On Learning Equilibria*

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Abstract

In this note we critically examine the results on learning equilibria, obtained by Bullard (1994) and Schönhofer (1999). In those papers it is shown that, in an overlapping generations model with fiat money, an increase in the money growth rate may lead to endogenous fluctuations. We suggest an alternative learning procedure, that models the same perceived law of motion, but which has more desirable stability properties.

1 Introduction

In recent years the limitations of the rational expectations hypothesis have become increasingly obvious. In particular, it has been perceived as unsatisfactory that this hypothesis endows economic agents with precise information about the structure of the economy and the beliefs of other agents as well as unbounded reasoning abilities to deal with this information. A number of authors have suggested that the rational expectations hypothesis still is valid as a description of long run behaviour, since economic agents learn over time and eventually arrive at a rational expectations steady state. The rational expectations hypothesis can therefore be supported by a learning story (see Lucas, 1976, Marcet and Sargent, 1989 and Evans and Honkapohja, 1999). In such a learning model the *bounded rational* agents are generally assumed to have no structural information about their economic environment other than time series observations on certain economic variables. They use these time series observations to make inferences about the economic environment. In his book on bounded rationality Sargent (1993, p.22) writes: "We can interpret the idea of bounded rationality

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broadly as a research program to build models populated by agents who behave like working economists or econometricians." Since the perceptions of agents influence their behaviour, the learning feeds back into the actual realizations of economic variables. Hence, the learning procedure itself is one of the determinants of the evolution of the economic variables. With respect to this learning procedure Bullard (1994, p.468) states:

"A common research question, asked increasingly often in the recent literature, is how this learning takes place, and more importantly, if it makes any difference for inferences from dynamic general equilibrium models whether the learning is explicitly modeled."

In his interesting paper Bullard shows that explicitly modelling agents as econometricians might create equilibria different from the rational expectations steady state. Some of these *learning equilibria* can be characterized by endogenous fluctuations in inflation rates and agents beliefs. Moreover, Schönhofer (1999) shows that *chaotic* learning equilibria exist. In this paper we show that the main results from these two papers arise from the estimation procedure that agents are supposed to use. In fact, for an estimation procedure that is more sensible from a statistical point of view, the dynamics of the model appear to be inherently stable. An important observation is that the perceived law of motion is the same for both procedures, the only difference is in the way this perceived law is estimated. The main point is that it is more sound from an econometric point of view to run a regression on a stationary time series than on a nonstationary time series. Since in our model agents want to predict the inflation rates and since the time series of price levels is nonstationary, the estimation procedure should be in terms of inflation rates instead of price levels.

The rest of the paper is organized as follows. Section 2 describes the overlapping generations model studied in Bullard (1994) and discusses the existence of learning equilibria. In Section 3 a learning procedure based upon inflation rates is introduced and the main stability results are given. Section 4 concludes.

2 Learning equilibria

We consider a standard two period overlapping generations model where in each period a generation is born that lives for two periods. The generation born in period t solves

$$\max_{c_0,c_1} U\left(c_0, c_1\right) \quad \text{subject to} \quad p_t c_0 + p^e_{t+1} c_1 \leq p_t w_0 + p^e_{t+1} w_1,$$

where $U : I\!R^2_+ \rightarrow I\!R$ is a strictly monotone, strictly quasi concave utility function, c_0 and c_1 are consumption in the first and second period of the agent's life and w_0 and w_1 are endowments in the first and second period of the agent's life. Furthermore, p_t is the price in period t and p_{t+1}^e is the price expected for period $t+1$. The optimization problem gives the optimal level of consumption in the first period of the agent's life as a function of expected inflation $\frac{p_{t+1}^e}{p_t}$, i.e. $c_0 = c_0\left(\frac{p_{t+1}^e}{p_t}\right)$. Optimal saving of the young generation is then given by the savings function

$$S\left(\frac{p_{t+1}^e}{p_t}\right) = w_0 - c_0\left(\frac{p_{t+1}^e}{p_t}\right).$$

From now on we will assume that the savings function is twice differentiable and positive, i.e. $S(\pi) > 0$ for all π (this corresponds to the *Samuelson* case, where people save when young).[1] The demand for real balances in period t is given by

$$\frac{M_t}{p_t} = S\left(\frac{p_{t+1}^e}{p_t}\right).$$

The only means of saving is money and the money stock M_t is controlled by the government and grows to finance government consumption. The monetary policy rule is

$$M_t = \vartheta M_{t-1}.$$

Combining the demand for real balances with the monetary policy rule, we get the following market clearing condition

$$S\left(\frac{p_{t+1}^e}{p_t}\right)p_t = \vartheta S\left(\frac{p_t^e}{p_{t-1}}\right)p_{t-1}.$$

In terms of gross inflation rates $\pi_t \equiv \frac{p_{t+1}}{p_t}$ this equilibrium condition becomes

$$\pi_{t-1} S\left(\pi_t^e\right) = \vartheta S\left(\pi_{t-1}^e\right). \tag{1}$$

At the monetary steady state, $\pi^* = \vartheta$, the inflation rate is equal to the money growth rate.

The model is closed by specifying the way in which agents form expectations about future inflation rates. Under rational expectations or perfect foresight we have $\pi_{t+1}^e = \pi_{t+1}$. It is well-known that for a downward sloping savings function the monetary steady state ϑ is unstable under perfect foresight. For nonmonotonic savings functions more complicated perfect foresight dynamics, such as cycles and chaotic fluctuations can occur (see e.g. Grandmont, 1985).

The assumption of perfect foresight requires that agents exactly know the market equilibrium equations as well as other agents' beliefs and are able to use this information to compute the market clearing prices for the future. An alternative approach is

[1] Since the savings function corresponds to an aggregate excess demand function, in principal any continuous function can be a savings function that is consistent with utility maximization, if we would extend the number of agents per generations to at least 2 (see Sonnenschein (1973)).

to assume that economic agents make inferences about their environment by means of a learning procedure. Such a procedure uses time series observations to make forecasts about the future development of variables. Consider the following example of such a learning procedure.

The generation born in period t has to make a forecast about the inflation rate for period $t+1$. Agents believe that the inflation rate is constant (which at the rational expectations steady state is indeed the case), that is, in terms of prices, they have the following *perceived law of motion*

$$p_t = \beta p_{t-1}, \tag{2}$$

The precise value of β, however, is unknown to the agents. Bullard (1994) assumes that agents run a least squares regression on prices in order to estimate this β and to be able to form predictions on the inflation rate. The least squares regression estimate for agents born in period t, using data available through time $t-1$, is

$$\beta_t = \frac{\sum_{s=1}^{t-1} p_{s-1} p_s}{\sum_{s=1}^{t-1} p_{s-1}^2}, \tag{3}$$

and hence their forecast of the inflation rate is $\pi_t^e = \beta_t$. Given this forecast, the *implied actual law of motion* for the price dynamics of the model becomes

$$p_t = \vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} p_{t-1}. \tag{4}$$

(3) and (4) together form an *expectations feedback system.* Realized prices influence perceptions agents have about their economic environment and these perceptions feed back into the actual dynamics and determine which prices will realize. The complete system (3-4) can be written as a recursive dynamical system by introducing the variable $g_t = p_{t-1}^2 \left[\sum_{s=1}^{t} p_{s-1}^2\right]^{-1}$. Written as a system of first-order difference equations, the learning model becomes

$$\begin{aligned}
\beta_{t+1} &= \beta_t + g_t \left[\vartheta \frac{S(\gamma_t)}{S(\beta_t)} - \beta_t\right], \\
\gamma_{t+1} &= \beta_t, \\
g_{t+1} &= \left[g_t^{-1} \left(\vartheta \frac{S(\gamma_t)}{S(\beta_t)}\right)^{-2} + 1\right]^{-1}.
\end{aligned} \tag{5}$$

The main result on learning equilibria is the following

Proposition 1 ***(Bullard, 1994)*** *Assume $\vartheta > 1$ and $S(.)$ is twice differentiable and downward sloping. Then (5) generically undergoes a Hopf bifurcation at the monetary steady state at that value ϑ^* of ϑ, for which*

$$\left(1 - \vartheta^{-2}\right) a(\vartheta) = 1,$$

where $a(\vartheta) = -\vartheta \frac{S'(\vartheta)}{S(\vartheta)}$ *is the* inflation elasticity of savings*, evaluated at the monetary steady state. For* $\vartheta < \vartheta^*$*, the monetary steady state is stable and for* $\vartheta > \vartheta^*$ *it is unstable.*

It will be useful for us to investigate the details of this result a little further. Consider the Jacobian matrix of (5) evaluated at the monetary steady state $(\beta^*, \gamma^*, g^*) = (\vartheta, \vartheta, 1 - \vartheta^{-2})$. This Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} \vartheta^{-2} + (1 - \vartheta^{-2}) a(\vartheta) & -(1 - \vartheta^{-2}) a(\vartheta) & 0 \\ 1 & 0 & 0 \\ \frac{\partial g}{\partial \beta} & \frac{\partial g}{\partial \gamma} & \vartheta^{-2} \end{pmatrix}. \tag{6}$$

One of the eigenvalues is equal to ϑ^{-2} and hence lies inside the unit circle for $\vartheta > 1$. The other two eigenvalues are eigenvalues are complex and lie on the unit circle when $(1 - \vartheta^{-2}) a(\vartheta) = 1$. Moreover, the eigenvalues cross the unit circle with positive speed as ϑ changes. The Hopf bifurcation described in the proposition leads to an invariant closed curve around the steady state of the learning dynamics. This closed curve can be attracting or repelling. Motion on the closed curve can be periodic and quasi-periodic.[2] Bullard (1994) calls these cycles "learning equilibria" since they correspond to equilibria of the learning dynamics, that are not equilibria under rational expectations. Their existence can therefore be attributed to the learning process. If ϑ increases further, the time series of the inflation rates can become even more complicated. Schönhofer (1999) gives, for a particular set of examples, numerical evidence for the existence of homoclinic orbits and chaos in the learning dynamics.

Let us now try to develop an intuition for the fact that the recursive least squares estimates do not converge to the monetary steady state. Ordinary least squares algorithms are so-called *decreasing gains* algorithms. Different observations receive the same weights in the regression which implies that, as time goes by and the number of observations increases, the impact of individual new observations becomes smaller. In (5) this gain is represented by the variable $g_t = p_{t-1}^2 / \sum_{s=1}^{t} p_{s-1}^2$. If price levels are bounded g_t will converge to 0 which, if it does not result in convergence to the monetary steady state, at least leads to ever smaller changes in the estimate of β. In the present model, however, price levels are unbounded and in fact, at the steady state they grow at a constant rate $\vartheta > 1$. This implies that the equilibrium value of the weight g_t is strictly positive, $g^* = 1 - \vartheta^{-2} > 0$. Hence, even after many observations, one new observation on the price level may lead to a significant change in the beliefs of the agents, which therefore keep on fluctuating, implying endogenous and persisting fluctuations in the inflation rates.

In fact, least squares learning on *price levels* is closely related to the *adaptive expectations rule* on *inflation rates.* Adaptive expectations (Nerlove, 1958) corresponds to updating the expectation in the direction of the last observation, i.e.,

[2] If the savings function is nonmonotonic similar phenomena occur. In that case, the monetary steady state may also lose stability throug a period-doubling bifurcation. This happens at that value ϑ^* for ϑ for which $\frac{\vartheta^2 - 1}{\vartheta^2 + 1} a(\vartheta) = -\frac{1}{2}$.

$\pi^e_{t+1} = \pi^e_t + \alpha(\pi_{t-1} - \pi^e_t)$, with $0 < \alpha \leq 1$. Notice that the weight α is constant, and adaptive expectations therefore correspond to a *constant gains* algorithm. Introducing adaptive expectations into (1) yields the following second order difference equation

$$\beta_{t+1} = \beta_t + \alpha\left(\vartheta\frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_t\right). \tag{7}$$

Notice that the only difference between (5) and (7) is that for the latter the weight α is constant whereas for the former it depends upon the realization of the prices. However, if the weight in (7) equals $\alpha = g^* = 1 - \vartheta^{-2}$, (7) has the same local stability properties as (5).[3] Hence, the learning scheme proposed by Bullard (1994) turns out to be closely related to adaptive expectations. Although the weight or gain g_t in (5) is not constant, it is certainly not (monotonically) decreasing over time.

3 An alternative procedure

In the previous section it was argued that the nonstationary nature of the price time series may lead to endogenous and persisting fluctuations in inflation rates. According to the perceived law of motion (2) agents believe that the systematic part of the inflation rate is constant. The residuals or forecast errors from the regression (3) turn out to be

$$e_t = p_t - \beta_{t-1}p_{t-1} = \left(\vartheta\frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1}\right)p_{t-1}.$$

Clearly, if the economy is not at the monetary steady state ϑ, the part between brackets does not vanish and the forecast errors grow without bound. This provides us with another intuition for the nonconvergence of the recursive system (5): because the forecast errors grow indefinitely (in absolute value), the estimates keep changing significantly, despite the fact that the weight attached to each individual observation decreases as time goes by. Each new observation can upset the current estimate and lead to a radical change in the estimated perceived law of motion which, of course, is an unsatisfactory property of an estimation procedure. Given the exploding forecast errors, and the fluctuations in the beliefs, agents will be inclined to change their estimation procedure. Apart from that, it seems to be not too sensible to run a least squares regression on an exploding time series. We can rewrite (2) into the following perceived law of motion

$$\pi_t = \beta. \tag{8}$$

[3] This follows from the fact that the upper 2×2 matrix of (6) (which is the relevant part) is equal to the Jacobian of (7).

Notice that the economic agents' perceptions underlying both (2) and (8) is that the inflation rate is constant. Now let us assume that agents try to learn β in (8) by running a least squares regression of inflation rates on a constant, which corresponds to averaging over past inflation rates, that is,

$$\beta_{t+1} = \frac{1}{t}\sum_{s=0}^{t-1}\pi_s = \frac{1}{t}\left[\sum_{s=0}^{t-2}\pi_s + \pi_{t-1}\right] = \left(1-\frac{1}{t}\right)\beta_t + \frac{1}{t}\pi_{t-1}.$$

The evolution of inflation rates and dynamics is then described by

$$\beta_{t+1} = \beta_t + \frac{1}{t}\left(\vartheta\frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_t\right). \tag{9}$$

This updating rule is closely related to (5) and (7), the main difference lying in the fact that the weight factor $1/t$ approaches 0 as t goes to infinity. Hence, the contribution of new observations will decrease over time and the stability properties of (9) turn out to be dramatically different from the stability properties of (5). In the following two subsections we will respectively investigate local and global stability of the monetary steady state ϑ for our learning process (9).

3.1 Local stability

In this section we will show that the monetary steady state ϑ of the dynamical system (9) is locally stable for almost any savings function. In particular, for all ϑ we have that if the initial inflation rates are sufficiently close to the monetary steady state, the inflation rates converge to this steady state. This local stability result comes in two steps. In the first step conditions are given under which the sequence $\{\beta_t\}$ is bounded, and in the second step it is shown that this bounded sequence will converge to the monetary steady state ϑ.

Proposition 2 *Assume $S(.)$ is differentiable and positive. Then for each $K > 2\vartheta$, there exists $T > 0$ such that if for $1 \le t \le T$ the condition*

$$\beta_t < K$$

holds, then the sequence $\{\beta_t\}$, generated by (9), is bounded.

Proof. Some preparatory remarks are made. Introduce $\delta_{t+1} = \beta_{t+1} - \beta_t$, the amount by which β_t changes at time $t+1$. Then

$$\delta_{t+1} = \frac{1}{t}\left(\frac{S(\beta_{t-1})}{S(\beta_t)}\vartheta - \beta_t\right).$$

Introduce

$$M = \frac{\max_{0\le\beta\le K} S(\beta)}{\min_{0\le\beta\le K} S(\beta)}.$$

If T is such that for $1 \leq t \leq T$, we have that $\beta_t < K$, then $|\delta_{t+1}|$ can be bounded as

$$|\delta_{t+1}| \leq \frac{1}{t} (M\vartheta + K) = \frac{K'}{t}. \tag{10}$$

Taking (10) as an implicit definition of K', the constant T is now chosen as

$$T = 3 \max \left\{ K' \frac{\max_{0 \leq \beta \leq K} S'(\beta)}{\min_{0 \leq \beta \leq K} S(\beta)}, K'\vartheta^{-1} \right\}.$$

In order to bound δ_{t+1} more sharply, the ratio $S(\beta_{t-1})/S(\beta_t)$ has to be estimated first. With Taylor's theorem, assuming that $\beta_{t-1}, \beta_t \leq K$:

$$\frac{S(\beta_{t-1})}{S(\beta_t)} = \frac{S(\beta_t - \delta_t)}{S(\beta_t)} = \frac{S(\beta_t) - S'(\xi)\,\delta_t}{S(\beta_t)} = 1 - \frac{S'(\xi)}{S(\beta_t)}\delta_t,$$

where $\xi \in (\beta_t - \delta_t, \beta_t)$; we use the convention that the open interval (a, b) is equal to (b, a), irrespective of wether $a < b$ or $b < a$. Hence

$$\left| \frac{S(\beta_{t-1})}{S(\beta_t)} - 1 \right| \leq \frac{\max_{0 \leq \beta \leq K} S'(\beta)}{\min_{0 \leq \beta \leq K} S(\beta)} \frac{K'}{t} \leq \frac{1}{3}\frac{T}{t}. \tag{11}$$

where (10) and the definition of T are used. The proof proceeds by an induction argument. Assume that $\beta_s < K$ for all $s \leq t$, where $t \geq T$. From (11) it follows that

$$\frac{2}{3} \leq \frac{S(\beta_{t-1})}{S(\beta_t)} \leq \frac{4}{3}. \tag{12}$$

Now, if $\beta_t > \frac{4}{3}\vartheta$, then combining estimate (12) with the definition of δ_{t+1} yields

$$\delta_{t+1} < \frac{1}{t}\left(\frac{4}{3}\vartheta - \beta_t\right),$$

and the right hand side is negative by assumption. Hence $\beta_{t+1} < \beta_t < K$ in this case.

Likewise it is shown that for $0 \leq \beta_t \leq \frac{2}{3}\vartheta$, $\frac{4}{3}\vartheta > \delta_{t+1} > 0$ and $0 \leq \beta_t < \beta_{t+1} < 2\vartheta < K$.

Consider now the third case, $\frac{2}{3}\vartheta \leq \beta_t \leq \frac{4}{3}\vartheta$. Combining (10), the fact that $t \geq T$, and the definition of T yields

$$|\delta_{t+1}| \leq \frac{K'}{T} \leq \frac{\vartheta}{3};$$

hence $\beta_{t+1} \leq \frac{5}{3}\vartheta < 2\vartheta$ and since $2\vartheta < K$ by assumption, it follows also in this case that $\beta_{t+1} < K$.

The induction is complete, and we have shown that $\beta_t < K$ for all $t \geq 0$. ■

Next we prove convergence of the bounded series.

Proposition 3 *The sequence* $\{\beta_t\}$*, generated by (9), with* $\beta_t \in [0, K]$ *for all* t*, converges to* ϑ*.*

Proof. Note that as a corollary of the proof of the preceding proposition, we obtain that (10) holds for all time t. Hence, $\delta_t \to 0$ as $t \to \infty$. By uniform continuity of S on $[0, K]$ it follows that

$$\frac{S(\beta_{t-1})}{S(\beta_t)} = \frac{S(\beta_{t-1})}{S(\beta_{t-1} + \delta_t)} \to 1,$$

as t tends to infinity. From this the following is inferred.

Choose $\varepsilon > 0$ arbitrarily. There is a $\widetilde{T} > 0$, such that

$$\left| \frac{S(\beta_{t-1})}{S(\beta_t)} - 1 \right| < \varepsilon,$$

for all $t > \widetilde{T}$, uniformly in β_t.

Three cases are distinguished. Assume first that $\beta_{t_0} > (1 + 2\varepsilon)\vartheta$ for some $t_0 > \widetilde{T}$. Then

$$\delta_{t+1} \leq \frac{1}{t}\left((1+\varepsilon)\vartheta - (1+2\varepsilon)\vartheta\right) \leq -\frac{\varepsilon\vartheta}{t}.$$

Were $\beta_t > (1 + 2\varepsilon)\vartheta$ for all $t \geq t_0$, then

$$(1+2\varepsilon)\vartheta < \beta_t = \beta_{t_0} + \sum_{s=t_0+1}^{t} \delta_s \leq \beta_{t_0} - \varepsilon\vartheta \sum_{s=t_0+1}^{t} \frac{1}{s}.$$

But the sum increases beyond all bounds as t tends to infinity, so this is an impossibility.

Likewise $\beta_t < (1 - 2\varepsilon)\vartheta$ cannot be satisfied for all $t \geq t_0$, for any t_0.

Moreover, since $|\delta_t| < 2\varepsilon\vartheta$ for t large enough, it follows that there exists some $t_1 \geq \widetilde{T}$ such that

$$(1-\varepsilon)\vartheta < \beta_{t_1} < (1+\varepsilon)\vartheta.$$

Let t_2 be the first t greater than t_1, such that $|\beta_{t_2} - \vartheta| > 2\varepsilon\vartheta$. Since $|\delta_t|$ is bounded by $2\varepsilon\vartheta$,

$$|\beta_{t_2} - \vartheta| \leq |\beta_{t_2} - \beta_{t_2-1}| + |\beta_{t_2-1} - \vartheta| \leq 2\varepsilon\vartheta + 2\varepsilon\vartheta = 4\varepsilon\vartheta.$$

But from t_2 onwards, β_t increases (or decreases) towards ϑ until it satisfies again $|\beta_t - \vartheta| < \varepsilon\vartheta$ (by the arguments given above). Hence, for all $t \geq t_1$ we have that

$$|\beta_t - \vartheta| \leq 4\varepsilon\vartheta.$$

Since ε was arbitrary, this proves the proposition. ■

Note that the main examples of savings functions studied in Bullard (1994) and Schönhofer (1999) satisfy the conditions of the proposition. Also observe that we do not require the savings function to be monotonically decreasing for our local stability results.

3.2 Global stability

Now that we have established that the monetary steady state is locally stable under our learning procedure, we like to say something about the global behaviour of the inflation rates. Will the inflation rates converge to the monetary steady state for any set of initial conditions? If the savings function is bounded away from zero the inflation rates themselves will also be bounded and the following Corollary is an immediate consequence from Proposition 3.

Corollary 4 *If $S(\beta)$ is differentiable, $c \in \mathbb{R}$, and*

$$S(\beta) \geq c > 0,$$

for all β, then the sequence $\{\beta_t\}$, generated by (9), converges to ϑ.

Notice that by the same argument the occurrence of bounded and persisting fluctuations in the inflation rate, as featured in the dynamical system (5), is excluded. The next proposition gives another class of savings functions for which global stability can be established.

Proposition 5 *Assume $S(\beta)$ is differentiable, $S(\beta) > 0$ for all $\beta \in [0, \infty)$ and that there is a constant K_0, such that $S(\beta)$ is monotonically decreasing for $\beta > K_0$. Assume furthermore that there is a constant $\alpha \in [0, 1]$, and a function $r(\beta)$ such that $S(\beta) = c\beta^{-\alpha} + r(\beta)$, with*

$$\left| \frac{r(\beta)}{c\beta^{-\alpha}} \right| \to 0 \qquad \text{as } \beta \to \infty. \tag{13}$$

Then the sequence $\{\beta_t\}_{t=0}^{\infty}$, determined by $\beta_0, \beta_1 > 0$, and the evolution equation:

$$\beta_{t+1} = \beta_t + \frac{1}{t} \left[\frac{S(\beta_{t-1})}{S(\beta_t)} \vartheta - \beta_t \right],$$

converges, as $t \to \infty$, to the limit ϑ.

Proof. All that needs to be done is to show that the sequence $\{\beta_t\}$ is bounded. First fix $K_1 > 3\vartheta$ such that for $\beta > K_1$:

$$\left| \frac{\beta^{\alpha} r(\beta)}{c} \right| < \frac{1}{2}. \tag{14}$$

Such a K_1 exists because of condition (13). Now take $K = \max\{K_0, K_1\}$. Two cases are distinguished:

1. $\beta_{t-1} \geq \beta_t > K$ and

2. $\beta_t \geq \beta_{t-1} > K$.

Case 1
If $K < \beta_t \leq \beta_{t-1}$, then $S(\beta_t) \geq S(\beta_{t-1})$ since $S(\beta)$ is monotonically decreasing for β larger than K, and we have

$$\frac{S(\beta_{t-1})}{S(\beta_t)}\vartheta - \beta_t \leq \vartheta - \beta_t < 0.$$

Hence $\beta_{t+1} < \beta_t$. This argument can be repeated until $\beta_{t+n} \leq K$, which brings us to case 3.

Case 2
If $K < \beta_{t-1} < \beta_t$, then:

$$\begin{aligned}
\beta_{t+1} &= \beta_t + \frac{1}{t}\left(\frac{c\beta_{t-1}^{-\alpha}}{c\beta_t^{-\alpha}} \frac{1 + c^{-1}r(\beta_{t-1})\beta_{t-1}^{\alpha}}{1 + c^{-1}r(\beta_t)\beta_t^{\alpha}} \frac{\vartheta}{\beta_t} - 1\right)\beta_t \\
&\leq \beta_t \left[1 + \frac{1}{t}\left(\frac{3\vartheta}{\beta_t^{1-\alpha}\beta_{t-1}^{\alpha}} - 1\right)\right],
\end{aligned}$$

where estimate (14) has been used. Since it has been assumed that $\beta_{t-1}, \beta_t > K > 3\vartheta$, we have $\beta_{t-1}^{\alpha} > (3\vartheta)^{\alpha}$ and $\beta_t^{1-\alpha} > (3\vartheta)^{1-\alpha}$and hence $\beta_t^{1-\alpha}\beta_{t-1}^{\alpha} > 3\vartheta$. It follows that $\beta_{t+1} < \beta_t$, and the situation of case 1 is obtained.

Case 3
Consider now the case that $\beta_t \leq K$. Then either $\beta_{t+1} \leq K$ and we again obtain the situation of this case, or $\beta_{t+1} > K$ and:

$$\beta_{t+1} \leq K + \frac{\max_{0\leq\beta\leq K} S(\beta)}{\min_{0\leq\beta\leq K} S(\beta)}\vartheta = K',$$

and the situation of case 4 is obtained.

Case 4
Finally, consider the case that $\beta_{t-1} \leq K < \beta_t$. We have either that $t = 1$ or $\beta_t \leq K'$, since $S(\beta_{t-2}) \leq \max_{0\leq\beta\leq K} S(\beta)$. From this we infer:

$$\beta_{t+1} \leq \max\{\beta_1, K'\} + \frac{\max_{0\leq\beta\leq K} S(\beta)}{\min_{0\leq\beta\leq K} S(\beta)}\vartheta = K'',$$

and we are in situation of one of the preceding cases. Either way, this discussion establishes that:

$$\beta_{t+1} \leq \max(\beta_0, \beta_1, K'').$$

Hence, the series $\{\beta_t\}$ is bounded and therefore, by Proposition 3, converges to the monetary steady state. ∎

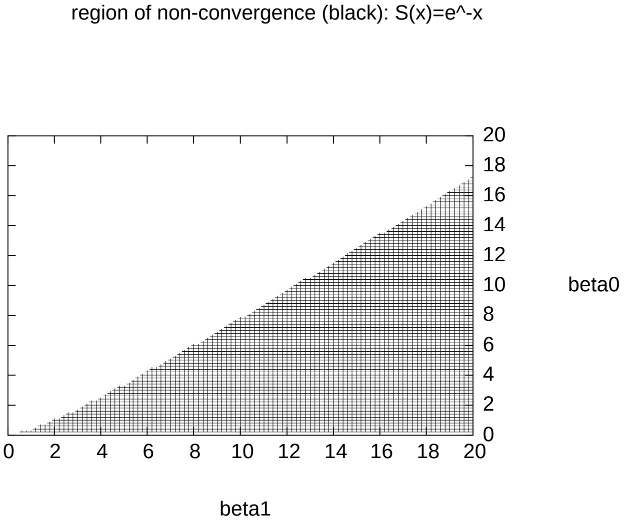

Figure 1: Region of nonconvergence in (β_0, β_1)-space for savings function $S(\beta) = \exp[-\beta]$ and $\vartheta = \sqrt{2}$.

Let us conclude this section with two examples where the monetary steady state is not globally stable. We consider two particular savings functions, $S(\beta) = \exp(-\beta)$ and $S(\beta) = \beta^{-3}$, respectively. Notice that they do not satisfy the assumptions required in Proposition 5, but from Propositions 2 and 3 it follows that for these savings functions the monetary steady state ϑ is locally stable. These savings function approach 0 very fast as β increases. Consider what happens if β_1 is much larger than β_0. In that case the term $S(\beta_0)/S(\beta_1)$ is very large, leading to a high value of β_2. Eventually, the inflation rates will run away to infinity and money loses its value. Figures 1 and 2 give, for these savings functions with $\vartheta = \sqrt{2}$, initial conditions β_0 and β_1, for which the inflation rates explode and run away to infinity. Notice that β_1 has to be sufficiently larger than β_0, for divergence to infinity to occur. These examples show that, although persistent endogenous fluctuations are impossible, it is, under certain circumstances, possible for the inflation rates to run off to infinity.

4 Concluding remarks

Departing from the theory of rational expectations introduces infinitely many degrees of freedom in modelling agents' beliefs. This "wilderness of bounded rationality" can be restricted by considering agents that, if not unboundedly rational, at least are trying to be "sensible" in predicting the future development of economic variables. That is, they should have a perceived law of motion that is reasonable, in some sense,

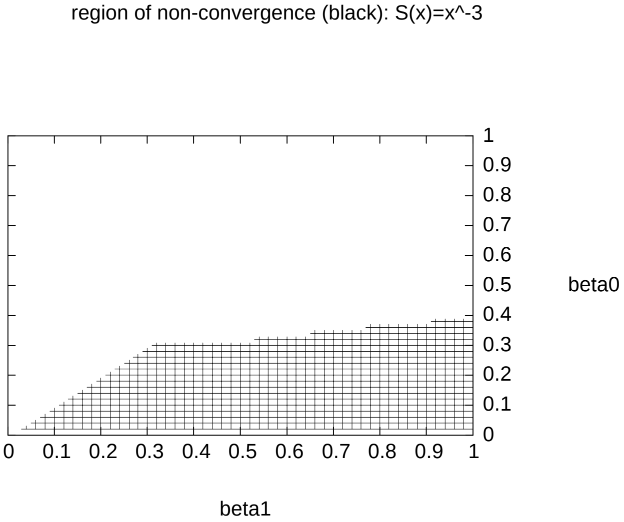

Figure 2: Region of non-convergence in (β_0, β_1)-space for savings function $S(\beta) = \beta^{-3}$ and $\vartheta = \sqrt{2}$.

and they should use the proper econometric techniques to estimate this perceived law of motion. The learning equilibria obtained by Bullard (1994) and Schönhofer (1999), however, are partly obtained from a somewhat misguided application of econometric techniques, that is, a regression is applied on a nonstationary price time series. In this paper we have shown that a more reasonable estimation technique (estimating the perceived law of motion on the basis of the stationary time series of inflation rates) induces convergence to the monetary steady state. Recall that the perceived laws of motion and therefore agents' beliefs are the same for both models.

Therefore, these learning equilibria are not the result of modelling agents as econometricians per se, but they are the result of modelling the agents as *naive* econometricians. Hence, Bullard (1994) was right in asserting that it is important how the learning process is modelled, in terms of the perceived law of motion *and* in terms of the estimation procedure that is used.

Examples where learning in a stationary environment does lead to chaotic equilibria are provided by Hommes and Sorger (1998) and Tuinstra (2000). In these models the perceived law of motion of the agents converges to some limit belief and given this limit belief prices keep fluctuating over some nontrivial attractor.

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