A Price Adjustment Process in a Model of Monopolistic Competition

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Abstract

We consider a price adjustment process in a model of monopolistic competition. Firms have incomplete information about the demand structure. When they set a price they observe the amount they can sell at that price and they observe the slope of the true demand curve at that price. With this information they estimate a linear demand curve. Given this estimate of the demand curve they set a new optimal price. We investigate the dynamical properties of this learning process. We find that, if the cross-price effects and the curvature of the demand curve are small, prices converge to the Bertrand-Nash equilibrium. The global dynamics of this adjustment process are analyzed by numerical simulations. By means of computational techniques and by applying results from homoclinic bifurcation theory we provide evidence for the existence of strange attractors.

Keywords: Bertrand competition, price adjustment, nonlinear oligopoly dynamics, bifurcation theory

JEL classification code: C72, D21, D43, E30

1 Motivation

Economic theory deals primarily with the existence of equilibria, i.e. prices at which market clearing occurs. The problem of how economic agents might coordinate on such an equilibrium has received much less attention than the problem of existence, but is nevertheless highly important. Indeed, if this problem cannot be solved satisfactorily, economic predictions and comparative statics based upon equilibrium analysis are of limited relevance. The tâtonnement process is the best known of the different

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adjustment processes that have been studied. It models the *law of supply and demand* and has the feature that if there is excess demand (supply) for a commodity its price increases (decreases). The tâtonnement process has been extensively studied in the literature. The classical references are Arrow and Hurwicz (1958) and Arrow, Block and Hurwicz (1959). The latter show that gross substitutability of the aggregate excess demand functions implies global stability of the (unique) equilibrium price vector. This condition is however rather strong, and a price adjustment mechanism which converges for almost all possible economies needs much more information on the aggregate excess demand functions, see e.g. Smale (1976) and Saari and Simon (1978). Moreover, since the tâtonnement process typically corresponds to a nonlinear dynamical system, all kinds of complicated price dynamics are possible for nonpathological economies (see e.g. Goeree, Hommes and Weddepohl, 1997 and Tuinstra, 2000).

The tâtonnement process suffers from some conceptual problems (for a critical treatment, see Schinkel, 2001). One of these problems surfaces when we realize that in a perfectly competitive economy all agents take prices as given in determining their consumption and production behaviour. Who is adjusting prices then? The traditional solution to this problem is the introduction of an *auctioneer* who sets prices. However, where all decisions of consumers arise from utility or profit maximization the behaviour of the auctioneer is postulated ad hoc. A consistent and realistic model of price adjustment requires that prices are set by economic agents, in a way that maximizes their utility or profit. In order to model this behaviour we have to consider equilibrium models with monopolistic competition.

In a seminal paper Negishi (1961) introduced monopolistic competition in a general equilibrium model by considering an economy with two types of producers: perfect competitors, who take prices as given, and imperfect competitors with enough market power to set some of the prices. Each imperfect competitor has some conjecture about the demand curves of the commodities for which it can influence the price. These conjectures are assumed to be linear functions of the own price. The only restriction on the conjectures is that they have to pass through the price-quantity combinations that correspond to the current state of the economy. The economy then is in equilibrium if all consumers and perfectly competitive firms maximize their utility and profits, given their conjectures and production sets, and if excess demand is zero. A problem with this *subjective* demand approach is that it has no predictive power: if conjectures of imperfectly competitive firms are not restricted any further almost any allocation can be an equilibrium. For example, if these conjectures are such that firms believe demand to be infinitely elastic at a certain price, the Walrasian equilibrium results. For this reason a number of people have tried to construct general equilibrium models of monopolistic competition that incorporate *objective* demand curves (see for example Gabszewicz and Vial, 1972 and Nikaido, 1975). An important drawback of the objective demand approach is the assumption that all firms are supposed to be able to construct a complete general equilibrium model in order to determine the objective demand curves. A compromise between the objective and subjective approaches would be to consider a subjective demand framework that incorporates more consistency conditions than Negishi (1961) does. Silvestre

(1977), for example, requires that in equilibrium not only subjective demand has to equal objective demand, but, at this equilibrium, the *slope* of the subjective demand curve also has to coincide with the slope of the objective demand curve. Gary-Bobo (1987) extends this approach to higher-order derivatives.

In this paper we use the subjective demand framework to analyse a simple price adjustment process where price movements result from profit-maximizing behaviour of firms. As a starting point we take a partial equilibrium model, with n firms, all being the single producer of a distinct commodity. As in Negishi (1961) we assume that firms, which do not know the demand they are facing, believe that the demand for their product only depends upon their own price and is linear in that price. A learning dynamics of the following type arises. In every period each firm estimates a demand curve for his commodity. This subjective demand curve leads to an optimal price which in turn leads to new information on the demand curve, leading to a new price, and so on. Notice that the demand curves are in general misspecified, since they do not incorporate information on prices for other products. Furthermore, firms estimate demand curves that are linear in the own price, whereas the objective demand curves might be nonlinear in the own price. Different learning procedures differ in the information that is gathered by firms and in the way that this information is used. A typical learning process is studied by Kirman (1995). Firms collect information on observed price and quantity combinations and use a recursive ordinary least squares algorithm to estimate a linear subjective demand curve. This estimated demand curve is used to determine a new optimal price, which leads to new information again. Simulations suggest that there is convergence on a so-called *conjectural* equilibrium, where firms are able to sell what they expected and where they maximize their profit, given their subjective demand curve. For a more general treatment of this kind of learning process, based upon Bayesian updating, see Schinkel, Tuinstra and Vermeulen (2000). This approach shares with the Negishi (1961) model the drawback that almost any combination of prices corresponds to a conjectural equilibrium for properly chosen belief parameters.

In this paper we consider a somewhat different learning procedure. Following Silvestre (1977), we assume that the slope of the demand curve plays an important role. In particular, we assume that when a firm sets a price it observes the amount it could have sold at that price and it observes the sensitivity of the demand curve with respect to small variations in that price, that is, it learns the derivative of the demand curve at the price it sets (it might, for example, observe this sensitivity through small price experiments). On the basis of this information a linear demand curve is estimated and a new optimal price is determined. In the next period, information about sales and the slope of the demand curve at the new price is gathered and this information gives a new estimated demand curve, and so on.

In general, the adjustment process introduced in this paper corresponds to a (multidimensional) nonlinear dynamical system. In this paper we provide a local stability analysis of the equilibrium of the learning process. Furthermore, we use numerical simulations to investigate the global dynamics of a duopoly version of the model. For this case different kinds of complicated phenomena emerge. We use bifurcation theory to get some insight into the global behaviour of the learning process. Moreover, by numerical simulations we provide evidence for the existence of a so-called *homoclinic bifurcation*, i.e. a tangency between the stable and unstable manifolds of the equilibrium point. The use of homoclinic bifurcation theory is rather new in economic theory. It has been applied, for example, by Brock and Hommes (1997) and Droste, Hommes and Tuinstra (2002) in evolutionary models with heterogenous agents and by de Vilder (1996) who studies the appearance of homoclinic bifurcations in a two-dimensional overlapping generations model with capital. In this paper we will provide a computational proof of the existence of a homoclinic orbit, which implies complicated price dynamics. An important observation is that these endogenous fluctuations are not the consequence of an atypical choice of demand curves. In fact, these fluctuations arise naturally in our model and therefore might be an explanation for business cycles observed in reality.

The outline for the rest of this paper is as follows. In Section 2 we discuss the partial equilibrium model and briefly review some well-known adjustment processes. In Section 3 the learning procedure is introduced and its local stability properties are investigated. Section 4 uses numerical simulations to study the global dynamical properties of the learning model, applied to a duopoly situation. Section 5 summarizes. Finally, Appendix A contains some proofs and Appendix B gives a brief outline of homoclinic bifurcation theory.

2 A partial equilibrium model

We consider a partial equilibrium model with heterogeneous commodities and imperfect competition. Let there be n firms, where each firm produces its own unique commodity and is the sole supplier of that commodity. The cost function of firm i is given by a nonnegative, continuous, nondecreasing and twice differentiable function $C_i : \mathbb{R}_+ \to \mathbb{R}_+$. Let $\mathsf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ be a price vector, where p_i is the price for the i'th commodity. For convenience, this price vector is sometimes written as (p_i, p_{-i}), where $\mathsf{p}_{-i} \equiv (p_1, \ldots, p_{i-1}, p_{i+1} \ldots, p_n)$. The demand for commodity i depends upon the prices of all commodities and is given by a demand function $D^i : \mathbb{R}^n_+ \to \mathbb{R}_+$, which is assumed to be nonnegative, continuous and twice differentiable in all its arguments, whenever it is strictly positive. First and second order derivatives of the demand functions are denoted by $D^i_j(\mathsf{p}) \equiv \frac{\partial D^i(\mathsf{p})}{\partial p_j}$ and $D^i_{jk}(\mathsf{p}) \equiv \frac{\partial^2 D^i(\mathsf{p})}{\partial p_j \partial p_k}$. We assume demand is nonincreasing in the own price $(D^i_i(\mathsf{p}) \leq 0)$ and that $D^i_j(\mathsf{p})$ and $D^j_i(\mathsf{p})$ have the same sign. If $D^i_j(\mathsf{p})$ and $D^j_i(\mathsf{p})$ are positive we call commodities i and j *substitutes*, if they are negative, we call them *complements*. Each firm chooses its own price in order to maximize profits, which are given by

$$\pi_i(p_i, \mathsf{p}_{-i}) = p_i D^i(\mathsf{p}) - C_i\left(D^i(\mathsf{p})\right).$$

The well-known *Bertrand-Nash* equilibrium can be defined as follows.

Definition 1 *A price vector* $\mathsf{p}^* = (p^*_1, \ldots, p^*_n)$ *is a* Bertrand-Nash equilibrium *if for each* $i = 1, \ldots, n$ *we have*

$$\pi_i\left(p^*_i, \mathsf{p}^*_{-i}\right) \geq \pi_i\left(p_i, \mathsf{p}^*_{-i}\right) \text{ for all } p_i.$$

At such an equilibrium no firm can increase its profits by unilaterally deviating from this equilibrium price level. An alternative formulation of the Bertrand-Nash equilibrium is in terms of *reaction functions.* The reaction function (or correspondence) for firm i gives the price that maximizes his profit, given the prices set by the other firms, that is, the reaction function is defined as

$$R^i\left(\mathsf{p}_{-i}\right) = \arg\max_{p_i} \pi\left(p_i, \mathsf{p}_{-i}\right).$$

It is easy to see that a Bertrand-Nash equilibrium corresponds to a fixed point of the map

$$\mathsf{R}\left(\mathsf{p}\right) = \begin{pmatrix} R^1\left(\mathsf{p}_{-1}\right) \\ \vdots \\ R^n\left(\mathsf{p}_{-n}\right) \end{pmatrix}.$$

To guarantee existence of a Bertrand-Nash equilibrium we have to assume quasiconcavity of the profit functions, a condition which does not follow from standard assumptions on the fundamentals of the economy, such as preferences, endowments or technology. In particular, this condition requires the demand function to be not "too convex". However, it is well known (see e.g. Sonnenschein, 1973) that demand functions can take almost any form and still be consistent with utility maximization. As is pointed out by Roberts and Sonnenschein (1977), lack of quasiconcavity of the profit functions may cause the reaction curves to exhibit discontinuities, possibly leading to nonexistence of the Bertrand-Nash equilibrium (nonpathological and robust examples can be found in e.g. Roberts and Sonnenschein, 1977, Friedman, 1983 and Bonanno, 1988).

A way out of this problem is to consider equilibria along the lines of Bonanno and Zeeman (1985) and Bonanno (1988). They study price setting oligopolies where all firms have constant marginal costs, that is $C_i\left(x_i\right) = c_i x_i$. Bonanno and Zeeman (1985) consider the case where producers only focus on the first order conditions for an optimum, that is, a price vector $\mathsf{p}^* = \left(p_1^*, \ldots, p_n^*\right)$ is called an *equilibrium* when

$$\frac{\partial \pi_i\left(\mathsf{p}^*\right)}{\partial p_i} = 0, \quad \text{for all } i. \tag{1}$$

They go on to show that, under some mild conditions, such an equilibrium always exists. However, (1) allows for the situation where some producer is at a local minimum of his profit function, since firms disregard second order conditions. Note that, if firms think they face a linear demand curve, they will believe they are at the global maximum of their profit function whenever the first order condition is satisfied. Bonanno (1988) defines a *local* Bertrand-Nash equilibrium as a price vector $\mathsf{p}^* = \left(p_1^*, \ldots, p_n^*\right)$ for which, besides (1) the second order condition for a local maximum is satisfied, that is,

$$\frac{\partial^2 \pi_i\left(\mathsf{p}^*\right)}{\partial p_i^2} < 0, \quad \text{for all } i.$$

At such a local Bertrand-Nash equilibrium no firm can improve his profit by deviating to a price in the *neighbourhood* of the equilibrium price. A sufficient condition for such a local Bertrand-Nash equilibrium to exist is that $D^i_{ii}(\mathsf{p}) \neq 2D^i(\mathsf{p}) / (p_i - c_i)^2$ for all i and p at which demand is positive. This condition implies that the first and second order derivative of the profit function never vanish together, which excludes the case that a local minimum and a local maximum of the profit function merge.

We are interested in how firms coordinate on a Bertrand-Nash equilibrium. Before we turn to our learning model, let us briefly discuss some well-known adjustment processes in this type of model. The best-known adjustment process is the *best-reply dynamics*, which is based upon the assumption that each individual firm believes that the other firms will not change their prices. It then sets a price which, given these fixed prices of its competitors, maximizes its profit. This price is specified by the reaction function. Implicitly it is assumed that each firm has full information about its demand curve and about the prices its competitors have set in the previous period. We assume reaction function are continuous and differentiable and a (not necessarily unique) Bertrand-Nash equilibrium p^* exists. The best-reply dynamics are given by

$$\begin{pmatrix} p_{1,t+1} \\ \vdots \\ p_{n,t+1} \end{pmatrix} = \begin{pmatrix} R^1(\mathsf{p}_{-1,t}) \\ \vdots \\ R^n(\mathsf{p}_{-n,t}) \end{pmatrix}. \tag{2}$$

These best-reply dynamics constitute an $n-$dimensional dynamical system. Another type of adjustment processes are *gradient processes*, see for example Furth (1986) and Bischi and Naimzada (1999). These gradient processes assume that firms change their price in the direction in which profit increases. That is, if marginal profit at the current price is positive, the firm increases its price and if marginal profit is negative, the firm decreases its price. In both cases the firm expects an increase in profits. However, since other firms also change their prices, it is very well possible that profits in fact decrease. In continuous time this process can be written as

$$\frac{dp_i}{dt} = K_i \frac{\partial \pi_i}{\partial p_i}, \qquad i = 1, \ldots, n, \tag{3}$$

where K_i is the speed with which the i'th producer changes the price for its product. Notice that an equilibrium of the gradient dynamics corresponds to a price vector where for each firm the first order condition (but not necesarrily the second order condition) for a profit maximum is satisfied. Now let us consider the stability properties of these two adjustment processes.

Proposition 2 *A Bertrand-Nash equilibrium* p^* *is locally stable under the best-reply dynamics (2) and under the gradient dynamics (3) if*

$$\left| \frac{\partial^2 \pi_i(\mathsf{p}^*)}{\partial p_i^2} \right| > \sum_{j \neq i} \left| \frac{\partial^2 \pi_i(\mathsf{p}^*)}{\partial p_i \partial p_j} \right|, \qquad \textit{for all } i. \tag{4}$$

If condition (4) is not satisfied, and if the reaction curves are nonmonotonic the best-reply dynamics can generate periodic and chaotic behaviour (see e.g. Rand, 1978, Bischi and Gardini, 1997 and Kopel, 1997 for similar complicated behaviour in the Cournot oligopoly model).

Gradient systems can also exhibit complicated behaviour. In fact, Corchón and MasCollel (1996) show that any set of functions can be generated by the gradient system with demand functions that satisfy nice properties such as gross substitutability, that is, for which the centralized tâtonnement process is globally stable.

3 The learning model

In this section we introduce an adjustment process for the partial equilibrium model discussed in the previous section. We assume firms do not know the demand curves for the products they are manufacturing but they believe these demand curves depend only upon their own price, and hence they are abstracting from any interdependence with prices set by other firms. Moreover, they believe that this relationship between price and demand is linear. Each firm estimates a demand curve on the basis of *local* information obtained in the previous period. This local information consists of two parts: a firm knows, given the price it charged in the previous period, how much it could have sold against that price, and it knows the sensitivity of the demand curve with respect to price changes at that price (that is, it knows the slope of the demand curve at that price, compare Silvestre, 1977), for example through experimentation or market research. This information is sufficient to identify a linear demand curve. On the basis of this perceived linear demand curve an optimal price is determined. In the next period this new price leads to new information about the location and slope of the true demand curve and hence to a new estimate of the perceived demand curve, leading to a new optimal price again, *ad infinitum*. Notice that the information a firm obtains about its demand curve not only varies from period to period because it charges different prices but also because the other firms may also change their prices. We are interested in the dynamical behaviour of this adjustment process.

We assume marginal costs are constant, $C_i(x_i) = c_i x_i$. Firm i's perceived demand curve is

$$d_i(p_i) = a_i - b_i p_i, \quad \text{with} \quad a_i, b_i > 0.$$

Given this perceived demand curve and the cost structure, firm i's (perceived) profit maximizing price can be determined as

$$p_i = \frac{a_i}{2b_i} + \frac{1}{2}c_i. \tag{5}$$

In each period firm i estimates the demand curve on the basis of information from the previous period. Firm i knows the price it charged in the previous period and the amount it (could have) sold, so it knows $(p_{it}, D^i(\mathsf{p}_t))$. Furthermore it knows the slope of the demand curve in this point, $D^i_i(\mathsf{p}_t)$. It follows that the estimated demand curve in period t then becomes

$$d_{it}(p_i) = a_{it} - b_{it}p_i = \left(D^i(\mathsf{p}_t) - D^i_i(\mathsf{p}_t)\,p_{it}\right) + D^i_i(\mathsf{p}_t)\,p_i.$$

From (5) we then find that the price will be

$$p_{i,t+1} = F_i(\mathsf{p}_t) = \frac{a_{it}}{2b_{it}} + \frac{1}{2}c_i = \frac{1}{2}(p_{it} + c_i) - \frac{1}{2}\frac{D^i(\mathsf{p}_t)}{D^i_i(\mathsf{p}_t)}. \tag{6}$$

It is easily verified that any fixed point $\mathsf{p}^* = (p_1^*, p_2^*, \ldots, p_n^*)$ of (6) satisfies the first order conditions for a Bertrand-Nash equilibrium, and hence corresponds to an equilibrium in the sense of Bonanno and Zeeman (1985), see the discussion in Section 2.

We are interested in the stability of these equilibria. We have the following result (recall that $D_{jk}^i(\mathsf{p}) \equiv \frac{\partial^2 D^i(\mathsf{p})}{\partial p_j \partial p_k}$).

Proposition 3 *A sufficient condition for local stability of an equilibrium* p^* *of (6) is*

$$\left| \frac{D^i(\mathsf{p}^*) D_{ii}^i(\mathsf{p}^*)}{\left(D_i^i(\mathsf{p}^*)\right)^2} \right| + \sum_{j \neq i} \left| \frac{D^i(\mathsf{p}^*) D_{ij}^i(\mathsf{p}^*) - D_i^i(\mathsf{p}^*) D_j^i(\mathsf{p}^*)}{\left(D_i^i(\mathsf{p}^*)\right)^2} \right| \leq 2, \text{ for all } i = 1, \ldots, n. \tag{7}$$

To get some intuition for condition (7) notice that the following two conditions together are sufficient (but not necessary) for condition (7) to hold

$$\left|D_i^i(\mathsf{p}^*)\right| \geq \sum_{j \neq i} \left|D_j^i(\mathsf{p}^*)\right| \text{ and } \left|D_i^i(\mathsf{p}^*)\right| \geq \sqrt{D^i(\mathsf{p}^*) \sum_{j=1}^{n} \left|D_{ij}^i(\mathsf{p}^*)\right|}.$$

The first of these says that the matrix of substitution effects has a dominant diagonal, the second condition says that the "curvature" (as measured by the second order derivatives) of the demand functions has to be small, as compared to the slope. Notice that there is a direct connection between these two conditions and the perceptions of the firms. The condition on the substitution effects corresponds to the perception of firms that prices of other firms do not matter and the condition on the curvature (or nonlinearity) of the demand curves corresponds to the perception of firms that the demand curve is linear. Hence, instability of the equilibrium is more likely to occur when perceptions of firms are less palatable.

As a last remark notice that if p^* is locally stable in our learning model then it satisfies the second order conditions for a local Bertrand-Nash equilibrium (compare Bonanno, 1988). That this is the case can be seen by rewriting the first and second order conditions for a maximum as

$$\frac{D^i(\mathsf{p}) D_{ii}^i(\mathsf{p})}{\left(D_i^i(\mathsf{p})\right)^2} \leq 2$$

and observing that this inequality is implied by (7). We end this section by applying our adjustment process to some typical and well known examples of demand functions.

Linear demand functions

Consider the following system of linear demand functions

$$\mathsf{D}(\mathsf{p}) = \alpha + \beta \mathsf{p}, \tag{8}$$

where α is a vector of intercepts and β is an $n \times n$ matrix. The parameter $\beta_{ii} < 0$ gives the sensitivity of demand for commodity i with respect to its own price and β_{ij} gives the sensitivity of demand for commodity i with respect to the price of commodity j.

First consider the existence of a Bertrand-Nash equilibrium. The first order conditions for an equilibrium are

$$-2\beta_{ii}p_i = \alpha_i - \beta_{ii}c_i + \sum_{j\neq i}\beta_{ij}p_j, \qquad i = 1,\ldots,n, \tag{9}$$

which in matrix notation becomes

$$-\begin{pmatrix} 2\beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & 2\beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & 2\beta_{nn} \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_n^* \end{pmatrix} = \begin{pmatrix} \alpha_1 - \beta_{11}c_1 \\ \alpha_2 - \beta_{22}c_2 \\ \vdots \\ \alpha_n - \beta_{nn}c_n \end{pmatrix}.$$

A nonnegative equilibrium price vector p^* exists for all $\mathsf{c} = (c_1,\ldots,c_n)' \geq \mathsf{0}$ and all $\alpha \geq \mathsf{0}$ if the matrix $(\beta + \beta_D)^{-1}$ exists and is nonpositive, where β_D is the diagonal matrix with β_{ii} as its i'th diagonal element. We have the following result.

Proposition 4 *If $\beta_{ii} < 0$ and $\beta_{ij} \geq 0$ for all i, j with $i \neq j$, then an equilibrium exists for all $\alpha \geq \mathsf{0}$ and all $\mathsf{c} \geq \mathsf{0}$ if*

$$2\left|\beta_{ii}\right| > \sum_{j\neq i}\left|\beta_{ij}\right|, \quad i = 1,\ldots,n. \tag{10}$$

Now assume a unique equilibrium exists and is given by

$$\mathsf{p}^* = -\left(\beta + \beta_D\right)^{-1}\left(\alpha - \beta_D\mathsf{c}\right).$$

The adjustment process (6) becomes

$$p_{i,t+1} = -\frac{\alpha_i - \beta_{ii}c_i + \sum_{j\neq i}\beta_{ij}p_{jt}}{2\beta_{ii}}, \qquad i = 1,\ldots,n. \tag{11}$$

Notice that in this case with linear demand functions, (11) coincides with the best reply dynamics, as can be seen from (9). Also, the linear specification (8) together with the linear perceived demand curves implies that each firm knows the slope of the demand curve (β_{ii} for firm i) but does not know the intercept since it neglects cross-price effects. The following stability property can easily be checked by looking at (11).

Corollary 5 *If a unique equilibrium $\mathsf{p}^* \geq 0$ exists and if the β_{ij}'s satisfy condition (10) the dynamical system is globally stable.*

Notice that condition (10) is weaker than diagonal dominance of the set of demand functions.

Loglinear demand functions

Another well known demand structure is given by so-called "constant elasticity" or loglinear demand functions. These correspond to

$$D^i(\mathsf{p}) = \alpha_i \prod_{j=1}^{n} p_j^{\beta_{ij}},$$

where $\beta_{ii} < -1$ for all i ensures the existence of a Bertrand-Nash equilibrium, which is

$$p_i = \frac{\beta_{ii}}{\beta_{ii}+1} c_i, \quad i = 1, \ldots, n.$$

Notice that this equilibrium does not depend upon other parameters than β_{ii} and c_i. Consider the adjustment process (6). This can be written as

$$p_{i,t+1} = \frac{1}{2}(p_{it} + c_i) - \frac{1}{2} \frac{\alpha_i \prod_{j=1}^{n} p_{jt}^{\beta_{ij}}}{\beta_{ii} p_{it}^{-1} \alpha_i \prod_{j=1}^{n} p_{jt}^{\beta_{ij}}} = \frac{1}{2}\left(1 - \frac{1}{\beta_{ii}}\right) p_{it} + \frac{1}{2} c_i.$$

Notice that the dynamics of the price of commodity i is independent of the dynamics of the other prices. The stability condition $\beta_{ii} < -1$ is always satisfied. Therefore the Bertrand-Nash equilibrium for the model with loglinear demand curves is globally stable in our adjustment process.

4 Endogenous Fluctuations

In the previous section we considered some examples where the learning procedure converged to the Bertrand-Nash equilibrium. In this section we study a typical example for which our learning process leads to more complicated dynamical phenomena. We assume $n = 2$ and $c_i = 0$ for $i = 1, 2$. Demand functions are given by

$$D^1(p_1, p_2) = \exp\left[-p_1^{\gamma} p_2^{\delta}\right] \text{ and } D^2(p_1, p_2) = \exp\left[-p_1^{\delta} p_2^{\gamma}\right]$$

where $\gamma > 0$ and $|\delta| < \gamma$. It is easily verified that the commodities are substitutes for $\delta < 0$ and complements for $\delta > 0$. The restriction $|\delta| < \gamma$ implies that the sensitivity of demand with respect to the own price is larger than the sensitivity of demand with respect to the other firm's price. The unique Bertrand-Nash equilibrium is[1]

$$p_1^* = p_2^* = \left[\frac{1}{\gamma}\right]^{1/(\gamma+\delta)}.$$

The adjustment process (6) becomes

$$\begin{aligned} p_{1,t+1} &= F_1(p_{1t}, p_{2t}) = \frac{1}{2} p_{1t} + \frac{1}{2}\frac{1}{\gamma} p_{1t}^{1-\gamma} p_{2t}^{-\delta} \\ p_{2,t+1} &= F_2(p_{1t}, p_{2t}) = \frac{1}{2} p_{2t} + \frac{1}{2}\frac{1}{\gamma} p_{1t}^{-\delta} p_{2t}^{1-\gamma}. \end{aligned} \tag{12}$$

[1] The profit function for firm 1 is $\pi_1(p_1, p_2) = p_1 \exp\left[-p_1^{\gamma} p_2^{\delta}\right]$. This gives a best reply function $p_1 = R_1(p_2) = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma}} p_2^{-\frac{\delta}{\gamma}}$ and symmetrically for firm 2. It can easily be checked that for the best-reply dynamics the eigenvalues of the Jacobian matrix evaluated at the Bertrand-Nash equilibrium are $\mu_1 = \mu_2 = -\frac{\delta}{\gamma}$, and hence, since $|\delta| < \gamma$, the best reply dynamics are stable.

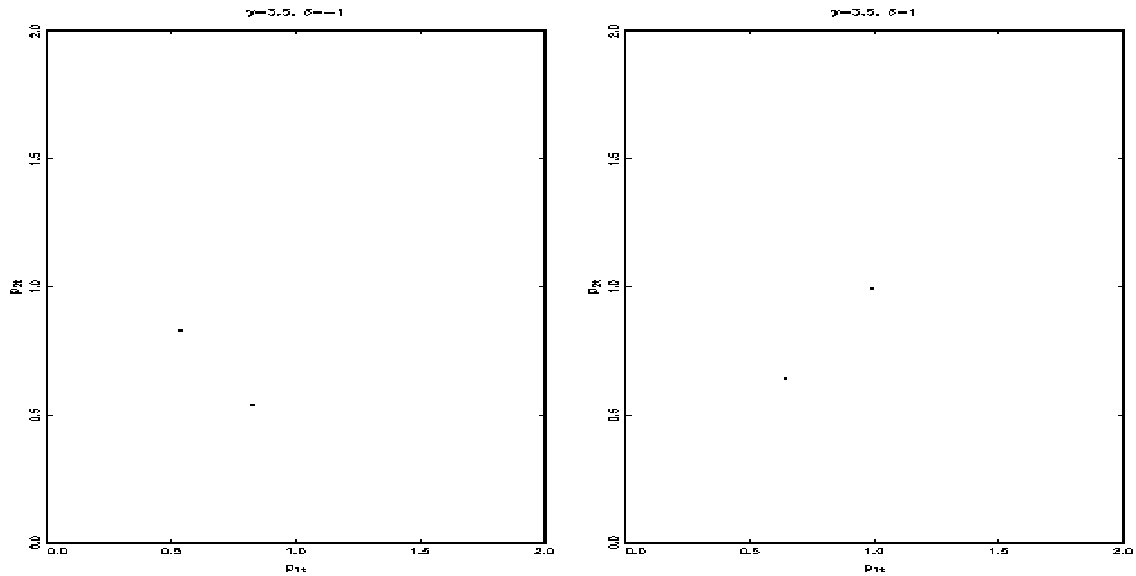

Figure 1: Stable period two orbits for the price adjustment process (12). Left panel: $\gamma = 3\frac{1}{2}$, $\delta = -1$ (goods are substitutes), right panel: $\gamma = 3\frac{1}{2}$, $\delta = 1$ (goods are complements).

This adjustment process exhibits many kinds of complicated dynamical features that are known from dynamical systems theory. Particularly, as the parameters γ and δ increase the Bertrand-Nash equilibrium becomes unstable and prices are attracted to a periodic orbit. Moreover, for high enough values of the parameters strange attractors exist. In the rest of this section we will investigate the dynamics of (12) in more detail. In Subsection 4.1 we will consider local stability of the Bertrand-Nash equilibrium and use local bifurcation theory to study what happens when this equilibrium becomes unstable. In Subsection 4.2 we will show the existence of strange attractors by providing a computational proof for the existence of a homoclinic intersection between the stable and unstable manifolds of the Bertrand-Nash equilibrium.

4.1 Local bifurcations

Due to the symmetry in the demand curves we have a symmetric adjustment process (for the analysis of symmetric dynamical systems, see Golubitsky, Stewart and Schaeffer, 1988 and for an economic application, see Tuinstra, 2000). Specifically, system (12) exhibits a so-called *reflection symmetry*: $F_1(p_2, p_1) = F_2(p_1, p_2)$. Of particular interest to us is the so-called *fixed point subspace*, which consists of all points in the state space that are invariant under the reflection symmetry. In our case this fixed-point subspace consists of all price vectors (p_1, p_2) with $p_1 = p_2$. To see this, observe that if $p_{1t} = p_{2t}$ then we will also have $p_{1s} = p_{2s}$ for all $s \geq t$. Hence, if prices get trapped in this fixed point subspace they will never get out.

With respect to the local stability of the Bertrand-Nash equilibrium we have

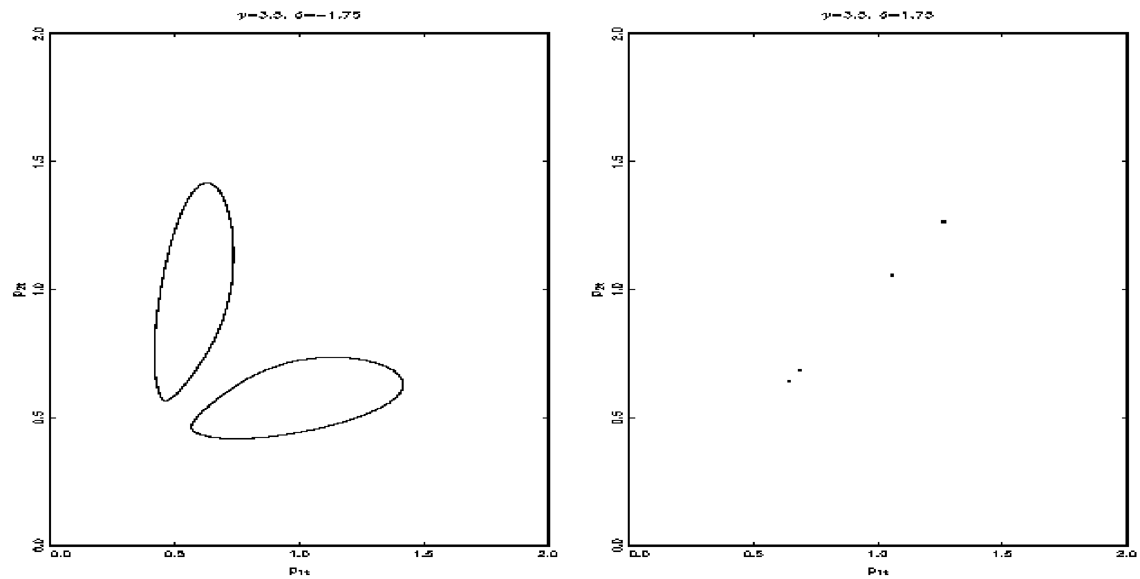

Figure 2: Left pannel: attractor consisting of two closed curves (created at Neimark-Sacker bifurcation of period two) for $\gamma = 3\frac{1}{2}$ and $\delta = -1.75$. Right panel: stable period four orbit (created at period-doubling bifurcation of period two) for $\gamma = 3\frac{1}{2}$ and $\delta = 1.75$.

Proposition 6 *The equilibrium (p_1^*, p_2^*) undergoes a period-doubling bifurcation for those values of γ and δ for which $\gamma + |\delta| = 4$. At these values for γ and δ a period two cycle is created. If $\delta < 0$ this period two cycle lies off the fixed point subspace and if $\delta > 0$, it lies in the fixed point subspace.*

In the previous section we indicated that local stability of the Bertrand-Nash equilibrium depends upon the curvature of the demand function with respect to the own price and the dependence of demand upon the prices of the other commodities. In the present example these two aspects are characterized by the parameters γ and δ, respectively. From Proposition 6 it follows that the Bertrand-Nash equilibrium is unstable when these effects are large. The structure of the resulting period two orbits is as follows. For $\delta < 0$ the period two orbit has the form $\left\{(p, q)', (q, p)'\right\}$ for some p and q and for $\delta > 0$, the period two orbit has the form $\left\{(p, p)', (q, q)'\right\}$ for some p and q. Hence, when goods are substitutes ($\delta < 0$) their prices move opposite to each other: when the price of good 1 is high the price of good 2 is low and vice versa. On the other hand, when goods are complements ($\delta > 0$), their prices move together: they are both high or both low.

Now we consider some numerical examples. Let $\gamma = 3\frac{1}{2}$. According to Proposition 6 we have a period-doubling bifurcation at $\delta = \pm\frac{1}{2}$. These period-doubling bifurcations are supercritical: for $|\delta|$ close to, but larger than $\frac{1}{2}$, an attracting period two orbit exists. Figure 1 shows these period two orbits for $\delta = -1$ and $\delta = 1$, respectively. Observe that they lie off and along the diagonal respectively, as indicated by Proposition 6. If we let δ increase further the amplitude of the period two orbits increases untill eventually these period two orbits become unstable through another

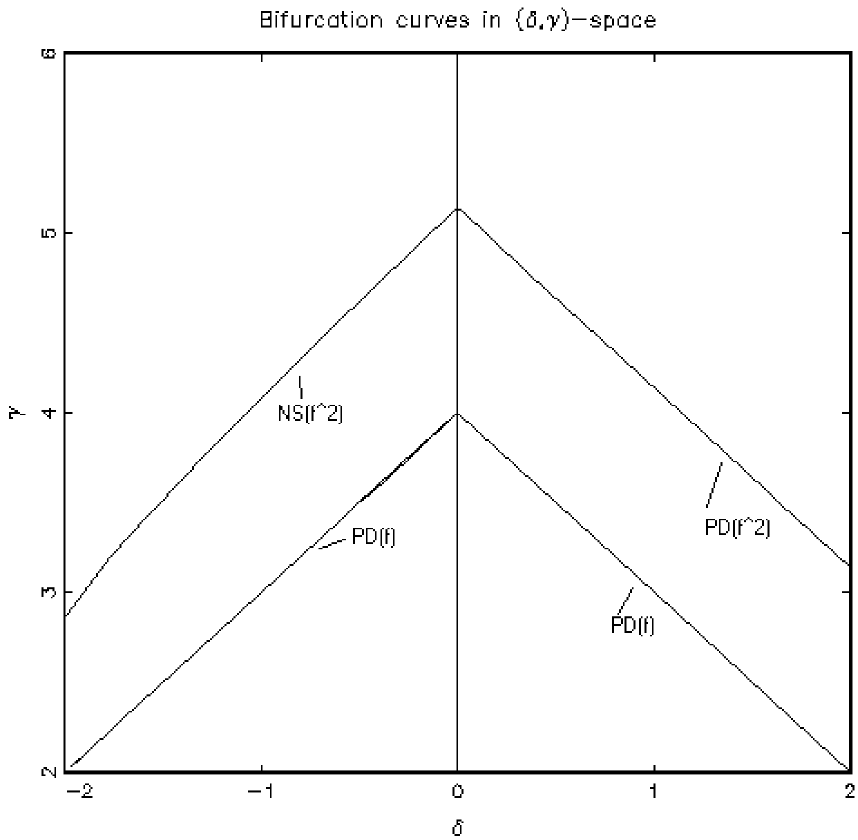

Figure 3: Bifurcation curves in (δ, γ)-space. The curves denoted $PD\,(f)$ give the parameter combinations at which a period-doubling bifurcation of the Bertrand-Nash equilibrium occurs. The curve denoted $PD\,(f^2)$ gives the parameter combinations at which a period doubling bifuration of the period two cycle occurs. The curve denoted $NS\,(f^2)$ gives the parameter combinations at which a Neimark-Sacker bifurcation of the period two orbit occurs.

bifurcation. For the case with $\delta < 0$ the period two orbit undergoes a Neimark-Sacker bifurcation at $\delta = \delta^{NS} \approx -1.519$. At this Neimark-Sacker bifurcation the period two orbit loses stability and an invariant closed curve around each of the period two points is created. These two invariant closed curves together form the new attractor for this system. The price dynamics might be periodic or quasi periodic. The resulting attractor is shown for $\delta = -1.7$ in the left panel of Figure 2. For the case with $\delta > 0$ the period two orbit undergoes another period doubling bifurcation at $\delta = \delta^{PD} \approx 1.643$. At this value of δ the original period two orbit loses stability and a stable period four orbit is created. This orbit also lies along the diagonal. The right panel of Figure 2 shows this period four orbit for $\delta = 1.7$.

Figure 3 shows a bifurcation plot for the parameters γ and δ. It shows the combinations of γ and δ for which the period-doubling bifurcation of the Bertrand-Nash equilibrium occurs and the combinations of γ and δ for which the Neimark-Sacker

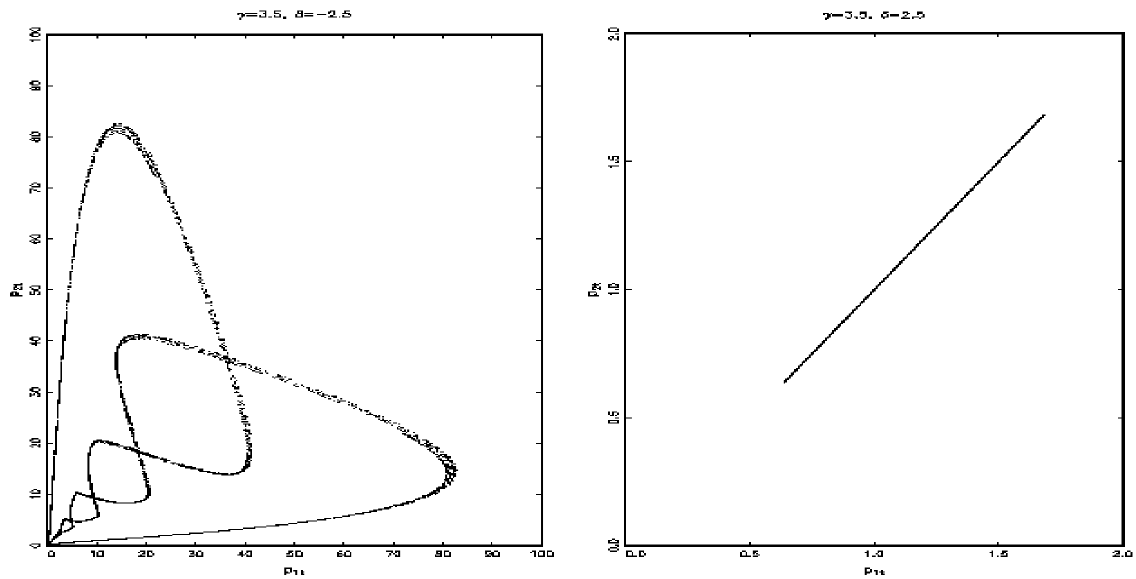

Figure 4: Left panel: attractor for price adjustment process (12) with $\gamma = 3\frac{1}{2}$ and $\delta = -2\frac{1}{2}$ (goods are substitutes). Right panel: attractor for price adjustment process (12) with $\gamma = 3\frac{1}{2}$ and $\delta = 2\frac{1}{2}$ (goods are complements).

and the period doubling bifurcations of the period two orbits occur, respectively.

4.2 Global bifurcations

We have seen that cyclic behaviour in the price adjustment process (12) is possible if the curvature of the demand curves (as measured by γ) or the interdependency between goods (as measured by δ) becomes high. In this section we show that even more complicated behaviour is possible. Consider again the case with $\gamma = 3\frac{1}{2}$. Figure 4 shows two attractors, one for $\delta = -2\frac{1}{2}$ and one for $\delta = 2\frac{1}{2}$. These attractors are more complicated than the ones from the previous section. First let us focus on the case where $\delta > 0$. The attractor in the right panel of Figure 4 emerges through a cascade of period doubling bifurcations along the diagonal (the first of these period doubling bifurcations were already discussed above). Figure 5 shows this cascade by means of a so-called bifurcation diagram. This bifurcation diagram displays the long-run behaviour of the price for good 1 for different values of δ.[2] Clearly, many period doubling bifurcations occur. These bifurcations eventually lead to a one dimensional chaotic attractor (shown in the right panel of Figure 4). Moreover, there is a window in the bifurcation diagram, (between $\delta \approx 3.026$ and $\delta \approx 3.218$) for which the dynamics converges to a stable period three cycle. Therefore, the so-called "period-three implies chaos"-result for one-dimensional maps (see e.g. Devaney, 1989) applies to the dynamical system (12) restricted to the diagonal. This result implies that cycles

[2] This bifurcation diagram is created in the following way. For each value of $\delta \in \{0.007, 0.014, \ldots, 3.5\}$, the dynamical system (12) is iterated 200 times for the initial condition $(p_{10}, p_{20}) = (p_1^*, p_2^* - 0.01)$ and the last 100 values of p_1 are then plotted.

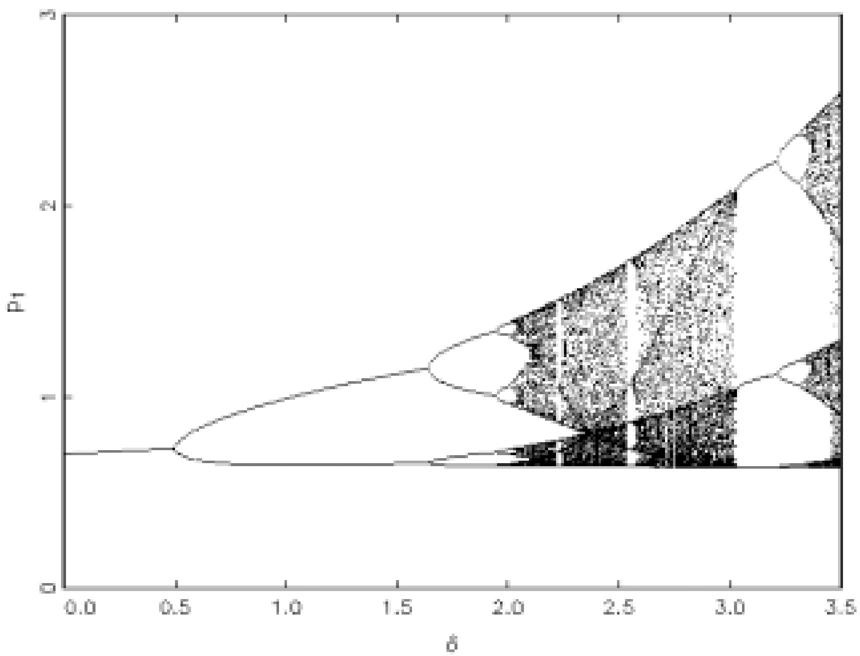

Figure 5: Bifurcation diagram for δ between 0 and $3\frac{1}{2}$, with $\gamma = 3\frac{1}{2}$.

of any period exist on the diagonal.

Now consider the attractor that emerges when $\delta < 0$. Figure 6 shows the time series associated with this attractor. Notice that periods of tranquility, where prices are close to the Bertrand-Nash equilibrium, are interchanged with periods of large fluctations in prices, first increasing and then falling back to their old level.

We will now provide some computational support for the conjecture that a strange attractor exists in this dynamical system. For that we use the theory of homoclinic bifurcations. For a comprehensive treatment of this theory we refer to Palis and Takens (1993). Appendix B contains a brief outline of the main results in this field. Let us denote the dynamical system (12) by $F(.)$. We know that for $\delta < 0$ and $\gamma - \delta > 4$ one of the eigenvalues lies outside the unit circle and hence the Bertrand Nash equilibrium is unstable. The other eigenvalue always lies in the unit circle. Therefore, the Bertrand-Nash equilibrium is a saddle point, with a stable and an unstable direction. The *stable and unstable manifolds* of the equilibrium $\mathsf{p}^* = (p_1^*, p_2^*)$ are defined as (see Palis and Takens, 1993, p.167)

$$W^s(\mathsf{p}^*) = \left\{\mathsf{p}_0 \in I\!R_+^2 | F^t(\mathsf{p}) \to \mathsf{p}^* \text{ for } t \to +\infty\right\},$$
$$W^u(\mathsf{p}^*) = \left\{\mathsf{p}_0 \in I\!R_+^2 | \exists \{\mathsf{p}_{-t}\}_{t \geq 0} \to \mathsf{p}^* \text{ with } \mathsf{p}_{-t+1} = F(\mathsf{p}_{-t})\right\}.$$

These manifolds may intersect each other in points different from the Bertrand-Nash equilibrium, that is, there may exist a point $\mathsf{q} \neq \mathsf{p}^*$ with $\mathsf{q} \in W^s(\mathsf{p}^*) \cap W^u(\mathsf{p}^*)$.

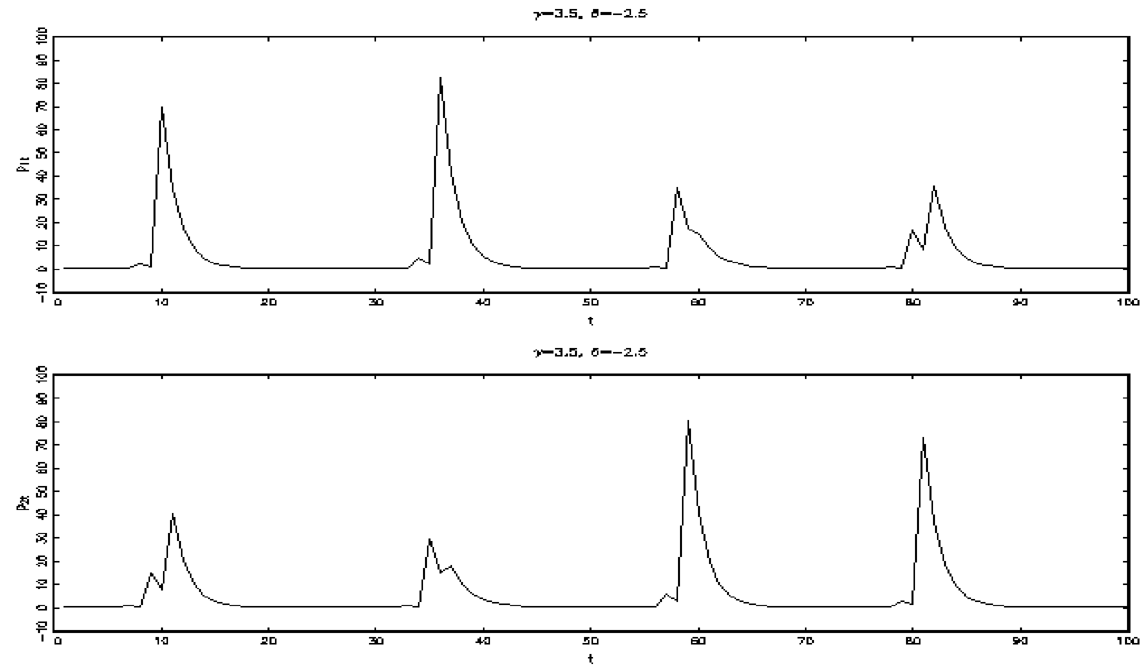

Figure 6: Time series for p_1 and p_2, respectively, generated by price adjustment process (12) with $\gamma = 3\frac{1}{2}$ and $\delta = -2\frac{1}{2}$.

Such a point q is called a *point of homoclinic intersection*. The existence of such a homoclinic intersection implies a very complicated structure of the unstable and stable manifolds, see Appendix B. For given γ, the map F, and therefore also the stable and unstable manifolds, depends upon the parameter δ. A *homoclinic bifurcation* is said to occur at $\delta = \delta^*$ when for $\delta < \delta^*$ there is no intersection between the unstable manifold $W^u(\mathsf{p}^*)$ and the stable manifold $W^s(\mathsf{p}^*)$, for $\delta = \delta^*$ there is a point of homoclinic tangency between $W^s(\mathsf{p}^*)$ and $W^u(\mathsf{p}^*)$ and for $\delta > \delta^*$ there is a point of transversal homoclinic intersection. Such a homoclinic bifurcation implies all kinds of complicated behaviour. In particular, when the equilibrium is dissipative at the homoclinic bifurcation (that is, if the product of eigenvalues of the Jacobian matrix evaluated at p^* is smaller than 1 in absolute value) there is chaotic dynamics on an invariant set for parameter values close to the bifurcation value.

We are interested in whether such a homoclinic bifurcation occurs in our model. First consider the stable manifold. The diagonal, which is the fixed point subspace of the reflection symmetry of system (12) is invariant under the adjustment process. Furthermore, it can be easily checked that all prices starting on this diagonal converge to the Bertrand-Nash equilibrium. The diagonal therefore belongs to the stable manifold. We approximate (part of) the unstable manifold by iterating a small part of the unstable eigenvector close to the equilibrium price under the map F. Figure 7 show this stable and unstable manifold for different values of δ. We find that a homoclinic bifurcation occurs at $\delta^* \approx -1.9448$. At this value of δ there is a homoclinic tangency between the stable and unstable manifold as can be seen in the upper right

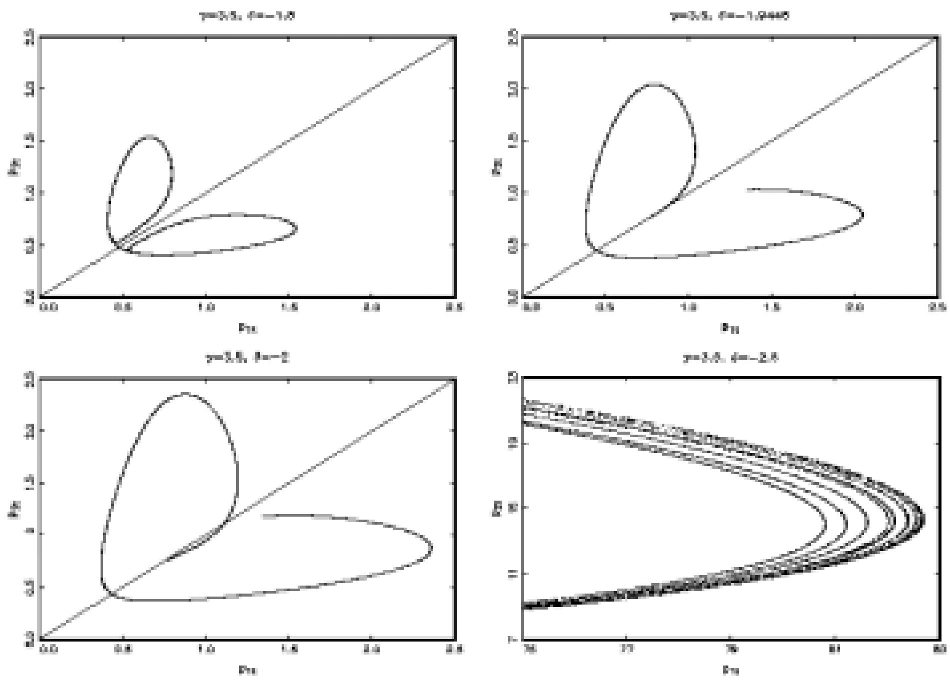

Figure 7: Stable and unstable manifolds for different values of δ. Upper left panel: $\delta = -1.8$. Upper right panel: $\delta = -1.9448$. Lower left panel: $\delta = -2$. Lower right panel: small portion of strange attractor for $\delta = -2.5$.

panel of Figure 7. Furthermore, the saddle point is dissipative at this value of δ. For lower values of δ there is no intersection between the stable and unstable manifold (see upper left panel of Figure 7) and for higher values of δ there is a transversal intersection between the stable and the unstable manifold (see lower left panel of Figure 7). From this we conclude that our model features complicated dynamics. The lower right panel of Figure 7 shows a small portion of the attractor for $\delta = -2\frac{1}{2}$ (which was shown in the left panel of Figure 4). This attractor clearly has a complicated fractal structure.

5 Summary

In this paper we addressed the classical problem of the stability of economic equilibrium. We have argued that a sensible adjustment process should incorporate producers who set prices. We have introduced a simple example of such an adjustment process, where firms do not know the demand they are facing, but try to learn this demand from past observations. In fact, the process uses information about the amount that could be sold against the price from the previous period and the sensitivity of demand at the price from the previous period (for example obtained by small price experiments) to estimate a demand curve. An important feature is that firms have mis-specified beliefs about the actual demand curves, since they assume that these demand curves only depend upon the own price and are linear in this own price.

The adjustment process appears to be stable for situations where the cross price effects and the curvature (or nonlinearity) of the demand functions are relatively small. However, when the cross price effects and curvature are not so small, endogenous fluctuations might emerge. Moreover, for a typical example we have shown the existence of homoclinic tangencies, which imply all kinds of complicated behaviour. Hence, in our model uncertainty about the economic environment explains the occurrence of business cycles.

The model we have discussed here has been a first step in the direction of a more realistic theory of price adjustment. It can be extended in a number of directions. First of all, one can imagine more sophisticated learning procedures where firms, for example, consider a subset of all prices instead of only their own price. Secondly, it would be interesting to consider learning procedures of this type in a general equilibrium framework, possibly allowing for trade at disequilibrium prices. Finally, we have focused on firms as myopic optimizers who are only interested in short term profit maximization. Ideally firms maximize some discounted stream of profits, which implies that each firm solves an optimal control problem which takes account of the trade off between the benefits of experimenting with prices in order to gain information on the demand curve and the resulting loss in short run profits. In the literature this is referred to as "sporadic price discrimination" or "active learning".

Appendix A

This appendix contains the proofs of the main results.

The following well-known result (see e.g. Atkinson, 1989) will prove to be helpfull for the local stability results.

Lemma 7 *Let λ be an eigenvalue of the matrix A and let $\|\cdot\|$ be any matrix norm. Then we have $|\lambda| \leq \|\mathsf{A}\|$.*

Proof. Let $\|\cdot\|_v$ be a vector norm compatible with matrix norm $\|\cdot\|$, i.e. $\|\mathsf{Az}\|_v \leq \|\mathsf{A}\| \|\mathsf{z}\|_v$ for all z, and notice that such a vector norm always exists. Now let x be the eigenvector corresponding to λ. We then have $\|\mathsf{Ax}\|_v = \|\lambda \mathsf{x}\|_v = |\lambda| \|\mathsf{x}\|_v$. By compatibility of $\|\cdot\|$ and $\|\cdot\|_v$, we have $\|\mathsf{Ax}\|_v \leq \|\mathsf{A}\| \|\mathsf{x}\|_v$. Hence, $|\lambda| \|\mathsf{x}\|_v \leq \|\mathsf{A}\| \|\mathsf{x}\|_v$, implying $|\lambda| \leq \|\mathsf{A}\|$. ■

Proof of Proposition 2. The equilibrium p^* is locally stable if the eigenvalues of the Jacobian matrix of (2) evaluated at p^* lie in the unit circle. This Jacobian matrix is

$$\mathsf{J} = \begin{pmatrix} 0 & R_2^1\left(\mathsf{p}_{-1}^*\right) & R_3^1\left(\mathsf{p}_{-1}^*\right) & \cdots & R_n^1\left(\mathsf{p}_{-1}^*\right) \\ R_1^2\left(\mathsf{p}_{-2}^*\right) & 0 & & & R_n^2\left(\mathsf{p}_{-2}^*\right) \\ \vdots & & \ddots & & \\ \vdots & & & 0 & R_n^{n-1}\left(\mathsf{p}_{-(n-1)}^*\right) \\ R_1^n\left(\mathsf{p}_{-n}^*\right) & R_1^n\left(\mathsf{p}_{-n}^*\right) & & R_{n-1}^n\left(\mathsf{p}_{-n}^*\right) & 0 \end{pmatrix},$$

where $R^i_j(\mathsf{p}_{-i}) \equiv \frac{\partial R^{\mathsf{i}}(\mathsf{p}_{-\mathsf{i}})}{\partial p_{\mathsf{j}}}$. By Lemma 7 the largest eigenvalue of A is always smaller, in absolute value, than the norm of A. Now consider the maximum row sum norm $\|\mathsf{A}\|_\infty = \max_i \sum_{j=1}^n |\mathsf{A}_{ij}|$, i.e. the largest row sum of absolute values of elements of A. The vector norm $\|\mathsf{x}\|_\infty = \max_i |\mathsf{x}_i|$ is compatible with this matrix norm. A sufficient condition for the Bertrand-Nash equilibrium to be locally stable under the best-reply dynamics then becomes $\sum_{j=1}^n \left|R^i_j\left(\mathsf{p}^*_{-i}\right)\right| < 1$ for all i. $R^i(\mathsf{p}_{-i})$ is implicitly defined by $\frac{\partial \pi_{\mathsf{i}}\left(R^{\mathsf{i}}(\mathsf{p}_{-\mathsf{i}}),\mathsf{p}_{-\mathsf{i}}\right)}{\partial p_{\mathsf{i}}} = 0$. Totally differentiating this equation with respect to p_j and solving for $R^i_j(\mathsf{p}_{-i}) \equiv \frac{\partial R^{\mathsf{i}}(\mathsf{p}_{-\mathsf{i}})}{\partial p_{\mathsf{j}}}$ gives

$$R^i_j(\mathsf{p}_{-i}) = -\frac{\partial^2 \pi_i(\mathsf{p})}{\partial p_i \partial p_j} \Bigg/ \frac{\partial^2 \pi_i(\mathsf{p})}{\partial p_i^2}.$$

The condition $\sum_{j=1}^n \left|R^i_j\left(\mathsf{p}^*_{-i}\right)\right| < 1$ then becomes equivalent with (4)

Now we turn to gradient system (3). The Jacobian matrix for this system is

$$J = \begin{pmatrix} K_1 \frac{\partial^2 \pi_1}{\partial p_1^2} & K_1 \frac{\partial^2 \pi_1}{\partial p_1 \partial p_2} & \cdots & K_1 \frac{\partial^2 \pi_1}{\partial p_1 \partial p_n} \\ K_2 \frac{\partial^2 \pi_2}{\partial p_2 \partial p_1} & K_2 \frac{\partial^2 \pi_2}{\partial p_2^2} & & \\ \vdots & & \ddots & \\ K_n \frac{\partial^2 \pi_n}{\partial p_n \partial p_1} & & & K_n \frac{\partial^2 \pi_n}{\partial p_n^2} \end{pmatrix}.$$

If for each row of this matrix, the diagonal elements are larger in absolute value than the sum of the absolute values of the off-diagonal elements, and if these diagonal elements are negative, then the matrix J satisfies diagonal dominance, which implies that J is negative definite. The eigenvalues of a negative definite matrix have negative real parts, and hence diagonal dominance implies local stability. It is easily verified that condition (4) together with the second order condition for a (local) Bertrand-Nash equilibrium implies diagonal dominance and hence local stability. ■

Proof of Proposition 3. To determine local stability we have to look at the Jacobian matrix J. The diagonal elements of the Jacobian are

$$J_{ii} = \frac{\partial F_i(\mathsf{p}^*)}{\partial p_i} = \frac{1}{2} \frac{D^i(\mathsf{p}^*)\, D^i_{ii}(\mathsf{p}^*)}{\left(D^i_i(\mathsf{p}^*)\right)^2}$$

The off-diagonal elements are

$$J_{ij} = \frac{\partial F_i(\mathsf{p}^*)}{\partial p_i} = \frac{1}{2} \frac{D^i(\mathsf{p}^*)\, D^i_{ij}(\mathsf{p}^*) - D^i_i(\mathsf{p}^*)\, D^i_j(\mathsf{p}^*)}{\left(D^i_i(\mathsf{p}^*)\right)^2}$$

Let λ be an eigenvalue of J, then, according to Lemma 7 we have $|\lambda| \leq \|\mathsf{J}\|$. Consider again $\|A\|_\infty = \max_i \sum_{j=1}^n |A_{ij}|$, then a sufficient condition for all eigenvalues of the Jacobian matrix to lie in the unit circle is

$$J_i = \sum_{j=1}^n |J_{ij}| \leq 1, \text{ for all } i = 1, \ldots, n,$$

and this is equal to condition (7). ■

Proof of Proposition 4. This is a straightforward application of Theorem 4.C.3 from Takayama (1985). ■

Proof of Corollary 5. Condition (10) follows directly from condition (7). Since the dynamical system is linear, local stability implies global stability. ■

Proof of Proposition 6. The Jacobian matrix of system (12) at any point (p_1, p_2) is

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{2}\frac{1}{\gamma}(1-\gamma)\, p_1^{-\gamma} p_2^{-\delta} & -\frac{1}{2}\frac{1}{\gamma}\delta p_1^{1-\gamma} p_2^{-\delta-1} \\ -\frac{1}{2}\frac{1}{\gamma}\delta p_1^{-\delta-1} p_2^{1-\gamma} & \frac{1}{2} + \frac{1}{2}\frac{1}{\gamma}(1-\gamma)\, p_1^{-\delta} p_2^{-\gamma} \end{pmatrix}.$$

Evaluating this Jacobian matrix in the equilibrium $p_1^* = p_2^* = \left[\frac{1}{\gamma}\right]^{1/(\gamma+\delta)}$ gives

$$J = \begin{pmatrix} 1 - \frac{1}{2}\gamma & -\frac{1}{2}\delta \\ -\frac{1}{2}\delta & 1 - \frac{1}{2}\gamma \end{pmatrix}$$

with eigenvalues $\mu_1 = 1 + \frac{1}{2}(\delta - \gamma)$ and $\mu_2 = 1 - \frac{1}{2}(\delta + \gamma)$, and corresponding eigenvectors: $v_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}'$ and $v_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}'$. First suppose $\delta < 0$. In that case the equilibrium is stable (unstable) for $\gamma - \delta < (>)\, 4$, and unstable for $\gamma + \delta > 4$. At $\gamma - \delta = 4$, μ_1 goes through -1.Now suppose $\delta > 0$. In that case the equilibrium is stable for $\gamma + \delta < (>)\, 4$, and unstable for $\gamma + \delta > 4$. At $\gamma + \delta = 4$, μ_2 goes through -1.

When one of the eigenvalues goes through -1 a period-doubling bifurcation occurs in the dynamical system restricted to the center manifold (see Guckenheimer and Holmes, 1983, p.158). The center manifold is locally reflection symmetric (Kuznetsov, 1995, Theorem 7.6). When $\mu_1 = -1$, the center manifold lies tangent to v_1 and a period two orbit which lies off the fixed point subspace with $p_1 = p_2$ is created. When $\mu_2 = -1$, the center manifold coincides with the fixed point subspace with $p_1 = p_2$ (which also equals the eigenspace corresponding to μ_2) and a period two orbit is created in that fixed point subspace. ■

Appendix B

This appendix contains a brief technical discussion of some of the concepts used in Section 4.2. Consider a differentiable two-dimensional map $F_\alpha : I\!R^2 \to I\!R^2$, where $\alpha \in I\!R$ is a parameter. Let p be a saddle fixed point, that is, let the Jacobian of F_α evaluated at p have two real eigenvalues where $0 < |\lambda_2| < 1 < |\lambda_1|$. We can now define the *stable and unstable manifolds* of the equilibrium p as

$$W^s(\mathsf{p}) = \left\{\mathsf{x} \in I\!R^2 | F^t(\mathsf{x}) \to \mathsf{p} \text{ for } t \to +\infty\right\},$$
$$W^u(\mathsf{p}) = \left\{\mathsf{x} \in I\!R^2 | F^t(\mathsf{x}) \to \mathsf{p} \text{ for } t \to -\infty\right\}.$$

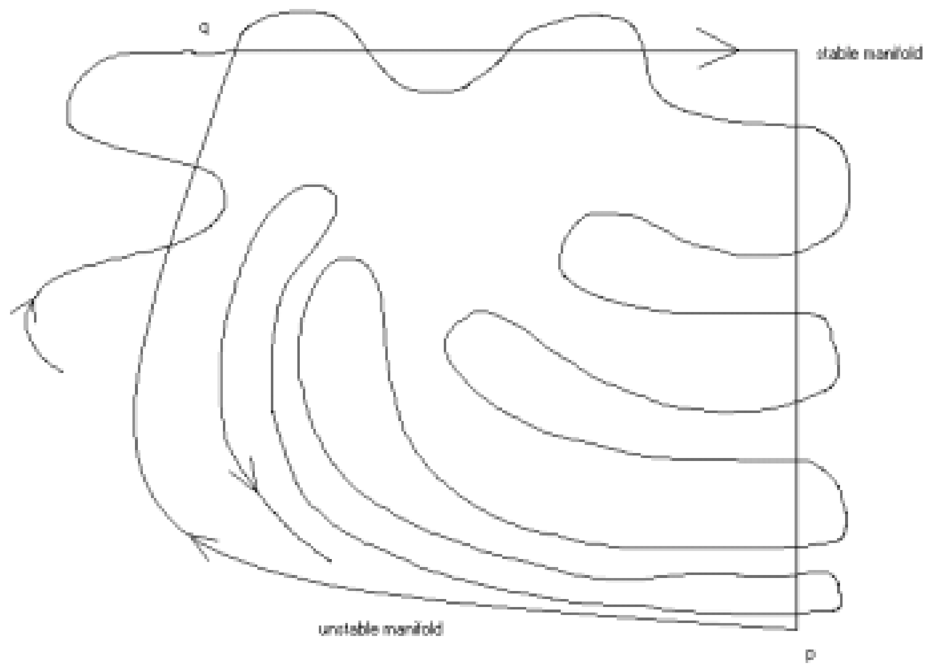

Figure 8: Shape of stable and unstable manifolds ($W^s(\mathsf{p})$ and $W^u(\mathsf{p})$) if there is a homoclinic intersection q.

When $F(.)$ is not a diffeomorphism and not invertible, the definition of the unstable manifold is a little bit more complicated (see Palis and Takens, 1993, p.167). The unstable manifold may then have selfintersections and the stable manifold may have different components.

The stable and unstable manifolds are tangent to the corresponding stable and unstable eigenvectors of the linearized system at the Bertrand-Nash equilibrium (this follows from the *stable manifold theorem* (see Guckenheimer and Holmes, 1983, p.18)). Notice that these manifolds are invariant under F, that is, $F(W^s(\mathsf{p})) = W^s(\mathsf{p})$ and $F(W^u(\mathsf{p})) = W^u(\mathsf{p})$. If F would be a linear system, these manifolds would correspond to the stable and unstable eigenvectors of the linearized system. However, for a nonlinear mapping the stable and unstable manifolds may have a more complicated structure. In fact, they can have intersections, that is, there may exist a point $\mathsf{q} \neq \mathsf{p}$ with $\mathsf{q} \in W^s(\mathsf{p}) \cap W^u(\mathsf{p})$. Such a point q is called a *point of homoclinic intersection.* Since the stable and unstable manifold are invariant under F, we must have that $F^t(\mathsf{q})$, $t = \pm 1, \pm 2, \ldots$, are also points of homoclinic intersection. The sequence of points $\{F^t(\mathsf{q})\}_{t=-\infty}^{t=+\infty}$ then is called a *homoclinic orbit.* The existence of such a homoclinic orbit implies a very complicated structure of the unstable and stable manifolds. Since the sequence $F^t(\mathsf{q})$ (by definition) converges to p when $t \to -\infty$, and all points lie in the stable and unstable manifold the unstable manifold accumulates onto itself infinitely often as $F^t(\mathsf{q})$ approaches p. The same holds for the stable manifold. Figure 8 shows the shape that the unstable and stable manifold have if

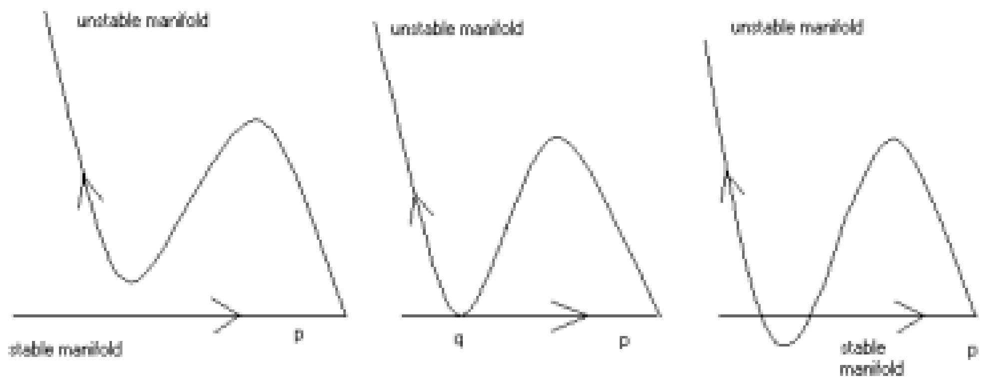

Figure 9: Homoclinic bifurcation. Left panel: stable and unstable manifold before the bifurcation ($\alpha < \alpha_0$). Middle panel: stable and unstable manifolds at the homoclinic bifurcation ($\alpha = \alpha_0$). Right panel: stable and unstable manifolds after the homoclinic bifurcation ($\alpha > \alpha_0$).

there is a point of homoclinic intersection.

The map F, and therefore also the stable and unstable manifolds, depends upon the parameter α. A *homoclinic bifurcation* is said to occur at $\alpha = \alpha_0$ when for $\alpha < \alpha_0$ there is no intersection between the unstable manifold $W^u(\mathsf{p})$ and the stable manifold $W^s(\mathsf{p})$, for $\alpha = \alpha_0$ there is a point of homoclinic tangency between $W^s(\mathsf{p})$ and $W^u(\mathsf{p})$ and for $\alpha > \alpha_0$ there is a point of transversal homoclinic intersection. Figure 9 shows the creation of such a homoclinic bifurcation as α increases.

Such a homoclinic bifurcation implies all kinds of complicated behaviour. First of all, there have to be wild oscillations of the unstable and stable manifolds as pointed out above. Also, the existence of homoclinic interesections implies the existence of so-called *horseshoes*.

Figure 10 given an example of such a horseshoe. The map G contracts the rectangle R in the vertical direction, then stretches it in the horizontal direction and finally folds it over itself, in a horseshoe-like fashion. Clearly, some of the points in R are mapped out of R and some of the points are mapped back into R. Iterating this map indefinitely one can find the set of points that will stay in R forever under the horeshoe map. The same thing can be done for the inverse map of G. The intersection Λ of these two sets, (that is, the set of all points that stay in R as $t \to \pm\infty$) is a so-called *Cantor* set of Lebesgue measure 0. This Cantor set has a fractal structure and dynamics on this Cantor set can be very complicated. In particular, Λ contains infinitely many unstable periodic points, and an uncountable set of aperiodic points (i.e. points which are neither periodic nor converge to a periodic orbit). Moreover, the map G has *sensitive dependence on initial conditions* with respect to initial states

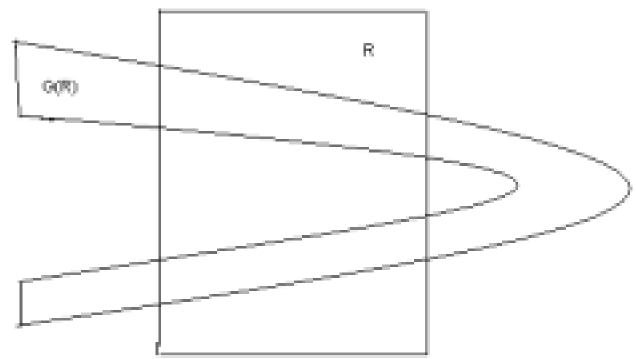

Figure 10: An example of a horseshoe. The rectangle R is mapped in a horseshoe-like fashion into $G(R)$.

in Λ, that is, there exists a $C > 0$, such that for all x_0, $y_0 \in \Lambda$, with $x_0 \neq y_0$

$$\lim_{n\to\infty} \sup |G^n(x_0) - G^n(y_0)| > C.$$

That is, any two initial conditions, no matter how close, will be separated from each other eventually. Sensitive dependence on initial conditions implies (long run) unpredictability of the corresponding dynamical system. However, the set Λ is very small and hence this *topological chaos* may only be relevant for the transient behaviour of the dynamics.

A homoclinic intersection therefore implies complicated behaviour for a small set of initial states. Other interesting dynamical phenomena occur in the interval $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, with $\varepsilon > 0$ small, when the equilibrium is *dissipative* at the homoclinic bifurcation, that is, if the product of eigenvalues of the Jacobian matrix evaluated at p is smaller than 1 in absolute value. Most importantly, it implies existence of Hénon-like strange attractors for an open interval of $\alpha-$values with positive Lebesgue measure (Benedicks and Carleson, 1991 and Mora and Viana, 1993). The Hénon map is given by $(x_{t+1}, y_{t+1}) = (1 - ax_t^2 + y_t, bx_t)$. Figure 11 shows the attractor for this Hénon map for $a = 1.4$ and $b = 0.3$. The right panel of Figure 11 shows a small portion of this attractor. Notice that the attractor has a fractal structure: its shape seems to repeat itself on a smaller scale. Also notice the resemblance with the lower right panel in Figure 7.

Other interesting phenomena occuring close to a homoclinic bifurctation are the coexistence of infinitely many stable cycles for a residual set of $\alpha-$values (this is called the Newhouse phenomenon, Newhouse, 1974,1979) and cascades of infinitely many period doubling and period halving bifurcations (Yorke and Alligood, 1983).

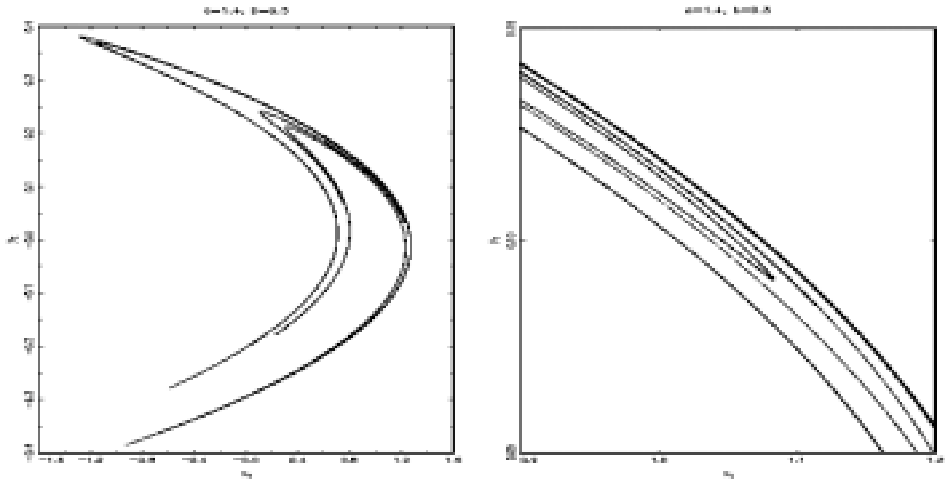

Figure 11: The Hénon map for parameters $a = 1.4$ and $b = 0.3$. The right panel shows an enlargement of a piece of the attractor from the left panel. This illustrates the fractal structure of the Hénon attractor.

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