# Bifurcations of Optimal Vector Fields

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**Abstract**

We study the structure of the solution set of a class of infinite-horizon dynamic programming problems with one-dimensional state spaces, as well as their bifurcations as problem parameters are varied. The solutions are represented as the integral curves of a multi-valued 'optimal' vector field on state space. Generically, there are three types of integral curves: stable points, open intervals that are forward asymptotic to a stable point and backward asymptotic to an unstable point, and half-open intervals that are forward asymptotic to a stable point and backward asymptotic to an indifference point; the latter are initial states to multiple optimal trajectories. We characterize all bifurcations that occur generically in one- and two-parameter families. Most of these are related to global dynamical bifurcations of the state-costate system of the problem.

*Keywords:* infinite horizon problems; multiple optimizers; indifference points; optimal vector fields; bifurcations

## 1 Introduction

The investigation of an economic dynamic optimization problem that features a globally attracting steady state reduces mostly to a quantitative quasi-static analysis of this state, determining the rates of change of the position of the steady state and the value of the objective functional as certain key parameters are varied. In contrast, if there are more than one attracting steady state in the system, or more generally, more than one attracting set, the question arises towards which of these the system is driven by the optimal policy. Put differently, in the presence of a single globally attracting steady state, optimal policies can differ only in degree; if there are multiple attracting states, they may also differ in kind.

Since the late 1970's, optimal policies that are qualitatively different have been found in many economic models: in growth theory they have been used to explain poverty traps (Dechert and Nishimura [5], Skiba [21]); in fisheries, they can model the coexistence of conservative versus overexploiting policies (Clark [4]); there are environmental models where both industry-promoting but polluting as well as ecologically conservative policies are optimal in the same

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model, depending on the initial state of the environment (Kiseleva and Wagener [13], Mäler *et al.* [14], Tahvonen and Salo [22], Wagener [24]); in migration studies, active relocation as well as no action policies occur in the same model (Caulkins *et al.* [2]); optimal advertising efforts may depend on the initial awareness level of a product (Sethi [18, 20]); the successful containment of epidemics may depend on the initial infection level (Rowthorn and Toxvaerd [17], Sethi [19]); in the control of illicit drug use, high law enforcement as well as low enforcement and treatment of drug users can depend on the initial level of drug abuse (Feichtinger and Tragler [6], Tragler *et al.* [23]); in R&D policies of firms, the optimal decision between high R&D expenditure investment to develop a technology versus low investment to phase a technology out may depend on the initial technology level (Hinloopen *et al.* [11]).

In all such models, there is for certain parameter configurations a critical state where both kinds of policy are simultaneously optimal, and where the decision maker is consequently indifferent between them. These points will be called *indifference points* in the following, though they go by many other names as well[1].

Usually, the presence of an indifference point is established numerically for a fixed set of parameter values of the model. In order to study the dependence of the qualitative properties of the optimal policies on the system parameters, it is possible in principle to do an exhaustive search over all parameter combinations. Such a strategy, while feasible, would however be very computing intensive.

A different approach is suggested by the theory of bifurcations of dynamical systems: to identify only those parameter configurations at which the qualitative characteristics of the solutions change. For instance, in Wagener [24] it was shown that indifference points disappear if a heteroclinic bifurcation of the state-costate system occurs. This mechanism, for which we propose the term *indifference-attractor bifurcation*, relates the change of the solution structure of the optimal control problem to a global bifurcation of the state-costate system.

The present article conducts a systematic study of the bifurcations of infinite horizon optimal control problems on the real line that are expected to occur in one- and two-parameter families. The theory developed here has already been applied in several places (Caulkins *et al.* [3], Graß [8], Kiseleva and Wagener [13], Wagener [24]).

# 2 Setting

## 2.1 Definitions

Let $X \subset \mathbb{R}$ be an open interval, and $U \subset \mathbb{R}^r$ a closed convex set with non-empty interior. Let $\rho > 0$ be a positive constant and $f : X \times U \to \mathbb{R}$, $g : X \times U \to \mathbb{R}$ be infinitely differentiable, or *smooth*, in the interior of $X \times U$, and such that all derivatives can be extended continuously to a neighborhood of $X \times U$. Finally, let $\xi \in X$.

Set

$$H = g(x, u) + pf(x, u)$$

and assume that

$$\frac{\partial^2 H}{\partial u^2}(x, p, u) < 0 \tag{1}$$

for all $(x, p, u) \in X \times \mathbb{R} \times U$

[1]For instance Skiba points, Dechert-Nishimura-Skiba points, Dechert-Nishimura-Sethi-Skiba points, regime switching thresholds, Maxwell points, shocks etc.

Consider the problem to maximize

$$J(x,u) = \int_0^\infty g(x(t),u(t))\mathrm{e}^{-\rho t}\mathrm{d}t \tag{2}$$

over the space of state-control trajectories (or programs) $(x,u)$ that satisfy

1. the function $u : [0,\infty) \to U$ is locally Lebesgue integrable over $[0,\infty)$ and essentially bounded; that is, $u \in L^\infty([0,\infty),U)$

2. the function $x : [0,\infty) \to X$ is absolutely continuous and satisfies

$$\dot{x} = f(x,u) \tag{3}$$

   almost everywhere;

3. the initial value of $x$ is given as $x(0) = \xi$.

This problem will be referred to as *infinite horizon problem* in the following. A solution $(x,u)$ to the problem is usually called a *maximiser* or a *maximising trajectory*.

**Assumption 2.1.** *In the infinite horizon problem, for every $\xi \in X$ there exists at least one maximizer $(x,u)$ satisfying $x(0) = \xi$.*

Maximizing trajectories enjoy the following time invariance property, which is commonly known as the dynamic optimization principle.

**Theorem 2.1.** *If the trajectory $(x(.),u(.))$ solves the infinite horizon problem with initial condition $\xi$, then for any $\tau > 0$, the time shifted trajectory $(x(\tau + .),u(\tau + .))$ solves the infinite horizon problem with initial condition $x(\tau)$.*

Define the *maximized Hamiltonian* as

$$\mathcal{H}(x,p) = \max_{u\in U}\{g(x,u) + pf(x,u)\}$$

Assumption (1) implies that the maximum is taken at a unique point $u = v(x,p)$, where $v$ depends smoothly on its arguments; consequently, the function $\mathcal{H}$ is smooth as well.

For a maximising state trajectory $x$, there exists a continuous costate trajectory $p$ satisfying the reduced canonical equations

$$\dot{x} = F_1 = \mathcal{H}_p, \qquad \dot{p} = F_2 = \rho p - \mathcal{H}_x, \tag{4}$$

which define the reduced canonical vector field $F = (F_1,F_2)$. Moreover, $x$ and $p$ satisfy the transversality condition

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}p(\tilde{x} - x) \leq 0 \tag{5}$$

for all admissible trajectories $\tilde{x}$. Trajectories of the state-costate equations (4) are classically called *extremal*. Extremal trajectories that satisfy the transversality condition (5) will be called *critical* in the following. Note that a noncritical trajectory cannot be a maximizer.

Recall that the *power set* $\mathcal{P}(S)$ of a set $S$ is the set of all subsets of $S$.

**Definition 2.1.** *The **optimal costate rule** is the set valued map $p^o : X \to \mathcal{P}(\mathbb{R})$ with the property that if $\eta \in p^o(\xi)$, then the solution of the reduced canonical equations with initial value*

$$(x(0), p(0)) = (\xi, \eta)$$

*maximises the integral $J$. Associated to it are the **optimal feedback rule***

$$u^o(x) = v(x, p^o(x)),$$

*and the **optimal vector field***

$$f^o(x) = \mathcal{H}_p(x, p^o(x)) = f(x, u^o(x))$$

*which are both set-valued as well.*

A map $x : [0, \infty) \to X$ is a trajectory of an optimal vector field if

$$\dot{x}(t) \in f^o(x(t))$$

for all $t \geq 0$.

The solution trajectories of an optimal vector field solve the associated maximization problem. Note that an optimal vector field is commonly called a 'regular synthesis' in the literature.

**Theorem 2.2.** *The sets $p^o(x(t))$ and $f^o(x(t))$ are single-valued for all $t > 0$.*

*Proof.* See Fleming and Soner [7], p. 44, corollary I.10.1. □

## 2.2 Indifference points

The following definition is one of the possible interpretations of the notion of 'Skiba point'.

**Definition 2.2.** *If $\xi \in X$ is such that there are maximizers $x_1$, $x_2$ of the infinite horizon problem with $x_1(0) = x_2(0) = \xi$ and $x_1(t) \neq x_2(t)$ for some $t \in [0, \infty)$, then $\xi$ is called an **indifference point**. The totality of indifference points form the **indifference set**; its complement in $X$ is the **domain of uniqueness**.*

In one-dimensional problems an indifference point is an initial point of two trajectories that have necessarily different long run behaviour. It is worthwile to note that this is not true for problems with higher dimensional state spaces, or for discrete time problems (see Moghayer and Wagener [15]).

**Definition 2.3.** *The $\omega$-**limit set** $\omega(x)$ of a state trajectory $x$ is given as*

$$\omega(x) = \{\xi \in X \,:\, x(t_i) \to \xi \text{ for some increasing sequence } t_i \to \infty\}.$$

Using $\omega$-limit sets, threshold points can be defined.

**Definition 2.4.** *A point $\xi \in X$ is a **threshold point**, if in every neighbourhood $N$ of $\xi$ there are two states $\xi_1, \xi_2 \in N$ that are initial states to state trajectories $x_1, x_2$ such that the respective $\omega$-limit sets are different: $\omega(x_1) \neq \omega(x_2)$.*

Threshold points are boundary points of basins of attractions.

**Definition 2.5.** *A set $B$ is the **basin of attraction** of another set $A$, the attractor, if for every $x \in B$ the $\omega$-limit set of $x$ is equal to $A$: $\omega(x) = A$ for all $x \in B$.*

Unlike the situation for 'ordinary' dynamical systems, a threshold point can be an element of one or more basins of attraction, and basins can overlap.

**Definition 2.6.** *A point $\xi \in X$ is an **indifference threshold** if it is both an indifference point and a threshold point.*

Equivalently, an indifference threshold is a point that is contained in more than one basin of attraction. In the literature, both threshold and non-threshold indifference points have been called 'Skiba points'. A more precise terminology seems to be desirable.

Dynamical systems on a one-dimensional state space that are defined by a vector field have typically two kinds of 'special' points: attractors and repellers, which are both steady states; the knowledge of these special points is sufficient to reconstruct the flow of the system qualitatively.

Analogously, an optimal one-dimensional vector field has *optimal attractors* and *optimal repellers*, which are both optimal equilibria; in addition it has indifference points. Again, the knowledge of the optimal equilibria and the indifference points is sufficient to reconstruct the qualitative features of the solution structure of the infinite horizon problem.

# 3 Bifurcations of optimal vector fields

The analysis of bifurcations of a parameterized family of optimal vector fields is performed in terms of the reduced canonical vector field, but it is perhaps worthwhile to point out that the latter is an auxiliary construct.

The optimal vector field defines a continuous time evolution on the state space, that is well defined for all positive times. When the state space is one-dimensional, the evolution has certain special properties: trajectories sweep out intervals that are bounded by optimal attractors and optimal repellers or indifference points. At a bifurcation, the qualitative structure of these trajectories changes. For instance, in a saddle-node bifurcation, an attractor and a repeller coalesce and disappear, together with the trajectory that joins them. Analogously, in an indifference-attractor bifurcation, an indifference point and an attractor coalesce and disappear, again together with the trajectory joining them. It is clearly impossible that a repeller and an indifference point coalesce, for the trajectory which should be joining them could have no $\omega$-limit point. However, there is a third possible bifurcation scenario: a repeller may turn into an indifference point. This also changes the solution structure, for the constant solution that remains in the repelling state has no equivalent in the situation with the indifference point.

The indifference-attractor bifurcation and the different kinds of indifference-repeller bifurcations have obviously no counterpart in the theory of dynamical systems: they are typical for optimization problems. Instances of indifference-attractor bifurcations have been analyzed in Wagener [24, 25].

## 3.1 Preliminary remarks.

If $N$ is a bounded interval of $\mathbb{R}$ with endpoints $a < b$, let the outward pointing 'vector' $\nu(x)$ be defined as

$$\nu(a) = -1, \quad \nu(b) = 1. \tag{6}$$

### 3.1.1 Notions from optimal control theory

The reduced canonical vector field $F$ of the infinite horizon problem under study is given as

$$F = (F_1, F_2) = (\mathcal{H}_p, \rho p - \mathcal{H}_x).$$

Assumption (1) implies that the strong Legendre-Clebsch condition

$$\mathcal{H}_{pp}(x, p) > 0 \tag{7}$$

holds for all $(x, p)$.

One of the implications of this condition is that eigenspaces of equilibria of the reduced canonical vector field are never vertical. More precisely, the following lemma holds.

**Lemma 3.1.** *If the strong Legendre-Clebsch condition holds, all eigenvectors $v$ of $DF$ can be written in the form $v = (1, w)$.*

*Proof.* The lemma is implied by the statement that if $\mathcal{H}_{pp} \neq 0$, then $e_2 = (0, 1)$ cannot be an eigenvector of

$$DF = \begin{pmatrix} \mathcal{H}_{px} & \mathcal{H}_{pp} \\ -\mathcal{H}_{xx} & \rho - \mathcal{H}_{px} \end{pmatrix}.$$

This is easily verified. □

**Lemma 3.2.** *Assume that the strong Legendre-Clebsch condition holds. If $v_1 = (1, w_1)$ and $v_2 = (1, w_2)$ are two eigenvectors of $DF$ with $\lambda_1 < \lambda_2$, then $w_1 < w_2$.*

*Proof.* The first component of the vector equation $DFv_i = \lambda_i v_i$ reads as

$$\mathcal{H}_{px} + \mathcal{H}_{pp} w_i = \lambda_i.$$

As $\mathcal{H}_{pp} > 0$, the lemma follows. □

The value of the objective $J$ over an extremal trajectory can be computed by evaluating the maximized Hamiltonian at the initial point (see for instance Skiba [21], Wagener [24]).

**Theorem 3.1.** *Let $(x(t), p(t))$ be a trajectory of the reduced canonical vector field $F$ that satisfies $\lim_{t\to\infty} \mathcal{H}(x(t), p(t))\mathrm{e}^{-\rho t} = 0$, and let $u(t) = v(x(t), p(t))$ be the associated control function. Then*

$$J(x, u) = \frac{1}{\rho}\mathcal{H}(x(0), p(0)).$$

### 3.1.2 Notions from dynamical systems

Recall the following notions from the theory of dynamical systems: two vector fields are said to be *topologically conjugate*, if all trajectories of the first can be mapped homeomorphically onto trajectories of the second; that is, by a continuous invertible transformation whose inverse is continuous as well.

An equilibrium $\bar{z}$ of a vector field $f$ is called *hyperbolic*, if no eigenvalue of $Df(\bar{z})$ is situated on the imaginary axis. The sum of the generalized eigenspaces associated to the hyperbolic eigenvalues is the hyperbolic eigenspace $E^h$, which can be written as the direct sum of the stable and unstable eigenspaces $E^s$ and $E^u$, associated to the stable and unstable eigenvalues respectively. The sum of the eigenspaces associated to the eigenvalues on the imaginary axis is the

neutral eigenspace $E^c$. The center-unstable and center-stable eigenspaces $E^{cu}$ and $E^{cs}$ are the direct sums $E^c \oplus E^u$ and $E^c \oplus E^s$ respectively.

The center manifold theorem (see Hirsch *et al.* [12]), ensures the existence of invariant manifolds that are tangent to the stable and unstable eigenspaces.

**Theorem 3.2** (center Manifold Theorem)**.** *Let $f$ be a $C^k$ vector field on $\mathbb{R}^m$, $k \geq 2$, and let $f(\bar{z}) = 0$. Let $E^u$, $E^s$, $E^c$, $E^{cu}$ and $E^{cs}$ denote the generalised eigenspaces of $Df(\bar{z})$ introduced above. Then there are $C^k$ manifolds $W^s$ and $W^u$ tangent to $E^s$ and $E^u$ at $\bar{z}$, and $C^{k-1}$ invariant manifold $W^c$, $W^{cu}$ and $W^{cs}$ tangent to $E^c$, $E^{cu}$ and $E^{cs}$ respectively at $\bar{z}$. These manifolds are all invariant under the flow of $f$; the manifolds $W^s$ and $W^u$ are unique, while $W^c$, $W^{cu}$ and $W^{cs}$ need not be.*

Invariant manifolds can be used to choose convenient coordinates around an equilibrium point of a vector field. For instance, let $f(0) = 0$, let $E^1$ and $E^2$ be two linear subspaces such that

$$E^1 \oplus E^2 = \mathbb{R}^m,$$

and let $W^1$ and $W^2$ be two invariant manifolds that are locally around $0$ parameterized as the graphs of functions

$$w^1 : E^1 \to E^2, \quad w^2 : E^2 \to E^1,$$

satisfying $Dw^1(0) = 0$, $Dw^2(0) = 0$. For a sufficiently small neighborhood $N$ of $0$ and for $(z_1, z_2) \in U \subset E^1 \times E^2$, define the coordinate transformation

$$(\zeta_1, \zeta_2) = (z_2 - w^1(z_1), z_1 - w^2(z_2)).$$

In the new coordinates, the vector field has necessarily the form

$$f(\zeta) = \begin{pmatrix} A_1\zeta_1 + \zeta_1\varphi_1(\zeta) \\ A_2\zeta_2 + \zeta_2\varphi_2(\zeta) \end{pmatrix},$$

where $\varphi_i(\zeta) \to 0$ as $\zeta \to 0$.

For a hyperbolic equilibrium of a vector field on the plane, a much stronger result is available, the $C^1$ liberalization theorem of Hartman (see Hartman [9, 10], Palis and Takens [16]).

**Theorem 3.3** (Hartman's $C^1$ linearisation theorem)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^2$ vector field in the plane, and let $z = 0$ be a hyperbolic equilibrium of $f$. Then there is a neighborhood $N$ of $0$ and coordinates $\zeta$ on $N$, such that*

$$f(\zeta) = Df(0)\zeta$$

*in these coordinates.*

## 3.2 Codimension one bifurcations

In this subsection, the codimension one bifurcations of optimal vector fields are treated: these are the bifurcations that cannot be avoided in one-parameter families. These are the indifference-repeller, the indifference-attractor and the saddle-node bifurcation.

It turns out that there are two configurations of the state co-state system that can give rise to indifference-repeller bifurcations of the optimal vector field; they are referred to as type 1 and type 2, respectively.

A general remark on notation: the codimension of a bifurcation will be denoted by a subscript, whereas the type is indicated, if necessary, by additional information in brackets. For instance, the abbreviation $\text{IR}_1(2)$ denotes a codimension one indifference repeller bifurcation of type 2.

### 3.2.1 IR$_1$(1) bifurcation

Consider the situation that the reduced canonical vector field $F$ has an equilibrium $e = (x_e, p_e) \in \mathbb{R}^2$ with eigenvalues $0 < \lambda^u < \lambda^{uu}$. Let $E^{uu}$ denote the eigenspace associated to $\lambda^{uu}$. As this eigenspace is invariant under the linear flow $DF(0)z$, by Hartman's liberalization theorem there is a one-dimensional differentiable curve $W^{uu}$, the *strong unstable* invariant manifold, that is invariant under $F$ and tangent to $E^{uu}$ at $e$.

**Definition 3.1.** *A point $e = (x_e, p_e)$ is a (codimension one) indifference repeller singularity of type 1, notation IR$_1$(1), of an optimization problem with reduced canonical vector field $F$, if the following conditions hold.*

1. *The eigenvalues $\lambda^u, \lambda^{uu}$ of $DF(e)$ satisfy $0 < \lambda^u < \lambda^{uu}$.*

2. *On some compact interval neighborhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that*
$$p^o(x) = \{p(x)\}.$$
*for all $x \in N$, and such that $p_e = p(x_e)$.*

3. *Let $W^{uu}$ denote the strong unstable manifold of $F$ at $e$, parameterized as the graph of $w : N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of $N$. There is exactly one $\bar{x} \in \partial N$ such that*
$$p(\bar{x}) = w(\bar{x}), \tag{8}$$
*whereas for $x \in \partial N$ and $x \neq \bar{x}$, we have that*
$$\nu(x)\Big(p(x) - w(x)\Big) < 0. \tag{9}$$

The definition is illustrated in Figure 1b.

**Theorem 3.4.** *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_1$(1) singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu(e_\mu)$ satisfy $0 < \lambda^u_\mu < \lambda^{uu}_\mu$. Let the strongly unstable manifold $W^{uu}_\mu$ of $e_\mu$ be parameterized as the graph $p = w(x, \mu)$ of a differentiable function $w : N \times \Gamma \to \mathbb{R}$.*

2. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*
$$p^o_\mu(x) = \{p(x, \mu)\}$$
*for all $x \in \partial N$ and all $\mu \in \Gamma$.*

3. *The function*
$$\alpha(\mu) = \nu(\bar{x})\big(p(\bar{x}, \mu) - w(\bar{x}, \mu)\big),$$
*for which $\alpha(0) = 0$ by (8), is defined on $N$ and satisfies*
$$D\alpha(0) \neq 0.$$

*Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \{x\}$$

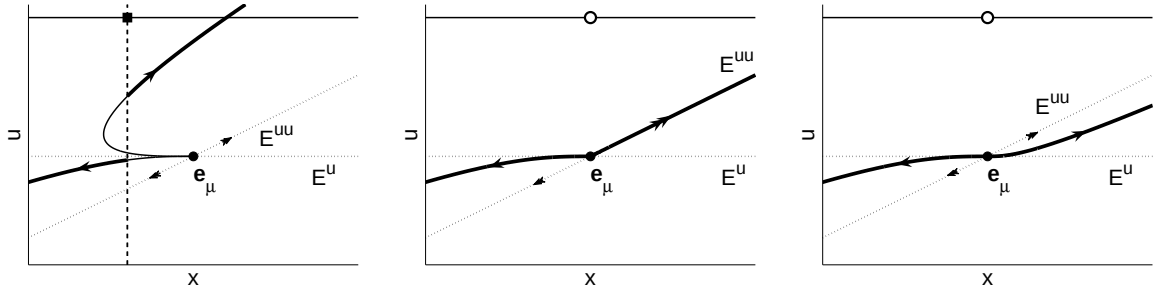
*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = \begin{cases} \{-1\}, & x < 0, \\ \{-1, 1\}, & x = 0, \\ \{1\}, & x > 0, \end{cases}$$

The theorem is illustrated in figure 1. Shown is a neighborhood of a repelling equilibrium of the state-costate equation. The dotted lines are the linear unstable eigenspaces of the equilibrium; the strongly unstable eigenspace corresponds to the line with the largest gradient. Approaching the equilibrium are two phase curves, drawn as solid black lines. The thick part of these curves denote the optimal costate rule.

The indifference point is marked as a vertical dashed line. At the top of the diagrams, the corresponding situation in the state space is sketched; solid black circles correspond to equilibria of the optimal vector field, squares to indifference points. In this case, all equilibria of the optimal vector field are repelling.

At the bifurcation, the relative position of the optimal trajectories and the strongly unstable manifold changes: for $\alpha(\mu) < 0$ the backward extension of the optimal trajectories are tangent to $E^u$ at either side of the equilibrium. This ensures that the equilibrium itself corresponds to an optimal repeller. For $\alpha(\mu) > 0$, the backward extensions are tangent to $E^u$ at the same side of the equilibrium. One of them necessarily intersects the line $x = x_e$, which implies that $e$ cannot be an optimal trajectory.




(a) $\alpha(\mu) < 0$ (b) $\alpha(\mu) = 0$ (c) $\alpha(\mu) > 0$

Figure 1: Before, at and after the indifference-repeller bifurcation point.

*Proof.* Let $E^{uu} = \mathbb{R}v^{uu}$ and $E^u = \mathbb{R}v^u$ be the eigenspaces spanned by the eigenvectors $v^{uu} = (1, w^{uu})$ and $v^u = (1, w^u)$ of $DF(r)$ corresponding to the eigenvalues $\lambda^{uu}$ and $\lambda^u$ respectively. Note that $w^{uu} > w^u$ as a consequence of lemma 3.2.

For a sufficiently small neighborhood of $e$ introduce $C^1$ linearising coordinates $\zeta = \zeta(z)$, with $C^1$ inverse $z = z(\zeta) = (x(\xi, \eta), p(\xi, \eta))$, such that $\zeta(e) = 0$, such that the linear map $D\zeta(0)$ maps $v^{uu}$ to $(1, 0)$ and $v^u$ to $(0, -1)$, and such that in these coordinates the vector field $F$ takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^{uu} & 0 \\ 0 & \lambda^u \end{pmatrix} \zeta.$$

As a consequence of these choices, the map $\zeta$ is orientation preserving and

$$x_\xi(0,0) = \pi_1 Dz(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_1 v^{uu} = 1 > 0,$$
$$x_\eta(0,0) = \pi_1 Dz(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\pi_1 v^{u} = -1 < 0,$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the first component, and where $x_\xi$ denotes partial derivation with respect to $\xi$ etc. By continuity, there is a neighborhood $V$ of $0$ such that

$$x_\xi > 0, \quad x_\eta < 0, \quad \text{and} \quad \det Dz > 0$$

on $V$.

Let $\bar{x}_i$, $i = 1, 2$ be such that

$$N = [\bar{x}_1, \bar{x}_2] \tag{10}$$

and set

$$\bar{z}_i = (\bar{x}_i, \bar{p}_i) = (\bar{x}_i, p(\bar{x}_i, \mu)), \tag{11}$$

as well as

$$\bar{\zeta}_i = (\bar{\xi}_i, \bar{\eta}_i) = \zeta(\bar{x}_i, \bar{p}_i).$$

Assume that $\bar{x} = \bar{x}_2$, that is

$$\alpha(\mu) = p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu);$$

the proof in the case $\bar{x} = \bar{x}_1$ is similar.

The trajectory $z_1(t) = (x_1(t), y_1(t))$ of $F$ through $\bar{z}_1$ has in linearising coordinates the form

$$\zeta_1(t) = \left(\bar{\xi}_1 \mathrm{e}^{\lambda^{uu} t}, \bar{\eta}_1 \mathrm{e}^{\lambda^{u} t}\right),$$

with $\bar{\xi}_1 < 0$ and $\bar{\eta}_1 > 0$ for all $\mu$. Note that it satisfies

$$\begin{aligned} \dot{x}_1 &= \frac{\mathrm{d}}{\mathrm{d}t} x(\zeta_1(t)) = x_\xi \bar{\xi}_1 \mathrm{e}^{\lambda^{uu} t} \lambda^{uu} + x_\eta \bar{\eta}_1 \mathrm{e}^{\lambda^{u} t} \lambda^{u} \\ &= \mathrm{e}^{\lambda^{u} t} \left( \bar{\eta}_1 \lambda^{u} x_\eta + \bar{\xi}_1 \lambda^{uu} \mathrm{e}^{(\lambda^{uu} - \lambda^{u}) t} x_\xi \right) \end{aligned}$$

As $\bar{\eta}_1 > 0$ , $x_\eta < 0$ and

$$\lim_{t \to -\infty} \mathrm{e}^{(\lambda^{uu} - \lambda^{u}) t} x_\xi(\zeta_1(t)) = 0,$$

it follows that there is a constant $T < 0$ such that $\dot{x}_1 < 0$ for all $t < T$ and all $\mu$ in a small neighborhood $\Gamma$ of $0$. If necessary by choosing $\varepsilon > 0$ smaller, it may be assumed that $T = 0$.

By assumption, the point

$$\bar{z}_2 = z(\bar{\xi}_2, \bar{\eta}_2)$$

can be written as

$$x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = \bar{p} = w(\bar{x}) + \alpha. \tag{12}$$

Note that for $\alpha = 0$, the point $\bar{p}$ is on $W^{uu}$, and therefore $\bar{\eta}_2 = 0$. To establish the dependence of $\eta$ on $\alpha$, derive first (12) with respect to $\alpha$ to obtain

$$x_\xi \left(\bar{\xi}_2\right)_\alpha + x_\eta \left(\bar{\eta}_2\right)_\alpha = 0, \quad p_\xi \left(\bar{\xi}_2\right)_\alpha + p_\eta \left(\bar{\eta}_2\right)_\alpha = 1.$$

Solving for $\left(\bar{\eta}_2\right)_\alpha$ yields

$$\left(\bar{\eta}_2\right)_\alpha = \frac{x_\xi}{\det Dz}$$

from which it follows that

$$(\bar{\eta}_2)_\alpha > 0.$$

The trajectory $z_2(t)$ through $\bar{z}_2$ has in linearising coordinates the form

$$\zeta_2(t) = \left(\bar{\xi}_2 \mathrm{e}^{\lambda^{uu} t}, \bar{\eta}_2 \mathrm{e}^{\lambda^u t}\right).$$

Note that $\bar{\xi}_2 > 0$ for all $\mu$, and that

$$\bar{\eta}_2 = \bar{\eta}_2(\alpha(\mu))$$

with $\bar{\eta}_2(0) = 0$. As before

$$\dot{x}_2 = \mathrm{e}^{\lambda^u t}\left(\lambda^u \bar{\eta}_2 x_\eta + \bar{\xi}_2 \lambda^{uu} \mathrm{e}^{(\lambda^{uu} - \lambda^u)t} x_\xi\right).$$

If $\alpha \leq 0$, then $\bar{\eta}_2 < 0$. From $x_\eta < 0$ it then follows that $\dot{x}_2 > 0$ for all $t$. By continuity, it follows also that $\dot{x}_2(0) > 0$ if $\alpha > 0$ is sufficiently small. Note however that

$$\lim_{t \to -\infty} \dot{x}_2 \mathrm{e}^{-\lambda^u t} = \bar{\xi}_2 \lambda^u x_\eta(0,0) < 0.$$

Consequently, for $\alpha > 0$ there is, by the intermediate value theorem, at least one $t < 0$ such that $x_2(t) < \bar{x}$ and $\dot{x}_2(t) = 0$. Let $t_*$ denote the largest of these $t$ if there are several.

Note that for $\alpha \leq 0$, the continuous curve formed by the union of the trajectories $z_1$, $z_2$ and the point $e$ intersects each leaf $\{x = const\}$ exactly once, and defines therefore a continuous function $x \mapsto p^o_\mu(x)$, which is necessarily the optimal costate map.

If $\alpha > 0$, then the trajectory $z_2$ is tangent to the leaf $L = \{x = x_2(t_*)\}$ at $z_{2*} = z_2(t_*)$, and $z_2$ cuts all other leaves $\{x = const\}$ transversally for $t^* < t \leq 0$. The leaf $L$ is cut by $z_1$ at $z_{1*}$. Since $\dot{x} = \mathcal{H}_p = 0$ at $z_{2*}$ and $\mathcal{H}$ is strictly convex in $p$, it follows that

$$\mathcal{H}(z_{2*}) < \mathcal{H}(z_{1*}).$$

Since $\xi_1(0) = \delta$, there is $t^* \in (t_*, 0)$ such that $x_2(t^*) = 0$. Again by convexity of $\mathcal{H}$ in $p$, it follows that

$$\mathcal{H}(z_2(t^*)) > \mathcal{H}(e) = \lim_{t \to -\infty} \mathcal{H}(z_1(t)).$$

Consequently there is $\tilde{t} \in (t_*, t^*)$ such that

$$\mathcal{H}(\tilde{z}_1) = \mathcal{H}(\tilde{z}_2),$$

where $\tilde{z}_i = (\tilde{x}, \tilde{p}_i) = (x_i(\tilde{t}), p_i(\tilde{t}))$, and $\tilde{x}$ is an indifference point by theorem 3.1. □

### 3.2.2 IR$_1$(2) bifurcation

An indifference-repeller singularity of type 2 occurs in certain situations when the dynamics of the repeller is a Jordan node. Specifically, consider the situation that the vector field $F$ on $\mathbb{R}^2$ has an equilibrium $e = (x_e, p_e)$, that its liberalization $DF(e)$ has two equal positive eigenvalues $\lambda_1 = \lambda_2 = \lambda > 0$, and such that its proper eigenspace $E^{pu}$ is only one-dimensional. By the Hartman theorem, there is a $C^1$ curve $W^{pu}$, the *proper unstable invariant manifold*, which is the image of $E^{pu}$ in general coordinates; trajectories $z(t)$ in $W^{pu}$ are characterized by the requirement that

$$\limsup_{t \to -\infty} \|z(t) - e\| \mathrm{e}^{-\lambda t} < \infty.$$

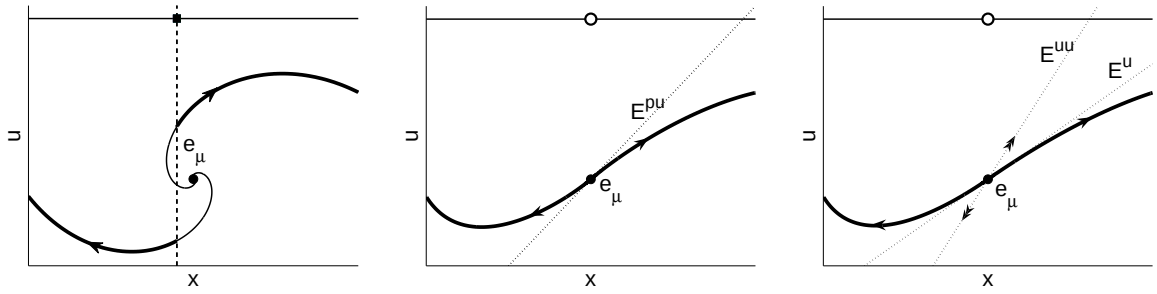
**Definition 3.2.** *A point $e = (x_e, p_e)$ is a (codimension one) indifference repeller singularity of type 2, notation IR$_1$(2), of an optimisation problem with reduced canonical vector field $F$, if the following conditions hold.*

1. *The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1, \lambda_2$ of $DF_0(e)$ satisfy $\lambda_1 = \lambda_2 = \frac{\rho}{2}$.*

2. *On some compact interval neighborhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that*
$$p^o(x) = \{p(x)\}.$$
*for all $x \in N$, and such that $p_e = p(x_e)$.*

3. *Let $W^{uu}$ denote the strong unstable manifold of $F$ at $e$, parameterized as the graph of $w : N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of $N$. For all $x \in \partial N$, we have that*
$$\nu(x)\Big(w(x) - p(x)\Big) > 0. \tag{13}$$

This singularity also gives rise to an indifference repeller bifurcation, as in the previous case, but through a different mechanism. See Figure 2: at bifurcation, the equilibrium of the reduced canonical vector field is a Jordan node. When the eigenvalues move off the real axis, it turns into a focus. This precludes the possibility of an optimal repeller. When the eigenvalues remain on the real axis but separate, two independent eigenspaces $E^{uu}$ and $E^u$ are generated. Condition (13) then implies the existence of an optimal repeller.




(a) $\alpha(\mu) < 0$ (b) $\alpha(\mu) = 0$ (c) $\alpha(\mu) > 0$

Figure 2: Before, at and after the type 2 indifference-repeller bifurcation point.

**Theorem 3.5.** *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an IR$_1$(2) singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$. Let $D(\mu)$ and $T(\mu)$ denote the trace and the determinant of $DF_\mu(e_\mu)$.*

2. *The function $\alpha : \Gamma \to \mathbb{R}$, defined by*
$$\alpha(\mu) = \frac{T(\mu)^2}{4} - D(\mu)$$
*and for which $\alpha(0) = 0$, satisfies*
$$D\alpha(0) \neq 0.$$

3. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*

$$p^o(x) = \{p(x, \mu)\}$$

*for all $x \in \partial N$ and all $\mu \in \Gamma$.*

*Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \begin{cases} -1 & x < 0, \\ \{-1, 1\} & x = 0, \\ 1 & x > 0. \end{cases}$$

*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = x.$$

*Proof.* There is a linear map $C_0$ such that

$$C_0^{-1} DF_0(e) C_0 = \begin{pmatrix} \frac{\rho}{2} & 1 \\ 0 & \frac{\rho}{2} \end{pmatrix}.$$

Arnol'd's matrix unfolding theorem (Arnold [1]) then implies that there is a family of maps $C(\alpha)$, smoothly depending on $\alpha$, such that $C(0) = C_0$ and such that

$$A_\alpha = C(\alpha)^{-1} DF_\mu(e_\mu) C(\alpha) = \begin{pmatrix} \frac{\rho}{2} & 1 \\ \alpha & \frac{\rho}{2} \end{pmatrix},$$

where $\alpha = \alpha(\mu)$. The eigenvalues of $DF_\mu(e_\mu)$ and consequently also those of $A_\alpha$ take the form

$$\lambda^u = \frac{\rho}{2} - \sqrt{\alpha}, \quad \lambda^{uu} = \frac{\rho}{2} + \sqrt{\alpha};$$

the corresponding eigenvectors of $A_\alpha$ take the form

$$v^u = (1, -\sqrt{\alpha}), \quad v^{uu} = (1, \sqrt{\alpha}).$$

Note that for $\alpha > 0$, these eigenvectors have the same ordering as the corresponding eigenvectors of $DF_\mu(e)$; cf. lemma 3.2. It follows that the matrix $C(\alpha)$ is necessarily orientation preserving for $\alpha > 0$ and, by continuity, for all other values of $\alpha$.

Define $\bar{x}_i$, $\bar{p}_i$ and $\bar{z}_i$ as in (10) and (11).

When $\alpha < 0$, the eigenvalues are complex, and the trajectories $z_1$ and $z_2$ emanating from $\bar{z}_1$ and $\bar{z}_2$ respectively spiral towards $e$ as $t \to -\infty$. Let $t_*$ be the largest $t \leq$ such that $\dot{x}_2(t) = 0$. Then necessarily

$$x_* = x_2(t_*) < x_e.$$

The trajectory $z_2$, restricted to $[t_*, 0]$, can be parameterized as the graph of a continuous function $p_2 : [x_*, \bar{x}_2] \to \mathbb{R}$. In the same way, if $t^* < 0$ is the largest $t$ such that $\dot{x}_1(t) = 0$, then $z_1$ restricted to $[t^*, 0]$ can be parameterized as the graph of the function $p_1 : [\bar{x}_1, x^*] \to \mathbb{R}$, where

$$x^* = x_1(t^*) > x_e.$$

Moreover, as $\mathcal{H}$ is strictly convex and $\mathcal{H}_p(x_*, p_2(x_*)) = 0$, it follows that

$$\mathcal{H}(x_*, p_2(x_*)) < \mathcal{H}(x_*, p_1(x_*));$$

likewise

$$\mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)).$$

By continuity, there is a point $\tilde{x} \in [x_*, x^*]$ such that

$$\mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})).$$

By Theorem 3.1, this is an indifference point.

Point 2 of Definition 3.2 implies that for $\mu = 0$, the sets $p^o(x)$ contain a single element $p(x)$ for all $x \in N$. Necessarily, the graph of $p$ is formed by two trajectories of $F_0$ as well as the equilibrium point $e$. These trajectories intersect the lines $\{x = const\}$ transversally and they are tangent to $E^{pu}$ at $x_e$; put differently, the graph of $p$ is tangent to the proper unstable manifold $W^{pu}$ at $e$. By point 3 of the definition, $p(x) > w(x)$ if $\bar{x}_1 < x < x_e$ and $p(x) < w(x)$ if $x_e < x < \bar{x}_2$.

For $\alpha(\mu) > 0$, there is a family $W^{uu}_\mu$ of strong unstable manifolds, depending continuously on $\mu$, and parameterized as the graph of a family of $C^1$ functions $w_\mu$ around $x_e$. In particular, if $N$ and $\Gamma$ sufficiently small

$$\nu(x)\,(w_\mu(x) - p(x, \mu)) > 0$$

for all $x \in \partial N$ and $\mu \in \Gamma$. By continuity, the backward trajectories through $\bar{z}_1$ and $\bar{z}_2$ intersect all lines $\{x = const\}$ transversally and are tangent to the weak unstable direction $E^u$ at $x_e$. But this implies that they form, together with the equilibrium $e$, the graph of a $C^1$ function $p_\mu$ that is defined on $N$, and for which

$$p^o_\mu(x) = \{p_\mu(x)\}.$$

□

### 3.2.3 $IA_1$ bifurcation

**Definition 3.3.** *A point $e = (x_e, p_e)$ is a (codimension one) indifference attractor singularity, notation $IA_1$, of an optimization problem with reduced canonical vector field $F$, if the following conditions hold.*

1. *The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda^s, \lambda^u$ of $DF(e)$ satisfy $\lambda^s < 0 < \lambda^u$.*

2. *On some compact interval neighborhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that*

   $$p^o(x) = \{p(x)\}.$$

   *for all $x \in N$, and such that $p_e = p(x_e)$.*

3. *Let $W^s$ and $W^u$ denote respectively the stable and the unstable manifold of $F$ at $e$, parameterized as the graph of functions $w^s, w^u : N \to \mathbb{R}$. If $\partial N = \{\bar{x}_1, \bar{x}_2\}$, then*

   $$p(\bar{x}_1) = w^u(\bar{x}_1), \quad p(\bar{x}_2) = w^s(\bar{x}_2). \tag{14}$$

Note that this definition does not require the points $\bar{x}_1$ and $\bar{x}_2$ to be ordered in a certain way.

**Theorem 3.6.** *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an $IA_1$ singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

1. *For all $\mu \in \Gamma$, there is $e_\mu \in \mathbb{R}^2$ such that $F_\mu(e_\mu) = 0$, and such that the eigenvalues of $DF_\mu(e_\mu)$ satisfy $\lambda^s_\mu < 0 < \lambda^u_\mu$. Let the stable and the unstable manifolds $W^s_\mu$ and $W^u_\mu$ of $e_\mu$ be parameterized as graph $p = w^s(x, \mu)$ and $p = w^u(x, \mu)$ of differentiable functions $w^s, w^u : N \times \Gamma \to \mathbb{R}$.*

2. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in its second argument, such that*

$$p^o(x) = \{p(x, \mu)\}$$

   *for all $x \in \partial N$ and all $\mu \in \Gamma$.*

3. *The function*

$$\alpha(\mu) = \nu(x_1)\big(p(x_1, \mu) - w(x_1, \mu)\big),$$

   *for which $\alpha(0) = 0$ by* (14)*, is defined on $\Gamma$ and satisfies*

$$D\alpha(0) \neq 0.$$

4. *For all $\mu \in \Gamma$ such that $\alpha(\mu) < 0$, the equality*

$$p(x_2, \mu) = w^s_\mu(x_2)$$

   *holds.*

*Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \begin{cases} \{-x\}, & x < 1, \\ \{-1, 1\}, & x = 1, \\ \{1\}, & x > 1, \end{cases}$$

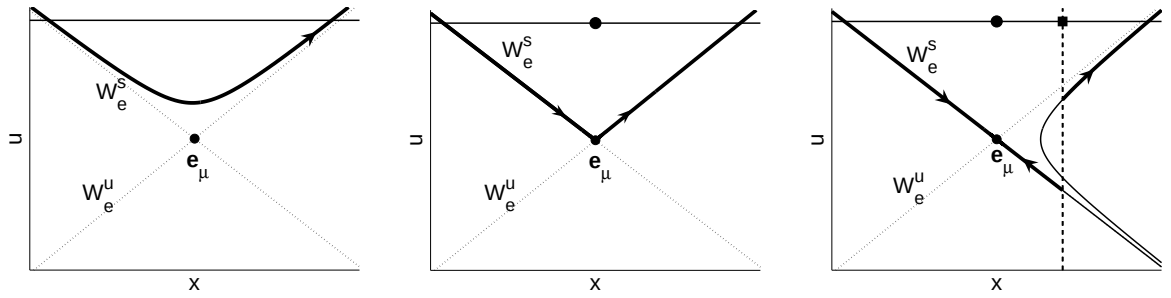
*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = \{1\}$$

The theorem is illustrated in Figure 3. As for the $\mathrm{IR}_1(1)$ bifurcation, at the bifurcation the relative position of the optimal trajectories and the 'most unstable' invariant manifold changes.

If $\alpha(\mu) < 0$, the backward extension of the optimal trajectory through the point $\bar{z}_1 = (\bar{x}_1, p(\bar{x}_1, \mu))$ has a vertical tangent at a certain point. Past this point, the trajectory cannot be optimal, even locally. It follows that $x_e$ is locally optimal. For $\alpha(\mu) > 0$, the trajectory through $\bar{z}_1$ intersects the line $x = x_e$. Theorem 3.1 then implies that the constant trajectory $e$ cannot be optimal at all in this case.

In many applications, the optimal trajectory through $\bar{z}_1$ is on the stable manifold of another equilibrium $e'$. For $\alpha(\mu) = 0$, we have also that $\bar{z}_1$ is in the unstable manifold of $e$, and the trajectory of $F$ through $\bar{z}_1$ then forms a *heteroclinic connection* between $e$ and $e'$. In this form, the indifference-attractor bifurcation was investigated in Wagener [24]. The present formulation in terms of the optimal costate rule is more general as it captures, for instance, also the situation that the optimal trajectory through $\bar{z}_1$ tends to infinity as $t \to \infty$ (cf. Hinloopen *et al.* [11]).




(a) $\alpha(\mu) < 0$ (b) $\alpha(\mu) = 0$ (c) $\alpha(\mu) > 0$

Figure 3: Before, at and after the indifference-attractor bifurcation point.

*Proof.* Restricted to a neighborhood of the saddle, in linearising coordinates the vector field $F_\mu$ takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^u & 0 \\ 0 & \lambda^s \end{pmatrix} \zeta.$$

The coordinates are chosen such that the coordinate transformation is orientation preserving; moreover, the direction of the axes is chosen such that

$$x_\xi > 0, \quad x_\eta < 0.$$

Note that the unstable and stable manifolds are in these coordinates equal to the horizontal and vertical coordinate axes respectively.

As in the proof of theorem 3.4, set $\bar{x}_i$, $\bar{p}_i$ and $\bar{z}_i$ as in (10) and (11).

Assume that $\bar{x}$ of point 2 of Definition 3.3 satisfies $\bar{x} = \bar{x}_2$; the opposite situation can be handled analogously. If $\bar{\xi}_2$ and $\bar{\eta}_2$ are defined as

$$x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \quad p(\bar{\xi}_2, \bar{\eta}_2) = w(\bar{x}) + \alpha,$$

then it follows as in the proof of theorem 3.4 that

$$(\bar{\eta}_2)_\alpha > 0$$

and $\bar{\eta}_2 = 0$ if $\alpha = 0$.

The trajectory $z_2(t) = (x_2(t), p_2(t))$ through $\bar{z}_2$ has in linearising coordinates the form

$$\zeta_2(t) = (\xi_1(t), \eta_1(t)) = \left(\bar{\xi}_2 \mathrm{e}^{\lambda^u t}, \bar{\eta}_2 \mathrm{e}^{\lambda^s t}\right).$$

It follows that

$$\dot{x}_2 = \mathrm{e}^{\lambda^s t} \left(\lambda^s \bar{\eta}_2 x_\eta + \lambda^u \bar{\xi}_2 x_\xi \mathrm{e}^{(\lambda^u - \lambda^s)t}\right). \tag{15}$$

If $\alpha(\mu) > 0$, then $\bar{\eta}_2 > 0$; as both $\lambda^s x_\eta > 0$ and $\lambda^u \bar{\xi}_2 x_\xi > 0$, it follows from (15) that $\dot{x}_2 > 0$ for all $t$. That is, the trajectory $z_2$ intersects each line $x = const$ exactly once, and therefore defines a $C^1$ function $x \mapsto p(x, \mu)$, which then necessarily satisfies

$$p^o_\mu(x) = \{p(x, \mu)\}$$

for all $x \in N$.

Consider now the case that $\alpha(\mu) < 0$. By equation (15), if $\alpha(\mu)$ and hence $\bar{\eta}_2$ is sufficiently close to 0, then $\dot{x}_2(0) > 0$. Let $T_\mu < 0$ be such that $\eta_2(T_\mu) = -\bar{\xi}_2$. Then

$$T_\mu = \frac{1}{\lambda^s} \log \frac{\bar{\xi}_2}{(-\bar{\eta}_2)}$$

and equation (15) yields

$$\begin{aligned} \dot{x}_2(T_\mu) \mathrm{e}^{-\lambda^s T_\mu} &= \lambda^s \bar{\eta}_2 x_\eta + \lambda^u \bar{\xi}_2 x_\xi \left( \frac{-\bar{\eta}_2}{\bar{\xi}_2} \right)^{1+\lambda^u/|\lambda^s|} \\ &= \lambda^s \bar{\eta}_2 x_\eta + o(\bar{\eta}_2). \end{aligned}$$

This is negative if $\bar{\eta}_2$, and hence $\alpha(\mu)$, is sufficiently close to 0. For such values of $\alpha$, there exists $t < 0$ such that $\dot{x}_2(t) = 0$. Let $t_*$ denote the largest value of $t$ with this property, and introduce

$$x_* = x_2(t_*).$$

For $x_* \leq x \leq \bar{x}_2$, the trajectory $(x_2(t), p_2(t))$ parametrises the graph of a function $p_2(x)$. As $\dot{x} = \mathcal{H}_p$ along trajectories, note that

$$\mathcal{H}_p(x_*, p_2(x_*)) = 0.$$

Let $p_1 : N \to \mathbb{R}$ be such that its graph parameterizes the stable manifold $W^s$ of $s$. Strict convexity of $\mathcal{H}$ implies the inequality

$$\mathcal{H}(x_*, p_2(x_*)) < \mathcal{H}(x_*, p_1(x_*)). \tag{16}$$

Define functions $V_1$ on $N$ and $V_2$ on $[x_*, \bar{x}_2]$ by

$$V_1(x) = \frac{\mathcal{H}(x, p_1(x))}{\rho}$$

and

$$V_2(x) = \frac{\mathcal{H}(x, p_1(x))}{\rho}.$$

Then

$$V_2(x_*) < V_1(x_*).$$

To establish the opposite inequality for some $x^* \in [x_*, \bar{x}_2]$, consider the situation for $\alpha(\mu) = 0$, when $\bar{z}_2 \in W^u$. Then $V_2$ is defined for all $x_s < x < \bar{x}_2$. Moreover,

$$\lim_{x \downarrow x_s} V_2(x) = V_1(x).$$

Note that since $V_i'(x) = p_i(x)$ and

$$p_2(x) > p_1(x)$$

for all $x_s < x < \bar{x}_2$, it follows that

$$V_2(x) - V_1(x) = \int_{x_s}^{x} (p_2(\sigma) - p_1(\sigma)) \, \mathrm{d}\sigma > 0$$

for all $x > x_s$. This implies in particular that

$$\mathcal{H}(x, p_2(x)) > \mathcal{H}(x, p_1(x))$$

for all $x > x_s$, if $\alpha(\mu) = 0$.

Fix $x^* \in (x_*, \bar{x}_2)$. Then for $\alpha(\mu) < 0$ sufficiently close to 0, by continuity

$$\mathcal{H}(x^*, p_2(x^*)) > \mathcal{H}(x^*, p_1(x^*)). \tag{17}$$

As a consequence of (16) and (17), there is $\tilde{x} \in (x_*, x^*)$ such that

$$\mathcal{H}(\tilde{x}, p_1(\tilde{x})) = \mathcal{H}(\tilde{x}, p_2(\tilde{x})).$$

By theorem 3.1, it follows that $\tilde{x}$ is an indifference point. □

### 3.2.4 The saddle-node bifurcation

The saddle-node bifurcation of dynamical systems has a natural counterpart as a bifurcation of optimal vector fields.

Recall that a family of vector fields $f_\mu : \mathbb{R}^m \to \mathbb{R}^m$ can be viewed as a single vector field $g : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ by writing

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = g(x, \mu) = \begin{pmatrix} f_\mu(x) \\ 0 \end{pmatrix}.$$

Consider the situation that for $\mu = 0$ the point $\bar{z}$ is an equilibrium of $f_0$, and that $Df_0(\bar{z})$ has a single eigenvalue 0. Then $Dg(\bar{z}, 0)$ has two eigenvalues zero and an associated two-dimensional eigenspace $E^c$. The center manifold theorem applied to $g$ implies that there is a differentiable invariant manifold $W^c$ of $g$ that is tangent to $E^c$ at $(\bar{z}, 0)$. The manifold $W^c$ can be viewed as a parameterized family of invariant manifolds $W^c_\mu$, which are defined for $\mu$ taking values in a full neighborhood of $\mu = 0$. Note that the center manifolds need not be unique.

**Definition 3.4.** *A point $e = (x_e, p_e)$ is a (codimension one) saddle-node singularity, notation $SN_1$, of an optimization problems with reduced canonical vector field $F$, if the following conditions hold.*

1. *The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1, \lambda_2$ of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.*

2. *There is a compact interval $N$ of $X$ containing $x_e$ and a function $p : N \to \mathbb{R}$ such that*

   $$p^o(x) = \{p(x)\}$$

   *for all $x \in N$, and such that the graph of $p$ is a center manifold $W^c$ of $F$ at $e$.*

3. *The restriction*

   $$F^c(x) = F_1(x_e + x, p(x_e + x)).$$

   *of $F$ to $W^c$ satisfies*

   $$F^c(0) = 0, \quad (F^c)'(0) = 0, \tag{18}$$

   *and*

   $$(F^c)''(0) \neq 0. \tag{19}$$

**Theorem 3.7.** *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ a $SN_1$ singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

1. *There is a function $p : N \times \Gamma \to \mathbb{R}$ such that*

$$p^o_\mu(x) = \{p(x, \mu)\}$$

*for all $(x, \mu) \in N \times \Gamma$.*

2. *For $\mu \in \Gamma$, the graphs of $x \mapsto p(x, \mu)$ form a family of center manifolds $W^c_\mu$ of $F$ at $e$.*

3. *If $F^c_\mu$ is*

$$F^c_\mu(x) = (F_\mu)_1 (x_e + x, p(x_e + x, \mu))$$

*then the function*

$$\alpha(\mu) = F^c_\mu(0)$$

*satisfies*

$$D\alpha(0) \neq 0.$$

*Then the optimal vector field $f^o_\mu$ restricted to $N$ is for $\mu \in \Gamma$ topologically conjugate to*

$$Y_\mu(x) = \{\alpha(\mu) - \sigma x^2\}$$

*where $\sigma \in \{-1, 1\}$ is given as*

$$\sigma = \operatorname{sgn}(F^c_0)''(0).$$

*Proof.* This is a direct consequence from the usual saddle-node bifurcation theorem. □

## 3.3 Codimension two bifurcations

Most codimension two situations are straightforward extensions of the corresponding codimension one bifurcations. The results in this subsection will in most cases be stated more briefly and less formally. An exception is made for the indifference-saddle-node bifurcation.

### 3.3.1 A model case: the $IR_2$(1,1) bifurcation

**Definition 3.5.** *A point $e = (x_e, p_e)$ is a (codimension two) indifference repeller singularity of type (1,1), notation $IR_2$(1,1), of an optimisation problem with reduced canonical vector field $F$, if all conditions of definition 3.1 hold, but with* (8) *and* (9) *replaced by the condition that*

$$p(x) = w(x)$$

*for all $x \in \partial N$.*

**Theorem 3.8.** *Consider a family of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an $IR_2$(1,1) singularity. Let all the conditions of Theorem 3.4 hold, excepting point 3, which is replaced by the following.*

*The function*

$$\alpha(\mu) = \big(\nu(x_1)\big(p(\bar{x}_1, \mu) - w(\bar{x}_1, \mu)\big), \nu(x_2)\big(p(\bar{x}_2, \mu) - w(\bar{x}_2, \mu)\big)\big)\,.$$

*for which* $\alpha(0) = (0,0)$*, is defined on* $\Gamma$ *and satisfies*

$$\operatorname{ran} D\alpha(0) = 2.$$

*Then the optimal vector field* $f^o$ *restricted to* $N$ *is topologically conjugate to*

$$Y(x) = x$$

*if* $\alpha_1(\mu) \leq 0$ *and* $\alpha_2(\mu) \leq 0$*, whereas it is conjugate to*

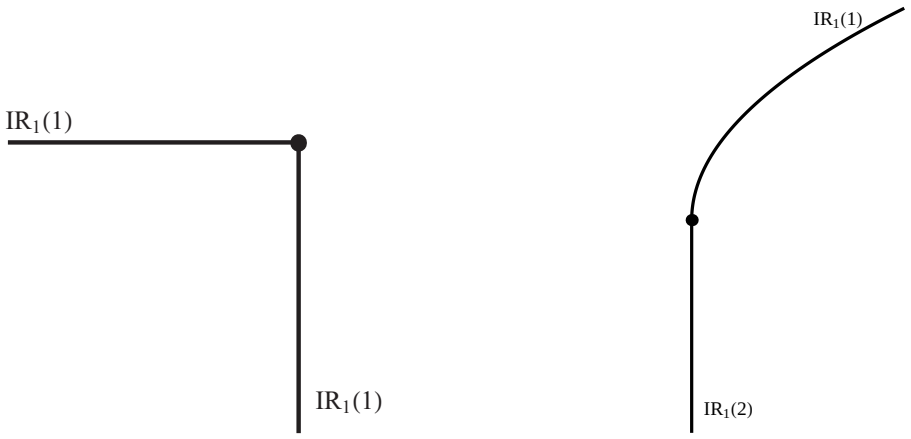
$$Y(x) = \begin{cases} -1 & x < 0, \\ \{-1, 1\} & x = 0, \\ 1 & x > 0. \end{cases}$$

*if* $\alpha_1(\mu) > 0$ *or* $\alpha_2(\mu) > 0$*. In particular, the curves* $\alpha_1(\mu) = 0$*,* $\alpha_2(\mu) < 0$ *and* $\alpha_2(\mu) = 0$*,* $\alpha_1(\mu) < 0$ *are codimension one indifference-repeller bifurcation curves.*

The proof is a simple modification of the proof of the codimension one case and is therefore omitted.

### 3.3.2 Other indifference-repeller and indifference-attractor bifurcations

Looking at the definition of the $IR_1(2)$ bifurcation, it is clear that bifurcations of higher codimension are obtained when condition (13) is violated at a boundary point. If this happens at one




(a) $IR_2(1,1)$ bifurcation diagram (b) $IR_2(1,2)$ bifurcation diagram

Figure 4: Indifference-repeller bifurcations of co-dimension two.

of the boundary points, a codimension two situation is obtained where an $IR_1(1)$ and an $IR_1(2)$ curve meet in a $IR_2(1,2)$ point. If it happens at both boundary points, a codimension three situation arises, denoted $IR_3$, where two $IR_1(1)$ and a $IR_1(2)$ surface meet. In order to avoid unnecessary repetions, the exact definitions for these bifurcations are not formulated; they can all be modeled on Definition 3.5 and Theorem 3.8. Their bifurcation diagrams are given in Figures 4(b) and 5.

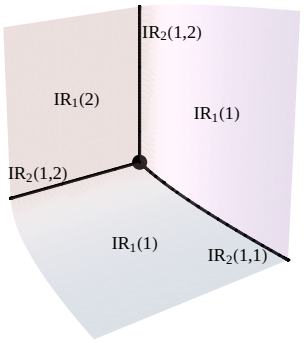



Figure 5: $\mathrm{IR}_3$ bifurcation diagram.

Likewise, a codimension two bifurcation is obtained if condition 14 is replaced by

$$p(x_1) = w^u(x_1), \quad p(x_2) = w^u(x_2). \tag{20}$$

This a two-sided or double indifference attractor bifurcation, denoted $\mathrm{DIA}_2$. Its bifurcation diagram is given in Figure 6.

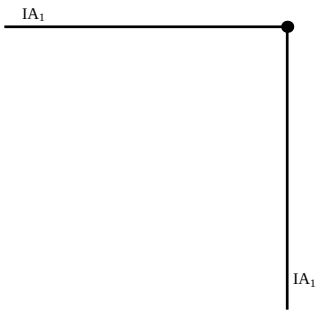



Figure 6: $\mathrm{DIA}_2$ bifurcation diagram.

### 3.3.3 Degenerate saddle-node bifurcations

The degenerate saddle-node bifurcations like the cusp ($\mathrm{SN}_2$), the swallowtail ($\mathrm{SN}_3$) etc. can be treated entirely analogously to the saddle-node itself.

### 3.3.4 The indifference-saddle-node bifurcation

The indifference-attractor and indifference-repeller bifurcations correspond to global bifurcations involving hyperbolic equilibria of the reduced canonical vector field; in contrast, the

saddle-node bifurcation corresponds to a local bifurcation. The final bifurcation to be considered is the indifference-saddle-node bifurcation, which corresponds to a global bifurcation involving a nonhyperbolic equilibrium.

**Definition 3.6.** *A point $e = (x_e, p_e)$ is a (codimension two) indifference-saddle-node singularity, notation ISN$_2$, of an optimisation problem with reduced canonical vector field $F$, if the following conditions hold.*

1. *The point $e$ is an equilibrium of $F$, such that the eigenvalues $\lambda_1$, $\lambda_2$ of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.*

2. *On some compact interval neighborhood $N$ of $x_e$, there is defined a continuous function $p : N \to \mathbb{R}$ such that*
$$p^o(x) = \{p(x)\}$$
*for all $x \in N$, and such that $p_e = p(x_e)$.*

3. *Let $W^u$ denote the unstable manifold of $F$ at $e$, parameterized as the graph of a function $w^u : N \to \mathbb{R}$. There is a unique $\bar{x} \in \partial N$ such that*
$$p(\bar{x}) = w^u(\bar{x}). \tag{21}$$

4. *There is a center manifold $W^c$ of $F$ at $e$, parameterized as the graph of $w^c : N \to \mathbb{R}$, such that for $x \in \partial N$ and $x \neq \bar{x}$, we have that*
$$p(x) = w^c(x). \tag{22}$$

5. *The restriction*
$$F^c(x) = F_1(x_e + x, w^c(x_e + x))$$
*of $F$ to $W^c$ satisfies*
$$F^c(0) = 0, \quad (F^c)'(0) = 0,$$
*and*
$$(F^c)''(0) \neq 0.$$

**Theorem 3.9.** *Consider a family of optimisation problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an ISN$_2$ singularity. Assume that there is a neighborhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

1. *There is a function $p : \partial N \times \Gamma \to \mathbb{R}$, differentiable in the second argument, such that*
$$p^o_\mu = \{p(x, \mu)\}$$
*for all $(x, \mu) \in \partial N \times \Gamma$, and such that*
$$\alpha_2(\mu) = p(\bar{x}, \mu) - p(\bar{x}, 0)$$
*satisfies*
$$D\alpha_2(0) \neq 0.$$

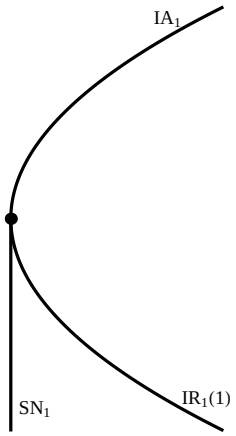



Figure 7: $\mathrm{ISN}_2$ bifurcation diagram.

2. *There is a family of center manifolds $W^c_\mu$, parameterized as the graphs of functions $x \mapsto w^c(x,\mu)$, such that $p(x,0) = w^c(x,\mu)$ if $x \in \partial N \backslash \{\bar{x}\}$.*

3. *Let $F^c_\mu$ be the restriction*

$$F^c_\mu(x) = (F_\mu)_1 (x_e + x, w^c(x_e + x, \mu))$$

*of $F$ to $W^c$. Then the function*

$$\alpha_1(\mu) = F^c_\mu(0)$$

*satisfies*

$$D\alpha_1(0) \neq 0.$$

4. *Let $\alpha(\mu) = (\alpha_1(\mu), \alpha_2(\mu))$. Then* $\operatorname{ran} D\alpha(0) = 2$.

*Then there is a differentiable functions $C(\alpha_2)$ such that $C(0) = C'(0) = 0$ and $C''(0) \neq 0$, and such that the problem has an indifference-attractor bifurcation if*

$$\alpha_1 = C(\alpha_2), \qquad \alpha_2 > 0,$$

*an indifference-repeller bifurcation if*

$$\alpha_1 = C(\alpha_2), \qquad \alpha_2 < 0,$$

*and a saddle-node bifurcation curve if*

$$\alpha_1 = 0, \qquad \alpha_2 < 0.$$

*Proof.* Assume without loss of generality that $(F^c)''(0) > 0$.

The system is first put, by an orientation preserving transformation, in coordinates $\zeta = (\xi, \eta)$ such that the center manifold $W^c_\mu$ corresponds to $\eta = 0$ for all $\mu$ close to $\mu = 0$, and the unstable manifold $W^u$ corresponds to $\xi = 0$ at $\mu = 0$. In these coordinates, the system, augmented by the parameter equation $\dot{\mu} = 0$, takes the form

$$\dot{\xi} = \alpha_1(\mu) + f_0(\xi, \mu) + \eta f_1(\zeta, \mu), \tag{23}$$

$$\dot{\eta} = \rho\eta \qquad\qquad + \eta g_1(\zeta, \mu), \tag{24}$$

$$\dot{\mu} = 0 \tag{25}$$

where by assumption $f_0(\xi,\mu) = c(\mu)\xi^2 + O(\xi^3)$ with $c(0) > 0$, and where $D\alpha_1(0) \neq 0$. These conditions imply that a saddle-node bifurcation occurs at $(\xi,\eta) = (0,0)$ if $\alpha_1(\mu) = 0$, generating a family of hyperbolic saddle and one of hyperbolic unstable equilibria of $F$.

The saddle equilibria have associated to them unique unstable invariant manifolds $W^u_\mu$; the unstable equilibria have associated to them strongly unstable manifolds $W^{uu}_\mu$, which are also unique. An indifference-attractor bifurcation occurs if $(x, p(x,\mu)) \in W^u_\mu$; an indifference-repeller bifurcation occurs if $(x, p(x,\mu)) \in W^{uu}_\mu$. The main thing to prove is that the manifolds $W^u_\mu$ and $W^{uu}_\mu$ can be parameterized as graphs of differentiable functions

$$x \mapsto w^u(x,\mu), \quad x \mapsto w^{uu}(x,\mu).$$

This is not automatic, for the function $w^u$ and $w^{uu}$ will not be differentiable as functions of $\mu$, having necessarily at $\mu = 0$ a singularity of the order $\sqrt{\mu}$.

In the following, it will however be shown that the closure of the invariant set

$$W = \bigcup_\mu W^u_\mu \cup W^{uu}_\mu \tag{26}$$

forms a differentiable manifold. From figure 8, it seems likely that $W$ can be described as the level set

$$W \ : \quad \alpha_1(\mu) = -f_0(\xi,\mu) + \eta w(\xi,\eta,\mu),$$

where $w$ is a function yet to be determined. The condition that $W$ is invariant under the flow of (23)–(25) leads to a first order partial differential equation for the function $w$; this equation is singular for $\eta = 0$.

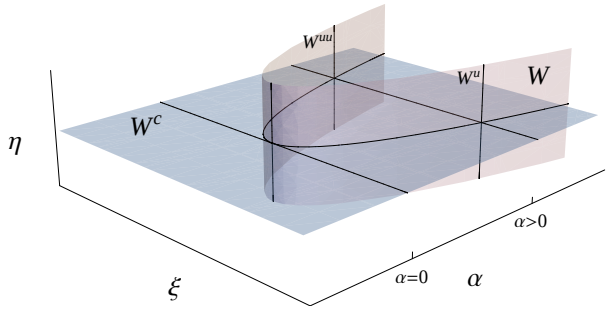



Figure 8: The manifold $W$.

To solve this equation using the method of characteristics, introduce $w = w(t)$ as an independent variable by setting

$$\eta w = \alpha_1 + f_0(\xi,\mu).$$

Deriving with respect to time and using equations (23)–(25) yields

$$\eta\dot{w} = -\dot{\eta}w - \frac{\partial f_0}{\partial \xi}\dot{\xi} = -w(\rho + g_1)\eta + \frac{\partial f_0}{\partial \xi}(w + f_1)\eta.$$

Dividing out $\eta$ formally, an equation for $\dot{w}$ is obtained. Together with equations (23)–(25), the following system is obtained:

$$\begin{aligned} \dot{\xi} &= \eta w + \eta f_1, \qquad & \dot{w} &= -\rho w - w g_1 - \frac{\partial f_0}{\partial \xi}(w + f_1), \\ \dot{\eta} &= \rho\eta + \eta g_1, \qquad & \dot{\mu} &= 0. \end{aligned}$$

Linearising the new system at $(\xi, \eta, w, \mu) = (0, 0, 0)$ yields

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{w} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \rho & & \\ & & -\rho & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ w \\ \mu \end{pmatrix}$$

Again invoking the center manifold theorem, we find that there is an invariant center-unstable manifold $W^{cu}$ that is tangent to the center-unstable eigenspace $E^{cu} = \{w = 0\}$. Let this manifold be parameterized, in a neighborhood of the origin, as

$$W^{cu} \ : \ w = w^{cu}(\xi, \eta, \mu).$$

Then $w^{cu}$ is the function we have been looking for.

A final note on $W$: as for $\mu = 0$ the unstable manifold $W^u$ is tangent to $\alpha_1 = 0$ at $\xi = 0$, the function $w$ in

$$W \ : \quad \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu), \tag{27}$$

has to satisfy $w = \xi^2 \tilde{w}$.

Indifference-attractor or indifference repeller bifurcations occur if $(\bar{x}, p(\bar{x}, \mu)) \in W$. The equations

$$x = \bar{x}, \quad p = p(\bar{x}, 0) + \alpha_2$$

take in $(\xi, \eta)$-coordinates the form

$$\xi = c_1\alpha_2 + O(\varepsilon^2 + \alpha_2^2), \quad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Note however that if $\mu = 0$, then $W$ is given by $\xi = 0$. Moreover, by assumption $p(\bar{x}, 0) \in W$; therefore the equations actually read as

$$\xi = \alpha_2 \left(c_1 + O(\varepsilon + \alpha_2)\right), \quad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Substitution in equation (27) yields the indifference-attractor and indifference-repeller bifurcation curves

$$\alpha_1 = c(\mu)c_1^2\alpha_2^2 + O(\alpha_2^3).$$

Taken together with the saddle-node curve

$$\alpha_1 = 0,$$

this yields the bifurcation diagram. Finally, note that if $p(\bar{x}, \mu) > w^{uu}(\bar{x})$, the saddle node bifurcation does not correspond to a bifurcation of the optimal vector field. □

### 3.3.5 DISN$_3$ bifurcation

It is possible that a 'double' ISN singularity, denoted DISN$_3$, occurs if conditions (21) and (22) of definition 3.6 are replaced by the condition that

$$p(x) = w^u(x)$$

for all $x \in \partial N$. This is clearly a codimension three situation.

# References


[1] ARNOLD, V. (1988). *Geometrical methods in the theory of ordinary differential equations*. Springer, New York.

[2] CAULKINS, J., FEICHTINGER, G., JOHNSON, M., TRAGLER, G. and YEGOROV, Y. (2005). Skiba thresholds in a model of controlled migration. *Journal of Economic Behavior & Organization*, **57** (4), 490–508.

[3] CAULKINS, J. P., FEICHTINGER, G., GRASS, D. and TRAGLER, G. (2007). Bifurcating DNS thresholds in a model of organizational bridge building. *Journal of Optimization Theory and Applications*, **133** (1), 19–35.

[4] CLARK, C. (1976). *Mathematical bioeconomics: the optimal management of renewable resources*. Wiley-Interscience, New York.

[5] DECHERT, W. D. and NISHIMURA, K. (1983). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory*, **31** (2), 332–354.

[6] FEICHTINGER, G. and TRAGLER, G. (2002). Skiba thresholds in optimal control of illicit drug use. In G. Zaccour (ed.), *Optimal control and differential games: essays in honor of Steffen Jørgensen, 1*, Kluwer Academic Publishers, pp. 3–22.

[7] FLEMING, W. and SONER, H. (2006). *Controlled Markov processes and viscosity solutions*. Springer, New York.

[8] GRASS, D. (2010). Numerical computation of the optimal vector field in a fishery model. *Working Paper, The Beijer International Institute of Ecological Economics*.

[9] HARTMAN, P. (1960). On local homeomorphisms of Euclidean spaces. *Boletine de la Sociedad Matematica Mexicana*, **5** (2), 220–241.

[10] — (1964). *Ordinary differential equations*. Wiley & Sons, New York.

[11] HINLOOPEN, J., SMRKOLJ, G. and WAGENER, F. O. O. (2010). From Mind to Market: A Global, Dynamic Analysis of R&D . *Working Paper, University of Amsterdam*.

[12] HIRSCH, M. W., PUGH, C. C. and SHUB, M. (1977). *Invariant manifolds*. Springer, Berlin.

[13] KISELEVA, T. and WAGENER, F. O. O. (2010). Bifurcations of optimal vector fields in the shallow lake model. *Journal of Economic Dynamics and Control*, **34** (5), 825–843.

[14] MÄLER, K. G., XEPAPADEAS, A. and DE ZEEUW, A. (2003). The economics of shallow lakes. *Environmental and Resource Economics*, **26**, 603–624.

[15] MOGHAYER, S. M. and WAGENER, F. O. O. (2009). Genesis of indifference thresholds and infinitely many indifference points in discrete time . *Working Paper, University of Amsterdam*.

[16] PALIS, J. and TAKENS, F. (1993). *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*. Cambridge University Press, Cambridge.

[17] ROWTHORN, R. and TOXVAERD, F. (2011). The optimal control of infectious diseases via prevention and treatment. *Working paper, University of Cambrige*.

[18] SETHI, S. P. (1977). Nearest feasible paths in optimal control problems: Theory, examples, and counterexamples. *Journal of Optimization Theory and Applications*, **23** (4), 563–579.

[19] — (1978). Optimal quarantine programmes for controlling an epidemic spread. *The Journal of the Operational Research Society*, **29** (3), 265–268.

[20] — (1979). Optimal advertising policy with the contagion model. *Journal of Optimization Theory and Applications*, **29** (4), 615–627.

[21] SKIBA, A. K. (1978). Optimal growth with a convex-concave production function. *Econometrica*, **46** (3), 527–539.

[22] TAHVONEN, O. and SALO, S. (1996). Nonconvexities in optimal pollution accumulation. *Journal of Environmental Economics and Management*, **31** (2), 160–177.

[23] TRAGLER, G., CAULKINS, J. and FEICHTINGER, G. (2001). Optimal dynamic allocation of treatment and enforcement in illicit drug control. *Operations Research*, **49** (3), 352–362.

[24] WAGENER, F. O. O. (2003). Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics and Control*, **27** (9), 1533–1561.

[25] — (2006). Skiba points for small discount rates. *Journal of Optimization Theory and Applications*, **128** (2), 261–277.