Some reflections on complexity economics research in the "SFI spirit" *

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Abstract

This chapter discusses some of our thoughts and reflections about complexity research in economics at the Santa Fe Institute and beyond in the last few decades. We cover a wide arrange of complexity economics topics: detecting chaos and nonlinearity in economic data, early warning signals, complex dynamics under bounded rationality and heterogeneous expectations, empirical validation of heuristics switching models, social interactions, experimental macroeconomics, managing positive feedback in complex systems and climate modeling. We discuss some policy implications and end with a short future perspective.

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1 Introduction

This chapter discusses some of our past research in complexity economics with some reflections on how what we call "The SFI Spirit" has pushed the two of us into research adventures that drive us to this day.

One of us (WB) has written chapters for all three previous conference volumes of *The economy as an evolving complex system* at the Santa Fe Institute and has been a fan of SFI's complexity efforts in economics since its beginning. This chapter puts forth some thoughts and reflections on the previous volumes and speculates about future research directions in complexity economics.

Another one of us (CH) has founded a complexity research institute CeNDEF (Center for Nonlinear Dynamics in Economics and Finance) at the University of Amsterdam 25 years ago, inspired by the SFI work. Our dean used to call CeNDEF the "SFI at the Amstel". This chapter discusses some CeNDEF complexity research over those years.

The chapter touches upon a wide range of complexity economics topics: detecting chaos and nonlinearity in economic and financial data, early warning signals in ecology, complex dynamics under bounded rationality and heterogeneous expectations, empirical validation of heuristics switching models, behavioral and experimental macroeconomics, how policy can manage positive feedback in complex systems and climate modeling.

2 Nonlinearity and chaos in economic data

Interest in nonlinearity and chaos in various time series data sets including economic and financial data flourished during the 80's and 90's (Dechert (1996)) and continues to this day. Part of the first volume of *The Economy as an Evolving Complex System* (Anderson, Arrow and Pines, Eds.,1988) is devoted to this topic. (See articles by Brock, Boldrin, Farmer and Sidorowich, Packard, Ruelle).

The possibility of noisy deterministic chaotic explanations of economic and financial time series data as well as time series data sets from other sciences attracted research, not only in economic theory (e.g. Grandmont (Ed.) Nonlinear Economic Dynamics (1987)), but also empirical research and formulation of various testing

methods.

Methods of testing for chaos were developed by physicists and mathematicians (e.g. Grassberger and Procaccia (1981), Crutchfield, Farmer, Packard, and Shaw (1986), Takens (1981), Theiler et al (1992)). Sugihara and May (1990) took a different approach, using nonlinear forecasting as a way of distinguishing chaos from noise.

Initial studies in economics and finance (Barnett and Chen in Barnett, Berndt, and White, Eds (1988), Brock and Sayers (1988), Scheinkman and LeBaron (1989)) raised interesting questions suggesting the possibility of attractors generated by deterministic dynamical systems that "looked random" but could be reconstructed (despite being pummeled by stochastic noise and unmeasured variables) well enough to give an incremental advantage in prediction over received methods of prediction. Grappling with such a challenge spurred many investigations and attracted a lot of publicity, e.g. David Berreby (1993) as well as generating articles in high profile more popular science outlets like Scientific American (Crutchfield, Farmer, Packard, and Shaw (1986)).

This work was adapted by econometricians into formal and applied statistical methods for testing for hidden structure where size and power against various alternatives could be evaluated by econometricians using, for example, tools from U-statistics theory (Brock, Dechert, Scheinkman, and LeBaron (1996), Brock, Hsieh and LeBaron (1991)). Brock's work on the BDS project and other explorations into nonlinearity benefitted much from interactions with Santa Fe Institute scholars.

The evidence in financial and macroeconomic data, especially financial returns data, for the hypothesis of chaos buffeted with dynamic noise is weak. The evidence for nonlinear dependence in the data, however, is strong and the BDS test has strong power against many nonlinear alternatives. At the same time economic theorists built New Classical general equilibrium models with rational expectations that could generate complex dynamics similar to the chaotic quadratic map (e.g. Grandmont (1985), Benhabib and Day (1982), Boldrin and Woodford (1990)). The patterns in the time series generated by these theoretical models however looked different from economic and financial data.

Empirical difficulties, both in calibrating new classical nonlinear endogenous business cycle models to economic data and in finding evidence for low-dimensional chaos

in economic and financial time series, thus prevented a full embracement and appreciation of complex nonlinear dynamics in economics in the 1980s and early 1990s (Hommes, 2013). More recent complexity economics research does not rely strongly on chaos, but rather on nonlinearities and near-unit root processes that may not generate sustained fluctuations on their own, but provide important nonlinear amplification mechanisms, consistent with behavior observed in laboratory experiments with human subjects and in empirical data, that generate near-unit root (temporary) bubble and crash like phenomena (see e.g. the recent survey in Hommes (2021))[1].

3 Early warning signals of regime changes

Are regime changes in ecology and other sciences predictable to some extent? One source of potential regime changes in ecosystems is impending bifurcations caused by slow changing forces that impact the dynamics of an ecosystem which can cause loss of stability or other structural changes. Kuznetsov (1995) is a basic textbook on bifurcations in differential and difference equations where there is a slow moving parameter that moves slowly relative to the dynamics of the differential and difference equations under study. Some bifurcations can admit Early Warning Signals (EWS) as explained below.

Consider the difference equation system perturbed by an independent and identically distributed stochastic process $\{e_t\}$:

$$x_{t+1} = f(x_t, a, \epsilon e_{t+1}), \qquad x_t \in R^n, a \in R. \tag{1}$$

Suppose a moves slowly relative to the time scale of the difference equation and suppose there is a steady state $\bar{x}(a) = f(\bar{x}(a), a)$. A bifurcation point a^* is where the qualitative behavior of (1) changes as a increases past a^*. Consider the case where the steady state $\bar{x}(a)$ loses local asymptotic stability at a^*. Is there some kind of EWS generated by (1) when a approaches a^*? Under appropriate sufficient conditions a local measure of variance increases as a approaches a^* (see Brock and Carpenter (2010) for a general treatment where x is an n-dimensional state vector

[1]Interestingly, empirical estimates of parameters of nonlinear behavioral heterogeneous expectations switching models, such as those discussed in Sections 5 and 9 of this chapter, typically do not lie in the chaotic parameter range, but are rather close to the first bifurcation towards instability and are thus consistent with near unit root behavior around a benchmark fundamental.

and some mathematics and Scheffer et al (2009) for a review of work up to 2009). Closely related to an increase in local variance is an increase in local autocorrelation as a approaches a^* (Scheffer et al (2009)).

The general method of analysis is to expand (1) in a Taylor series around the steady state $\bar{x}(a)$ and $\epsilon = 0$, collect the terms in $dx_t \equiv x_t - \bar{x}(a)$ and ϵ, compute the local variance covariance matrix of dx_t using the $n \times n$ derivative matrix $\partial f(\bar{x}(a), a, 0)/\partial x$, and study the behavior of the quadratic form built out of the local variance-covariance matrix as a approaches bifurcation point a^* (Brock and Carpenter (2010) and the Appendix to Biggs et al (2009) cited in Brock and Carpenter (2010)).

We consider the work on EWS part of the general literature on complexity because it involves studying bifurcations and patterns that can surround some classes of impending bifurcations (but not all). A large literature has grown out of the study of EWS and has been found many applications. Application to observational data in search of EWS is fraught with problems e.g. O'Brien et al (2023). Some of the issues raised by O'Brien et al (2023) are less of a concern in field studies where there is more control over noise, observed and unobserved confounders, etc than in studies using observational data (Carpenter et al (2011)).

4 Complex dynamics under heterogeneous expectations

In a world of bounded rationality there is an abundance of nonlinearity and complexity due to learning and interactions of heterogeneous agents. Our joint route in complexity started with our "rational route to randomness" paper Brock and Hommes (1997) introducing heterogeneous expectations in a cobweb model, where agents switch between different forecasting rules depending upon their relative performance. Agents can choose between a freely available simple rule, naive expectations, and a perfectly rational rule at positive information gathering costs. This model generates complicated, chaotic price fluctuations with irregular switching between a stable phase of close to the steady state fluctuations when the cheap naive rule dominates and an unstable phase with large price fluctuations with agents at some point switching back to rational expectations pushing prices back close to the steady state. The paper

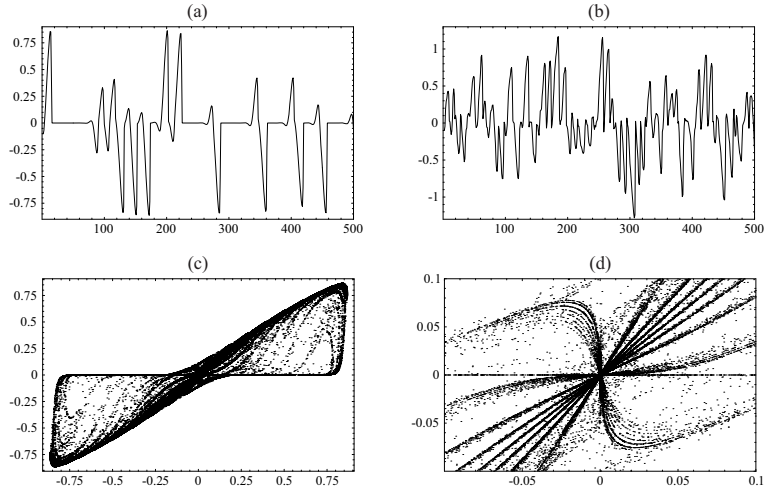


Figure 1: Chaotic (top left) and noisy chaotic (top right) bubble and crash dynamics in deviations from fundamental a behavioral asset pricing model with fundamentalists versus trend-extrapolators. The dynamics is characterized by a strange attractor (bottom left) with fractal structure (see enlargement bottom right) .

showed that the system exhibits a "rational route to randomness", that is, a bifurcation route to complex dynamics when agents become more “rational” in the sense that they become more sensitive to differences in fitness and switch faster to better, i.e. more profitable strategies.

In a follow-up paper Brock and Hommes (1998) applied the same nonlinear switching mechanism to a standard asset pricing model, with the price derived from myopic mean-variance maximizing agents with heterogeneous expectations. This paper got inspiration from the Santa Fe artificial stock market agent-based simulation model (Arthur et al.,1997; LeBaron et al., 1999; see also Brock’s (1997) contribution "Asset price behavior in complex environments", in "The Economy as an Evolving Complex System, Vol. II). Our goal was to simplify a rather complex agent-based simulation model and build an analytically tractable framework that could be used for empirical testing. Agents switch between fundamentalists’ strategies, assuming that asset prices are fully determined by economic fundamentals, and technical analysts, extrapolating observed patterns in past prices. Switching between strategies is based on the relative strategy performance. This simple behavioral asset pricing model exhibits complicated (chaotic) bubble and crash dynamics of asset prices, as illustrated in Figure 1. Along the bubbles trend-following strategies dominate and amplify prices,

with agents switching back to fundamentalists' strategies when prices deviate too far from fundamental value.

5 Empirical validation of two type switching model

Are these (chaotic) price fluctuation with temporary bubbles and crashes empirically relevant? Boswijk et al. (2007) estimated a simple 2-type switching model using yearly price-to-dividend or price-to-earnings S&P500 data. The simple 2-type switching model is of the form

$$R^* x_t = n_t \phi_1 x_{t-1} + (1 - n_t)\phi_2 x_{t-1} + \epsilon_t \qquad R^* = \frac{1+r}{1+g} > 1, \tag{2}$$

where x_t is the price deviation from the fundamental benchmark (discounted sum of expected future dividends or earnings), n_t and $1 - n_t$ are the time-varying fractions of the two types, fundamentalists and trend-followers, $\phi_1 x_{t-1}$ and $\phi_2 x_{t-1}$ are the forecasted deviations x_{t+1} by types 1 and 2, ϵ_t is a noise term and R^* depends on the interest rate r and the growth rate g of dividends or earnings.

Boswijk et al. (2007) find *behavioral heterogeneity* in the data, that is, the estimation supports two type of rules: fundamentalists using a mean-reverting rule with $\hat{\phi}_1 = 0.762$ and trend-followers with a coefficient $\hat{\phi}_2 = 1.135$, believing that the price deviation from fundamental will increase. Notice that the pricing equation (2) of the 2-type model is in fact a nonlinear AR(1) model with a time varying AR(1) coefficient, because the fractions n_t and $1-n_t$ are time-varying and depend on past relative performance and prices. Define the *market sentiment* or the *exuberance index*

$$\phi_t = \frac{n_t \phi_1 + (1 - n_t)\phi_2}{R^*}. \tag{3}$$

When the fundamentalists dominate $\phi_t < 1$ and there is mean reversion in prices. If on the other hand $\phi_t > 1$, the market is explosive and a (temporary) bubble may arise due to coordination on trend-following behavior. In this behavioral asset pricing model bubbles are triggered by shocks to fundamentals and amplified by the nonlinear switching mechanism to trend-extrapolation.

Hommes and Veld (2017) estimated a 2-type switching model using quarterly data 1950-2016 and obtained similar results, as illustrated in Figure 2. The market

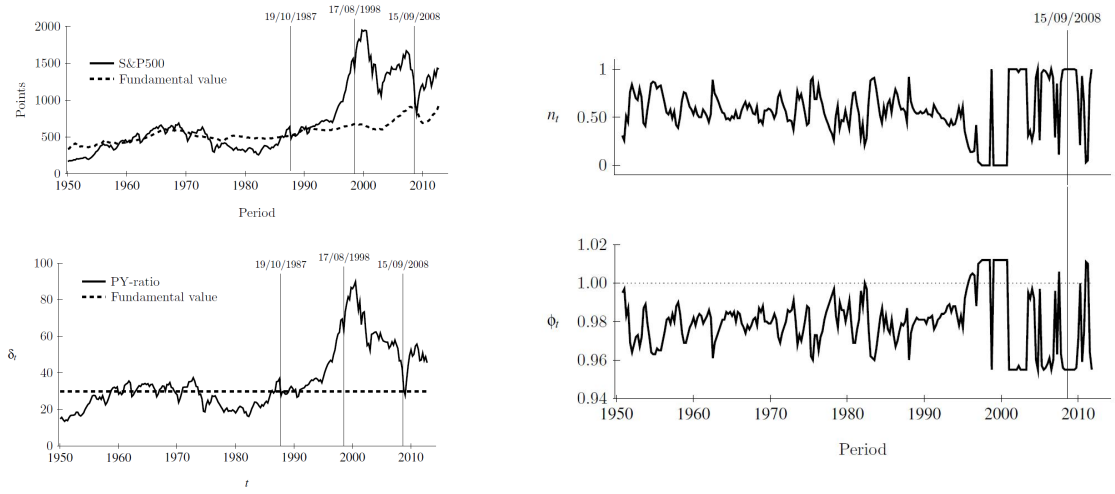


Figure 2: Top left: S&P500 together with benchmark fundamental (discounted sum of expected future dividends, dotted line). Bottom left: price-to-dividend ratio of S&P500 and its fundamental value around 30 (dotted line). Top right: Estimated fractions n_t of fundamentalists, and bottom right: time varying market sentiment or exuberance index.

sentiment fluctuates between 0.95 and 1.01, so price deviations from fundamental follow a near-unit root process. In the late 1990s and early 2000s the exuberance index exceeds 1 during the dot-com bubble. The behavioral story of the model is that the dot-com bubble was triggered by a positive shock (good news about a new internet technology) and subsequently amplified by trend-following behavior.

Exuberance in Housing Markets

Following a similar approach Bolt et al. (2019) estimated a 2-type housing model with fundamentalists versus trend-followers using housing prices from OECD countries. Many of these countries exhibit housing bubbles lasting several years, sometimes more than a decade. Emenogu et al. (2021) applied the same model to housing prices of Canadian cities. They computed a *heatmap* for housing prices in Canadian cities as shown in Figure 3. The green color indicates a relatively calm housing market with HPEI below 0.95; yellow indicates HPEI between 0.95 and 1. Red indicates that HPEI exceeds 1, implying higher risk for a bubble to appear. Over the years several Canadian cities have been identified as exuberant, for example Toronto and Hamilton around 2016-2017.

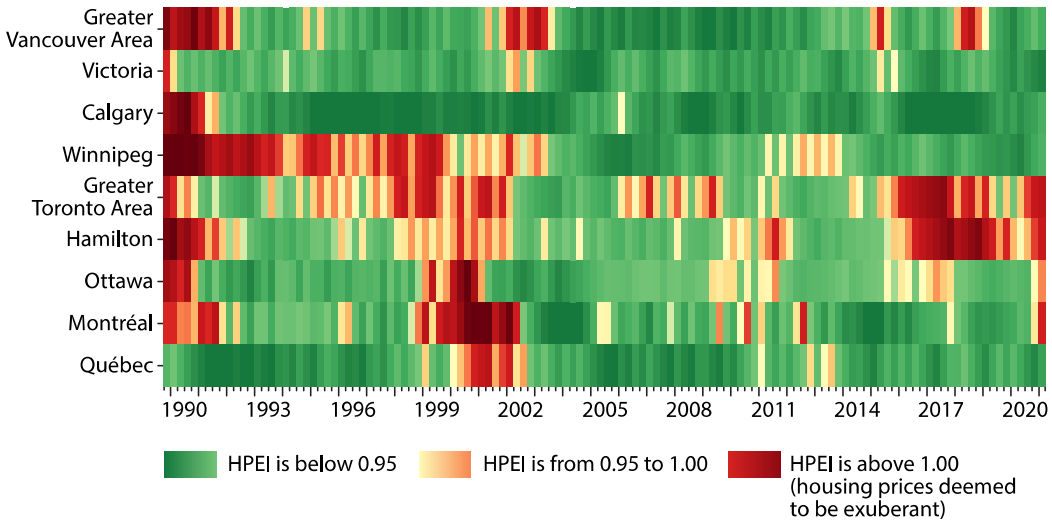


Figure 3: Heatmap of housing priice exuberance index (HPEI) measuring the stability or instability of housing prices in nine Canadian cities.

6 Social interactions

The economics of social interactions is another example of a complex system. This system has multiple equilibria, ingredients of network economics, non-cooperative equilibrium analysis, and even some mathematics of statistical mechanics. It has interesting econometric issues using data in separating endogenous social interactions where policy interventions can increase welfare of society. Brock and Durlauf in The Economy as an Evolving Complex System III (2006) review this area and show how multiple equilibria arise naturally. Using a multinomial logit discrete choice model similar to that in Brock and Hommes (1997), the probability of choosing choice k out of L choices is given by

$$p(k) = \exp(\beta h_k + \beta J p^e(k)) / \sum_{j=1}^{L} \exp(\beta h_j + \beta J p^e(j)), \quad k = 1, 2, \cdots, L. \tag{4}$$

Here h_k, $\beta \geq 0$, $J \geq 0$ and $p^e(k)$ are utility of choice k when $J = 0$, intensity of choice, social interaction parameter, and point expectation held by the representative agent on the probability of choice k. We impose rational point expectations, $p^e(k) = p(k)$, $k = 1, 2, \cdots, L$, apply Brouwer's fixed point theorem to obtain the equilibrium equations

$$p(k) = \exp(\beta h_k + \beta J p(k)) / \sum_{j=1}^{L} \exp(\beta h_j + \beta J p(j)), \quad k = 1, 2, \cdots, L. \tag{5}$$

It is obvious that the solution to (5) is unique when $\beta = 0$, and the solution is unique for all $\beta \geq 0$, when $J = 0$. Brock and Durlauf (2006) show if the product βJ is large enough multiple equilibria appear. We refer the reader to Brock and Durlauf (2006) for the details.

Brock and Durlauf (2006) show how difficult the econometric identification problem of policy-relevant social interactions is in using observational data sets because of unobserved variables, group selection bias, and other confounders. Brock and Durlauf (2006) and its references give a reader's guide to the voluminous literature in this area as well to the policy relevance of positive social interactions and the existence of social multipliers that can be exploited by policy.

7 Experimental macro

Laboratory experiments with human subjects have become a standard tool to test macroeconomic theory in the lab; see e.g the extensive surveys by Duffy (2016), Arifovic and Duffy (2018) and Hommes (2021). A macro experiment is a group experiment, where individual and aggregate behavior and their interactions are tested simultaneously in a controlled experimental environment. A macro experiment thus collects individual as well as aggregate data from the lab. For example, expectation formation has been extensively tested in so-called *learning-to-forecast experiments* (LtFEs), where subjects only task is to forecast a price for say 50 periods in an environment with expectations feedback, where realized price depends on the average or median forecast of a group of individuals. Key questions in the LtFEs are (1) Does the price converge to a rational expectations equilibrium, and (2) If not, which alternative theory of expectations explains individual and aggregate laboratory data? In many of these experiments prices do not converge to rational equilibrium, but rather fluctuate and follow bubble and crash patterns (e.g. Hommes et al., 2005). Some examples are shown in Figure 4. Hommes et al. (2021) show that the coordination on a bubble and crash pattern is robust against group size and not only occurs for small groups of 6-10 subjects, but also for large group sizes up to 100.

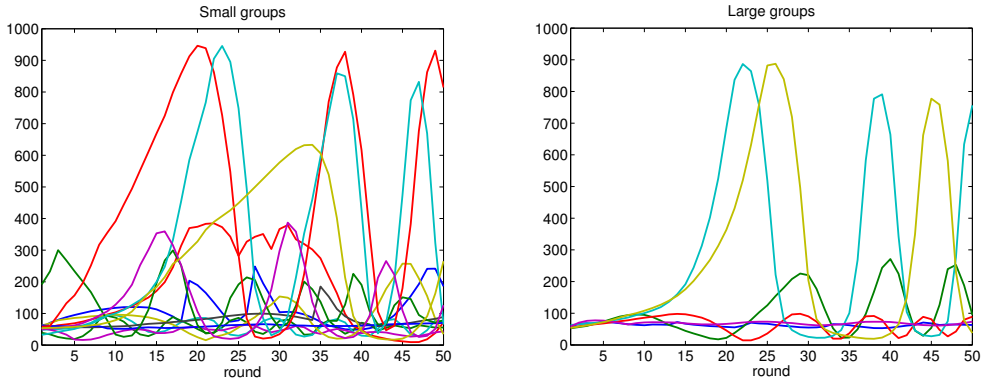


Figure 4: Experimental asset markets in LtFEs (Hommes et al., 2021). Often experimental markets do not converge to the rational expectations fundamental price (here 65), but rather exhibit bubble and crash dynamics. Coordination of expectations on large bubbles and crashes arises in small groups of 6 subjects (left plot) as well as in large groups of up to 100 subjects (right plot).

8 Positive versus negative feedback

The type of expectations feedback in the system is an important driver of its stability or instability. Positive (negative) feedback means that the price increases (decreases) when average forecast increases. Positive feedback corresponds to an environment with strategic complementarity and negative feedback to an environment with strategic substitutability. Positive feedback is typical in a speculative asset market, where price increases when investors become more optimistic. Negative feedback is common in supply driven commodity markets where optimistic producers produce more leading to lower market prices.

Heemeijer et al.,(2009) ran simple LtFEs with positive versus negative feedback using the linear pricing rules:

$$\text{negative feedback:} \qquad p_t = 60 - 0.95(\bar{p}_t^e - 60) + \epsilon_t$$

$$\text{positive feedback:} \qquad p_t = 60 + 0.95(\bar{p}_t^e - 60) + \epsilon_t,$$

where $\bar{p}_t^e$ is the average forecast of a group of subjects and ϵ_t is a small noise term. The linear pricing rules have the same RE equilibrium 60 and only differ in the sign of their slopes (or eigenvalues) $+0.95$ vs -0.95.

Figure 5 shows that the behaviors under positive vs negative feedback are strikingly different. Under negative feedback the market is stable with prices and expectations converging to the rational outcome 60. Under positive feedback prices fluctuate

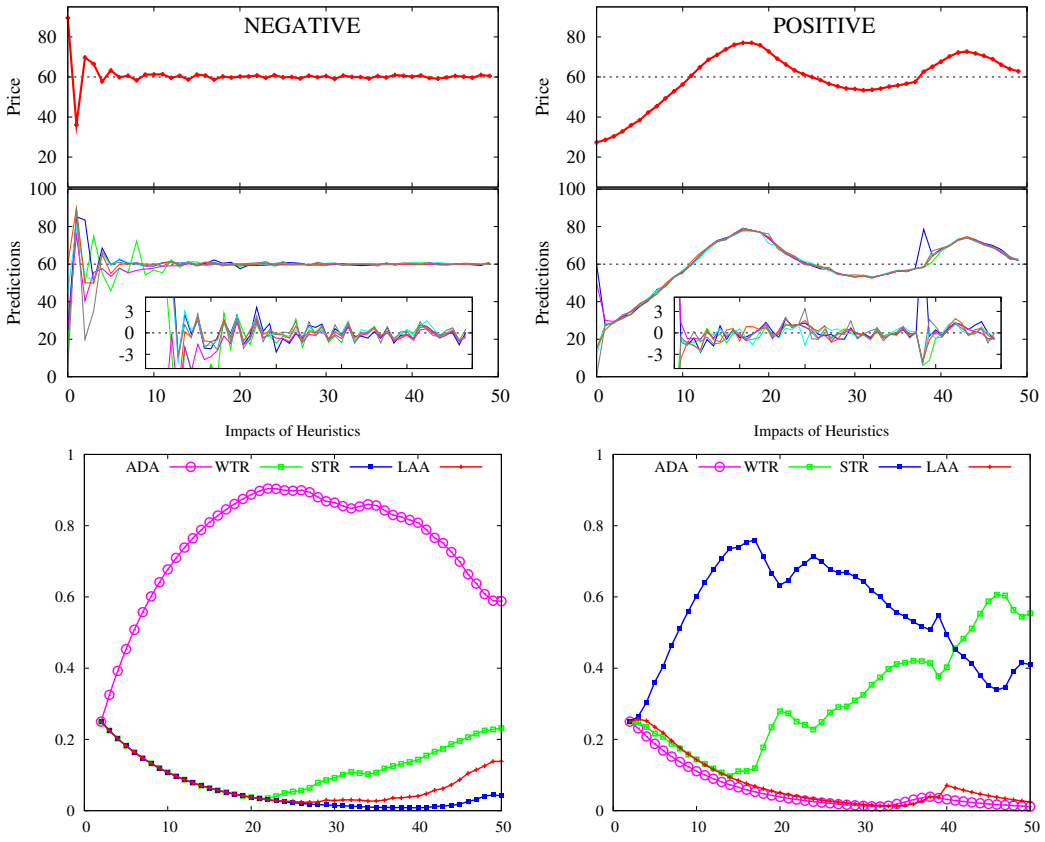


Figure 5: Negative feedback experiments (top left plots) are stable and converge quickly to the rational outcome. Positive feedback experiments (top right plots) are unstable, with drifting expectations and fluctuating prices. The HSM shows that subjects coordinate on adaptive expectations under negative feedback (bottom left plot). In contrast, under positive feedback (bottom right plot) coordination on trend-extrapolating behavior amplifies price fluctuations.

and expectations coordinate on trend-following behavior. Which theory of expectations can explain the different behaviors of positive versus negative feedback systems. The fact that the behavior in the lab is different for different treatments suggests that heterogeneity is important for expectations.

9 Heuristics Switching Model

Anufriev and Hommes (2012) proposed a Heuristics Switching Model (HSM), an extension of the nonlinear switching mechanism of Brock and Hommes (1997), to explain the different types of behavior observed in laboratory experiments: monotonic convergence, oscillatory convergence, explosive behavior and persistent oscillations. The idea of the HSM is that there are four heuristics that participants are choosing from to make their predictions in each time step. The heuristics are an Adaptive heuristic (ADA), a Weak Trend-following Rule (WTR), a Strong Trend-following Rule (STR) and tan Anchoring and Adjustment rule (LAA):

$$p^e_{1,t+1} = 0.65p_{t-1} + 0.35p^e_{1,t} \quad \text{(ADA)}$$

$$p^e_{2,t+1} = p_{t-1} + 0.4(p_{t-1} - p_{t-2}) \quad \text{(WTR)}$$

$$p^e_{3,t+1} = p_{t-1} + 1.3(p_{t-1} - p_{t-2}) \quad \text{(STR)}$$

$$p^e_{4,t+1} = 0.5(p^{av}_{t-1} + p_{t-1}) + (p_{t-1} - p_{t-2}). \quad \text{(LAA)}$$

Agents switch between these four heuristics based upon their relative performance. They use the same measure that is used to determine the pay-off of a forecast: .

$$U_{t-1,i} = \frac{100}{1 + |p_{t-1} - p^e_{t-1,i}|} + \eta U_{t-2,i}, \quad (6)$$

where $p_{t-1,i}$ is the previous forecast of rule i and η is a memory parameter. The HSM uses a discrete choice model with asynchronous updating to determine what heuristics the participants will use in the next time step. The intensity of choice parameter β determines how fast participants will switch to the most successful rule, and the parameter δ determines how many participants are switching strategies in an individual time step. The fraction of participants using strategy i to make their next

prediction is then given by

$$n_{t,i} = \delta n_{t-1,i} + (1-\delta)\frac{e^{\beta U_{t-1,i}}}{\sum_j e^{\beta U_{t-1,j}}}. \tag{7}$$

HSM applied to positive vs negative feedback experiment

The HSM can now be applied to the positive-negative feedback experiment discussed before,. The result is illustrated in Figure 5 (bottom plots). Under negative feedback the HSM shows that subjects coordinate on adaptive expectations (bottom left plot). In contrast, under positive feedback (bottom right plot) coordination on trend-extrapolating behavior amplifies price fluctuations. The behavioral HSM provides a clear explanation of why negative feedback markets are stable and positive feedback markets are unstable. Under negative feedback trend-following rules perform poorly and agents coordinate on adaptive expectations leading to stable dynamics. In contrast, under (strong) positive feedback trend-extrapolating rules perform well and coordination on trend-following behavior amplifies fluctuations.

Bao and Hommes (2019) ran an experimental housing market with positive feedback due to speculative demand for housing and negative feedback due to supply of housing. They show that the housing market is stable as long as the overall positive feedback is weak (an eigenvalue of $\lambda = 0.7$). When the positive feedback becomes stronger (eigenvalue $\lambda = 0.85$) the experimental housing market fluctuates and eventually shows coordination on trend-extrapolative behavior and (explosive) housing bubbles under strong positive feedback (eigenvalue $\lambda = 0.95$).

10 Policy: managing positive feedback

These results suggest an important role for policy in managing the instability of complex economic systems. If negative feedback leads to stable behavior and (strong) positive feedback to unstable behavior policy should in general add negative feedback to the system to prevent coordination on trend-extrapolating behavior. We illustrate this by discussing an experiment by Assenza et al. (2021) in the New Keynesian

framework. The NK model is given by the simple three equations system:

$$\begin{aligned} y_t &= y^e_{t+1} - \varphi(i_t - \pi^e_{t+1}) + \epsilon_t \quad \textbf{output gap} \\ \pi_t &= \lambda y_t + \rho \pi^e_{t+1} + \upsilon_t \qquad \textbf{inflation} \\ i_t &= \text{Max}\{\overline{\pi} + \phi_\pi(\pi_t - \overline{\pi}), 0\} \quad \textbf{monetary policy rule} \end{aligned}$$

Here y_t is the output gap, π_t inflation and i_t the interest rate set by the central bank. The monetary policy rule is an inflation targeting rule with inflation target $\bar{\pi}$ and policy coefficient ϕ_π. An increase of ϕ_π means a more aggressive inflation targeting rule.

The NK model exhibits *positive feedback* from expectations of inflation π^e_{t+1} and output gap y^e_{t+1}, but negative negative feedback from the MP inflation targeting rule Assenza et al. (2021) tested in the lab whether a more aggressive inflation targeting rule could stabilize expectations, inflation and output when expectations are formed in the lab. Some of their results are shown in Figure 6

Under weak inflation targeting ($\phi_\pi = 1$) the system is highly unstable and inflation explodes. Under strong inflation targeting ($\phi_\pi = 1.5$) the system is stable and inflation and output gap converge to RE equilibrium. Adding more negative feedback through a more aggressive inflation targeting rule thus stabilizes inflation (and output gap).

The HSM fits the individual and aggregate experimental data well. What is the behavioral explanation of the HSM? Under weak inflation targeting agents coordinate on a strong trend-extrapolating rule amplifying fluctuations and inflation (and output gap) explode. Under a stronger inflation targeting policy trend-following behavior can not survive competition with other heuristics and agents abandon trend-following rules and rather coordinate on adaptive expectations so that inflation (and output gap) stabilize.

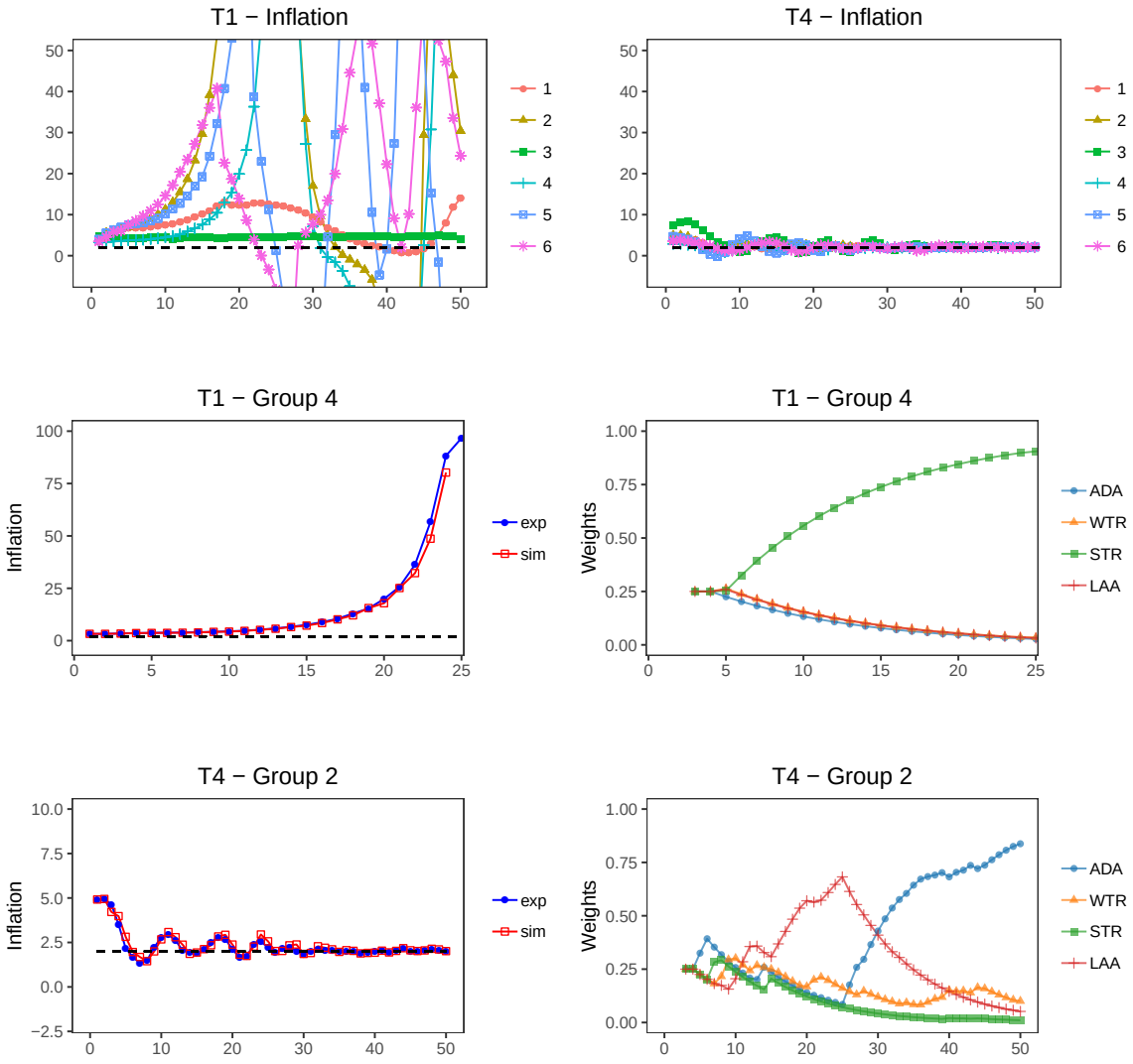


Figure 6: A stronger inflation targeting rule adds negative feedback to the system preventing coordination on trend-extrapolating behavior and thus stabilizing the NK system (Assenza et al., 2021).

11 Climate Modeling

An example of a complex system generating headlines is the climate system. The majestic review of Ghil and Lucarini (2020) opens with the sentences,

"*The climate is a forced, dissipative, nonlinear, complex, and heterogeneous system that is out of thermodynamic equilibrium. The system exhibits natural variability on many scales of motion, in time as well as space, and it is subject to various external forcings, natural as well as anthropogenic.*"

Their review covers chaos, critical transitions, bi-stability, Hopf bifurcations, teleconnections, homoclinic and heteroclinic bifurcations, bifurcation tree (Figure 31), and much more, all germane to the understanding of climate dynamics at a range of spatial and temporal scales.

Like climate sciences, climate economists work with a hierarchy of models ranging from simple energy balance models to spatial diffusion models illustrating poleward energy transport, but not yet encapsulating the full range of complexity. Climate economists have yet to build economic models that build in the spatial and temporal dynamics respecting the complexity as reviewed by Ghil and Lucarini (2020). We would like to tackle this challenge ourselves. In view of space shortage, we do not review the received work by climate economists here. We content ourselves with just mentioning a few here which the reader can google up (William Nordhaus, Yongyang Cai, Kenneth Judd, Thomas Lontzek, Michael Barnett, Lars Hansen, Anastasios Xepapadeas).

A major policy concern as the world tries to design policies to cope with climate change is the multiple layers of uncertainty that appear in the coupled economic and climate system. An example of recent work that accounts for multiple layers of uncertainty in addition to risk (where probabilities are known) is Barnett et al (2022). Accounting for multiple layers of uncertainty strengthens the case for strong policy action now rather than waiting until we learn more about the unknown unknowns in the coupled climate and economic dynamical system. For more discussion of climate (change) models see the Chapters in part IV Climate & Sustainability in this Volume.

12 Future perspective

We have discussed a variety of examples of complexity economics research and there are many more not covered here. SFI has played a key role in generating complexity ideas and providing us with the SFI spirit driving us.

Here, we have emphasized rather stylized models of complexity that are (partly) analytically tractable. More complex simulation models, such as agent-based models and climate (change) change models are an important complementary tool of complexity to study empirically relevant phenomena. For example, Hommes et al., (2024ab, including the chapter by Hommes, Kozicki, Poledna and Zhang (2024) in this Volume, use a detailed ABM of the Canadian economy at scale 1:100 for monetary policy analysis. For more on ABMs in macro and central banking see e.g. the Chapters Dawid et al. (2024) and Borsos et al. (2024). We conclude by saying that both stylized models and detailed ABMs should become part of the complexity toolbox for policy analysis.

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