interval. Informally, a function ϕ_i is constructed of sections of real analytic functions, but with the possibility of discontinuities where two adjacent sections are pieced together; at such an interface the value of ϕ_i is not defined, but one-sided derivatives exist. The set of such strategies is denoted $\mathscr{S}_i$.

In the remainder of this subsection, we fix an $(N-1)$-tuple ϕ_{-i} of Markovian strategies. When using Markovian strategies, players set their actions as $a_i(t) = \phi_i(y(t))$. Given an action schedule a_i, and a strategy profile ϕ_{-i}, the system evolves according to

$$\dot{y}(t) = f_i(y(t), a_i(t)) = f(y(t), a_i(t), \phi_{-i}(y(t))), \tag{4}$$

where we introduced the dynamics $f_i(x, q_i) = f(x, q_i, \phi_{-i}(x))$ facing player i. In optimal control problems, it can occur that the optimal policy is described by a discontinuous Markovian policy rule. Hence, when there are several players present, the best response of a player may be a discontinuous Markovian strategy. A description of the game therefore has to take into account the possibility that discontinuous strategies are being played.

When Markovian strategies are not required to be continuous, f_i may have discontinuities, so that the evolution equation (4) may not have classical solutions, or may have a multiplicity of solutions. In the remainder of this section we generalise our notion of solution and describe how payoffs are adapted to that notion.

4.2 Coverings and dynamics

Fix an integer $J > 0$ and points $x_{\min} = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_{J-1} < \bar{x}_J = x_{\max}$: the $\bar{x}_j$ are the possible locations of the discontinuities. Introduce a *covering* $\mathscr{X} = \mathscr{X}(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_J) = \{\mathcal{X}_j\}_{j=1}^J$ of $\mathcal{X}$, where the $\mathcal{X}_j$ are the closed intervals $[\bar{x}_{j-1}, \bar{x}_j]$. Then the state space is the union of the $\mathcal{X}_j$ as $\mathcal{X} = \bigcup_{j=1}^J \mathcal{X}_j$. For $j \in \{1, \ldots, J-1\}$, the *interface* $\mathcal{I}_j$ between $\mathcal{X}_j$ and $\mathcal{X}_{j+1}$ is the intersection $\mathcal{X}_j \cap \mathcal{X}_{j+1} = \{\bar{x}_j\}$. The set $\mathcal{I} = \bigcup_{j=1}^{J-1} \mathcal{I}_j$ is the union of all interfaces.

Let $\mathscr{F}^n_{\mathscr{X}}$ be the space of functions $\phi : \mathcal{X} \to \mathbb{R}^n$ with interface points given by $\mathscr{X}$. Precisely $\phi \in \mathscr{F}^n_{\mathscr{X}}$ if for each j, the restriction ϕ_j of ϕ to $\mathring{\mathcal{X}}_j$ is a real analytic function and ϕ_j and its derivative ϕ'_j can be extended continuously to the closed interval $\mathcal{X}_j$. Two functions $\phi, \psi \in \mathscr{F}^n_{\mathscr{X}}$ are considered to be identical if they coincide on all open intervals $\mathring{\mathcal{X}}_j$. The space $\mathscr{F}^n$ is the union of all the $\mathscr{F}^n_{\mathscr{X}}$ for different number and locations of interfaces

$$\mathscr{F}^n = \bigcup_{J=1}^{\infty} \bigcup_{\bar{x}_0 < \ldots < \bar{x}_J} \mathscr{F}^n_{\mathscr{X}(\bar{x}_0,\ldots,\bar{x}_J)}.$$

In terms of these spaces, the set of full strategy profiles is given as

$$\mathscr{S} = \left\{\phi \in \mathscr{F}^N \,:\, \phi(x) \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_N \text{ for all } x \in \mathcal{X}\right\}.$$

The strategy spaces $\mathscr{S}_i$ for player i and $\mathscr{S}_{-i}$ for all players except player i are defined analogously, with Q_i respectively Q_{-i} replacing $Q_1 \times \ldots \times \mathcal{Q}_N$. We also declare strategy spaces $\mathscr{S}_{\mathscr{X}} = \mathscr{S} \cap \mathscr{F}^N_{\mathscr{X}}$ with given interface points.

Definition 2. *A* Markovian strategy *of player i is a function $\phi_i \in \mathscr{S}_i$. A* (full) strategy profile *is an N-tuple of Markovian strategies $\phi \equiv (\phi_1, \ldots, \phi_N) \in \mathscr{S}$. The strategy profile of all players except player i is denoted $\phi_{-i} = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N) \in \mathscr{S}_{-i}$.*

The *local dynamics* for player i are given, for $j \in \{1, \ldots, J\}$ and $x \in \mathcal{X}_j$, by

$$f_{i,j}(x, q_{i,j}) = f(x, q_{i,j}, \phi_{-i,j}(x)),$$

where $q_{i,j}$ is the *local action*. Note that $f_{i,j}$ is conditional on $\phi_{-i,j}$; we do not explicitly indicate this in order to avoid notational clutter.

We introduce $F_{i,j}(x) = f_{i,j}(x, \mathcal{Q}_i)$ for $x \in \mathcal{X}_j$ and $F_{i,j}(x) = \emptyset$ for $x \in \mathcal{X} \backslash \mathcal{X}_j$. Then we define the set-valued map $F_i : \mathcal{X} \rightsquigarrow \mathbb{R}$ in terms of the $F_{i,j}(x)$ as

$$F_i(x) = \overline{\text{co}} \left(\bigcup_{j=1}^{J} F_{i,j}(x) \right).$$

Using F_i we can formulate the announced extension of our original notion of trajectory.

Definition 3. *Given a strategy profile $\phi_{-i} \in \mathscr{S}_{-i}$ and an state $x \in \mathcal{X}$, a* (state) trajectory *of F_i with initial state x is an absolutely continuous function $y : \mathcal{T} \to \mathcal{X}$ such that $y(0) = x$ and $\dot{y}(t) \in F_i(y(t))$ for almost all $t \in \mathcal{T}$.*

Formulating the dynamics in terms of F_i hides the actions. We give an equivalent formulation that shows them. The *indicator function* $\mathbf{1}_S$ of a set S is given as $\mathbf{1}_S(x) = 1$ if $x \in S$ and $\mathbf{1}_S(x) = 0$ if $x \notin S$.

Definition 4. *The* dynamics *for player i are a function $\mathbf{f}_i : \mathcal{X} \times \mathcal{Q}_i^J \to \mathbb{R}$ given as*

$$\mathbf{f}_i(x, q_i) = \begin{cases} \sum_{j=1}^{J} \mathbf{1}_{\mathcal{X}_j}(x) f_{i,j}(x, q_{i,j}) & \text{if } x \in \mathcal{X} \backslash \mathcal{I}, \\ \mu_{i,j}(q_i) f_{i,j,-} + (1 - \mu_{i,j}(q_i)) f_{i,j,+} & \text{if } x = \bar{x}_j \in \mathcal{I}, \end{cases}$$

where $(f_{i,j,-}, f_{i,j,+}) = (f_{i,j}(\bar{x}_j, q_{i,j}), f_{i,j+1}(\bar{x}_j, q_{i,j+1}))$ and $\mu_{i,j}(q_i) = f_{i,j,+}/(f_{i,j,+} - f_{i,j,-})$ if $f_{i,j,+} f_{i,j,-} \leq 0$ and $(f_{i,j,+}, f_{i,j,-}) \neq (0, 0)$, and $\mu_{i,j}(q_i) = 0$ otherwise.

Note that $\mu_{i,j}(q_i)$ is chosen such that $\mathbf{f}_i(\bar{x}_j, q_i) = 0$ for the push-push situation $f_{i,j,-} \geq 0$ and $f_{i,j,+} \leq 0$ as well as for the pull-pull situation $f_{i,j,-} \leq 0$ and $f_{i,j,+} \geq 0$.

Definition 5. *An* action schedule *of player i is a vector-valued function*

$$a_i(t) = (a_{i,1}(t), \dots, a_{i,J}(t)),$$

with $a_{i,j} \in L^\infty(0, \infty; \mathcal{Q}_i)$ the local action schedules*. The set of action schedules of player i is denoted $\mathcal{A}_i$. If y is a state trajectory such that*

$$\dot{y}(t) = \mathbf{f}_i(y(t), a_i(t)) \tag{5}$$

for almost all $t \in \mathcal{T}$, then (y, a_i) is called a trajectory–action pair*.*

The local action schedule $a_{i,j}$ is the active action schedule if the trajectory y is in the relevant part $\mathcal{X}_j$ of the state space.

The next proposition is a selection result stating that every state trajectory is generated by some action schedule. It is a direct corollary of Barles et al. (2013, Theorem 2.1).

Proposition 4.1. *If y is a state trajectory, there exists an action schedule a_i such that (y, a_i) is a trajectory–action pair.*

The following result is a converse to Proposition 4.1: every action schedule generates a trajectory. The argument is straightforward and therefore omitted.

Proposition 4.2. *For every $\phi_{-i} \in \mathscr{S}_{-i}$, $x \in \mathcal{X}$, and $a_i \in \mathcal{A}_i$, there is a state trajectory y with initial state x such that (y, a_i) is a trajectory–action pair.*

Specifying an action schedule and an initial state does not uniquely determine a state evolution: if the initial state is at an interface, and the two active actions are pulling the state away from the interface, then both a trajectory that remains at the interface for a positive amount of time and a trajectory that veers away immediately are valid state trajectories. Such a pull-pull situation that goes on for a positive amount of time is inherently unstable and would be immediately resolved by the slightest perturbation. We call trajectories that do not display this behaviour 'regular'.

Definition 6. *Let (y, a_i) be a trajectory–action pair. If for almost all $t \in \mathcal{T}$ such that $y(t) \in \mathcal{I}_j$ for some $j \in \{1, \dots, J-1\}$ we have*

$$f_{i,j}(y(t), a_{i,j}(t)) \geq 0 \quad \textit{and} \quad f_{i,j+1}(y(t), a_{i,j+1}(t)) \leq 0, \tag{6}$$

then the trajectory–action pair is called regular*.*

Given x, a_i and ϕ_{-i}, the set of all trajectories y such that (y, a_i) is a trajectory–action pair, respectively a regular trajectory–action pair, is denoted $\mathcal{Y}_{x,a_i,\phi_{-i}}$, respectively $\mathcal{Y}^{\mathrm{reg}}_{x,a_i,\phi_{-i}}$.

4.3 Payoffs and boundary conditions

Analogously to the local dynamics, we define for $j \in \{1, \dots, J\}$ and $x \in \mathcal{X}_j$ the *local flow payoffs* $u_{i,j}(x, q_{i,j}) = u_i(x, q_{i,j})$, which are equal to the flow payoffs if $x \in \mathring{\mathcal{X}}_j$. At an interface, the payoffs are given by a weighted average of the left hand and the right hand payoff, where the weights are the same as for the dynamics.

Definition 7. *The* flow payoff *for player i is a function $\mathbf{u}_i : \mathcal{X} \times \mathcal{Q}_i^J \to \mathbb{R}$ defined as*

$$\mathbf{u}_i(x, q_i) = \begin{cases} \sum_{j=1}^J \mathbf{1}_{\mathcal{X}_j}(x) u_{i,j}(x, q_{i,j}) & \text{if } x \in \mathcal{X} \backslash \mathcal{I}, \\ \mu_{i,j}(q_i) u_{i,j}(x, q_{i,j}) + (1 - \mu_{i,j}(q_i)) u_{i,j+1}(x, q_{i,j+1}) & \text{if } x = \bar{x}_j \in \mathcal{I}. \end{cases}$$

Total welfare is integrated discounted felicity:

Definition 8. *Given a trajectory–action pair (y, a_i) with initial state x, the* total payoff *from the pair for player i is given by*

$$U_i(y, a_i) = \int_0^\Theta \exp(-\rho_i t) \mathbf{u}_i(y(t), a_i(t)) \, \mathrm{d}t + \exp(-\rho_i \Theta) \beta_i(y(\Theta)).$$

The value *at the initial state x of the profile ϕ_{-i} to player i is*

$$V_i(x) = \sup_{\mathcal{A}_i} \sup_{\mathcal{Y}_{x,a_i,\phi_{-i}}} U_i(y, a_i)$$

where the first supremum is taken over the action schedules a_i, and the second over the set of trajectories y for player i, given x, a_i, and ϕ_{-i}.

The regular value V_i^{reg} *is defined analogously, with the set $\mathcal{Y}_{x,a_i,\phi_{-i}}$ of trajectories replaced by the set $\mathcal{Y}^{\text{reg}}_{x,a_i,\phi_{-i}}$ of regular trajectories.*

An action schedule a_i for which the supremum is realised is called a best response of player i*. The set of all trajectories, respectively all regular trajectories, that are associated to a best response is denoted $\mathcal{Y}^*_{x,\phi_{-i}}$, respectively $\mathcal{Y}^{\text{reg},*}_{x,\phi_{-i}}$.*

As a consequence of Proposition 4.2, the value V_i is finite for all x. An important technical result will be to show that the condition $(u_i)_x < 0$ implies that V_i and V_i^{reg} are identical.

4.4 Markovian best responses and MPE

We next consider which trajectory–action pairs are compatible with Markov strategy profiles. Let therefore a full strategy profile ϕ be given, as well as a covering $\mathscr{X}$ adapted to it. The evolution equation reads then as

$$\dot{y}(t) = f(y(t), \phi(y(t))) \equiv f^\phi(y(t)). \tag{7}$$

As before, solutions are not well-defined at discontinuities of f^ϕ. A solution, or state trajectory, of (7) is therefore defined as an absolutely continuous function $y(t)$ that satisfies the differential inclusion

$$\dot{y}(t) \in F^\phi(y(t)) \tag{8}$$

almost everywhere, where the set-valued map F^ϕ is given as $F^\phi(x) = \{f^\phi(x)\}$ if x is not an interface and $F^\phi(x) = \overline{\text{co}}\{f_j^\phi(x), f_{j+1}^\phi(x)\}$ for $x = \bar{x}_j$. The differential inclusion reduces to (7) if $y(t)$ is not at an interface point, whereas at an interface $x = \bar{x}_j$ we have

$$\begin{aligned}\dot{y}(t) = \mu_{i,j}^\phi(x) f(y(t), \phi_{1,j}(x), \ldots, \phi_{N,j}(x)) \\ + (1 - \mu_{i,j}^\phi(x)) f(y(t), \phi_{1,j+1}(x), \ldots, \phi_{N,j+1}(x)),\end{aligned}$$

where we have introduced the *feedback weights* $\mu_{i,j}^\phi(x) = \mu_{i,j}(\phi_i(x))$.

The standard feedback requirement

$$a_{i,j}(t) = \phi_{i,j}(y(t)) \tag{9}$$

for all $j \in \{1, \ldots, J\}$, and for almost all $t \in \mathcal{T}$ such that $y(t) \in \mathcal{X}_j$, is sufficient to ensure that a trajectory–action (y, a_i) pair of (5) has the property that y is a state trajectory of (7).

Definition 9. *Given an initial state x, a full strategy profile ϕ, and a state trajectory y of* (7) *with $y(0) = x$, a* Markovian action schedule $a_i = a_i^\phi$ *induced by ϕ is an action schedule such that* (9) *holds almost everywhere. The set of Markovian action schedules for player i, initial state x, and induced by ϕ is denoted $\mathcal{MA}_{i,x,\phi}$.*

A full strategy profile ϕ and an initial state x uniquely specify the resulting state trajectory, except at pull-pull interfaces $\mathcal{I}_j$ where $f_{i,j}(x, \phi_{i,j}(x)) \leq 0$ and $f_{i,j+1}(x, \phi_{i,j+1}(x)) \geq 0$, with at least one of the inequalities strict. At such points, there are infinitely many trajectories that remain at the interface for an initial time interval of positive length, before moving either to the right or to the left. Even restricting to regular trajectories does not fully eliminate the multiplicity: if both inequalities are strict, there is one regular trajectory that moves immediately to the right, and another that moves immediately to the left.

We now define player i's payoffs and optimal trajectories when restricted to Markovian action schedules.

Definition 10. *Given an initial state $x \in \mathcal{X}$ and a strategy profile $\phi_{-i} \in \mathscr{S}_{-i}$ of the remaining players, the* value of the strategy ϕ_i *to player i is*

$$V_i^\phi(x) = \sup_{\mathcal{MA}_{i,x,\phi}} \sup_{\mathcal{Y}_{x,a_i^\phi,\phi_{-i}}} U(y, a_i^\phi). \tag{10}$$

Player i's best response is Markovian if, for any initial state x, the player cannot do better than choose a Markovian action schedule:

Definition 11. *A* Markovian best response *by player i to a strategy profile ϕ_{-i} is a strategy ϕ_i such that $V_i(x) = V_i^{\phi}(x)$ for all $x \in \mathcal{X}$.*

Given an initial state x, a strategy profile ϕ_{-i}, and a Markovian best response ϕ_i, a Markovian best response trajectory *for player i is a trajectory y with $y(0) = x$, such that, if a_i^{ϕ} is the action schedule induced by ϕ, the pair (y, a_i^{ϕ}) realises the supremum in* (10)*. The set of Markovian best response trajectories for player i is denoted by $\mathcal{MT}^*_{i,x,\phi}$.*

Finally, we define the game and our equilibrium concept.

Definition 12. *The tuple $\Gamma = (N, \mathcal{X}, \mathcal{Q}_1, \ldots, \mathcal{Q}_N, f, u_1, \ldots, u_N, \rho_1, \ldots, \rho_N)$ defines a* differential game*. If $\mathcal{Q}_i = \mathcal{Q}_j$, $u_i = u_j$ and $\rho_i = \rho_j$ for all i, j, the game is* symmetric*.*

Definition 13. *A* stationary Markov-perfect Nash equilibrium*, or MPE, of the differential game Γ is a strategy profile $\phi \in \mathscr{S}$ such that, first, for any player i, the strategy ϕ_i is a Markovian best response to ϕ_{-i} and, second, the set of* Markovian equilibrium trajectories *$\mathcal{Y}^*_{x,\phi} = \bigcap_{i=1}^N \mathcal{Y}^*_{i,x,\phi_{-i}} \cap \mathcal{MT}^*_{i,x,\phi}$ is nonempty for every $x \in \mathcal{X}$.*

An MPE is continuous *if all ϕ_i are continuous; otherwise it is* discontinuous*. A* symmetric *MPE is an MPE of a symmetric differential game such that $\phi_i = \phi_j$ for all i, j.*

The set of equilibrium trajectories $\mathcal{Y}^*_{x,\phi}$ may contain multiple elements. This gives rise to a problem of trajectory selection, akin to equilibrium selection: different players could choose different trajectories that are consistent with the same strategy profile ϕ. We sidestep this question by assuming that the players are able to coordinate on a joint best response trajectory, which indeed always exists for a symmetric MPE.

5 Results

This section states the main results of our article. The first result shows the well-behavedness of the best-response correspondence for almost all strategy profiles. We start by making precise what we mean "almost all".

Let S be a subset of a complete metric linear space $\mathscr{V}$[7]. The set S is *nowhere dense* if its complement is open and dense; it is *shy* if there exists a Borel set S' containing S and a

[7]The space $\mathscr{F}^n_{\mathscr{X}}$ is a complete metric linear space: a metric is constructed as follows. For a compact set $K \subset \mathcal{X}_j$, let $|\phi|_K = \inf\{C : \max_{x \in K} |\phi^{(k)}(x)| \leq C^{1+k} k!\}$. Let moreover $|\phi|_{j,\infty} = \max_{x \in \mathcal{X}_j} |\phi_j(x)|$ be the max-norm on $\mathcal{X}_j$, where ϕ_j is the extension to $\mathcal{X}_j$ of the restriction of ϕ to $\mathring{\mathcal{X}}_j$. Let $K_{n,j} = [\bar{x}_{j-1} + 1/n, \bar{x}_j - 1/n]$. Introduce the distance

$$d_j(\phi, \psi) = \max\left\{ |\phi - \psi|_{j,\infty}, |\phi' - \psi'|_{j,\infty}, \sum_{n=1}^{\infty} 2^{-n} \frac{|\phi - \psi|_{K_{n,j}}}{1 + |\phi - \psi|_{K_{n,j}}} \right\}.$$

Then $d(\phi, \psi) = \max_{1 \leq j \leq J} d_j(\phi, \psi)$ is a metric on $\mathscr{F}^n_{\mathscr{X}}$, and $\mathscr{F}^n_{\mathscr{X}}$ is complete with respect to d.

measure on $\mathscr{V}$ that takes a finite value on some compact set, such that $\text{meas}(S' - v) = 0$ for all $v \in \mathscr{V}$. Shy sets generalise measure zero sets: if $\mathscr{V}$ is finite dimensional, a set is shy if and only if its Lebesgue measure is 0 (Hunt et al., 1992).

Theorem 1. *Let Γ be a differential game for which Assumptions 1, 2, and 3 hold. For every covering $\mathscr{X}$ there is a nowhere dense and shy set $\mathscr{E} \subset \mathscr{S}_{\mathscr{X},-i}$ such that Markovian best response mapping $\mathscr{B}_i : \mathscr{S}_{\mathscr{X},-i} \backslash \mathscr{E} \to \mathscr{S}_i$ is well-defined: for every strategy profile $\phi_{-i} \in \mathscr{S}_{\mathscr{X},-i} \backslash \mathscr{E}$, there is exactly one Markovian best response $\phi_i \in \mathscr{S}_i$ by player i.*

The theorem is proved in Appendices A–D. Appendix A shows that the value function V_i of player i is non-increasing. Appendix B introduces the notion of viscosity solution to the Hamilton–Jacobi–Bellman equation of player i. In Appendix C, we show first that the value function satisfies a number of additional properties: it is left continuous and continuous everywhere except for a finite number of points which we characterise. Then we show that the value function is the unique viscosity solution to the Hamilton–Jacobi–Bellman equation in the class of functions with these properties. Appendix D shows that V_i is differentiable on a dense set; using centre manifold theory, we strengthen this to piecewise real analytic. From this the result follows.

The theorem shows that our specification of a differential game, and the Markovian strategy space $\mathscr{S}_{\mathscr{X},-i}$, are well-formed in the sense that each player will have a best response in $\mathscr{S}_i$ to any profile of the other players' strategies in the complement of the shy set $\mathscr{E}$. Our specification thus connects the literature on differential games to the standard notion of pure-strategy Nash equilibria in static games, with the 'actions' being policy rules ϕ_i. The payoffs for each player are then defined for all initial states x—this is just the value function—and, for any i, a strategy is preferred if it yields a higher payoff for every initial state. The payoffs and best responses in this policy rule game are well-defined, except in the shy set $\mathscr{E}$.

Another implication of the theorem is that, while we had to set up in Section 4 the rather convoluted technical apparatus for dealing with potentially non-Markovian best responses, ultimately the best responses turn out to be Markovian, so that for applications it suffices to rely on the simpler Markovian best responses and the associated Filippov dynamics (8).

Explicit conditions can be formulated for a strategy profile ϕ_{-i} to be in the domain $\mathscr{S}_{\mathscr{X},-i} \backslash \mathscr{E}$ of the best response map: these are of evident importance for applying Theorem 1. One such condition is given in Appendix D as Corollary 1.

As the best response is piecewise analytic, it can be characterised by classical conditions in the regions of analyticity, and by compatibility conditions at the interfaces. The second main result of this article, Theorem 2, formulates such conditions in terms of the value V_i^{ϕ} of strategy ϕ_i to player i, which is the payoff to player i if the strategy profile ϕ is played. Similarly $f_j^{\phi}(x)$ are the local dynamics under the profile ϕ.

Theorem 2. *Assume the same conditions hold as for Theorem 1, and let $\mathscr{E}$ be the shy sets given by that theorem. Let $\phi \in \mathscr{S}$ be such that, for $\mathscr{X}$ a covering adapted to ϕ_{-i}, we have $\phi_{-i} \in \mathscr{S}_{\mathscr{X},-i} \backslash \mathscr{E}$.*

Then $\phi_i = \mathscr{B}(\phi_{-i})$ if and only if the following hold.

(i) Maximum principle: If $x \in \mathring{\mathfrak{X}}_j$ and V_i^ϕ is differentiable at x, then $\phi_i(x)$ maximises

$$q_i \mapsto u_i(x, q_i) + (V_i^\phi)'(x) f_{i,j}(x, q_i) \qquad \text{on } \mathfrak{Q}.$$

(ii) Monotonicity: V_i^ϕ is decreasing and left-continuous.

(iii) Boundary values: If $x = \bar{x}_0$, either $V_i^\phi(x) \geq \beta_i(x)$ or $f_{i,1}(x, q) \geq 0$ for all $q \in \mathfrak{Q}$; if $x = \bar{x}_J$, either $V_i^\phi(x) \geq \beta_i(x)$ or $f_{i,J}(x, q) \leq 0$ for all $q \in \mathfrak{Q}$.

(iv) Value discontinuities: If V_i^ϕ is not continuous at x, then $x = \bar{x}_j \in \mathfrak{I}$, $f_j^\phi(x) \leq 0$ and $f_{i,j+1}(x, q) \geq 0$ for all $q \in \mathfrak{Q}$.

(v) Value at interface steady states: For $x = \bar{x}_j \in \mathfrak{I}$, let

$$C_{0,j} = \{(q_{i,j}, q_{i,j+1}) \,:\, \mu_{i,j}(q_i) f_{i,j}(x, q_{i,j}) + (1 - \mu_{i,j}(q_i)) f_{i,j+1}(x, q_{i,j+1}) = 0\}\,.$$

Then $\rho V_i^\phi(x) \geq \max_{C_{0,j}} [\mu_{i,j}(q_i) u_i(x, q_{i,j}) + (1 - \mu_{i,j}(q_i)) u_i(x, q_{i,j+1})]$.

(vi) Regularity at strong push–push steady states: If $x = \bar{x}_j \in \mathfrak{I}$ is such that

$$\lim_{z \uparrow x} f_{i,j}^\phi(z) > 0 > \lim_{z \downarrow x} f_{i,j+1}^\phi(z),$$

then V_i^ϕ is differentiable at x.

The conditions can be interpreted. Condition (i) is standard. Condition (ii) follows from the fact that the stock is a public bad, and says that there are no strategic incentives so perverse as to make the stock locally a 'good' for player i. Suppose this were the case and, for intuition, consider flow felicity functions without a bliss point in terms of the control variable. Player i would set the maximal emission rate to grow the stock as fast as possible, at least until the value peaks. But then their flow utility will have been decreasing, as emissions have been constant but damages from the stock have been increasing.

Condition (iii) states that, on the edge of the state space, either a player can exit and take the associated payoff, or exit is impossible.

The remaining conditions place restrictions on the best response where the other players' dynamics are discontinuous. Condition (iv) says that a discontinuity in value is only possible at points where at least one of the other players' strategies is discontinuous, in such a way that player i is unable to control the dynamics back to the region of low stock if they ever end up on the high side of the discontinuity. Condition (ii) then implies the value can only have a downward (not upward) discontinuity at such a point.

Condition (v) ensures that the value at an interface point is at least the value that can be obtained by stabilising the dynamics at that point. Finally, Condition (vi) follows from the fact that value is continuous. If a player's best response is to be pushed strictly towards a stabilisation point, they end up at the same point whether approaching from the left or the right, and very close to the stabilisation point the continuity of the payoffs implies that the marginal value of the stock does not depend on the direction of approach.

The proof of Theorem 2 again uses viscosity theory, and is detailed in Appendix E. It shows that the HJB equation of player i has the player's value function as unique viscosity solution. The necessary conditions placed on player i's strategy ϕ_i then ensure that the function V_i^{ϕ} is a viscosity solution of the HJB equation, and therefore equals the value function V_i. This then establishes that ϕ_i is a Markovian best response.

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Appendix

A Singular and regular value functions are identical

As the remainder of the article treats the dynamic optimisation problem of a single player, given the strategic choices of the other players, we drop the index i, unless we are explicitly referring to the game. Also we assume throughout that Assumptions 1, 2, and 3 hold, without explicitly mentioning the fact.

In this section, we shall show that the value function is decreasing.

A.1 Notation

If $S \subset \mathbb{R}$ is a measurable subset, $|S|$ denotes the Lebesgue measure of S. For bounded continuous functions $h : S \to \mathbb{R}$, we set $\|h\|_\infty = \sup_{z \in S} |h(z)|$.

We have that the local dynamics f_j and the local payoffs u_j are bounded, real analytic on $\mathring{\mathcal{X}}_j$, and can be continuously extended, together with their derivatives, to the closed interval $\mathcal{X}_j$. The global dynamics $\mathbf{f}$ and payoffs $\mathbf{u}$ equal their local counterparts f_j and u_j on $\mathring{\mathcal{X}}_j$ and appropriately weighted convex combinations of f_j and f_{j+1}, respectively u_j and u_{j+1}, on interfaces $\mathcal{I}_j$. We therefore have $\|\mathbf{f}\|_\infty = \|f\|_\infty$ and $\|\mathbf{u}\|_\infty = \|u\|_\infty$.

A.2 Existence of optimal action schedules

Proposition A.1. *For every $x \in \mathcal{X}$, there is a trajectory–action pair (y^*, a^*) such that $y^*(0) = x$ and $V(x) = U(y^*, a^*)$.*

The proof of this proposition is given in Appendix F.1; it follows from Assumption 2.

A.3 Equality of singular and regular value

A sufficient condition for uniqueness of solutions to the Hamilton–Jacobi–Bellman equation is equality of value and regular value function (Barles et al., 2013, 2014).

The object of this section is to show this equality for our context. Our first result is a consequence of the assumption $u_x < 0$: given a fixed emission strategy, payoffs decrease with increasing initial pollution stock levels.

Proposition A.2. *Let (y, a) and $(\tilde{y}, a)$ be trajectory–action pairs with the same action schedule and initial points $x, \tilde{x} \in \mathcal{X}$, $\tilde{x} < x$. If $\tilde{y}(t) \leq y(t)$ for all $t \in \mathcal{T}$, then $U(\tilde{y}, a) > U(y, a)$.*

Proof. This follows from the facts that $\tilde{y}(t) < y(t)$ in a neighbourhood of $t = 0$ and that $\mathbf{u}(x, q, \lambda)$ is decreasing in $x \in \mathcal{X}$. □

This idea is used to show that the value function decreases; details are given in Section F.2.

Proposition A.3. *The value function is decreasing.*

To prove equality of value and regular value function, we are going to exhibit for every trajectory–action pair a regular trajectory–action pair generating an outcome that is at least as good. First we show that such a regular trajectory either almost never is at its initial state, or it is there always.

Proposition A.4. *Let (y, a) be a trajectory–action pair with initial state x. Then there is a trajectory–action pair $(\tilde{y}, \tilde{a})$ such that $U(\tilde{y}, \tilde{a}) \geq U(y, a)$ and either $\tilde{y}(t) > x$ for all $t \in \mathcal{T}$, or $\tilde{y}(t) < x$ for all $t \in \mathcal{T}$, or $\tilde{y}(t) = x$ and $\tilde{a}(t)$ constant for all $t \in \mathcal{T}$.*

The proof is given in Section F.3.

If a trajectory–action pair is always at an interface steady state where the regularity condition (6) is not satisfied, that is, at a 'pull-pull' steady state in the terminology of Barles et al. (2013), there is a second trajectory–action pair with the same action schedule and the same initial condition such that the trajectory is always to the left of that steady state, and such that the pair has a higher total payoff. This is the heart of the following result.

Proposition A.5. *For every non-regular trajectory–action pair there is a regular trajectory–action pair with a higher total payoff.*

The proof consists in replacing all singular pull–pull trajectory segments by regular trajectories going to the left. The details are given in Section F.4.

Proposition A.6. *The value function V and the regular value function V^{reg} are equal.*

Proof. Take $x \in \mathcal{X}$ and find a trajectory–action pair (y, a) such that $y(0) = x$ and $V(x) = U(y, a)$. By Proposition A.5 there is a regular trajectory–action pair $(\tilde{y}, \tilde{a})$ such that $\tilde{y}(0) = x$ and $U(\tilde{y}, \tilde{a}) \geq U(y, a)$. We have that $V(x) = U(y, a) \leq U(\tilde{y}, \tilde{a}) \leq V^{\mathrm{reg}}(x) \leq V(x)$, and hence $V(x) = V^{\mathrm{reg}}(x)$. □

B Viscosity solutions

The value function of the optimisation problem is the unique viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equation. We introduce the appropriate notions.

For $j \in \{1, \dots, J\}$, $x \in \mathcal{X}_j$ and $p \in \mathbb{R}$, the *local Hamilton function* is

$$H_j(x, p) = \max_{q \in \mathcal{Q}} \left[u(x, q) + p f_j(x, q) \right].$$

Assumption 2 implies that the function $q \mapsto u(x, q) + p f_j(x, q)$ has a unique maximiser $q^* = q_j^*(x, p)$ in $\mathcal{Q}$. If $q^* \in \mathring{\mathcal{Q}}$, then $u_q(x, q^*) + p (f_j)_q(x, q^*) = 0$.

We also define local Hamilton functions H_0 and H_{J+1} at the boundary. For this, we introduce functions $f_0(x, q)$, $f_{J+1}(x, q)$ and $n(x)$ as follows. If $x = x_{\min}$, we set $f_0(x, q) = f_1(x, q)$ and $n(x) = -1$; if $x = x_{\max}$, we set $f_{J+1}(x, q) = f_J(x, q)$ and $n(x) = 1$. If for $x \in \partial\mathcal{X}$ and $j \in \{0, J + 1\}$ we have $n(x) f_j(x, q) > 0$ for some $q \in \mathcal{Q}$, then exit from $\mathcal{X}$ is possible at x and we set $H_j(x, p) = \rho\beta(x)$. If exit is not possible at x, we set $H_j(x, p) = -\infty$.

The *Hamilton function* of the optimisation problem is

$$\mathbf{H}(x, p) = \begin{cases} H_j(x, p) & x \in \mathring{\mathcal{X}}_j,\ j \in \{1, \dots, J\}, \\ \max\{H_j(x, p), H_{j+1}(x, p)\} & x \in \mathcal{I}_j,\ j \in \{0, \dots, J\}. \end{cases}$$

Let $C_{0,j} = \{q : \mathbf{f}(\bar{x}_j, q) = 0\}$ be the set of actions stabilising interface point $\bar{x}_j \in \mathcal{I}$. The *interface Hamilton function* is then given as

$$H_j^{\mathcal{I}}(\bar{x}_j) = \max_{q \in C_{0,j}} \left[\mathbf{u}(\bar{x}_j, q) \right].$$

We set $H_j^{\mathcal{I}}(\bar{x}_j) = -\infty$ if the set $C_{0,j}$ is empty.

Let $\mathcal{Z} \subset \mathbb{R}$. For a function $W : \mathcal{Z} \to \mathbb{R}$, the *upper semi-continuous envelope* is

$$W^*(x) = \lim_{\delta \downarrow 0} \sup\{W(z) : z \in \mathcal{Z},\ |z - x| \leq \delta\}.$$

The *lower semi-continuous envelope* W_* is defined analogously, with inf replacing sup.

We have that $\mathbf{H}^*(x,p) = \mathbf{H}(x,p)$ for all (x,p), while

$$\mathbf{H}_*(x,p) = \begin{cases} H_j(x,p) & x \in \mathring{\mathcal{X}}_j,\ j \in \{1,\dots,J\}, \\ \min\{H_j(x,p), H_{j+1}(x,p)\} & x \in \mathcal{I}_j,\ j \in \{0,\dots,J\}. \end{cases}$$

The *superdifferential* $D^+W(x)$ of a bounded upper semicontinuous function $W : \mathcal{Z} \to \mathbb{R}$ at a point x is the set

$$D^+W(x) = \left\{ p \in \mathbb{R} : \limsup_{\substack{z \to x \\ z \in \mathcal{Z}}} \frac{W(z) - W(x) - p(z-x)}{|x-z|} \le 0 \right\}.$$

The *subdifferential* $D^-W(x)$ of a bounded lower semicontinuous function W on $\mathcal{Z}$ at x is defined similarly, with sup replaced by inf and $\le$ by $\ge$ (see e.g. Bardi and Capuzzo-Dolcetta, 2008, Chapter V). We have that $p \in D^+W(x)$ if and only if there is a continuously differentiable function ψ such that $\psi'(x) = p$ and $W - \psi$ restricted to $\mathcal{X}$ has a local maximum at x. An analogous characterisation exists for subdifferentials.

Definition 14. *The function $W : \mathcal{X} \to \mathbb{R}$ is a* viscosity supersolution *of the Hamilton–Jacobi–Bellman equation*

$$\rho W(x) - \mathbf{H}(x, W'(x)) = 0 \tag{11}$$

if for all $x \in \mathcal{X}$ and all $p \in D^-W_*(x)$ we have

$$\rho W_*(x) - \mathbf{H}_*(x,p) \ge 0, \tag{12}$$

and if for all $j \in \{1,\dots,J-1\}$ and $x \in \mathcal{I}_j$, we have

$$\rho W_*(x) - H^{\mathcal{I}}(x) \ge 0. \tag{13}$$

The function W is a viscosity subsolution *of* (11) *if for all $x \in \mathcal{X}$ and all $p \in D^+W^*(x)$*

$$\rho W^*(x) - \mathbf{H}^*(x,p) \le 0. \tag{14}$$

Finally, W is a viscosity solution *of* (11) *if it is both a supersolution and a subsolution.*

Note that (13) is only a condition for being a supersolution, not a subsolution.

Theorem B.1. *The value function V is a viscosity solution of the Hamilton–Jacobi–Bellman equation* (11)*.*

Proof. The statement is local, and the proof is a combination of known results. See Proposition III.2.8 of Bardi and Capuzzo-Dolcetta (2008) for (12) and (14) if $x \in \mathring{\mathcal{X}}_j$; Theorem 2.5 of Barles et al. (2013) for the case that $x \in \mathcal{I}_j$, $j \in \{1,\dots,J-1\}$, as well as for (13); and Theorem V.4.13 of Bardi and Capuzzo-Dolcetta (2008) for the boundary case $x \in \mathcal{I}_j$ if $j \in \{0, J\}$. □

C Unicity of solutions

A central question is whether the dynamics are controllable at an interface.

Definition 15. *The dynamics are* left (right) controllable *at a point* $\bar{x}_j \in \mathcal{I} \cup \partial X$*, if there is an open interval* I *containing* 0 *such that*

$$I \subset f_j(\bar{x}_j, \mathcal{Q}) \qquad \Big(I \subset f_{j+1}(\bar{x}_j, \mathcal{Q})\Big).$$

If the dynamics are both left and right controllable at $\bar{x}_j$*, they are* controllable.

If the dynamics are not controllable at $\bar{x}_j$, we introduce the closed sets $\mathcal{X}_{j,-} = \{x \in \mathcal{X} : x \leq \bar{x}_j\}$ and $\mathcal{X}_{j,+} = \{x \in \mathcal{X} : x \geq \bar{x}_j\}$, and we decompose the optimisation problem into two coupled optimisation problems on $\mathcal{X}_{j,-}$ and $\mathcal{X}_{j,+}$ respectively.

Following Barles et al. (2014), we introduce the state-constrained value function V_-^{sc} and V_+^{sc} for, respectively, the optimisation problems where the state restriction $y(t) \in \mathcal{X}_{j,-}$, respectively $y(t) \in \mathcal{X}_{j,+}$, is required to hold for all t.

Additionally, we introduce $V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)$ for the state-constrained value function of the optimisation problem with the restriction $y(t) = \bar{x}_j$ for all t, where only action schedules that satisfy $\mathbf{f}(\bar{x}_j, a) = 0$ are admitted. Finally for $k \in \{-, \mathcal{I}, +\}$ we set $V_k^{\mathrm{sc}} = -\infty$ if there is no trajectory starting at x that satisfies the particular state constraint.

The following result establishes necessary and sufficient conditions for the continuity of the value function interfaces. To formulate it, we introduce the notions of one-sided semi-repellers and semi-attractors.

Definition 16. *A point* $\bar{x}_j \in \mathcal{I} \cup \partial X$ *is a* left semi-repeller *if* $f_j(\bar{x}_j, q) \leq 0$ *for all* $q \in \mathcal{Q}$*, and a* right semi-repeller *if* $f_{j+1}(\bar{x}_j, q) \geq 0$ *for all* $q \in \mathcal{Q}$*.*

Likewise, $\bar{x}_j \in \mathcal{I} \cup \partial X$ *is a* left semi-attractor *if* $f_j(\bar{x}_j, q) \geq 0$ *for all* $q \in \mathcal{Q}$*, and a* right semi-attractor *if* $f_{j+1}(\bar{x}_j, q) \leq 0$ *for all* $q \in \mathcal{Q}$*.*

Proposition C.1. *Let* $\bar{x}_j \in \mathcal{I} \cup \partial \mathcal{X}$ *and* V *the value function. Then the following hold.*

(i) $V(\bar{x}_j) = \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j), V_+^{\mathrm{sc}}(\bar{x}_j)\}$.

(ii) If the dynamics are left (right) controllable at $\bar{x}_j$*, then* V *is left (right) Lipschitz continuous in a left (right) neighbourhood of* $\bar{x}_j$*.*

(iii) If $\bar{x}_j$ *is a right semi-attractor, then* V *is right continuous at* $\bar{x}_j$ *and we have* $V(\bar{x}_j) = \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)\}$.

(iv) If $\bar{x}_j$ *is a left semi-attractor, then* V *is left continuous at* $\bar{x}_j$ *and we have* $V(\bar{x}_j) = \max\{V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j), V_+^{\mathrm{sc}}(\bar{x}_j)\}$.

(v) If $\bar{x}_j$ *is a left semi-repeller, then* V *is left continuous at* $\bar{x}_j$ *and* $V(\bar{x}_j) = V_-^{\mathrm{sc}}(\bar{x}_j)$.

(vi) The value function V *is not continuous at* $\bar{x}_j$ *if and only if* $\bar{x}_j$ *is a right semi-repeller and* $V_+^{\mathrm{sc}}(\bar{x}_j) < \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)\}$.

The asymmetry in the result is a consequence of the fact that $u_x < 0$. The proof of this result is given in Appendix F.5.

The result motivates the following definition.

Definition 17. *Let $\mathscr{X}$ be a covering of $\mathcal{X}$, and let $\mathbf{f}$ be a dynamics defined on $\mathcal{X}$. The class $\mathscr{G}$ consists of functions $W : \mathcal{X} \to \mathbb{R}$ such that*

(i) W *is decreasing;*
(ii) W *is left continuous everywhere;*
(iii) W *is continuous on* $\mathring{\mathcal{X}}_j$ *for all* $j \in \{1, \dots, J\}$*;*
(iv) if W *is not continuous at* $\bar{x}_j$*, then* $j \in \{1, \dots, J-1\}$ *and this point is a right semi-repeller under the dynamics* $\mathbf{f}$*.*

The following result is central.

Theorem C.1. *The value function V is the unique viscosity solution of the Hamilton–Jacobi–Bellman equation* (11) *in the class $\mathscr{G}$.*

The theorem is a consequence of the following comparison result.

Proposition C.2. *Let $v \in \mathscr{G}$ and $w \in \mathscr{G}$ be respectively a supersolution and a subsolution of* (11)*. Then $v(x) \geq w(x)$ for all $x \in \mathcal{X}$ where v and w are continuous.*

To prove this proposition, we need a number of technical results. The first gives the subsolution version of condition (13) for supersolutions. The subsolution version either holds, or an alternative property must be true. The result is adapted from Barles et al. (2013, Theorem 3.3): its proof is given in Section F.6.

Proposition C.3. *Let $w \in \mathscr{G}$ be a subsolution of* (11) *and $\bar{x} = \bar{x}_j \in \mathcal{I}$.*

Then either of the following two statements holds.

A. $\rho w(\bar{x}) - H^{\mathcal{I}}(\bar{x}) \leq 0$

B. *(i) If w is continuous at $\bar{x}$, then there is a constant $\eta > 0$, an index $\ell \in \{j, j+1\}$, and a sequence $x_k \to \bar{x}$ such that $x_k \in \mathcal{X}_\ell$ for all k, $w(x_k) \to w(\bar{x})$ as $k \to \infty$, and for each k there is a trajectory–action pair (y_k, a_k) such that $y_k(0) = x_k$, $y_k(t) \in \mathcal{X}_\ell$ for all $t \in [0, \eta]$ and*

$$w(x_k) \leq \int_0^\eta u(y_k(t), a_k(t)) \exp(-\rho t)\, \mathrm{d}t + w(y_k(\eta)) \exp(-\rho \eta)$$

(ii) If w is not continuous at $\bar{x}$, the previous statement holds with $\ell = j$.

The next result, proved in Section F.7, settles the continuous case of Proposition C.2.

Proposition C.4. *Let $v \in \mathscr{G}$ be a continuous viscosity supersolution, and $w \in \mathscr{G}$ a continuous viscosity subsolution of the HJB equation* (11)*. Then $v \geq w$ in $\mathcal{X}$.*

For the discontinuous case, we need the following technical result. Its proof is a standard viscosity test function argument, given in Section F.8.

Proposition C.5. *Let $j \in \{1, \dots, J-1\}$, $\mathcal{X}_{j,+} = \{x \in \mathcal{X} : x \geq \bar{x}_j\}$ and $v, w \in \mathscr{G}$ such that v and w are respectively a supersolution and a subsolution of* (11) *on the set $\mathcal{X}_{j,+} \backslash \{\bar{x}_j\}$, and either v or w is discontinuous at $\bar{x}_j$. Then v and w are also, respectively, a supersolution and a subsolution of* (11) *on $\mathcal{X}_{j,+}$.*

Proof of Proposition C.2. The proof proceeds by induction.

Let $x_{\min} \leq \hat{x}_1 < \hat{x}_2 < \ldots < \hat{x}_{L-1} < x_{\max}$ be the right semi-repellers of the dynamics; then $\hat{x}_\ell = \bar{x}_{i_\ell}$ for $0 \leq i_0 < \ldots < i_\ell < J$. Introduce for $\ell = 1, 2, \dots, L$ the intervals $\mathcal{X}^{(\ell)} = [\hat{x}_{L-\ell}, x_{\max}]$, as well as extensions v_ℓ, w_ℓ to $\mathcal{X}^{(\ell)}$ of the respective restrictions of v and w to $\mathcal{X}^{(\ell)} \backslash \{\hat{x}_{L-\ell}\}$ such that v_ℓ and w_ℓ are right continuous at $\hat{x}_{L-\ell}$: as $v, w \in \mathscr{G}$ these extensions are well-defined and unique. Proposition C.5 implies that v_ℓ and w_ℓ are respectively a supersolution and a subsolution of (11) on $\mathcal{X}^{(\ell)}$.

The induction hypothesis is that the inequality $v_\ell(x) \geq w_\ell(x)$ holds for all $x \in \mathcal{X}^{(\ell)}$ where v_ℓ and w_ℓ are continuous.

The functions v_1 and w_1 are continuous on $\mathcal{X}^{(1)}$. Proposition C.4 then implies the induction hypothesis for $\ell = 1$.

Assuming that the hypothesis is true for $\ell \geq 1$, consider $\Delta_{\ell+1} = w_{\ell+1} - v_{\ell+1}$. By the hypothesis, $\Delta_{\ell+1} \leq 0$ on $\mathcal{X}^{(\ell)} \backslash \{\hat{x}_{L-\ell}\}$. If $\Delta_{\ell+1} \leq 0$ on $\mathcal{X}^{(\ell+1)}$, there is nothing left to prove. If not, then $\Delta_{\ell+1}$ takes a positive maximum $M = \Delta_{\ell+1}(\bar{x}) > 0$ at a point $\bar{x} \in [\hat{x}_{L-(\ell+1)}, \hat{x}_{L-\ell}]$, as the interval is compact and $\Delta_{\ell+1}$ restricted to this interval is continuous.

If $\bar{x} \neq \hat{x}_{L-\ell}$, the same arguments used in the proof of Proposition C.4 can be used to derive a contradiction to the statement that $M > 0$. Hence we may assume that $\bar{x} = \hat{x}_{L-\ell}$. As this is an interface point, Proposition C.3, on which Proposition C.4 is based, applies.

If $v_{\ell+1}$ and $w_{\ell+1}$ are continuous at $\bar{x}$, the argument of Proposition C.4 for interface points again produces a contradiction.

If $v_{\ell+1}$ is discontinuous at $\bar{x}$, but $w_{\ell+1}$ is not, we have that $M = w_{\ell+1}(\bar{x}) - v_{\ell+1}(\bar{x}) > 0$ and, since $\Delta_{\ell+1} \leq 0$ if $x > \bar{x}$, also that $\lim_{x \downarrow \bar{x}} (w_{\ell+1}(x) - v_{\ell+1}(x)) \leq 0$, which implies that $v_{\ell+1}(\bar{x}) < \lim_{x \downarrow \bar{x}} v_{\ell+1}(\bar{x})$. But then $v_{\ell+1}$ cannot be an element of $\mathscr{G}$.

Finally, if $w_{\ell+1}$ is discontinuous at $\bar{x}$, the argument of Proposition C.4 for interface points holds again, as in Alternative B the sequence elements x_k satisfy $x_k \leq \bar{x}$ for all k.

We conclude that the induction hypothesis also holds for $\ell + 1$, and therefore for all $1 \leq \ell \leq L$. This completes the proof. □

Proof of Theorem C.1. By Theorem B.1, V is a viscosity solution to (11), and according to Proposition C.1, it is in $\mathscr{G}$.

Assume $W \in \mathscr{G}$ is another viscosity solution to (11). As V is a supersolution and W is a subsolution, by Proposition C.2 we have that $V \geq W$ at all points in $\mathcal{X}$ where V and W are continuous. Interchanging V and W yields also that $V \leq W$ at all points of continuity. As V and W are both left continuous everywhere, this shows that $V = W$ for all $x \in \mathcal{X}$. □

D Existence of the best response map

This section shows that the *best response map* $\phi_i = \mathscr{B}_i(\phi_{-i})$ is well-defined for all profiles ϕ_{-i} with adapted covering $\mathscr{X}$ that are in the complement of a set $\mathscr{E}_{\mathscr{X},-i}$, and it shows that the latter set is 'shy'—small in a topological as well as a measure-theoretical sense. The map $\mathscr{B}_i$ gives the Markovian best response of player i to the strategy profile ϕ_{-i} of the other players. That is, given the profile ϕ_{-i}, the strategy ϕ_i is the feedback strategy such that for every initial state the Markovian action schedule induced by ϕ_i maximises the total payoff U_i, given the dynamics $\mathbf{f}_i$.

The construction starts with a general profile $\phi_{-i} \in \mathscr{S}_{-i}$. In the first step the value function of player i is shown to be differentiable on a dense set of points. The second step improves the regularity: the value function has to be real analytic on non-constant optimal state orbits, and compact intervals not containing interface points intersect with only finitely many of these orbits. Restricted to such a compact interval, the set of points at which the value function is not real analytic is shown to be discrete, hence finite. The associated strategy can fail to be in $\mathscr{S}_i$ only if the points of non-analyticity accumulate on an interface point: the final step of the proof is to show that this only occurs for a shy set of profiles ϕ_{-i}.

D.1 Notations

In this section we work in a fixed interval $\mathcal{X}_j$. We therefore fix $j \in \{1, \dots, J\}$ and drop this index for the sake of readability. Hence, in the whole section, unless announced differently, $f(x,q)$ stands for $f_j(x,q_j)$, which in turn stands for $f_{i,j}(x,q_{i,j})$, etc.

We introduce a number of auxiliary quantities. The functions $p_\ell, p_u : \mathcal{X} \to \mathbb{R}$ are given as

$$p_\ell(x) = -u_q(x,q_\ell)/f_q(x,q_\ell), \quad p_u(x) = -u_q(x,q_u)/f_q(x,q_u).$$

We have sets

$$\mathcal{P}_\ell = \{(x,p) \,:\, p \leq p_\ell(x)\}, \quad \mathcal{P}_u = \{(x,p) \,:\, p \geq p_u(x)\}, \quad \text{and} \quad \mathcal{P}_{\text{int}} = \mathcal{X} \times \mathbb{R} \backslash (\mathcal{P}_\ell \cup \mathcal{P}_u).$$

With these definitions, the maximiser $q = q^*(x,p)$ of $u(x,q) + pf(x,q)$ equals q_ℓ if $(x,p) \in \mathcal{P}_\ell$, q_u if $(x,p) \in \mathcal{P}_u$, and it takes a value in $\mathring{\mathcal{Q}}$ if $(x,p) \in \mathcal{P}_{\text{int}}$.

The boundaries of these sets are the *switching manifolds*

$$\mathcal{S}_\ell = \{(x,p) \,:\, p = p_\ell(x)\}, \quad \mathcal{S}_u = \{(x,p) \,:\, p = p_u(x)\}, \quad \mathcal{S} = \mathcal{S}_\ell \cup \mathcal{S}_u.$$

D.2 Differentiability of the value function in a dense set

We partition the interior of $\mathcal{X}$ into three sets $\mathcal{X}_L(0)$, $\mathcal{X}_M$ and $\mathcal{X}_N$, based on whether $H(x,p)$ has a single minimiser with respect to p or not, and, if not, whether $H(x,p) = \rho V(x)$ is locally solvable for p or not.

Definition 18. *Let $c \geq 0$ and $k = 0, 1, 2 \ldots$. Introduce the sets*

(i) $\mathcal{D}_x = \{x \,:\, V$ *is k times differentiable at* $x\}$*;*

(ii) $\mathcal{X}_L(c) = \{x \,:\, H_p(x,p) < -c$ *if* $p \leq p_\ell(x)$ *and* $H_p(x,p) > c$ *if* $p \geq p_u(x)\}$*;*

(iii) $\mathcal{X}_M = \{x \,:\, \exists$ *a unique p s.t.* $\rho V(x) = H(x,p)$ *and* $H_p(x,p) \neq 0\}$*;*

(iv) $\mathcal{X}_N = \{x \,:\, \exists\, p$ *s.t.* $p \leq p_\ell(x)$ *or* $p \geq p_u(x)$*,* $\rho V(x) = H(x,p)$*, and* $H_p(x,p) = 0\}$*.*

The sets $\mathcal{X}_L(0)$, $\mathcal{X}_M$ and $\mathcal{X}_N$ are mutually disjoint and satisfy $\mathcal{X}_L(0) \cup \mathcal{X}_M \cup \mathcal{X}_n = \mathring{\mathcal{X}}$.

Proposition D.1. *The value function V is differentiable almost everywhere on $\mathcal{X}_L(0)$, and it is real analytic on $\mathcal{X}_M \cup \mathring{\mathcal{X}}_N$.*

Proof. Take $c > 0$. As V is continuous in $\mathring{\mathcal{X}}$ and a supersolution of the Hamilton–Jacobi–Bellman equation, for $x \in \mathcal{X}_L(c)$ and $p \in D^- V(x)$, we have for every $q \in \mathcal{Q}$ that

$$\rho V(x) \geq H(x,p) \geq u(x,q) + p f(x,q),$$

which implies, since $|V(x)| \leq \|u\|_\infty/\rho$ for all x, that

$$p f(x,q) \leq \rho V(x) - u(x,q) \leq 2\|u\|_\infty.$$

Taking $q = q_u$, and using that $f(x,q_u) = H_p(x,p_u(x)) > c$, we find $p < 2\|u\|_\infty/c$. Similarly, for $q = q_\ell$, we have $f(x,q_\ell) = H_p(x,p_\ell(x)) < -c$ and $p > -2\|u\|_\infty/c$. We conclude that if $p \in D^- V(x)$, then $|p| < 2\|u\|_\infty/c$.

Bardi and Capuzzo-Dolcetta (2008, Remark II.5.16) now implies that V is Lipschitz continuous on $\mathcal{X}_L(c)$ with Lipschitz constant $2\|u\|_\infty/c$; Rademacher's theorem (Clarke et al., 1998, Chapter 3, Corollary 4.19) subsequently ensures almost everywhere differentiability of V on $\mathcal{X}_L(c)$, and hence on $\mathcal{X}_L(0) = \bigcup_{c>0} \mathcal{X}_L(c)$, proving the first part of the statement.

Next, consider $x \in \mathcal{X}_M$. By the implicit function theorem the solution $p = \kappa(x,w)$ of $w = H(x,p)$ is locally real analytic.

Assume that $H_p(x,\kappa(x,w)) > 0$, the other situation being similar. As $x \in \mathcal{X}_M$, this implies that $H_p(x,p) \geq 0$ for all p. If $p \in D^+ V(x)$, the subsolution property of V implies

$$H(x,\kappa(x,\rho V(x))) = \rho V(x) \leq H(x,p)$$

and therefore $p \geq \kappa(x, \rho V(x))$, by convexity of $H(x,p)$ in p; similarly, the supersolution property implies for $p \in D^- V(x)$ that $p \leq \kappa(x, \rho V(x))$. Again using Bardi and Capuzzo-Dolcetta (2008, Remark II.5.16), it follows that $V(x)$ is a classical solution of $V'(x) = \kappa(x, \rho V(x))$. Since κ is real analytic, it follows that V is a real analytic solution of $\rho V = H(x, V')$ in $\mathcal{X}_M$.

Finally, for $x \in \mathring{\mathcal{X}}_N$, $\rho V(x) = u(x, q_\ell)$ or $\rho V(x) = u(x, q_u)$, and V is again real analytic. This proves the second part. □

As a corollary of this result, we obtain

Proposition D.2. *The set $\mathcal{D}_1$ is dense in $\mathring{\mathcal{X}}$.*

Proof. If this were not the case, there is a point $\bar{x} \in \partial \mathcal{X}_N$ such that V is not differentiable for any point in an open interval I with positive length containing $\bar{x}$. As $\bar{x}$ is a boundary point of $\mathcal{X}_N$, there is a point $\tilde{x} \in I \backslash \mathcal{X}_N$. Hence the intersection $I \cap (\mathcal{X}_L(0) \cup \mathcal{X}_M)$ is nonempty. But this intersection is open, and therefore it contains a positive measure subset of points where V is differentiable, which is a contradiction. □

D.3 Canonical trajectories

To extend the domain of differentiability of the value function, we show that differentiability is carried forward along optimal orbits by the costate dynamics. This result is closely related to the Pontryagin Maximum Principle in the finite horizon context, the difference being that we here have initial values rather than terminal values for the costate equation. The result, whose proof is given in Section F.9, is an adaptation of Cannarsa and Frankowska (1991, Theorem 3.3) to the present context.

Proposition D.3. *Let (y^*, a^*) be an optimal trajectory–action pair with initial point $x \in \mathcal{D}_1$ and let $T \geq 0$ be such that $y^*(t) \in \mathring{\mathcal{X}}$ for all $0 \leq t \leq T$.*

Let moreover p^* be the solution of

$$\dot{p}(t) = \rho p(t) - u_x(y^*(t), a^*(t)) - p(t) f_x(y^*(t), a^*(t)), \qquad p(0) = V'(x). \tag{15}$$

Then for every $0 \leq t \leq T$ we have that $y^*(t) \in \mathcal{D}_1$, $V'(y^*(t)) = p^*(t)$, and $a^*(t) = q^*(y^*(t), p^*(t))$.

Proposition D.3 can be expressed in the more familiar form that an optimal trajectory necessarily satisfies the *canonical equations*

$$\dot{y}(t) = H_p(y(t), p(t)), \qquad \dot{p}(t) = \rho p(t) - H_x(y(t), p(t)),$$

with $y(0) = x$, $p(0) = V'(x)$. This motivates the following definition:

Definition 19. *The* canonical vector field $X : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^2$ *is given as* $X(x,p) = (H_p(x,p), \rho p - H_x(x,p))$*. A* canonical trajectory *is a trajectory of* X*. An* optimal canonical trajectory *is a canonical trajectory* (y,p) *such that* y *is an optimal trajectory.*

The next proposition, proved in Section F.10, states properties of non-constant optimal trajectories: only at their initial states the value function may be non-differentiable; they are monotone; and they correspond to optimal canonical trajectories, which moreover converge to steady states of X in $\mathcal{X}$, or leave the interval $\mathcal{X}$ $(= \mathcal{X}_j)$.

Proposition D.4. *Let* (y^*, a^*) *be a non-constant optimal trajectory–action pair.*

(i) $y^*(t) \in \mathcal{D}_1$ *for all* $t > 0$ *such that* $y^*(t) \in \mathring{\mathcal{X}}$*;*
(ii) there is a unique initial point (x, p_0) *such that the canonical trajectory* (y,p) *through this point satisfies* $y^*(t) = y(t)$ *for all* $t \geq 0$*;*
(iii) either $\dot{y}^*(t) > 0$ *for all* $t \geq 0$*, or* $\dot{y}^*(t) < 0$ *for all* $t \geq 0$*;*
(iv) either $T = \inf\{t > 0 : y^*(t) \notin \mathcal{X}\}$ *is finite, or there is a point* $(\bar{x}, \bar{p})$ *such that* $(y^*(t), p^*(t)) \to (\bar{x}, \bar{p})$ *as* $t \to \infty$*.*

D.4 Markovian best responses

This section constructs the piecewise real analytic Markovian best response strategy. For the remainder of this section, we reinstate the full indexed notation.

Definition 20. *An* optimal orbit *is an interval* $I \subset \mathcal{X}$ *such that there is* $\mathcal{T} = [0, \infty)$ *or* $\mathcal{T} = \mathbb{R}$ *and a state trajectory* $y : \mathcal{T} \to I$ *with the property that for every* $x \in I$ *there is* $\tau \in \mathcal{T}$ *such that* $y_\tau(t) = y(\tau + t)$ *satisfies* $y_\tau(0) = x$ *and* y_τ *is an optimal trajectory. If I consists of a single point, it is an* optimal steady state*; if I has positive length, it is an* optimal non-constant orbit*.*

The next two results, proved in Section F.11, give the structure of the set of optimal non-constant orbits in $\mathcal{X}_j$: there are at most countably many, and they can only accumulate on the end points of $\mathcal{X}_j$. Moreover, restricted to the interior of a non-constant orbit, a best response exists and is real analytic.

Proposition D.5. *A compact interval* $C \subset \mathring{\mathcal{X}}_j$ *intersects finitely many optimal non-constant orbits.*

Definition 21. *Let* (y,p) *be a canonical trajectory. A* switching point *of this trajectory is a point* $(y(t_0), p(t_0)) \in \mathcal{S}_b$*,* $b \in \{\ell, u\}$*, such that in every neighbourhood of* t_0 *there are* t_1, t_2 *with* $(y(t_1), p(t_1)) \in \mathcal{P}_{int}$ *and* $(y(t_2), p(t_2)) \in \mathcal{P}_b$*. Switching points for non-constant optimal state trajectories and optimal orbits are defined as switching points of their associated canonical trajectories.*

For the remainder of the section, we introduce for an interval $I \subset \mathcal{X}$ the notation $\mathscr{S}_i(I)$ for the strategy space $\mathscr{S}_i$ with the interval $\mathcal{X}$ replaced by I.

Proposition D.6. *If the open interval $I \subset \mathcal{X}$ is part of a non-constant optimal orbit, does not contain switching points and satisfies $\phi_{i,j}(x) = q_i^*(x, V_i'(x))$ for all $x \in I$, then V_i is real analytic on $J = I \cap \mathring{\mathcal{X}}_j$ and $\phi_{i,j} \in \mathscr{S}_i(J)$ for all j.*

By piecing together the results for constant and non-constant optimal orbits, we obtain the existence of Markovian best responses.

Proposition D.7. *Let $C \subset \mathring{\mathcal{X}}_j$ be a compact interval. Then V is piecewise real analytic on C, the function defined as $\phi_{i,j}(x) = q_i^*(x, V_i'(x))$ for all $x \in \mathcal{D}_1 \cap C$ can be extended to a strategy $\phi_{i,j} \in \mathscr{S}_i(C)$, and $\phi_{i,j}$ is the unique Markovian best response to $\phi_{-i,j}$ on C. Each point of non-analyticity of $\phi_{i,j}$ is contained in the closure of some non-constant orbit.*

Proof. By Proposition D.5, the interval C intersects only finitely many non-constant optimal orbits $I_1, \ldots, I_m$. The complement of the intersection consists of finitely many compact intervals $N_1, \ldots, N_n$ containing only optimal steady states.

If $x \in \mathring{N}_k$, then $V_i(x) = u_i(x, \bar{q}_i(x))/\rho$, where $q = \bar{q}_i(x)$ solves $f_{i,j}(x, q) = 0$. By the implicit function theorem, the function $\bar{q}_i$, and hence V_i and $\phi_{i,j}$, are real analytic in this set, showing the last statement of the proposition. Proposition D.6 implies that V_i is real analytic on $\mathring{I}_k \cap \mathring{\mathcal{X}}_j$ and $\phi_{i,j} \in \mathscr{S}_i(\mathring{I}_k \cap \mathring{\mathcal{X}}_j)$ for each k. We therefore conclude that V_i is piecewise real analytic on C and $\phi_{i,j} \in \mathscr{S}_i(C)$.

It is clear that $\phi_{i,j}$ is uniquely determined: it remains to show that it is a best response. Let V_i be differentiable at a point $x \in \mathcal{X}_j$, and let (y^*, a^*) be the optimal trajectory–action pair with initial point x. Then Proposition D.3 implies that

$$a_{i,j}^*(t) = q_{i,j}^*(y^*(t), V_i'(y^*(t)))$$

for all $t \geq 0$. This implies the compatibility condition (9). □

Proof of Theorem 1. Take $\phi_{-i} \in \mathscr{S}_{-i}$, and let $\mathscr{X}$ be a covering adapted to ϕ_{-i}. For the duration of this proof, we write $\mathscr{F} = \mathscr{F}_{\mathscr{X}}^{N-1}$. A profile $\phi_{-i} \in \mathscr{S}_{\mathscr{X},-i} = \mathscr{F} \cap \mathscr{S}_{-i}$ then satisfies $\phi_{-i,j} \in \mathscr{S}_{-i}(\mathcal{X}_j)$. Let $\phi_{i,j} : \mathring{\mathcal{X}}_j \to \mathbb{R}$ be the unique Markovian best response to ϕ_{-i} on compact subsets of $\mathring{\mathcal{X}}_j$, given by Proposition D.7.

If $\phi_{i,j}$ has infinitely many points of non-analyticity, by Propositions D.6 and D.7 they have to be either endpoints of non-constant optimal orbits or switching points. Every compact subinterval of $\mathring{\mathcal{X}}_j$ contains only finitely many of these: hence points of non-analyticity have to accumulate on a boundary point of $\mathcal{X}_j$. In particular, at such a boundary point the canonical vector field either is tangent to $\mathcal{P}_\ell$ or $\mathcal{P}_u$ or vanishes. We shall show that the set $\mathscr{E} \subset \mathscr{S}_{-i}$ of profiles ϕ_{-i} such that $\phi_{i,j}$ has this latter property for some j is shy.

The space $\mathscr{F}$ is a complete metric linear space. Let $w^{(1)}(x) = 1$ and $w^{(2)}(x) = \prod_{j=0}^{J}(x - \bar{x}_j)$ and introduce $\psi^{(k)} \in \mathscr{F}$ as $\psi^{(k)}(x) = (w^{(k)}(x), \ldots, w^{(k)}(x))$, for $k = 1, 2$. Denote by $\mathscr{L}$

the two-dimensional subspace of $\mathscr{F}$ spanned by the $\psi^{(k)}$, and let meas be the Lebesgue measure on $\mathscr{L}$. Let $X_{\phi_{-i,j}}$ be the canonical vector field on $\mathcal{X}_j \times \mathbb{R}$ associated to $\phi_{-i,j}$.

Let $S_0 = \Big\{\phi_{-i} \in \mathscr{F} \,:\, X_{\phi_{-i}} = 0 \text{ at a boundary point of } \mathcal{X}_j\Big\}$. For $b \in \{\ell, u\}$ and $n_b(x)$ the normal vector to $\mathcal{S}_b$ at $(x, p_b(x))$ pointing out of $\mathcal{P}_{\text{int}}$, let

$$SW_b = \Big\{\phi_{-i} \in \mathscr{F} \,:\, n_b \cdot X_{\phi_{-i}} = 0 \text{ at a boundary point of } \mathcal{X}_j\Big\}.$$

Note that $\mathscr{E} \subset S_0 \cup SW_\ell \cup SW_u \equiv \mathscr{N}$.

We prove shyness for SW_b, the proof for S_0 being similar. Take $\phi_{-i} \in \mathscr{S}_{\mathscr{X},-i}$ and $\hat{\phi}_{-i} \in SW_b$. The point $\hat{\phi}_{-i} - \phi_{-i}$ is in $\mathscr{L}$ if $\hat{\phi}_{-i} = \phi_{-i} + \lambda_1\psi^{(1)} + \lambda_2\psi^{(2)}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. We have $\hat{\phi}_{-i} \in SW_b$ if

$$\begin{aligned} 0 = n_b \cdot X_{\hat{\phi}_{-i}} = n_{b,1} f(x, q_i^*, \hat{\phi}_{-i}) + n_{b,2}\Big(\rho p_{i,b}(x) - (u_i)_x(x, q_i^*)\Big) \\ - n_{b,2} p_{i,b}(x)\Big(f_x(x, q_i^*, \hat{\phi}_{-i}) + f_{q_{-i}}(x, q_i^*, \hat{\phi}_{-i})\hat{\phi}'_{-i}\Big). \end{aligned} \tag{16}$$

For $\bar{x} \in \partial\mathcal{X}_j$ we have $(\psi^{(1)})'(\bar{x}) = 0$ and $\psi^{(2)}(\bar{x}) = 0$. Equation (16) reduces to

$$0 = n_b \cdot X_{\phi_{-i}+\lambda_1\psi^{(1)}} - \lambda_2 n_{b,2}(\bar{x}) p_{i,b}(\bar{x}) w'(\bar{x}) \sum_{k\neq i} \frac{\partial f}{\partial q_k}(\bar{x}, q^*(\bar{x}, p_i, b(\bar{x})), \phi_{-i}(\bar{x})). \tag{17}$$

Consider first the situation that $p_{i,b}(\bar{x}) \neq 0$. Since $n_{b,2}(\bar{x})$ and $w'(\bar{x})$ are both nonzero, and $\frac{\partial f}{\partial q_k} > 0$ for all k by Assumption 1, the solutions (λ_1, λ_2) of equation (17) are located on a graph $\lambda_2 = \lambda_2(\lambda_1)$. There are $2J$ such graphs, one for every endpoint of every $\mathcal{X}_j$. Hence the set $SW_b \cap \phi_{-i} + \mathscr{L}$ has measure zero in this case.

If $p_{i,b}(\bar{x}) = 0$, equation (17) reads as $n_{b,1} f(x, q_i^*, \phi_{-i} + \lambda_1 w^{(1)}) - n_{b,2}(u_i)_x(x, q_i^*) = 0$. If $n_{b,1} = 0$, this has no solution by Assumption 2; if $n_{b,1} \neq 0$, by Assumption 1, this equation has a unique constant solution $\lambda_1 = \lambda_1(\lambda_2)$. As before, we infer that $SW_b \cap \phi_{-i} + \mathscr{L}$ has measure zero, and we conclude that SW_b is shy.

Unions and subsets of shy sets are shy (Hunt et al., 1992). Hence $\mathscr{N}$ and $\mathscr{E} \subset \mathscr{N}$ are shy. The complement of a shy set is dense. As $\mathscr{N}$ is closed, it follows that its complement is also open, and that $\mathscr{N}$, and therefore $\mathscr{E}$, are nowhere dense.

Take $\phi_{-i} \in \mathscr{S}_{\mathscr{X},-i} \backslash \mathscr{N}$. Then the strategy profile $\phi_{-i,j}$ has only finitely many points of non-analyticity on $\mathring{\mathcal{X}}_j$. It remains to show that the limit of $\phi_{i,j}(x)$ and $\phi'_{i,j}(x)$ exist as x tends to the boundary of $\mathcal{X}_j$.

Let $I \subset \mathcal{X}_j$ be a maximal open interval on which $\phi_{i,j}$ is real analytic and that contains a boundary point $\bar{x}$ of $\mathcal{X}_j$. Set $\bar{p} = \lim_{x\to\bar{x}, x\in I} V'(x)$. If $(x, V'(x))$ is contained in $\mathcal{P}_b$ for all $x \in I$, $b = \ell, u$, the best response $\phi_{i,j}(x)$ equals q_b and can be extended to a continuously differentiable function on the closure of I.

If $(x, V'(x)) \in \mathcal{P}_{\text{int}}$, and if $\bar{z} = (\bar{x}, \bar{p})$ is not a steady state, we have, since $(X_{\phi_{-i,j}})_1(\bar{x}) \neq 0$, that there is a continuously differentiable extension of the canonical trajectory through

$\bar{z}$ to a neighbourhood of $\bar{z}$, giving rise to a continuously differentiable extension of $\phi_{i,j}$ to $\bar{x}$. If $\bar{z}$ is a steady state, the graph $(x, V'(x))$ is tangent to a, necessarily one-dimensional, eigenspace of $DX_{\phi_{-i,j}}(\bar{z})$ at $\bar{z}$ and can be extended as a continuously differentiable function as well. This shows that $\phi_i \in \mathscr{S}_i$. □

Corollary 1. *Let $\mathscr{P} \subset \mathscr{S}_{-i}$ be the set of profiles ϕ_{-i} such that for every j the canonical vector field $X_{\phi_{-i,j}}$ of player i has only finitely many steady states and finitely many switching points. Then for every $\phi_{-i} \in \mathscr{P}$ there is a unique best response $\phi_i \in \mathscr{S}_i$.*

E Characterisation of the best response map

We continue not to indicate the player index i. This section provides the proof of Theorem 2. Before giving this proof, we need to collect information about V^ϕ.

Introduce for $x \in \mathcal{X}_j$ the functions

$$f_j^\phi(x) = f_j(x, \phi_j(x)), \quad u_j^\phi(x) = u_i(x, \phi_j(x)), \quad H_j^\phi(x,p) = u_j^\phi(x) + p f_j^\phi(x).$$

In terms of these functions, set

$$\mathbf{u}^\phi(x) = \begin{cases} \sum_{j=1}^J \mathbf{1}_{\mathcal{X}_j}(x) u_j^\phi(x), & x \in \mathcal{X} \backslash \mathcal{I}, \\ \mu_j^\phi(x) u_j^\phi(x) + (1 - \mu_j^\phi(x)) u_{j+1}^\phi(x), & x = \bar{x}_j \in \mathcal{I}, \end{cases}$$

and

$$\mathbf{f}^\phi(x) = \begin{cases} \sum_{j=1}^J \mathbf{1}_{\mathcal{X}_j}(x) f_j^\phi(x), & x \in \mathcal{X} \backslash \mathcal{I}, \\ \mu_j^\phi(x) f_j^\phi(x) + (1 - \mu_j^\phi(x)) f_{j+1}^\phi(x), & x = \bar{x}_j \in \mathcal{I}. \end{cases}$$

Then the value V^ϕ accruing to player i under the profile ϕ has the following properties, proved in Appendix F.12.

Proposition E.1. *Assume $\phi \in \mathscr{S}$ and $i \in \{1, \ldots, N\}$. Then there is a covering $\mathscr{X} = \{\mathcal{X}_j\}_{j=1}^J$ of X, with $\mathcal{X}_j = [x_{j-1}, x_j]$, adapted to ϕ, and a finite set $\mathcal{E} \subset \mathcal{X}$, such that the following hold.*

(i) $\|V^\phi\|_\infty \le \max\{\|u\|_\infty/\rho, \|\beta\|_\infty\}$.

(ii) If $x = \bar{x}_0$, then either $V^\phi(x) = \beta(x)$ or $f_1^\phi(x) \ge 0$. Likewise, if $x = \bar{x}_J$, then either $V^\phi(x) = \beta(x)$ or $f_J^\phi(x) \le 0$.

(iii) If $x = \bar{x}_j \in \mathcal{I}$, V^ϕ is continuous at x, and either $f_j^\phi(x) < 0 < f_{j+1}^\phi(x)$ or $f_j^\phi(x) > 0 > f_{j+1}^\phi(x)$, then $\rho V^\phi(x) = \mathbf{u}^\phi(x)$.

(iv) The function V^ϕ is continuous in $\mathcal{X} \backslash \mathcal{I}$ and real analytic in $\mathcal{X} \backslash (\mathcal{I} \cup \mathcal{E})$.

(v) If $x \in \mathcal{X}_j \backslash (\mathcal{I} \cup \mathcal{E}_j)$, then $\rho V^\phi(x) = H_j^\phi(x, (V^\phi)'(x))$, while if $x \in \mathcal{E}_j$, then $f_j^\phi(x) = 0$ and $\rho V^\phi(x) = u_j^\phi(x)/\rho$.

(vi) *If* $x \in \mathcal{X}$ *and* $z \uparrow x$ $(z \downarrow x)$, *there is a value* $p_-, (p_+) \in \mathbb{R} \cup \{-\infty, \infty\}$, *such that* $(V^\phi)'(z) \to p_-$ $((V^\phi)'(z) \to p_+)$.

Proof of Theorem 2. We have to show that Conditions (i)–(vi) of the theorem imply that V^ϕ is in class $\mathscr{G}$ and that it is a viscosity solution of (11). Theorem C.1 then implies that $V^\phi = V$, and hence that $\phi = \phi_i$ is a best response to ϕ_{-i}.

Conversely, we have to show that if ϕ is a best response, then Conditions (i)–(vi) hold true.

E.1 Notations

We recall the notation $q_j^*(x,p)$ for the maximiser of $q \mapsto u(x,q) + pf_j(x,q)$ over $\mathcal{Q}$, the local Hamilton functions $H_j(x,p) = u(x, q_j^*(x,p)) + pf_j(x, q_j^*(x,p))$, as well as $p_{j,b}(x) = -u_q(x,q_b)/(f_j)_q(x,q_b)$ for $b \in \{\ell, u\}$.

For a given x, we write the left and right limits of $(V^\phi)'$ at x as

$$p_- = \lim_{z\uparrow x}(V^\phi)'(z) \quad \text{and} \quad p_+ = \lim_{z\downarrow x}(V^\phi)'(z).$$

Proposition E.1(vi) ensures that these limits exist everywhere in $\mathcal{X}$, if we allow the possibility that the limits take the values $-\infty$ or ∞.

For $x \in \mathcal{I}_j$, introduce the further abbreviations $H_-(p) = H_j(x,p)$ and $H_+(p) = H_{j+1}(x,p)$.

E.2 Sufficiency

Assume that Conditions (i)–(vi) hold. Proposition E.1(iv), as well as Conditions (ii) and (iv) imply that $V^\phi \in \mathscr{G}$. We have to show that it is a viscosity solution of (11) for every $x \in \mathcal{X}$.

Subdifferentials and superdifferentials. For any point $x \in \mathcal{X}$ where V^ϕ is continuous, if $p_- < p_+$, then $D^-V^\phi(x) = [p_-, p_+]$ and $D^+V^\phi(x) = \emptyset$; similarly, if $p_+ < p_-$, then $D^-V^\phi(x) = \emptyset$ and $D^+V^\phi(x) = [p_+, p_-]$; finally if $p_- = p_+ = p$, then $D^-V^\phi(x) = D^+V^\phi(x) = \{p\}$. The final situation occurs if and only if V^ϕ is differentiable at x.

For a point $x \in \mathcal{X}$ at which V^ϕ is not continuous, Condition (ii) implies that

$$D^-(V^\phi)_*(x) = (-\infty, p_+] \quad \text{and} \quad D^+(V^\phi)^*(x) = (-\infty, p_-].$$

Interior of $\mathcal{X}_j$. Take first $x \in \mathring{\mathcal{X}}_j$ for $j \in \{1, \ldots, J\}$: by Condition (iv) the function V^ϕ is continuous at x.

If V^ϕ is differentiable at x, set $p = (V^\phi)'(x)$: then Condition (i) and Proposition E.1(v) imply

$$H_j(x,p) = u(x,\phi(x)) + pf_j(x,\phi(x)) = H_j^\phi(x,p) = \rho V^\phi(x). \tag{18}$$

If V^ϕ is not differentiable at x, then $p_- \neq p_+$. Since ϕ is continuous at x, we have that $\phi(x) = q_j^*(x,p_-) = q_j^*(x,p_+)$, and therefore either $p_-, p_+ \le p_{j,\ell}(x)$ or $p_-, p_+ \ge p_{j,u}(x)$. Take $p \in D^-V^\phi(x) \cup D^+V^\phi(x)$: one of the two sets is empty. Then $q_j^*(x,p) = \phi(x) = q_b$, with $b \in \{\ell, u\}$. By Proposition E.1(v) we have $f_j(x,q_b) = 0$ and $\rho V^\phi(x) = u(x,q_b)$.

It follows that

$$\rho V^\phi(x) = u(x,q_b) + pf_j(x,q_b) = u(x,q_j^*(x,p)) + pf_j(x,q_j^*(x,p)) = H_j(x,p). \tag{19}$$

Equations (18) and (19) together show that (12) and (14) hold for all $x \in \mathring{\mathfrak{X}}_j$.

Interface points at which V^ϕ is continuous. Take $x = \bar{x}_j \in \mathfrak{I}$ with V^ϕ is continuous at x. If $D^-(V^\phi)(x)$ is nonempty, we have to show that

$$\rho V^\phi(x) \ge \min\{H_-(p), H_+(p)\}$$

for all $p \in D^-(V^\phi)(x)$. By continuity, $\rho V^\phi(x) = H_-(p_-) = H_+(p_+)$. Assume there is a point $\hat{p} \in (p_-, p_+)$ such that

$$H_-(p_-) = \rho V^\phi(x) < H_-(\hat{p}) \quad \text{and} \quad H_+(p_+) = \rho V^\phi(x) < H_+(\hat{p}).$$

By convexity of H_- and H_+, it follows that $\hat{f}_- := (H_-)_p(\hat{p}) > 0$ and $\hat{f}_+ := (H_+)_p(\hat{p}) < 0$. Hence there are $\lambda_-, \lambda_+ > 0$ such that $\lambda_- + \lambda_+ = 1$ and $\lambda_-\hat{f}_- + \lambda_+\hat{f}_+ = 0$. Set $u_- = u_j(x, q_j^*(x,\hat{p}))$ and $u_+ = u_{j+1}(x, q_{j+1}^*(x,\hat{p}))$. Condition (v) then implies that

$$\begin{aligned}\rho V^\phi(x) &\ge \lambda_- u_- + \lambda_+ u_+ = \lambda_- u_- + \lambda_+ u_+ + \hat{p}(\lambda_-\hat{f}_- + \lambda_+\hat{f}_+) \\ &= \lambda_- H_-(\hat{p}) + \lambda_+ H_+(\hat{p}) \ge \min\{H_-(\hat{p}), H_+(\hat{p})\} > \rho V^\phi(x),\end{aligned}$$

a contradiction, which proves (12) in this situation.

Next, assume that $D^+V^\phi(x)$ is nonempty. We have to show that

$$\rho V^\phi(x) \le \max\{H_-(p), H_+(p)\}$$

for all $p \in D^+(V^\phi)(x) = [p_+, p_-]$. Assume, as before, that the relation does not hold for some $\hat{p} \in (p_+, p_-)$, that is

$$H_-(p_-) = \rho V^\phi(x) > H_-(\hat{p}) \quad \text{and} \quad H_+(p_+) = \rho V^\phi(x) > H_+(\hat{p}).$$

Convexity now implies that $f_- := (H_-)_p(p_-) > 0$ and $f_+ := (H_+)_p(p_+) < 0$. Since $f_- = f_j(x, \phi_j(x))$ and $f_+ = f_{j+1}(x, \phi_{j+1}(x))$, Condition (vi) implies that V^ϕ is differentiable at

x, and we have therefore that $p_- = p_+$ and $\rho V^\phi(x) = H_-(p) = H_+(p)$ for all $p \in D^+V^\phi(x)$. Hence (14) holds for this case.

Interface points at which V^ϕ is not continuous. The next situation to consider is $x = \bar{x}_j \in \mathcal{I}$ such that V^ϕ is not continuous at x.

To show that (12) holds in this case, assume that there is $\hat{p} \in D^-(V^\phi)_* = (-\infty, p_+]$ for which $\rho(V^\phi)_*(x) < \min\{H_-(\hat{p}), H_+(\hat{p})\}$. Then we have in particular that $\hat{p} < p_+$ and $H_+(p_+) = \rho(V^\phi)_*(x) < H_+(\hat{p})$. Convexity of H_+ implies that $(H_+)_p(\hat{p}) < 0$. This however contradicts Condition (iv), which implies that $(H_+)_p(p) \geq 0$ for all p.

Turning to (14), assume there is $\hat{p} \in D^+(V^\phi)^* = (-\infty, p_-]$ for which

$$\rho(V^\phi)^*(x) > \max\{H_-(\hat{p}), H_+(\hat{p})\}.$$

Then $\hat{p} < p_-$ and $H_-(p_-) = \rho(V^\phi)^*(x) > H_-(\hat{p})$. Invoking convexity of H_-, we obtain that $(H_-)_p(p_-) = f_j(x, \phi_j(x)) > 0$. Again, this is incompatible with Condition (iv).

Boundary points. We only consider the situation that $x = \bar{x}_0$, the other being entirely analogous. At x, we have that $D^+V^\phi(x) = [p_+, \infty)$ and $D^-V^\phi(x) = (-\infty, p_+]$. It follows from Condition (iv) that V^ϕ is continuous at x.

To prove (12) at x, assume that $V^\phi(x) < \beta(x)$. Condition (iii) then implies that $f_1(x, q) \geq 0$ for all $q \in \mathcal{Q}$. In particular $f(x, q^*(x, p)) = (H_+)_p(p) \geq 0$ for all p and $H_+(p)$ is non-decreasing in p. Since $\rho V^\phi(x) - H_+(p_+) = 0$ by continuity, it follows that $\rho V^\phi(x) - H_+(p) \geq 0$ for all $p \in (-\infty, p_+] = D^-V^\phi(x)$, which implies (12).

To show (14) at x, assume that $V^\phi(x) > \beta(x)$. By Proposition E.1(ii) and Condition (i), we have $f^\phi(x) = f(x, q^*(x, p_+)) = (H_+)_p(p_+) \geq 0$, and, by convexity of $H_+(p)$, it follows that $H_+(p) \geq H_+(p_+)$ for all $p > p_+$, implying that $\rho V(x) - H_+(p) \leq \rho V(x) - H_+(p_+) = 0$ for all $p \in D^+V^\phi(x)$.

Finally, Condition (v) implies (13). This concludes the proof of the sufficiency part.

E.3 Necessity

To prove the necessity of Conditions (i)–(vi) — reinstating the player index i for a moment — assume that ϕ_i is the best response to ϕ_{-i}. Then $V_i^\phi = V_i$ is the viscosity solution of (11).

Maximum principle. If $x \in \mathring{\mathcal{X}}_j$ and V is differentiable at x, then $D^-V(x) = D^+V(x) = \{V'(x)\}$, and (12) and (14) imply that $\rho V(x) = H_j(x, V'(x))$. Moreover, since $V = V^\phi$, we also have that $V'(x) = (V^\phi)'(x) =: p$ and $H_j(x, p) = H_j^\phi(x, p)$, which is equivalent to

$$u(x, \phi_j(x)) + pf(x, \phi_j(x), \phi_{-j}(x)) = \max_q \left(u(x, q) + pf(x, q, \phi_{-j}(x))\right),$$

and therefore implies Condition (i).

Monotonicity. Condition (ii) follows from Proposition A.3.

Boundary values. We show Condition (iii) for $x = \bar{x}_0$, the other case being analogous.

If $\rho V(x) < \rho\beta(x) = H_-(p)$, by (12) we have that $\rho V(x) - H_+(p) \geq 0$ for all $p \in D^- V(x) = (-\infty, p_+]$. By convexity of H_+, this implies that $(H_+)_p = f_+(x, q^*(x,p)) \geq 0$ for all $p \leq p_+$, which implies in particular that $f_+(x, q_\ell) \geq 0$, and hence $f_+(x,q) \geq 0$ for all $q \in \mathcal{Q}$.

Value discontinuities. To show Condition (iv), note that, since $V \in \mathscr{G}$, if V fails to be continuous at x, then $x = \bar{x}_j \in \mathcal{J}$ and $f_+(x,q) \geq 0$ for all $q \in \mathcal{Q}$. It therefore remains to show that $f_-^\phi(x) = (H_-)_p(x,p) \leq 0$.

Proposition C.1(vi) implies that $V^*(x) = \max\{V_-^{\mathrm{sc}}(x), V_{\mathcal{J}}^{\mathrm{sc}}(x)\}$. If $V^*(x) = V_{\mathcal{J}}^{\mathrm{sc}}(x)$, then according to (13), we have $\rho V_*(x) \geq H_j^{\mathcal{J}}(x) = \rho V_{\mathcal{J}}^{\mathrm{sc}}(x) = \rho V^*(x)$, and V is actually continuous at x, which is ruled out by hypothesis. So assume that $V^*(x) = V_-^{\mathrm{sc}}(x)$, then for all $p \in D^+ V^*(x) = (-\infty, p_-]$ we have that $H_-(x, p_-) = \rho V^*(x) \leq H_-(x,p)$, and consequently $f^\phi(x) = (H_-)_p(x, p_-) \leq 0$, which had to be proved.

Value at interface steady states. Condition (v) is a direct consequence of (13).

Strong push–push steady state. To show Condition (vi), let x be a strong push–push steady state. By hypothesis, we have $f_-^\phi > 0 > f_+^\phi$. Let $\lambda \in (0,1)$ be such that $\lambda f_-^\phi + (1-\lambda) f_+^\phi = 0$. Then $\rho V(x) = \lambda u_- + (1-\lambda) u_+$. We also have $\rho V(x) = H_-(p_-) = H_+(p_+)$. Combining these equalities, we see that

$$
\begin{aligned}
0 &= \lambda H_-(p_-) + (1-\lambda) H_+(p_+) - ((1-\lambda) u_+ + \lambda u_-) \\
&= \lambda p_- f_-^\phi + (1-\lambda) p_+ f_+^\phi = \lambda (p_- - p_+) f_-^\phi.
\end{aligned}
$$

As $\lambda \neq 0$ and $f_-^\phi \neq 0$, we infer that $p_- = p_+ = p^*$ and V_i^ϕ is differentiable at x, proving Condition (vi). This completes the proof of Theorem 2. □

F Technical Appendix

F.1 Proof of Proposition A.1

Proof. Take $T > 0$ and $x \in \mathcal{X}$. According to the Dynamic Programming Principle (Bardi and Capuzzo-Dolcetta, 2008, Proposition III.2.5: although the assumptions are not fulfilled in our context, the proof carries over), we have

$$
V(x) = \sup \left(\int_0^\theta \mathbf{u}(y(s), a(s)) \exp(-\rho s)\, \mathrm{d}s + V(y(\theta)) \exp(-\rho\theta) \right),
$$

where $\theta = \min\{T, \Theta\}$ with Θ the exit time of y from $\mathcal{X}$, and where the supremum is taken over trajectory–action pairs (y, a) with $y(0) = x$.

Let (θ_k, y_k, a_k) be a sequence of time–trajectory–action triples with $y_k(0) = x$ such that

$$\int_0^{\theta_k} \mathbf{u}(y_k(s), a_k(s)) \exp(-\rho s) \, \mathrm{d}s + V(y_k(\theta_k)) \exp(-\rho\theta_k) \to V(x)$$

as $k \to \infty$, where $\theta_k = \min\{T, \Theta_k\}$ and Θ_k the exit time of y_k from $\mathcal{X}$. Introduce

$$w_k(t) = \int_0^{\min\{t,\theta_k\}} \mathbf{u}(y_k(s), a_k(s)) \exp(-\rho s) \, \mathrm{d}s + V(y_k(\theta_k)) \exp(-\rho\theta_k).$$

Then $w_k(T) \to V(x)$ as $k \to \infty$ and $\dot{w}_k(t)$ is measurable for $t \in [0, T]$. Extend (y_k, a_k) to $[0, T]$ by setting $(y_k(t), a_k(t)) = (y_k(\theta_k), a_k(\theta_k))$ if $\theta_k < t \leq T$. As θ_k is bounded, after restricting to a subsequence we may assume that $\theta_k \to \bar{\theta}$ as $k \to \infty$.

Introduce set-valued maps $\Phi_j : [0, T] \times \mathcal{X} \rightsquigarrow \mathbb{R}^2$ by setting

$$\Phi_j(t, z) = \{(\eta_0, \eta) \,:\, -\|u\|_\infty \leq \eta_0 \leq u_j(z, q) \exp(-\rho t), \ \eta = f_j(z, q), \ q \in \mathcal{Q}\}$$

if $(t, z) \in [0, T] \times \mathcal{X}_j$ and $\Phi_j(t, z) = \emptyset$ everywhere else. The sets $\Phi_j(t, z)$ are compact and convex by Assumption 2. Define $\Phi : [0, T] \times \mathcal{X} \rightsquigarrow \mathbb{R}^2$ by setting

$$\Phi(t, z) = \overline{\mathrm{co}} \left(\bigcup_{j=1}^{J} \Phi_j(t, z) \right).$$

Then $\Phi(t, z)$ is also compact and convex. Moreover, for all (t, z) it satisfies Property (Q) of Cesari (1983), that is,

$$\Phi(t, z) = \bigcap_{\delta > 0} \overline{\mathrm{co}} \left(\bigcup_{\|(\tilde{t}, \tilde{z}) - (t, z)\| < \delta} \Phi(\tilde{t}, \tilde{z}) \right).$$

We have that $(\dot{w}_k(t), \dot{y}_k(t)) \in \Phi(t, y_k(t))$ for all k and almost all $t \in [0, \theta_k]$, hence $|\dot{w}_k(t)| \leq \|u\|_\infty$ and $|\dot{y}_k(t)| \leq \|f\|_\infty$ almost everywhere on $[0, T]$. It follows that the (w_k, y_k) are equicontinuous, and a subsequence converges uniformly to a limit (w, y) on $[0, T]$. After relabelling, we may assume that the sequence itself converge uniformly to (w, y). By Cesari (1983, Theorem 8.6.i), it follows that $(\dot{w}(t), \dot{y}(t)) \in \Phi(t, y(t))$ for almost all $t \in [0, \bar{\theta}]$. Moreover, as the w_k converge uniformly, we have $w(T) = V(x)$. By the Filippov selection theorem (Vinter, 2000, Theorem 2.3.13), there is a measurable action schedule a such that $\dot{y}(t) = \mathbf{f}(y(t), a(t))$ almost everywhere on $[0, \bar{\theta}]$ and such that

$$V(x) = \int_0^{\bar{\theta}} \mathbf{u}(y(t), a(t)) \exp(-\rho t) \, \mathrm{d}t + V(y(\bar{\theta})) \exp(-\rho\bar{\theta}).$$

Set $y^*(t) = y(t)$ for $t \in [0, \bar{\theta}]$.

If $\bar{\theta} = T$, we repeat the argument with $x = y^*(T)$ and setting $(y^*(t), a^*(t)) = (y(t - T), a(t - T))$ for $t \in (T, 2T]$. Continuing inductively, we construct a trajectory–action pair (y^*, a^*) defined on an interval $[0, \Theta]$ such that $V(x) = U(y^*, a^*)$. $\square$

F.2 Proof of Proposition A.3

Proof. Take $x, \tilde{x} \in \mathcal{X}$ such that $\tilde{x} < x$, and let a be such that $V(x) = U(y, a)$. Let $\Theta = \inf\{t : y(t) \notin \mathcal{X}\}$. Construct a real-valued function $\tilde{a}_0(t)$ such that $u(y(t), \tilde{a}_0(t)) = \mathbf{u}(y(t), a(t))$ for all $0 \leq t \leq \Theta$: that is, if $y(t) \in \mathcal{X}\backslash\mathcal{I}$, set $\tilde{a}_0(t) = \sum_{j=1}^{J} a_j(t)\mathbf{1}_{\mathcal{X}_j}(y(t))$; if $y(t) \in \mathcal{I}_j$, take $\tilde{a}_0(t)$ such that

$$u(y(t), \tilde{a}_0(t)) = \mu_j(a(t))u(y(t), a_j(t)) + (1 - \mu_j(a(t)))u(y(t), a_{j+1}(t));$$

finally, if $t > \Theta$, set $\tilde{a}_0(t)$ to an arbitrary constant value in $\mathcal{Q}$.

Using $\tilde{a}_0$, we define an action schedule $\tilde{a}$ by setting $\tilde{a}_j(t) = \tilde{a}_0(t)$ for all t and all $j \in \{1, \dots, J\}$. By Proposition 4.2 there is a state trajectory $\tilde{y}(t)$ such that $\tilde{y}(0) = \tilde{x}$ and $(\tilde{y}, \tilde{a})$ is a state–action pair.

Let $\tau = \min\{t : \tilde{y}(t) = y(t)\}$ and $\tilde{\Theta} = \inf\{t : \tilde{y}(t) \notin \mathcal{X}\}$. If $\tau \leq \min\{\Theta, \tilde{\Theta}\}$, we have for all $0 \leq t < \tau$ that $\tilde{y}(t) < y(t)$ and therefore $\mathbf{u}(\tilde{y}(t), \tilde{a}(t)) = u(\tilde{y}(t), \tilde{a}_0(t)) > u(y(t), \tilde{a}_0(t)) = \mathbf{u}(y(t), a(t))$, while for $t \geq \tau$, the trajectory-control pairs and their felicity flows are equal.

Take now $\tau > \min\{\Theta, \tilde{\Theta}\}$. For $\Theta < t < \tilde{\Theta}$, Assumption 3 implies that $\mathbf{u}(\tilde{y}(t), \tilde{a}(t)) > \rho\beta(y(\Theta))$, while for $\tilde{\Theta} < t < \Theta$, it implies $\rho\beta(\tilde{y}(\tilde{\Theta})) > \mathbf{u}(y(t), a(t))$. Finally, if $t \geq \max\{\Theta, \tilde{\Theta}\}$, we have $\beta(\tilde{y}(\tilde{\Theta})) \geq \beta(y(\Theta))$. This proves the result. □

F.3 Proof of Proposition A.4

Proof. Let $\sigma = \exp(-\rho t)\,\mathrm{d}t$ be the Borel measure on $[0, \infty)$ defined by $\sigma([t_1, t_2]) = (\exp(-\rho t_1) - \exp(-\rho t_2))/\rho$. The set $\{t \in \mathcal{T} : y(t) \neq x\}$ can be written as the union of at most countably many intervals $I_k = (t_{1,k}, t_{2,k})$ such that $\sigma(I_k) > 0$ and $y(t_{1,k}) = y(t_{2,k}) = x$, where K is the number of such intervals, and one interval $\hat{I} = (\hat{t}, \infty)$ such that $y(\hat{t}) = x$, which may be empty. Let $I_0 = [0, \infty)\backslash \left(\bigcup_{k=1}^{K} I_k \cup \hat{I}\right)$: this set is measurable, possibly of measure 0.

For $0 \leq k \leq K$ such that $\sigma(I_k) > 0$, introduce

$$v_k \equiv \frac{1}{\sigma(I_k)} \int_{I_k} \mathbf{u}(y(t), a(t)) \exp(-\rho t)\,\mathrm{d}t;$$

if $\sigma(\hat{I}) > 0$, set $\hat{v} \equiv (1/\sigma(\hat{I})) \int_{\hat{I}} w(t) \exp(-\rho t)\,\mathrm{d}t$ with $w(t) = \mathbf{u}(y(t), a(t))$ if $\hat{t} < t \leq \Theta$ and $w(t) = \beta(y(\Theta))$ if $t > \Theta$; finally $v_0 = 0$ if $\sigma(I_0) = 0$. Then

$$\bar{U} \equiv \int_0^{\Theta} \mathbf{u}(y(t), a(t)) \exp(-\rho t)\,\mathrm{d}t + \exp(-\rho\Theta)\beta(y(\Theta)) = \hat{v}\sigma(\hat{I}) + \sum_{k=0}^{K} v_k\sigma(I_k).$$

As $\sigma([0, \infty)) = 1/\rho$, either there exists $k \in \{0, \dots\}$ such that $v_k \geq \rho\bar{U}$ and $\sigma(I_k) > 0$, or $\hat{v} \geq \rho\bar{U}$ and $\sigma(\hat{I}) > 0$.

Assume first that the first alternative holds for $k > 0$, and $y(t) > x$ for $t \in I_k$. Set $\Delta = t_{2,k} - t_{1,k} > 0$ and construct a trajectory-action pair by setting for $\ell = 0, 1, 2, \ldots$

$$(\tilde{y}(t), \tilde{a}(t)) = (y(t_{1,k} + t - \ell\Delta), a(t_{1,k} + t - \ell\Delta)), \qquad \text{if} \quad \ell\Delta \le t < (\ell + 1)\Delta$$

We have

$$\begin{aligned}
U(\tilde{y}, \tilde{a}) &= \int_0^\infty \mathbf{u}(\tilde{y}(t), \tilde{a}(t)) \exp(-\rho t)\, \mathrm{d}t \\
&= \sum_{\ell=0}^{\infty} \int_{\ell\Delta}^{(\ell+1)\Delta} \mathbf{u}(y(t_{1,k} + t - \ell\Delta), a(t_{1,k} + t - \ell\Delta)) \exp(-\rho t)\, \mathrm{d}t \\
&= \exp(\rho t_{1,k}) \int_{t_{1,k}}^{t_{2,k}} \mathbf{u}(y(s), a(s)) \exp(-\rho s)\, \mathrm{d}s \sum_{\ell=0}^{\infty} \exp(-\rho\ell\Delta) \\
&= \frac{1 - \exp(-\rho\Delta)}{\rho} v_k \frac{1}{1 - \exp(-\rho\Delta)} = v_k/\rho \ge \bar{U}.
\end{aligned}$$

Moreover $\tilde{y}(t) > x$ for almost all $t \ge 0$. Hence we have constructed the required trajectory. The argument for the situation that $y(t) < x$ for $t \in I_k$ is entirely analogous.

If the first alternative holds for $k = 0$, then the set C_0 of constant actions q such that $\mathbf{f}(x, q) = 0$ is non-empty. As C_0 is compact, there is a maximiser $\bar{q}$ of $\mathbf{u}(x, q)$ restricted to C_0. Let $(\tilde{y}, \tilde{a})$ be the trajectory–action pair $\tilde{y}(t) = x$, $\tilde{a}(t) = \bar{q}$ for all t. Then $\mathbf{u}(y, a) \le \mathbf{u}(\tilde{y}, \tilde{a})$ for all $t \in I_0$, and

$$\begin{aligned}
\rho\bar{U} &\le v_0 = \frac{1}{\sigma(I_0)} \int_{I_0} \mathbf{u}(\tilde{y}(t), \tilde{a}(t)) \exp(-\rho t)\, \mathrm{d}t = \frac{1}{\sigma(I_0)} \int_{I_0} \mathbf{u}(x, \bar{q}) \exp(-\rho t)\, \mathrm{d}t \\
&= \mathbf{u}(x, \bar{q}) = \rho U(\tilde{y}, \tilde{a}),
\end{aligned}$$

completing the construction of the trajectory also in this situation.

Finally, if $\hat{v} \ge \rho\bar{U}$ and $\sigma(\hat{I}) > 0$, then $(\tilde{y}(t), \tilde{a}(t)) = (y(\hat{t} + t), a(\hat{t} + t))$ achieves a higher payoff than $\bar{U}$. □

F.4 Proof of Proposition A.5

Proof. Let $\Delta = \min_{j \ne k} |\bar{x}_j - \bar{x}_k|$, and let $M = \|\mathbf{f}\|_\infty > 0$. Introduce for a trajectory–action pair $\pi = (y, a)$ the exit time $\Theta(\pi) = \inf\{t \ge 0 : y(t) \notin \mathcal{X}\}$, the time interval $\mathcal{T}(\pi) = [0, \Theta(\pi)]$, and the set $S_j(\pi)$ of singular pull-pull events as

$$S_j(\pi) \equiv \{t \in \mathcal{T}(\pi) \,:\, y(t) = \bar{x}_j,\ f_j(y(t), a_j(t)) < 0\,,\ f_{j+1}(y(t), a_{j+1}(t)) > 0\}\,.$$

If π is not regular, the union $\bigcup_j S_j(\pi)$ has positive Lebesgue measure.

Let π be a given trajectory–control pair. For $\ell = 1, 2, \ldots$, we shall inductively construct a sequence $\pi^{(\ell)} = (y^{(\ell)}, a^{(\ell)})$ of trajectory–action pairs such that $\pi^{(0)} = \pi$, $S_j(\pi^{(\ell)}) \cap [0, \ell\Delta/M)$ has measure zero for every j, and $U(\pi^{(\ell+1)}) \ge U(\pi^{(\ell)})$ for all $\ell \ge 0$.

Assume that $\pi^{(\ell)}$ has already been constructed. Let

$$\tau = \inf\left\{t_1 \in \mathcal{T}(\pi^{(\ell)}) \,:\, S_j(\pi^{(\ell)}) \cap [0, t_1] \text{ has positive measure for some } j\right\}.$$

If $\tau \geq (\ell+1)M/\Delta$, then we set $\pi^{(\ell+1)} = \pi^{(\ell)}$ and the induction step is completed.

If $\tau < (\ell+1)M/\Delta$, then $y^{(\ell)}(\tau) = \bar{x}_j$ for some j, and we set $\pi_\tau(t) = \pi^{(\ell)}(t-\tau)$.

By Proposition A.4, there is a trajectory–action pair $\tilde{\pi} = (\tilde{y}, \tilde{a})$ such that either $\tilde{y}(t) < \bar{x}_j$ for all $t \geq 0$, or $\tilde{y}(t) > \bar{x}_j$ for all $t \geq 0$, or $\tilde{y}(t) = \bar{x}_j$ for all $t \geq 0$, as well as $U(\tilde{\pi}) \geq U(\pi_\tau)$. In the first two cases, $\tilde{y}(t) \notin \mathcal{I}\backslash\mathcal{I}_j$ for all $0 \leq t < \Delta/M$, as for those values of t we have $|y(t) - y(0)| \leq Mt < \Delta$. In these situations we set $\pi^{(\ell+1)}(t) = \pi^{(\ell)}(t)$ for $0 \leq t \leq \tau$ and $\pi^{(\ell+1)}(t) = \tilde{\pi}(t-\tau)$ for $t \geq \tau$. Then

$$\begin{aligned} U(\pi^{(\ell+1)}) &= \int_0^\tau \mathbf{u}(y^{(\ell)}(t), a^{(\ell)}(t)) \exp(-\rho t)\,\mathrm{d}t + \exp(-\rho\tau)U(\tilde{\pi}) \\ &\geq \int_0^\tau \mathbf{u}(y^{(\ell)}(t), a^{(\ell)}(t)) \exp(-\rho t)\,\mathrm{d}t + \exp(-\rho\tau)U(\pi^{(\ell)}) = U(\pi^{(\ell)}). \end{aligned}$$

In the third case, according to Proposition A.4, we may assume that $\tilde{\pi}$ is generated by a constant action schedule $\tilde{a}(t) = q$ for all $t \geq 0$. If $\tilde{\pi}$ is a regular trajectory, then we define $\pi^{(\ell+1)}$ as in the first two cases. If $\tilde{\pi}$ is singular, then in particular $f_j(\bar{x}_j, q_j) < 0$. Consider the trajectory–action pair $(z, \tilde{a})$ that satisfies $z(0) = \bar{x}_j$ and $\dot{z}(t) = f_j(z(t), q_j)$ for $0 \leq t < M/\Delta$. As before, we have that $z(t) \notin \mathcal{I}$ for $0 < t < M/\Delta$ and, as $f_j(\bar{x}_j, q_j) < 0$, we also have that $z(t) < \bar{x}_j = y(t)$ for all $t > 0$. By Proposition A.2, it follows that $U(z, \tilde{a}) > U(y, \tilde{a})$. Setting $\pi^{(\ell+1)}(t) = \pi^{(\ell)}(t)$ for $0 \leq t \leq \tau$ and $\pi^{(\ell+1)}(t) = (z(t-\tau), \tilde{a}(t-\tau))$ for $t \geq \tau$, and noting that also in this case $U(\pi^{(\ell+1)}) \geq U(\pi^{(\ell)})$ finishes the inductive step.

The induction either breaks off at the ℓ'th step and produces a regular trajectory, as indicated, or it continuous indefinitely. In the latter case, we set $\bar{\pi}(t) = \lim_{\ell\to\infty} \pi^{(\ell)}(t)$. Then $S_j(\bar{\pi})$ has measure zero for all j and $\bar{y}$ is regular also in this case. □

F.5 Proof of Proposition C.1

Proof of Proposition C.1. Throughout the proof, we write $\bar{x}$ for $\bar{x}_j$, f_- for f_j and f_+ for f_{j+1} etc. In particular $\mathcal{X}_+$ denotes $\mathcal{X}_{j+1}$ and not $\mathcal{X}_{j,+}$. We fix $\bar{q}$ such that $u(\bar{x}, \bar{q}) \geq u(\bar{x}, q)$ for all $q \in \mathcal{Q}$.

Statement (i) is a direct corollary of Proposition A.4.

To prove (ii), assume that the dynamics are right controllable at $\bar{x}$: the argument for left controllability is analogous.

By controllability and continuity of f_+, there are $\delta, m > 0$ such that $[\bar{x}, \bar{x}+\delta] \in \mathcal{X}_+$ and $[-m, m] \subset f_+(\bar{x}, \mathcal{Q})$ for all $\bar{x} \leq x \leq \bar{x}+\delta$. Take $x_1, x_2 \in [\bar{x}, \bar{x}+\delta]$ as well as $\sigma \in \{-m, m\}$ such that $y(t) = x_1 + \sigma t$ satisfies $y(0) = x_1$ and $y(\tau) = x_2$ if $\tau = |x_2 - x_1|/m$.

As $|\dot{y}(t)| = m$ and $y(t) \in [\bar{x}, \bar{x}+\delta]$ for $0 \le t \le \tau$ there is $a(t)$ such that $\dot{y}(t) = f(y(t), a(t))$ for all $0 \le t \le \tau$. Then

$$V(x_1) \ge \int_0^\tau u_+(y(t), a(t)) \exp(-\rho t)\, \mathrm{d}t + V(x_2) \exp(-\rho\tau).$$

As $|V(x)| \le \|\mathbf{u}\|_\infty/\rho$ for all x, we obtain

$$V(x_1) - V(x_2) \ge -|\exp(-\rho\tau) - 1|\|\mathbf{u}\|_\infty/\rho - \|\mathbf{u}\|_\infty \tau \ge -2\|\mathbf{u}\|_\infty \tau.$$

Interchanging the roles of x_1 and x_2, and using the definition of τ, then gives

$$|V(x_1) - V(x_2)| \le \frac{2\|\mathbf{u}\|_\infty}{m} |x_1 - x_2|.$$

For (iii), let $\bar{x}$ be a right semi-attractor: then $f_+(\bar{x}, q) \le 0$ for all $q \in \mathcal{Q}$.

Choosing (y, c) such that $y(0) = \bar{x}$ and $a_-(t) = a_+(t) = \bar{q}$ for all $t \in \mathcal{T}$ implies first that $y(t) \le \bar{x}$ for all $t \in \mathcal{T}$, as $\bar{x}$ is a right semi-attractor. Since $u_x(x, \bar{q}) < 0$ for all x, we then have $u(y(t), \bar{q}) \ge u(\bar{x}, \bar{q})$ for all $t \ge 0$ and hence $V(\bar{x}) \ge u(\bar{x}, \bar{q})/\rho$.

Take $x \ge \bar{x}$, and let now the pair (y, c) be such that $y(0) = x$ and $V(x) = U(y, c)$. Introduce $\theta = \inf \left\{ t \in \mathcal{T} \,:\, y(t) \notin \mathring{\mathcal{X}}_{j,+} \right\}$. We have

$$\begin{aligned} V(x) &= \int_0^{\min\{\theta,\Theta\}} u(y(t), a_+(t)) \exp(-\rho t)\, \mathrm{d}t \\ &\qquad + V(\bar{x}) \exp(-\rho\theta) \mathbf{1}_{\{t\,:\,t \le \Theta\}}(\theta) + V(y(\Theta)) \exp(-\rho\Theta) \mathbf{1}_{\{t\,:\,t > \Theta\}}(\theta) \\ &\le \int_0^{\min\{\theta,\Theta\}} u(\bar{x}, a_+(t)) \exp(-\rho t)\, \mathrm{d}t \\ &\qquad + V(\bar{x}) \exp(-\rho\theta) \mathbf{1}_{\{t\,:\,t \le \Theta\}}(\theta) + V(y(\Theta)) \exp(-\rho\Theta) \mathbf{1}_{\{t\,:\,t > \Theta\}}(\theta) \\ &\le \int_0^{\min\{\theta,\Theta\}} u(\bar{x}, \bar{q}) \exp(-\rho t)\, \mathrm{d}t + V(\bar{x}) \exp(-\rho \min\{\theta, \Theta\}) \le V(\bar{x}). \end{aligned}$$

This shows right upper semi-continuity of V at $\bar{x}$.

Proposition IV.3.4 of Bardi and Capuzzo-Dolcetta (2008) implies that the value function is also lower semi-continuous at $\bar{x}$. This then establishes right continuity.

Finally, if there is a trajectory starts at $\bar{x}$ and remains in $\mathcal{X}_{j,+}$ for all $t \ge 0$, it must be equal to $y(t) = \bar{x}$. Hence $V_+^{\mathrm{sc}}(\bar{x}) \le V_{\mathcal{J}}^{\mathrm{sc}}(\bar{x})$, which shows the second part of the statement.

For (iv), let $\bar{x}$ be a left semi-attractor.

If $f_-(\bar{x}, q_\ell) > 0$, then there are $m > 0$ and $\delta > 0$ such that $f_-(z, q) > m$ for all $z \in [\bar{x} - \delta, \bar{x}]$ and all $q \in \mathcal{Q}$. Fix $x \in [\bar{x} - \delta, \bar{x}]$, and let (y, a) be a trajectory–action pair such that $y(0) = x$ and $V(x) = U(y, a)$. Then there is $0 < \tau < |x - \bar{x}|/m$ such that $y(t) < \bar{x}$ for all

$0 < t < \tau$ and $y(\tau) = \bar{x}$. This implies

$$V(x) - V(\bar{x}) = \left(\int_0^\tau \mathbf{u}(y(t), a(t)) \exp(-\rho t)\,\mathrm{d}t + (\exp(-\rho\tau) - 1)V(\bar{x})\right)$$
$$\leq 2\|u\|_\infty \tau = \frac{2\|u\|_\infty}{m}|x - \bar{x}|,$$

which shows that V is left upper semi-continuous at $\bar{x}$. Proposition IV.3.4 of Bardi and Capuzzo-Dolcetta (2008) again ensures left continuity.

If $f_-(\bar{x}, q_\ell) = 0$, take $\varepsilon > 0$ and $\delta > 0$. Let T be the unique solution of $\exp(-\rho T)\|u\|_\infty/\rho = \varepsilon/2$; this solution is positive if $\varepsilon > 0$ is sufficiently small. Let $L_f > 0$ be such that $|f_-(z, q_\ell)| \leq L_f|z - \bar{x}|$ for all $-\delta < z - \bar{x} < 0$, and let $\delta_1 = \exp(-L_f T)\delta$.

Take $x \in (\bar{x} - \delta_1, \bar{x})$, and let y be any state trajectory with $y(0) = x$. Set $\tau = \inf\{t \geq 0 : y(t) \in \mathcal{X}_+\}$, and let $\theta = \min\{\tau, T\}$. Using the Gronwall inequality, we have that $-\delta < -\exp(L_f t)|x - \bar{x}| \leq y(t) - \bar{x} \leq 0$ for all $0 \leq t \leq \theta$.

Let now (y, a) be a trajectory–action pair such that $y(0) = x$ and $V(x) = U(y, a)$. To obtain an estimate for the payoff on the time interval $[0, \theta]$, we split it in a part $\mathcal{T}_1$ where the state moves quickly to the right, which restricts the amount of time it can spend in this set, and a part $\mathcal{T}_2 \cup \mathcal{T}_3$ where it moves slowly to the right, or not at all, restricting the value of $a_-(t)$ from above, and hence the payoff.

Take $\eta > 0$ and form the partition $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ of the interval $[0, \theta]$, where $\mathcal{T}_1 = \{t : \dot{y}(t) > \eta\}$, $\mathcal{T}_2 = \{t : 0 \leq \dot{y}(t) \leq \eta\}$, $\mathcal{T}_3 = \{t : \dot{y}(t) < 0\}$, and $\mathcal{T}_4 = \{t : y \text{ is not differentiable at } t\}$. Note that $\mathcal{T}_4$ is a set of measure zero.

Clearly

$$y(\theta) - x = \int_0^\theta \dot{y}(t)\,\mathrm{d}t = \int_{\mathcal{T}_1} + \int_{\mathcal{T}_2} + \int_{\mathcal{T}_3} \dot{y}(t)\,\mathrm{d}t.$$

Since $-\exp(L_f\theta)|x - \bar{x}| \leq y(\theta) - \bar{x} \leq 0$, the measure $|\mathcal{T}_1|$ of the first partitioning set satisfies

$$\begin{aligned}
\eta|\mathcal{T}_1| &\leq \int_{\mathcal{T}_1} \dot{y}(t)\,\mathrm{d}t \leq \int_{\mathcal{T}_1} \dot{y}(t)\,\mathrm{d}t + \int_{\mathcal{T}_2} \dot{y}(t)\,\mathrm{d}t = y(\theta) - x - \int_{\mathcal{T}_3} \dot{y}(t)\,\mathrm{d}t \\
&\leq |y(\theta) - x| - \int_{\mathcal{T}_3} \dot{y}(t)\,\mathrm{d}t \leq |y(\theta) - \bar{x}| + |\bar{x} - x| - \int_{\mathcal{T}_3} f_-(y(t), q_\ell)\,\mathrm{d}t \\
&\leq (1 + \exp(L_f\theta))|x - \bar{x}| + \int_{\mathcal{T}_3} L_f|y(t) - \bar{x}|\,\mathrm{d}t \\
&\leq (1 + \exp(L_f\theta) + L_f\theta\exp(L_f\theta))\,|x - \bar{x}| =: C_1|x - \bar{x}|.
\end{aligned}$$

Consequently, the integral of the discounted flow payoff evaluated over $\mathcal{T}_1$ is bounded by

$$\int_{\mathcal{T}_1} u(y(t), a(t)) \exp(-\rho t)\,\mathrm{d}t \leq \|u\|_\infty|\mathcal{T}_1| \leq \frac{C_1\|u\|_\infty}{\eta}|x - \bar{x}|.$$

To estimate the payoff evaluated over $\mathcal{T}_2 \cup \mathcal{T}_3$, we need an upper bound on $a_-(t)$. Let

$L_u, \ell_f > 0$ be such that $|u(z,q) - u(\bar{x}, q_\ell)| \le L_u(|z - \bar{x}| + |q - q_\ell|)$ for all (z,q), and $0 < \ell_f < \frac{\partial f_-}{\partial q}(\bar{x}, q)$ for all $q \in \mathcal{Q}$: such constants exist as a consequence of Assumptions 1 and 2 and the compactness of $\mathcal{Q}$. We have for $t \in \mathcal{T}_2 \cup \mathcal{T}_3$ that

$$\begin{aligned}\eta \ge \dot{y}(t) = f_-(y(t), a_-(t)) &\ge f_-(y(t), q_\ell) + \ell_f(a_-(t) - q_\ell)\\ &\ge -L_f|y(t) - \bar{x}| + \ell_f(a_-(t) - q_\ell);\end{aligned}$$

in the last inequality we used that $f(\bar{x}, q_\ell) = 0$. Hence

$$a_-(t) - q_\ell \le (\eta/\ell_f) + (L_f/\ell_f)|y(t) - \bar{x}|$$

and

$$\begin{aligned}|u(y(t), a_-(t)) - u(\bar{x}, q_\ell)| &\le L_u\left(|y(t) - \bar{x}| + |a_-(t) - q_\ell|\right)\\ &\le L_u\left(1 + \frac{L_f}{\ell_f}\right)|y(t) - \bar{x}| + \frac{L_u}{\ell_f}\eta.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\mathcal{T}_2 \cup \mathcal{T}_3} u(y(t), a(t)) \exp(-\rho t)\,\mathrm{d}t\\ &\le \int_0^\theta \Big(u(\bar{x}, q_\ell) + L_u(1 + L_f/\ell_f)|y(t) - \bar{x}| + L_u\eta/\ell_f\Big)\exp(-\rho t)\,\mathrm{d}t\\ &\le (1 - \exp(-\rho\theta))u(\bar{x}, q_\ell)/\rho + C_2|x - \bar{x}| + C_3\eta,\end{aligned}$$

where $C_2 = L_u(1 + L_f/\ell_f)\exp(L_f T)$ and $C_3 = TL_u/\ell_f$.

Combining these estimates yields

$$\begin{aligned}V(x) &\le \int_{\mathcal{T}_1} + \int_{\mathcal{T}_2} + \int_{\mathcal{T}_3} u(y(t), a(t))\,\mathrm{d}t + \exp(-\rho\theta)V(y(\theta))\\ &\le (1 - \exp(-\rho\theta))u(\bar{x}, q_\ell)/\rho + \exp(-\rho\theta)V(y(\theta))\\ &\quad + \frac{C_1\|u\|_\infty}{\eta}|x - \bar{x}| + C_2|x - \bar{x}| + C_3\eta.\end{aligned}$$

Choose $\eta = \varepsilon/(6C_3)$ and $|x - \bar{x}| < \min\{\delta_1, \varepsilon^2/(36C_1C_3\|u\|_\infty), \varepsilon/(6C_2)\}$, and recall that $V_-^{\mathrm{sc}}(\bar{x}) = u(\bar{x}, q_\ell)/\rho \le V(\bar{x})$, to obtain

$$V(x) \le (1 - \exp(-\rho\theta))V(\bar{x}) + \exp(-\rho\theta)V(y(\theta)) + \varepsilon/2.$$

If $\theta = T$, then $\exp(-\rho\theta)V(y(\theta)) \le \varepsilon/2$ and $V(x) \le V(\bar{x}) + \varepsilon$, showing that V is left upper semi-continuous at $\bar{x}$. If $\theta = \tau$, then $V(y(\theta)) = V(\bar{x})$ and $V(x) \le V(\bar{x}) + \varepsilon/2$, again showing that V is left upper semi-continuous at $\bar{x}$. As lower semi-continuity is assured, it follows that V is left continuous at $\bar{x}$.

A trajectory starting at $\bar{x}$ and remaining in $\mathcal{X}_-$ for all $t \ge 0$ must satisfy $y(t) = \bar{x}$ for all t: therefore $V_-^{\mathrm{sc}}(\bar{x}) \le V_{\mathcal{J}}^{\mathrm{sc}}(\bar{x})$.

Proceeding to (v), let $\bar{x}$ be a left semi-repeller, and let (y, a) be a trajectory–action pair such that $y(0) = \bar{x}$, $a_-(t) = a_+(t) = \bar{q}$ and $y(t) \in \mathcal{X}_-$ for all $t \geq 0$. Then $V_-^{\mathrm{sc}}(\bar{x}) \geq U(y, a) \geq u(\bar{x}, \bar{q})/\rho \geq V_J^{\mathrm{sc}}(\bar{x})$. Let $(\tilde{y}, \tilde{a})$ be any trajectory–action pair such that $\tilde{y}(0) = \bar{x}$ and $\tilde{y}(t) \in \mathcal{X}_+$ for all $t \geq 0$. Then

$$u(\bar{x}, \bar{q}) - u(\tilde{y}(t), \tilde{a}_+(t)) \geq u(\bar{x}, \bar{q}) - u(\bar{x}, \tilde{a}_+(t)) + u(\bar{x}, \tilde{a}_+(t)) - u(\tilde{y}(t), \tilde{a}_+(t)) \geq 0,$$

as $\bar{q}$ maximises $u(\bar{x}, \cdot)$ and as $u(x, q)$ is decreasing in x. This implies $V_-^{\mathrm{sc}}(\bar{x}) \geq u(\bar{x}, \bar{q})/\rho \geq V_+^{\mathrm{sc}}(\bar{x})$. We conclude that $V(\bar{x}) = V_-^{\mathrm{sc}}(\bar{x})$, and hence that V is left continuous at $\bar{x}$.

Finally, we prove (vi) The sufficiency of the condition is clear. To show necessity, we combine (ii), (iv) and (v) to infer that V is always left-continuous. Statements (ii) and (iii) imply that it can only fail to be right-continuous if $\bar{x}$ is a right semi-repeller and $V_+^{\mathrm{sc}}(\bar{x}) < V(\bar{x})$. □

F.6 Proof of Proposition C.3

First, we formulate superoptimality and suboptimality principles at interfaces.

Lemma F.1. *Let $\bar{x} = \bar{x}_j$ be an interface point, $v : \mathcal{X} \to \mathbb{R}$ a supersolution and $w : \mathcal{X} \to \mathbb{R}$ a subsolution of* (11)*, such that $v, w \in \mathscr{G}$. Choose $\xi_1 \in \mathring{\mathcal{X}}_j$, and let $\tau_j = \inf\{t \geq 0 : y(t) \notin (\xi_1, \bar{x})\}$ be the exit time from $(\xi_1, \bar{x})$. Then for all $t \geq 0$ and all $x \in (\xi_1, \bar{x})$, we have for $\theta_j = \min\{t, \tau_j\}$ that*

$$v(x) \geq \sup_a \left(\int_0^{\theta_j} \mathbf{u}(y(s), a(s)) \exp(-\rho s) \, \mathrm{d}s + \exp(-\rho\theta_j) v(y(\theta_j)) \right) \tag{20}$$

and

$$w(x) \leq \sup_a \left(\int_0^{\theta_j} \mathbf{u}(y(s), a(s)) \exp(-\rho s) \, \mathrm{d}s + \exp(-\rho\theta_j) w(y(\theta_j)) \right). \tag{21}$$

If v or w are, respectively, continuous at $\bar{x}$, $\xi_2 \in \mathring{\mathcal{X}}_{j+1}$ and $\tau_{j+1} = \inf\{t \geq 0 : y(t) \notin (\bar{x}, \xi_2)\}$, then for all $t \geq 0$ and $x \in (\bar{x}, \xi_2)$, the inequalities (20) *or* (21) *hold, respectively, with θ_j replaced by $\theta_{j+1} = \min\{t, \tau_{j+1}\}$.*

Proof. As the hypotheses imply that v and w restricted to $[\xi_1, \bar{x}]$ and $[\bar{x}, \xi_2]$ are continuous, equation (20) is implied by Bardi and Capuzzo-Dolcetta (2008, Remark III.2.34), and (21) by Bardi and Capuzzo-Dolcetta (2008, Remark IV.3.16). □

Proof of Proposition C.3. Write H_- for H_j and H_+ for H_{j+1}; then we have that $\mathbf{H}(\bar{x}, p) = \max\{H_-(\bar{x}, p), H_+(\bar{x}, p)\}$. Introduce $\varphi(p) = \rho w(\bar{x}) - \mathbf{H}(\bar{x}, p)$.

Examine first the situation that $\varphi(p) \leq 0$ for all p. As $\varphi(p)$ is an affine function of p if $p < p_\ell(\bar{x})$ or $p > p_u(\bar{x})$, a maximiser $\bar{p}$ of φ exists. The function φ is concave and maximal at $\bar{p}$, hence 0 is an element of the subgradient of $-\varphi(\bar{p})$, which is the closed convex hull of the derivatives $(H_-)_p(\bar{x},\bar{p})$ and $(H_+)_p(\bar{x},\bar{p})$ (Aubin, 1993, Corollary 4.4). Using the fact that $(H_\pm)_p(x,p) = f_\pm(x, q^*(x,p))$, and setting $\bar{q}^* = q^*(\bar{x},\bar{p})$, this implies that there is $0 \le \lambda \le 1$ such that

$$\mathbf{H}(\bar{x},\bar{p}) = \lambda H_-(\bar{x},\bar{p}) + (1-\lambda)H_+(\bar{x},\bar{p}), \qquad \lambda f_-(\bar{x},\bar{q}^*) + (1-\lambda) f_+(\bar{x},\bar{q}^*) = 0, \tag{22}$$

$\lambda = \mu(\bar{q}^*,\ldots,\bar{q}^*)$ and $(\bar{q}^*,\bar{q}^*) \in C_0$, where C_0 is the set of controls stabilising $\bar{x}$. Using (22), as well as the definition of $H^{\mathfrak{I}}$, we obtain

$$\begin{aligned} 0 \ge \varphi(\bar{p}) &= \rho w(\bar{x}) - (\lambda H_-(\bar{x},\bar{p}) + (1-\lambda)H_+(\bar{x},\bar{p})) \\ &= \rho w(\bar{x}) - \lambda u(\bar{x},\bar{q}^*) - (1-\lambda)u(\bar{x},\bar{q}^*) - \bar{p}(\lambda f_-(\bar{x},\bar{q}^*) + (1-\lambda) f_+(\bar{x},\bar{q}^*)) \\ &\ge \rho w(\bar{x}) - H^{\mathfrak{I}}(\bar{x}). \end{aligned}$$

In this case the alternative A holds true.

Consider now the second situation, that there is $\bar{p}$ such that $\varphi(\bar{p}) > 0$. Let $\varepsilon > 0$ and set

$$\psi_\varepsilon(x) = w(\bar{x}) + \bar{p}(x-\bar{x}) + \frac{(x-\bar{x})^2}{2\varepsilon^2}.$$

Now $\bar{x}$ cannot maximise $w - \psi_\varepsilon$ for any $\varepsilon > 0$, for if it did, $\psi_\varepsilon'(\bar{x}) = \bar{p} \in D^+w(\bar{x})$, which would imply, as w is a subsolution, that $\varphi(\bar{p}) \le 0$.

For every $\varepsilon > 0$ let x_ε denote a maximiser of $w - \psi_\varepsilon$. Necessarily $x_\varepsilon \ne \bar{x}$ and $0 = w(\bar{x}) - \psi_\varepsilon(\bar{x}) \le w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon)$, which implies, first, with $\sigma = (x_\varepsilon - \bar{x})/|x_\varepsilon - \bar{x}|$, that

$$(x_\varepsilon - \bar{x})^2 + 2\sigma\varepsilon^2\bar{p}|x_\varepsilon - \bar{x}| \le 2\varepsilon^2(w(x_\varepsilon) - w(\bar{x})) \le 4\varepsilon^2\|w\|_\infty;$$

then $(|x_\varepsilon - \bar{x}| + \sigma\varepsilon^2\bar{p})^2 \le \varepsilon^2\left(4\|w\|_\infty + \varepsilon^2\bar{p}^2\right)$; and finally $|x_\varepsilon - \bar{x}| \le C\varepsilon$, where $C = (4\|w\|_\infty + \varepsilon^2\bar{p}^2)^{\frac{1}{2}} - \varepsilon\sigma\bar{p}$. So $x_\varepsilon \to \bar{x}$ as $\varepsilon \to 0$. In particular, if $\varepsilon > 0$ is sufficiently small, x_ε is in a neighbourhood of $\bar{x}$ containing only a single interface point, namely $\bar{x}$.

We can say more about x_ε if w is discontinuous at $\bar{x}$. As w is left continuous and non-increasing, there is $\zeta > 0$ such that $w(x) \le w(\bar{x}) - \zeta$ if $x > \bar{x}$. Since w is non-increasing, for $\bar{x} < x \le \bar{x} + C\varepsilon$, with C defined as above, we have that

$$w(x) - \psi_\varepsilon(x) \le -\zeta - \bar{p}(x-\bar{x}) - \frac{(x-\bar{x})^2}{2\varepsilon^2} \le -\zeta + C|\bar{p}|\varepsilon < 0$$

if $\varepsilon > 0$ is sufficiently small. Since $w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \ge 0$, it follows that $x_\varepsilon \le \bar{x}$ if $\varepsilon > 0$ is sufficiently small.

We select a sequence $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$, an index $\ell \in \{j, j+1\}$, and a sequence of maximisers x_k of $w - \psi_{\varepsilon_k}$ such that $x_k \in \mathfrak{X}_\ell$ for all k; by the previous remark, $x_k \in \mathfrak{X}_j$ for all k if w is discontinuous at $\bar{x}$. Actually, we can pick $\xi_1 \in \mathring{\mathfrak{X}}_j$ and $\xi_2 \in \mathring{\mathfrak{X}}_{j+1}$ such that either $x_k \in I_j \equiv (\xi_1, \bar{x})$ or $x_k \in I_{j+1} \equiv (\bar{x}, \xi_2)$ for all k, if necessary by discarding a finite number of the initial x_k.

Let $t > 0$ be sufficiently small such that any trajectory y starting at x_k satisfies $\xi_1 < y(s) < \xi_2$ for all $0 \le s \le t$. Let $\tau_\ell(y) = \inf\{s \ge 0 : y(s) \notin I_\ell\}$. By Lemma F.1, we have for $\theta_\ell(y) = \min\{t, \tau_\ell(y)\}$ that

$$w(x_k) \le \sup_a \left(\int_0^{\theta_\ell(y)} u(y(s), a(s)) \exp(-\rho s) \, \mathrm{d}s + \exp(-\rho\theta_\ell) w(y(\theta_\ell)) \right). \tag{23}$$

For every k, let (y_k, a_k) be a trajectory–action pair starting at x_k that realises the supremum on the right hand side of (23), and let $\theta_\ell^{(k)} = \theta_\ell(y_k)$.

If the alternative B holds, we are done. So assume that it does not hold. Then $\theta_\ell^{(k)} \to 0$ as $k \to \infty$ and $\min\{t, \theta_\ell^{(k)}\} = \theta_\ell^{(k)}$ for k sufficiently large.

Note that $y_k(\theta_\ell^{(k)}) = \bar{x}$. From the fact that $w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \ge 0$, we derive $w(x_k) \ge \psi_{\varepsilon_k}(x_k) \ge w(\bar{x}) + \bar{p}(x_k - \bar{x})$. Combining this with (23) then yields

$$\begin{aligned}
0 &\le \int_0^{\theta_\ell^{(k)}} u(y_k(s), a_k(s)) \exp(-\rho s) \, \mathrm{d}s + (\exp(-\rho\theta_\ell^{(k)}) - 1) w(\bar{x}) - \bar{p}(x_k - \bar{x}) \\
&= \int_0^{\theta_\ell^{(k)}} u(y_k(s), a_k(s)) \exp(-\rho s) \, \mathrm{d}s + (\exp(-\rho\theta_\ell^{(k)}) - 1) w(\bar{x}) \\
&\qquad + \bar{p} \int_0^{\theta_\ell^{(k)}} f(y_k(s), a_k(s)) \, \mathrm{d}s \\
&\le \int_0^{\theta_\ell^{(k)}} \max_{q \in \mathcal{Q}} \left[u(y_k(s), q) \exp(-\rho s) + \bar{p} f(y_k(s), q) \right] \, \mathrm{d}s + (\exp(-\rho\theta_\ell^{(k)}) - 1) w(\bar{x}).
\end{aligned}$$

Dividing by $\theta_\ell^{(k)}$ and taking the limit $k \to \infty$ then yields $0 \le H_\ell(\bar{x}, \bar{p}) - \rho w(\bar{x})$, implying that $\varphi(\bar{p}) = \rho w(\bar{x}) - \max\{H_j(\bar{x}, \bar{p}), H_{j+1}(\bar{x}, \bar{p})\} \le 0$, contradicting the choice of $\bar{p}$. □

F.7 Proof of Proposition C.4

Proof. The continuous function $w - v$ takes on the compact set $\mathcal{X}$ a maximum M at a point $\bar{x}$. Assume that $M > 0$, as otherwise the lemma is proved.

If $\bar{x}$ is neither an interface point nor a boundary point of $\mathcal{X}$, the proof uses the classical "doubling of variables" technique, (see Bardi and Capuzzo-Dolcetta, 2008, Theorem II.3.1) to derive a contradiction.

If $\bar{x} \in \partial\mathcal{X}$, say $\bar{x} = \bar{x}_0$, the case $\bar{x} = \bar{x}_J$ being similar, then (12) and (14) imply either $w(\bar{x}) \le \beta(\bar{x}) \le v(\bar{x})$, contradicting $M > 0$, or that one of the following holds: $\rho w(\bar{x}) - H_1(\bar{x}, p) \le 0$ for all $p \in D^+ w(\bar{x})$, or $\rho v(\bar{x}) - H_1(\bar{x}, p) \ge 0$ for all $p \in D^- w(\bar{x})$. The argumentation proceeds then as in the proof of Bardi and Capuzzo-Dolcetta (2008, Theorem V.4.16).

Hence we only have to consider the situation that $\bar{x}$ is an interface point. According to Proposition C.3, one of two alternatives can obtain. If Alternative A is true, then we have $\rho w(\bar{x}) \leq H^{\mathcal{I}}(\bar{x}) \leq \rho v(\bar{x})$, as the second inequality is implied by (13). This implies that $w(\bar{x}) - v(\bar{x}) = M \leq 0$, a contradiction.

If Alternative B holds, there is $\eta > 0$, $\ell \in \{j, j+1\}$, and a sequence $x_k \to \bar{x}$, such that $x_k \in \mathcal{X}_\ell$ for all k, and for each k there is a trajectory–action pair (y_k, a_k) such that $y_k(0) = x_k$, $y_k(t) \in \mathcal{X}_j$ for all $t \in [0, \eta]$, and

$$w(x_k) \leq \int_0^\eta u(y_k(t), a_k(t)) \exp(-\rho t)\, \mathrm{d}t + w(y_k(\eta)) \exp(-\rho\eta). \tag{24}$$

Moreover, from (20) we obtain that for k sufficiently large

$$v(x_k) \geq \int_0^\eta u(y_k(t), a_k(t)) \exp(-\rho t)\, \mathrm{d}t + v(y_k(\eta)) \exp(-\rho\eta). \tag{25}$$

Combining (24) and (25) yields

$$w(x_k) - v(x_k) \leq (w(y_k(\eta)) - v(y_k(\eta))) \exp(-\rho\eta) \leq M \exp(-\rho\eta).$$

Taking the limit $k \to \infty$ yields then $M \leq M \exp(-\rho\eta) < M$, again a contradiction. We conclude that necessarily $M \leq 0$. □

F.8 Proof of Proposition C.5

Proof. We give the proof for the subsolution case; the supersolution case is similar.

Set $\bar{x} = \bar{x}_j$. By hypothesis, the subsolution property holds for all $x \in \mathcal{X}_{j,+} \backslash \{\bar{x}\}$. Assuming that the statement of the proposition is false, there is a C^1 function ψ such that, firstly, $\psi(\bar{x}) = \bar{w}(\bar{x})$, secondly $\bar{w}(x) - \psi(x)$ restricted to $\mathcal{X}_{j,+}$ is maximal at $\bar{x}$, and finally

$$\rho \bar{w}(\bar{x}) - H_+(\bar{x}, \psi'(\bar{x})) > 0, \tag{26}$$

where $H_+ = H_{j+1}$. Introduce $\Delta(y) = \bar{w}(\bar{x} + y) - \psi(\bar{x} + y) - y^2$. Then Δ is continuous for $y \geq 0$, maximal at $y = 0$, and $\Delta(0) = 0$. Continuity implies that for every $n > 0$ there is $\xi_n > 0$ such that $\Delta(\xi_n) > -1/n$. On the other hand, if $y \geq 2/\sqrt{n}$, then $\Delta(y) \leq -y^2 \leq -4/n$. It follows that $0 < \xi_n < 2/\sqrt{n}$.

Set $\varepsilon_n = \xi_n/n$. The function

$$\Delta(y) - \varepsilon_n/y = \bar{w}(\bar{x} + y) - \Big(\psi(\bar{x} + y) + y^2 + \varepsilon_n/y\Big)$$

satisfies $\Delta(\xi_n) - \varepsilon_n/\xi_n \geq -2/n$ and $\Delta(y) - \varepsilon_n/y \leq -4/n$ if $y \geq 2/\sqrt{n}$. Hence it takes its maximum at a point $0 < y_n < 2/\sqrt{n}$, and, setting $x_n = \bar{x} + y_n$, we have

$$p_n \equiv \psi'(x_n) + 2y_n - \varepsilon_n/y_n^2 \in D^+\bar{w}(x_n).$$

As y_n maximises $\Delta(y) - \varepsilon/y$, we have, first, that $\Delta(y_n) - \varepsilon_n/y_n \geq \Delta(\xi_n) - \varepsilon_n/\xi_n \geq -2/n$, and, second, that $0 < \varepsilon_n/y_n \leq \Delta(y_n) + 2/n \leq 2/n$. Consequently, if $n \to \infty$ we obtain

$$p_n y_n = \psi'(x_n) y_n + 2y_n^2 - \varepsilon_n/y_n \to 0. \tag{27}$$

Since $v, w \in \mathscr{G}$ and one of these functions is discontinuous at $\bar{x}$, the point $\bar{x}$ is a right semi-repeller. In particular this implies that $(H_+)_p(\bar{x}, p) = f_+(\bar{x}, p) \geq 0$ for all p.

Writing $q_n = q^*(x_n, p_n)$, there are $0 < \theta_n^{(1)}$, $\theta_n^{(2)} < 1$ such that

$$\begin{aligned} H_+(x_n, p_n) &= u(\bar{x}, q_n) + u_x(\bar{x} + \theta_n^{(1)} y_n, q_n) y_n + p_n[f_+(\bar{x}, q_n) + (f_+)_x(\bar{x} + \theta_n^{(2)} y_n, q_n) y_n] \\ &= H_+(\bar{x}, p_n) + r_n \leq H_+(\bar{x}, \psi'(x_n) + 2y_n) + r_n, \end{aligned}$$

where we have set $r_n = u_x(\bar{x} + \theta_n^{(1)} y_n, q_n) y_n + (f_+)_x(\bar{x} + \theta_n^{(2)} y_n, q_n) p_n y_n$, and where we have used that $p_n \leq \psi'(x_n) + 2y_n$ as well as the fact that $H_+(\bar{x}, p)$ is nondecreasing in p. As u_x and $(f_+)_x$ are bounded, equation (27) implies that $r_n \to 0$ as $n \to \infty$.

Since w is a subsolution, we have $\rho w(x_n) \leq H_+(x_n, p_n) \leq H_+(\bar{x}, \psi'(x_n) + 2y_n) + r_n$. Taking the limit $n \to \infty$ then yields $\rho \bar{w}(\bar{x}) \leq H_+(\bar{x}, \psi'(\bar{x}))$, contradicting (26). □

F.9 Proof of Proposition D.3

We need a technical result about linearisations (e.g. Cannarsa et al., 2015, Lemma 2.3).

Lemma F.2. *Let $g(t, x)$ be measurable in t and continuously differentiable in x. For $x \in \mathring{\mathfrak{X}}$, denote by $y(t; x)$ the solution to $\dot{y}(t) = g(t, y(t))$, $y(0) = x$. Assume that for $x_0 \in \mathring{\mathfrak{X}}$ there is $T > 0$ such that $y(t; x_0) \in \mathring{\mathfrak{X}}$ for all $t \in [0, T]$. Let Φ be the absolutely continuous solution of the linear system*

$$\dot{\Phi}(t) = g_x(t, y(t; x_0))\Phi(t), \quad \Phi(0) = 1.$$

Then for all x in a neighbourhood of x_0 in $\mathring{\mathfrak{X}}$, we have for $t \in [0, T]$

$$y(t; x) = y(t; x_0) + \Phi(t)(x - x_0) + o_t(|x - x_0|),$$

with $o_t(|x - x_0|)/|x - x_0| \to 0$ as $x \to x_0$, uniformly in t.

In the proof below, the *lower Dini directional derivative* is used, which for a continuous function $W(x)$ is defined as $\partial^- W(x; \xi) = \liminf_{h \downarrow 0}(W(x + h\xi) - W(x))/h$. Unlike an ordinary derivative, this derivative exists for all x and ξ. Clearly, if W is differentiable at x, then $\partial^- W(x; \xi) = W'(x)\xi$ for all ξ.

Proof of Proposition D.3. Let $I = y^*([0, T])$ be the orbit of the optimal trajectory. If y^* is constant, then I consists of a single point and V is differentiable on all of I. If y^* is nonconstant, then I has positive length and by Proposition D.2 the value function V is differentiable on a dense subset $S \subset I$.

We first establish a relation between the derivatives V' on different points in S using a linearisation argument. Then we show that V' restricted to S is continuous, which will finally imply that V' exists everywhere in I.

For $z \in \mathring{\mathfrak{X}}$, let $y(t;z)$ and $\Phi(t)$ be, respectively, the solutions of $\dot{y}(t;z) = f(y(t;z), a^*(t))$ and $y(0;z) = z$, and of

$$\dot{\Phi}(t) = f_x(y(t;z), a^*(t))\Phi(t), \quad \Phi(0) = 1.$$

Choose $\xi \in \mathbb{R}$ arbitrarily, and take $h > 0$ such that $y(t; x + h\xi) \in \mathring{\mathfrak{X}}$ for all $t \in [0, T]$. By the optimality principle,

$$V(x + h\xi) \geq \int_0^t \exp(-\rho s) u(y(s; x + h\xi), a^*(s)) \, \mathrm{d}s + V(y(t; x + h\xi)) \exp(-\rho t).$$

For the optimal pair (y^*, c^*), we have

$$V(x) = \int_0^t \exp(-\rho s) u(y^*(s), a^*(s)) \, \mathrm{d}s + V(y^*(t)) \exp(-\rho t).$$

Differentiability of V at x implies

$$\begin{aligned}
V'(x)\xi &= \liminf_{h \downarrow 0} (V(x + h\xi) - V(x))/h \\
&\geq \liminf_{h \downarrow 0} \Big(h^{-1} \int_0^t \exp(-\rho s) \Big(u(y(s; x + h\xi), a^*(s)) - u(y^*(s), a^*(s)) \Big) \, \mathrm{d}s \\
&\qquad\qquad + \exp(-\rho t) \frac{V(y(t; x + h\xi)) - V(y^*(t))}{h} \Big) \\
&= \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s)) \Phi(s) \xi \, \mathrm{d}s + \exp(-\rho t) \partial^- V(y^*(t); \Phi(t)\xi),
\end{aligned}$$

where in the last equality Lemma F.2 has been used.

For $t \in [0, T]$ such that $y^*(t) \in S$, we find

$$V'(x)\xi \geq \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s)) \Phi(s) \xi \, \mathrm{d}s + \exp(-\rho t) V'(y^*(t)) \Phi(t) \xi.$$

Taking successively $\xi = 1$ and $\xi = -1$ yields

$$V'(x) = \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s)) \Phi(s) \, \mathrm{d}s + \exp(-\rho t) V'(y^*(t)) \Phi(t). \tag{28}$$

As $\Phi(t) \neq 0$, for all $t \in [0, T]$ we define a function $\hat{p}(t)$ by the relation

$$V'(x) = \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s)) \Phi(s) \, \mathrm{d}s + \exp(-\rho t) \hat{p}(t) \Phi(t).$$

Clearly $\hat{p}(0) = V'(x)$. Differentiating this relation with respect to t shows moreover that $\hat{p}$ satisfies (15), and consequently that $\hat{p}(t) = p^*(t)$ for all t. We then infer from (28) that $p^*(t) = V'(y^*(t))$ whenever $y^*(t) \in S$.

Take $z \in S$, and consider a sequence $z_n \in S$ such that $z_n \to z$ as $n \to \infty$. Find a sequence t_n such that $z_n = y^*(t_n)$. If necessary after passing to a subsequence we may assume—as $[0, T]$ is compact—that $t_n \to \tau$, and therefore $V'(z_n) = V'(y^*(t_n)) = p(t_n) \to p(\tau) = V'(z)$ as $n \to \infty$. Hence V' is continuous on S, and can uniquely be extended to a continuous function on I, which implies that V is continuously differentiable on I.

To prove the last statement, differentiate the equality

$$V(x) = \int_0^t \exp(-\rho s) u(y^*(s), a^*(s))\,\mathrm{d}s + \exp(-\rho t) V(y^*(t))$$

with respect to t and divide by $\exp(-\rho t)$ to obtain

$$\rho V(y^*(t)) = u(y^*(t), a^*(t)) + V'(y^*(t)) f(y^*(t), a^*(t)).$$

Since V is a supersolution and $p^*(t) = V'(y^*(t))$, we have $\rho V(y^*(t)) \geq H(y^*(t), p^*(t))$, which reads as

$$u(y^*(t), a^*(t)) + V'(y^*(t)) f(y^*(t), a^*(t)) \geq \max_q (u(y^*(t), q) + p^*(t) f(y^*(t), q)),$$

the last statement of the proposition. □

F.10 Proof of Proposition D.4

We begin by proving the first three statements of Proposition D.4.

Proof of Proposition D.4, Statements (i)–(iii).

(i) If y^* is not constant, then $\dot{y}^*(0) \neq 0$ and y^* is locally invertible on an interval $[0, \varepsilon_0)$. Consequently for every $0 < \varepsilon < \varepsilon_0$ there is $0 < t_1 < \varepsilon$ such that $y^*(t_1) \in \mathcal{D}_1$, as $\mathcal{D}_1$ is dense. But then $y^*(t) \in \mathcal{D}_1$ for all $t \geq t_1$ by Proposition D.3. As $\varepsilon > 0$ was arbitrary, this shows that $y^*(t) \in \mathcal{D}_1$ for all $t > 0$ such that $y^*(t) \in \mathring{\mathcal{X}}$.

(ii) Let $\tau > 0$ be such that $y^*(\tau) \in \mathcal{D}_1$. The trajectory (y^τ, p^τ) starting at the point $(y^*(\tau), V'(y^*(\tau)))$ satisfies $y^\tau(t) = y^*(\tau + t)$: in particular, we have $y^*(0) = y^\tau(-\tau)$ and $p_0 = p^\tau(-\tau)$.

(iii) Monotonicity of y^* has been shown in, e.g., Wagener (2003). □

In the proof of the last statement of Proposition D.4, and several other results, we shall use the invariant manifold theorem (Hirsch et al., 1977; Takens and Vanderbauwhede, 2010), which for a planar vector field ensures the existence of invariant curves that are tangent to the eigenspaces of a steady state.

More precisely, let $\bar{\zeta}$ be a steady state of a real analytic planar vector field $Y : \mathbb{R}^2 \to \mathbb{R}^2$, whose linearisation $DY(\bar{\zeta})$ has eigenvalues λ_1 and λ_2. If $\lambda_1 < 0 < \lambda_2$, there are unique and real analytic invariant manifolds W^{s} and W^{u} tangent, respectively, to the eigenspace E_1

of λ_1 and E_2 of λ_2 at the steady state, the *stable* and *unstable* manifolds. If $\lambda_1 = 0 < \lambda_2$, there exists a, not necessarily unique, C^∞ invariant manifold W^{cs} tangent to E_1 at the steady state, the *centre-stable* manifold, and a unique real analytic unstable manifold W^{u} tangent to E_2. If $0 < \lambda_1 < \lambda_2$, then there exists a unique and real analytic invariant manifold W^{uu} tangent to E_2 at the steady state, called the *strongly unstable* manifold. If $0 < \lambda_1 < \lambda_2$, all trajectories not on W^{uu} are tangent to E_1 and can be parametrised as the graph of a C^1 function $w : E_1 \to E_2$.

The eigenvalues and eigenspaces that correspond to a given invariant manifold are denoted by the same superscript: e.g. the centre-stable manifold W^{cs} is tangent to the centre-stable eigenspace E^{cs} of λ^{cs}.

We are mostly concerned with centre-stable manifolds. These manifolds are in general not unique and only infinitely often differentiable, not real analytic. However, the following result provides a condition for unicity and analyticity of the centre-manifold.

Theorem F.1 (Aulbach (1986))**.** *Let $\bar{\zeta}$ be a steady state of a real analytic vector field $Y : N \to \mathbb{R}^2$, where N is a neighbourhood of $\bar{\zeta}$ in $\mathbb{R}^2$. Let $\lambda_1 = 0$ and $\lambda_2 > 0$ be the eigenvalues of $DY(\bar{\zeta})$, and let E_1 and E_2 be the corresponding eigenspaces.*

If every neighbourhood of $\bar{\zeta}$ contains a fixed point of Y different from $\bar{\zeta}$, then there is a disk $D \subset N$ of positive radius, centred at $\bar{\zeta}$, and a unique analytic local centre–stable manifold $W^{\text{cs}} \subset D$, tangent to E_1, such that all points on W^{cs} are steady states of Y.

The next result solves the Hamilton–Jacobi–Bellman equation if the action schedule takes a corner value q_b, $b \in \{\ell, u\}$, if we set $g(x) = f(x, q_b)$ and $v(x) = u(x, q_b)$.

Proposition F.1. *Let $g(x)$ and $v(x)$ be real analytic, and let $\bar{x}$ be such that $g(\bar{x}) = 0$. Consider for $\rho > 0$ the differential equation*

$$\rho V(x) - v(x) - V'(x)g(x) = 0. \tag{29}$$

(i) Equation (29) has bounded solutions V, all of which satisfy $V(\bar{x}) = v(\bar{x})/\rho$.
(ii) If $g'(\bar{x}) < \rho$, each solution is continuously differentiable and $V'(\bar{x}) = v'(\bar{x})/(\rho - g'(\bar{x}))$.
(iii) If $g'(\bar{x}) < 0$, the solution V is unique and real analytic.

Proof. For g identically zero $V(x) = v(x)/\rho$ is the unique real analytic solution of (29).

Otherwise $\bar{x}$ is an isolated zero of g. Let $N = (x_1, x_2)$ be an open interval containing $\bar{x}$. Restricted to $N \times \mathbb{R}$, the graph of a differentiable solution V of (29) is a union of orbits of

$$\dot{y} = g(y), \quad \dot{w} = \rho w - v(y)$$

as $w(t) = V(y(t))$. This system has a unique steady state $(\bar{x}, \bar{w}) = (\bar{x}, v(\bar{x})/\rho)$. The linearisation $\begin{pmatrix} g'(\bar{x}) & 0 \\ -v'(\bar{x}) & \rho \end{pmatrix}$ at steady state has eigenvalues $\lambda_1 = g'(\bar{x})$ and $\lambda_2 = \rho > 0$, and corresponding eigenspaces $E_1 = \mathbb{R}\begin{pmatrix} \rho - g'(\bar{x}) \\ v'(\bar{x}) \end{pmatrix}$ and $E_2 = \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $\lambda_1 > 0$, the steady state is a repeller. For N sufficiently small there are two trajectories $(y_i(t), w_i(t))$, $i = 1, 2$, converging to the steady state as $t \to -\infty$, with $y_i(t_i) = x_i$ for some t_i. As $\dot{y}_i(t) \neq 0$ if $t < t_i$, these trajectories yield a continuous solution on N by setting $V(y_i(t)) = w_i(t)$ and $V(\bar{x}) = v(\bar{x})/\rho$. If $\lambda_1 \leq 0$, the centre-stable manifold W^{cs} of the steady state is tangent to E_1 and the graph of a bounded solution. This shows (i).

If $\lambda_1 < \rho$, the manifold E_2 is invariant, and any bounded solution trajectory not on E_2 is on a manifold W_1 tangent to E_1 at the steady state, which is the graph of a C^1 solution V to (29). The gradient of V at the steady state is the inclination of the eigenspace E_1, which evaluates to $V'(\bar{x}) = v'(\bar{x})/(\rho - g'(\bar{x}))$, showing (ii).

If $\lambda_1 < 0$, the manifold W^1 is the stable manifold of the steady state, which is unique and real analytic, completing the proof. □

Lemma F.3. *Let (y, p) be a canonical trajectory such that $y(t) \to \bar{x}_\pm$ as $t \to \pm\infty$. Then $p(t) \to \bar{p}_\pm$ as $t \to \pm\infty$, with $\bar{p}_\pm \in \mathbb{R} \cup \{-\infty, \infty\}$.*

Proof. We have to show that the set $\{t \geq 0 : p(t)\}$ has at most one accumulation point.

Assume that $\bar{p}_1 < \bar{p}_2$ are two distinct accumulation points. Then for all $n > 0$ there are $t_{1,n} < t_{2,n} < t_{1,n+1}$ such that $t_{1,n}$, $t_{2,n} \to \infty$ and $p(t_{1,n}) \to \bar{p}_1$, $p(t_{2,n}) \to \bar{p}_2$ as $n \to \infty$. Taking $p \in (\bar{p}_1, \bar{p}_2)$, for all $n > 0$ there are $t_{1,n} < t_{3,n} < t_{2,n} < t_{4,n} < t_{1,n+1}$ such that $p(t_{3,n}) = p(t_{4,n}) = p$ and $\dot{p}(t_{3,n}) > 0$ and $\dot{p}(t_{4,n}) < 0$. We conclude that the second component X_2 of the canonical vector field, and hence X itself, must vanish for all $(\bar{x}, p)$ with $p \in [\bar{p}_1, \bar{p}_2]$. As $\partial X_1/\partial p \neq 0$ on $\mathcal{P}_{\text{int}}$, we must have that $\bar{p}_1 \geq p_u(\bar{x})$ or $\bar{p}_2 \leq p_\ell(\bar{x})$. But for $(\bar{x}, p) \in \mathcal{P}_\ell \cup \mathcal{P}_u$, the conditions $X_2 = (\rho - f_x)p - u_x = 0$ and $\partial X_2/\partial p = (\rho - f_x) = 0$ imply $u_x = 0$, contradicting Assumption 2. The argument for $\{t \leq 0 : p(t)\}$ is analogous. □

Proof of Proposition D.4, Statement (iv). If the first alternative does not hold, we have $y^*(t) \in \mathcal{X}$ for all $t \geq 0$. Let (y, p) be the canonical trajectory such that $y^* = y$, whose existence is guaranteed by (ii). Then (iii) implies that $y(t) \to \bar{x} \in \mathcal{X}$ as $t \to \infty$.

Lemma F.3 implies that $p(t)$ converges to a limit as $t \to \infty$, or diverges to ∞, or to $-\infty$. Proposition A.3 rules out the second possibility. We prove the result by showing that the third possibility cannot occur either.

If $p(t) \to -\infty$ as $t \to \infty$, there is t_1 such that $(y(t), p(t)) \in \mathcal{P}_\ell$ for all $t \geq t_1$. Introduce $v(x) = u(x, q_\ell)$ and $g(x) = f(x, q_\ell)$. Then $H(x, p) = v(x) + pg(x)$ and $\dot{y}(t) = g(y(t))$ for all $t \geq t_1$. In particular $g(\bar{x}) = 0$ and $g'(\bar{x}) \leq 0$.

Set $I = y((t_1, \infty))$. By (i), on I the value function is differentiable and satisfies (29). Proposition F.1 implies that $V'(\bar{x}) = v'(\bar{x})/(\rho - g'(\bar{x})) \geq v'(\bar{x})/\rho$. As $p(t) \to V'(\bar{x})$ for $t \to \infty$, we have reached the desired contradiction. □

F.11 Proofs of Propositions D.5 and D.6

First we note an implication of real analyticity. For $b \in \{\ell, u\}$ and $z \in \mathcal{S}_b$, let $n_b(z)$ be the unit normal vector to $\mathcal{S}_b$ pointing out of $\mathcal{P}_{\text{int}}$ at z, and let $\omega_b(x) = n_b(x, p_b(x)) \cdot X(x, p_b(x))$.

Lemma F.4. *Let $b \in \{\ell, u\}$ and let $C \subset \mathring{\mathcal{X}}$ be a compact set.*

(i) If the cardinality of $\{x \in C : f(x, q_b) = 0\}$ is infinite, then $f(x, q_b) = 0$ for all $x \in \mathcal{X}$.

(ii) If the cardinality of $\{x \in C : \omega_b(x) = 0\}$ is infinite, then $\mathcal{S}_b$ is invariant under X.

Proof. Both assertions follow from the fact that a real analytic function whose zeros have an accumulation point vanishes identically. □

Proof of Proposition D.5. Assume the statement is false. Then for every $n > 0$ there are initial points $\tilde{x}_n$ located in different non-constant optimal orbits. We may assume that they are ordered in an increasing or decreasing sequence: we prove the result for the increasing case, the other being similar. By Proposition D.4.(ii) there are $\tilde{p}_{n,0}$ such that the non-constant trajectory starting at $\tilde{x}_n$ is the state component of the optimal canonical trajectory $(\tilde{y}_n, \tilde{p}_n)$ starting at $(\tilde{x}_n, \tilde{p}_{n,0})$. Proposition D.4.(iv) implies that this canonical trajectory converges to a point $(\bar{x}_n, \bar{p}_n)$ with $\bar{x}_n \in (\tilde{x}_n, \tilde{x}_{n+1})$; necessarily $\bar{p}_n \leq 0$.

Introduce $T_n = \min\{\tau \geq 0 : (\tilde{y}_n(t), \tilde{p}_n(t)) \in \mathcal{P}_b \text{ for all } t \geq \tau\}$. If $0 < T_n < \infty$ infinitely often, or after relabelling, for all $n > 0$, then the points $(\bar{x}_n, \bar{p}_n)$ are steady states of X in $\mathcal{P}_b$ and $f(\bar{x}_n, q_b) = 0$; applying Lemma F.4, it follows that $f(x, q_b) = 0$ for all x. But then $f_x(x, q_b) = 0$ for all x as well, and $X_2(x, p) = \rho p - u_x(x, q_b)$ if $(x, p) \in \mathcal{P}_b$. As $X_2(\bar{x}_n, \bar{p}_n) = 0$, it follows that $\omega_b(\bar{x}_n) < 0$. On the other hand, for $\hat{x}_n = \tilde{y}_n(T_n)$, we have $\omega_b(\hat{x}_n) \geq 0$. Hence ω_b vanishes in the interval $(\tilde{x}_n, \tilde{x}_{n+1})$; Lemma F.4 then implies that $\mathcal{S}_b$ is invariant. This however contradicts that $T_n > 0$.

If $T_n = 0$ infinitely often, it follows as above that $f(x, q_b) = 0$ for all x and $\tilde{y}_n(t) = \tilde{x}_n$ for all t, contradicting that $\tilde{x}_n$ is located in a non-constant optimal orbit.

Consider next the situation that $T_n = \infty$ for all n sufficiently large and $(\bar{x}_n, \bar{p}_n) \notin \mathcal{P}_{\text{int}}$. This implies that $(\bar{x}_n, \bar{p}_n) \in \mathcal{S}_b$. Since these points are steady states, $\omega_b(\bar{x}_n) = 0$, and by Lemma F.4 the set $\mathcal{S}_b$ is invariant and $f(x, q_b) = 0$ for all x. But then $\mathcal{S}_b$ is the centre–stable manifold for $(\bar{x}_n, \bar{p}_n)$, and there is no non-constant trajectory that tends to $(\bar{x}_n, \bar{p}_n)$ as $t \to \infty$, which contradicts the choice of $(\tilde{y}_n, \tilde{p}_n)$.

We are left with the situation that $(\bar{x}_n, \bar{p}_n) \in \mathcal{P}_{\text{int}}$ for n sufficiently large. A subsequence of these points converges to a steady state $(\bar{x}, \bar{p}) \in \overline{\mathcal{P}_{\text{int}}}$ of X. Hence by the Aulbach theorem, the invariant centre-stable manifold of $(\bar{x}, \bar{p})$ consists of steady states and contains $(\bar{x}_n, \bar{p}_n)$ for n sufficiently large, again contradicting the fact that $(\tilde{y}_n, \tilde{p}_n)$ is non-constant. □

Proposition F.2. *A compact set $C \subset \mathring{\mathcal{X}}$ contains only finitely many switching points of any optimal orbit.*

Proof. Let I be a non-constant optimal orbit, $b \in \{\ell, u\}$, $y : \mathcal{T} \to I$ a state trajectory parametrising I, and p such that (y, p) is the canonical trajectory associated to y.

Assume that there are infinitely many switching points in I. There is an increasing sequence $t_1 < t_2 < \dots$ in $\mathcal{T}$ such that $(y(t_{2k-1}), p(t_{2k-1})) \in \mathcal{P}_{\text{int}}$ and $(y(t_{2k}), p(t_{2k})) \in \mathcal{P}_b$ for all $k > 0$. Consequently, there are times $t_1 < \hat{t}_1 < t_2 < \hat{t}_2 < \dots$ such that $(y(\hat{t}_n), p(\hat{t}_n)) = (\hat{x}_n, p_b(\hat{x}_n)) \in \mathcal{S}_b$, and such that $\omega_b(\hat{x}_{2k-1}) \geq 0$ and $\omega_b(\hat{x}_{2k}) \leq 0$ for all k. Hence ω_b vanishes in the interval $(\hat{x}_n, \hat{x}_{n+1})$ for every $n \geq 0$. By Lemma F.4, the set S_b is invariant, which contradicts the existence of switching points. □

Proof of Proposition D.6. If $x_0 \in \mathring{I} \cap \mathring{\mathcal{X}}_j$, then there is a non-constant optimal trajectory $y : \mathcal{T} \to I$, with $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = [0, \infty)$, such that $x_0 = y(t_0)$ with $t_0 \in \mathring{\mathcal{T}}$. By Proposition D.4, V is differentiable in a neighbourhood of x_0 and there is a canonical trajectory (y, p) such that $V'(y(t)) = p(t)$ for all $t > 0$ and such that $\dot{y}(t)$ does not change sign. Moreover, if x_0 is not a switching point, then $(y(t), p(t))$ is real analytic for t close to t_0, since it is locally the trajectory of a real analytic vector field X. Hence we can solve $x = y(t)$ as $t = y^{-1}(x)$ around x_0, and obtain that $V'(x) = p(t) = p(y^{-1}(x))$ is real analytic.

It remains to show that ϕ can be extended to a differentiable function on an open interval containing I. Let $\bar{t} \in \partial\mathcal{T}$: that is, $\bar{t} \in \{0, -\infty, \infty\}$. Proposition D.4.(ii) and Lemma F.3 imply that $(y(t), p(t))$ converges either to $(\bar{x}, \infty)$, or to $(\bar{x}, -\infty)$, or to a finite limit $(\bar{x}, \bar{p})$, as $t \to \bar{t}$. In the first and second case we respectively have $\phi(y(t)) = q^*(y(t), p(t))$ $q^*(y(t), p(t)) = q_u$ and $q^*(y(t), p(t)) = q_\ell$ for t in a neighbourhood of $\bar{t}$, and it is clear that ϕ can be differentiably extended.

In the third case $(y(t), p(t))$ tends to a steady state $\bar{z} = (\bar{x}, \bar{p})$ of the canonical vector field as $t \to \bar{t}$. Lemma F.2 implies that there is $0 < t_1 < \bar{t}$ such that $z(t) = (y(t), p(t))$ does not pass through a switching point for $t \in (t_1, \bar{t})$. If $z(t) \in \mathcal{P}_b$ for $b \in \{\ell, u\}$ for $t \in N$, then $\phi(y(t)) = q_b$ for those values of t, and we conclude as above. If $z(t) \in \mathcal{P}_{\text{int}}$ for all $t \in N$, the trajectory is tangent to an eigenspace E of $DX_I(\bar{z})$, where X_I is the real analytic extension to $\mathcal{X} \times \mathbb{R}$ of the restriction of X to $\mathcal{P}_{\text{int}}$. The fact that $H_{pp}(\bar{z}) > 0$ implies, first, that none of these eigenspaces is vertical, and, second, if $DX_I(\bar{z})$ has an eigenvalue with algebraic multiplicity 2, then the geometric multiplicity is 1.

As in all cases the eigenspaces are one-dimensional non-vertical lines, it follows that $V''(y(t)) = X_2(z(t))/X_1(z(t))$ converges to the inclination $\bar{w}$ of E with respect to the horizontal axis as $t \to \bar{t}$. Consequently $\phi'(y(t)) = q_x^*(y(t), p(t)) + q_p^*(y(t), p(t))V''(y(t))$ converges to $q_x^*(\bar{z}) + q_p^*(\bar{z})\bar{w}$. □

F.12 Proof of Proposition E.1

Proof. The dynamics take the form $\dot{y}(t) = \mathbf{f}^{\phi}(y(t))$. Given a trajectory y with initial state x, the payoff at x is

$$V_i^{\phi}(x) = \int_0^{\Theta} \exp(-\rho t)\mathbf{u}^{\phi}(y(t))\,\mathrm{d}t + \exp(-\rho\Theta)\beta(y(\Theta)).$$

Properties (i), (ii) and (iii) are immediate.

As f_j^{ϕ} is real analytic on $\mathring{\mathcal{X}}_j$, we either have that $f_j^{\phi}(x) = 0$ for all $x \in \mathcal{X}_j$, or the set $\mathcal{E}_j = \left\{x \in \mathcal{X}_j \,:\, f_j^{\phi}(x) = 0\right\}$ is finite. We set $\mathcal{E} = \cup_j \mathcal{E}_j$. If f_j^{ϕ} is identically zero, then $V_i^{\phi}(x) = u_{i,j}^{\phi}(x)/\rho$ is real analytic on $\mathcal{X}_j$. If not, take $x \in \mathcal{X}_j \backslash \mathcal{E}_j$. As $f_j^{\phi}(x) \neq 0$, we have

$$\begin{aligned} V_i^{\phi}(x + f_j^{\phi}(x)t + o(t)) - V_i^{\phi}(x) &= V_i^{\phi}(y(t)) - V_i^{\phi}(x) \\ &= (\exp(\rho t) - 1)V_i^{\phi}(x) - \exp(\rho t)\int_0^t \exp(-\rho s)\mathbf{u}_i^{\phi}(y(s))\,\mathrm{d}s, \end{aligned}$$

which on dividing by t and taking the limit $t \to 0$ yields, first, that the limit of the left hand expression exists, and, second, that it equals

$$\left(V_i^{\phi}\right)'(x)\, f_j^{\phi}(x) = \rho V_i^{\phi}(x) - u_{i,j}^{\phi}(x). \tag{30}$$

Note that the graph of solutions of (30) consists of trajectories of the dynamical system

$$\dot{y} = f_j^{\phi}(y), \quad \dot{v} = -u_{i,j}^{\phi}(y) + \rho v. \tag{31}$$

Then Proposition F.1 implies that V_i^{ϕ} is continuous on $\mathring{\mathcal{X}}_j$ and real analytic on $x \in \mathring{\mathcal{X}}_j \backslash \mathcal{E}$, showing (iv). Now (v) is also straightforward.

Let $\bar{x}$ be such that $f_j^{\phi}(\bar{x}) = 0$. If $(f_j^{\phi})'(\bar{x}) < \rho$, V_i^{ϕ} is differentiable at $\bar{x}$ by Proposition F.1. If $(f_j^{\phi})'(\bar{x}) \geq \rho$, trajectories of (31) tending to $(\bar{x}, u_{i,j}^{\phi}(\bar{x})/\rho)$ are tangent to an eigenspace of the linearisation of (31) at the steady state, showing that the limit of $(V_i^{\phi})'(x)$ as $x \to \bar{x}$ exist, even if it is possibly infinite. This shows (vi). □