Heterogeneous Agent Models: two simple examples.

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Abstract These notes review two simple heterogeneous agent models in economics and finance. The first is a cobweb model with rational versus naive agents introduced in Brock and Hommes (1997). The second is an asset pricing model with fundamentalists versus technical traders introduced in Brock and Hommes (1998). Agents are boundedly rational and switch endogenously between different trading strategies, based upon an evolutionary fitness measure given by realized past profits. Evolutionary switching creates a nonlinearity in the dynamic models. Rational routes to randomness, that is, bifurcation routes to complicated dynamical behavior occur when agents become more sensitive to differences in evolutionary fitness.

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1 Introduction

The key difference between economics and the natural sciences is perhaps the fact that decisions of economic agents today depend upon their expectations or *beliefs* about the future. For example, in financial markets an overoptimistic estimate of future growth of ICT industries may contribute to an excessively rapid growth of stock prices and indices and might lead to over valuation of stock markets worldwide. Any dynamic economic system is in fact an *expectations feedback* system. A theory of expectation formation is therefore a crucial part of any economic model or theory.

Since its introduction in the sixties by Muth (1961) and its popularization in macroeconomics by Lucas (1971), the rational expectations hypothesis (REH) has become the dominating expectation formation paradigm in economic theory. According to the REH all agents are rational and take as their subjective expectation of future variables the objective prediction by economic theory. In a rational expectations model agents have perfect knowledge about the (linear) market equilibrium equations and use these to derive their expectations. Although many economists nowadays view rational expectations as something unrealistic, it is still viewed as an important benchmark. Despite a rapidly growing literature on *bounded rationality*, where agents use learning models for their expectations, it seems fair to say that at this point no generally accepted alternative theory of expectations is available.

In finance, the REH is intimately related to the Efficient Market Hypothesis (EMH). There are weak and strong forms of the EMH, but when economists speak of financial markets as being efficient, they usually mean that they view asset prices and returns as the outcome of a competitive market consisting of rational traders, who are trying to maximize their expected returns. The main reason why financial markets must be efficient is based upon an *arbitrage* argument (e.g. Fama (1970)). If markets were not efficient, then there would be unexploited profit opportunities, that could and would be exploited by rational traders. For example, rational traders would buy (sell) an underpriced (overpriced) asset, thus driving its price back to the correct, fundamental value. In an efficient market, there can be *no forecastable structure* in asset returns, since any such structure would be exploited by rational traders and therefore would be doomed to disappear. Rational agents thus process information quickly and this is reflected immediately in asset prices. The value of a risky asset is completely determined by its *fundamental price*, equal to the present discounted value of the expected stream of future dividends. In an efficient market, all traders are rational and changes in asset prices are completely random, solely driven by unexpected 'news' about changes in economic fundamentals.

In contrast, Keynes already questioned a completely rational valuation of assets, arguing that investors sentiment and mass psychology ('*animal spirits*') play a significant role in financial markets. Keynes used his famous *beauty contest* as a parable to financial markets. In order to predict the winner of a beauty contest, objective beauty is not all that important, but knowledge or prediction of others' perceptions of beauty is much more relevant. Keynes argued that the same may be true for the fundamental price of an asset: '*Investment based on genuine long-term expectation is so difficult as to be scarcely practicable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes*' (Keynes, 1936, p.157). In Keynes view, stock prices are thus not governed by an objective view of 'fundamentals', but by 'what average opinion expects average opinion to be'.

New classical economists have viewed 'market psychology' and 'investors sentiment' as being *irrational* however, and therefore inconsistent with the REH. For example, Friedman (1953) argued that irrational speculative traders would be driven out of the market by rational traders, who would trade against them by taking long opposite positions, thus driving prices back to fundamentals. In an efficient market, 'irrational' speculators would simply loose money and therefore fail to survive evolutionary competition.

In a perfectly rational EMH world *all* traders are rational and it is *common knowledge* that all traders are rational. In real financial markets however, traders are different, especially with respect to their expectations about future prices and dividends. A quick glance at the financial pages of newspapers is sufficient to observe that *difference of opinions* among financial analysts is the rule rather than the exception. In the last decade, a rapidly increasing number of structural heterogeneous agent models have been introduced in the economics and finance literature, see for example Arthur et al. (1997), Brock (1993, 1997), Brock and Hommes (1997, 1998), Brock and LeBaron (1996), Chiarella (1992), Chiarella and He (2000), Dacorogna et al. (1995), Day and Huang (1990), DeGrauwe et al. (1993), De Long et al. (1990ab), Farmer (1998), Farmer and Joshi (2002), de Fontnouvelle (2000), Frankel and Froot (1988), Gaunersdorfer (2000), Gaunersdorfer and Hommes (2000), Kirman (1991), Kirman and Teyssière (2000), Kurz (1997), LeBaron

(2000), LeBaron et al. (1999), Lux (1995, 1997), Lux and Marchesi (1999, 2000), Wang (1994) and Zeeman (1974), as well as many more references in these papers. Some authors even talk about a *Heterogeneous Market Hypothesis*, as a new alternative to the Efficient Market Hypothesis. In all these heterogeneous agent models different groups of traders, having different beliefs or expectations, co-exist. Two typical trader types can be distinguished. The first are rational, '*smart money*' traders or *fundamentalists*, believing that the price of an asset is determined completely by economic fundamentals. The second typical trader type are '*noise traders*', sometimes called *chartists* or *technical analysts*, believing that asset prices are not determined by fundamentals, but that they can be predicted by simple technical trading rules based upon patterns in past prices, such as trends or cycles.

In a series of papers, Brock and Hommes (1997a,b, 1998, 1999), henceforth BH, propose to model economic and financial markets as *Adaptive Belief Systems (ABS)*, where agents are heterogeneous and switch between different trading strategies. In these notes we review two simple heterogeneous agent models, the cobweb model with sophisticated rational versus simple naive traders and a standard asset pricing model with fundamentalists versus chartists. These models are discussed in detail in Brock and Hommes (1997a) and Brock and Hommes (1998); see also the survey in Hommes (2001). An ABS is an evolutionary competition between trading strategies. Different groups of traders have different expectations about future prices. For example, one group might be fundamentalists, believing that asset prices return to their fundamental equilibrium price, whereas another group might be chartists, extrapolating patterns in past prices. Traders choose their trading strategy according to an evolutionary 'fitness measure', such as accumulated past profits. Agents are boundedly rational, in the sense that most traders choose strategies with higher fitness. BH introduce the notion of *Adaptive Rational Equilibrium Dynamics (ARED)*, an endogenous coupling between market equilibrium dynamics and evolutionary updating of beliefs. Current beliefs determine today's equilibrium prices, generating new, adapted beliefs which in turn lead to new equilibrium prices tomorrow, etc.. In an ARED, equilibrium prices and beliefs co-evolve over time. Most of the heterogeneous agent literature is computationally oriented. An ABS may be seen as a tractable theoretical framework for the computationally oriented 'artificial stock market' literature, such as the Santa Fe artificial stock market of Arthur et al. (1997) and LeBaron et al. (1999). A convenient feature of an ABS is that the model can be formulated in terms of *deviations* from a benchmark fundamental. In fact, the perfectly rational EMH benchmark is nested within an ABS as a special case. An ABS may thus be used for experimental and empirical testing whether deviations from a suitable RE benchmark are significant.

The coupling between market equilibrium and updating of strategies makes the evolutionary system highly *nonlinear*. A common finding is the occurrence of a *rational route to randomness*, that is, a bifurcation route to complicated price fluctuations as traders become more sensitive to differences in evolutionary fitness. These notes are organized as follows. Section 2 reviews the cobweb model with sophisticated rational versus naive producers. Section 3 reviews the asset pricing model with heterogeneous beliefs and presents examples with two, three and four different trader types. Finally, section 4 concludes.

2 The cobweb model

The cobweb model describes fluctuations of equilibrium prices in an independent market for a non-storable consumption good. The good takes one time period to produce, so that producers must form price expectations one period ahead. Applications of the cobweb model mainly concern agricultural markets, such as the classical examples of cycles in hog or corn prices. Supply $S(p_t^e)$ is a function of producer's next period expected price, p_t^e and is derived from expected profit maximization:

$$S(p_t^e) \;=\; \text{argmax}_{q_t}\,\{p_t^e\, q_t - c(q_t)\} \;=\; (c')^{-1}(p_t^e), \tag{2.1}$$

where $c(\cdot)$ is an increasing, convex cost function. The supply curve thus coincides with the inverse of the marginal cost curve. The expected price may be some function of (publically known) past prices, that is, $p_t^e = H(\vec{P}_{t-1})$, where $\vec{P}_{t-1} = (p_{t-1}, p_{t-2}, \cdots, p_{t-L})$ denotes a vector of past prices of lag-length L, and $H(\cdot)$ is called a *predictor* or *forecasting rule*.

Consumer demand D depends upon the current market price p_t. The demand curve D may be derived from utility maximization under a budget constraint, but for our purposes it is not necessary to specify preferences explicitly. We will assume that consumer demand is decreasing in the market price. If beliefs are homogeneous, i.e., all producers use the same predictor, market equilibrium price dynamics in the cobweb model is given by

$$D(p_t) \;=\; S(H(\vec{P}_{t-1})), \qquad \text{or} \qquad p_t \;=\; D^{-1}(S(H(\vec{P}_{t-1}))). \tag{2.2}$$

The actual equilibrium price dynamics thus depends upon the demand curve D, the supply curve S as well as the predictor H used by the producers. If all producers have rational expectations or perfect foresight, that is, their prediction coincides exactly with the realized price, $H^R(\vec{P}_{t-1}) = p_t$, price dynamics become extremely simple: $p_t = p^*$ in all periods, where p^* is the (unique) price corresponding to the intersection of demand and supply. If, on the other hand, all producers use the naive predictor $H^N(\vec{P}_{t-1}) = p_{t-1}$, that is, the forecast coincides with the last observation, the price dynamics is given by $p_t = D^{-1}(S(p_{t-1}))$, which is the familiar textbook cobweb system. If demand D is decreasing and supply S is increasing, price dynamics in the cobweb model with naive expectations is simple. If $-1 < S'(p^*)/D'(p^*) < 0$ prices converge to the stable steady state p^*; otherwise, prices diverge away from the steady state and either converge to a stable 2-cycle or exhibit unbounded up and down oscillations.[1]

2.1 Heterogeneous beliefs

Brock and Hommes (1997a), henceforth BH97a, studied heterogeneity in expectation formation by introducing the concept of *Adaptive Rational Equilibrium Dynamics (ARED)*, a coupling between market equilibrium dynamics and adaptive predictor selection. The ARED is an evolutionary dynamics between competing prediction strategies.

[1]Note that for other predictors such as adaptive expectations or linear predictors with two or three lags, price fluctuations in the cobweb model can become much more complicated. In particular, chaotic price oscillations may arise even when both demand and supply are *monotonic* (Hommes (1994,1998)).

Agents can choose between different prediction strategies and update their beliefs over time according to a publically available 'fitness' or 'performance' measure such as (a weighted sum of) past realized profits. Prediction strategies with higher fitness in the recent past are selected more often than those with lower fitness. Market equilibrium in the cobweb model with heterogeneous beliefs is determined by

$$D(p_t) = \sum_{j=1}^{J} n_{j,t-1} S(H_j(\vec{P}_{t-1})), \tag{2.3}$$

where H_j, $1 \leq j \leq J$, represents the forecasting strategy of type j and $n_{j,t-1}$ is the fraction of agents using strategy j at the beginning of period t.

BH97a present a detailed analysis of the cobweb model with two trader types and linear demand and supply. Demand is linearly decreasing and given by[2]

$$D(p_t) = a - dp_t, \qquad d > 0. \tag{2.4}$$

The supply curve is linear and given by

$$S(p_t^e) = sp_t^e, \qquad s > 0, \tag{2.5}$$

or equivalently, producer's cost function is quadratic and given by $c(q) = q^2/(2s)$.

Agents can either buy a sophisticated, rational expectations (perfect foresight) forecast at positive per period information costs $C \geq 0$, or freely obtain the simple, naive forecast. The two forecasting rules are thus given by

$$H_1(\vec{P}_{t-1}) = \quad p_t, \tag{2.6}$$

$$H_2(\vec{P}_{t-1}) = \quad p_{t-1}. \tag{2.7}$$

Market equilibrium in the cobweb model with rational versus naive expectations and linear demand and supply is given by

$$a - dp_t = n_{t-1}^R sp_t + n_{t-1}^N sp_{t-1}, \tag{2.8}$$

where n_{t-1}^R and n_{t-1}^N denote the fractions of producers using the rational respectively naive predictor, at the beginning of period t. Notice that producers using the rational expectations predictor have perfect foresight, and therefore must have perfect knowledge about the market equilibrium equation (2.8), including past prices as well as the fractions of both groups. Consequently, rational agents have perfect knowledge about the beliefs of all other agents. The difference C between the per period information costs for rational and naive expectations represents an extra effort cost producers incur over time when acquiring this perfect knowledge. It is straightforward to solve (2.8) explicitely for the market equilibrium price

$$p_t = \frac{a - n_{t-1}^N sp_{t-1}}{d + n_{t-1}^R s}. \tag{2.9}$$

[2]See Goeree and Hommes (2000) for an analysis of the cobweb model with rational versus naive expectations in the case of nonlinear (but monotonic) demand and supply.

The cobweb model with rational versus naive expectations, may be seen as an analytically tractable, stylized two predictor model in which rational expectations represents a costly sophisticated (and stabilizing) predictor, and naive expectations represent a cheap 'habitual rule of thumb' (but potentially destabilizing) predictor. It is interesting to note that other two predictor cases, such as fundamentalists (expecting prices to return to the rational expectations fundamental steady state price p^*) versus adaptive expectations yield similar results.

To complete the model, we have to specify how the fractions of traders using rational respectively naive expectations are determined. These fractions change over time, and are updated according to a publically available 'performance' or 'fitness' measure associated to each predictor. Here, we take the most recent realized net profit as the performance measure for predictor selection.[3] For the rational expectations forecasting strategy (2.6) and linear supply (2.5), realized profit in period t is given by

$$\pi_t^R \;=\; p_t\, S(p_t) - c(S(p_t)) = \frac{s}{2} p_t^2. \tag{2.10}$$

The *net* realized profit for rational expectations is thus given by $\pi_t^R - C$, where C is the per period information cost that has to be paid for obtaining the perfect forecast. For the naive predictor (2.7) and linear supply (2.5) the realized net profit in period t is given by

$$\pi_t^N \;=\; p_t\, S(p_{t-1}) - c(S(p_{t-1})) = \frac{s}{2} p_{t-1}(2p_t - p_{t-1}). \tag{2.11}$$

The fractions of the two groups are determined by the Logit discrete choice model probabilities. Anderson, de Palma and Thisse (1992) contains an extensive discussion and motivation of discrete choice modelling in various economic contexts; BH97a provide motivation of discrete choice models for selecting prediction strategies. The fraction of agents using the rational expectations predictor in period t equals

$$n_t^R = \frac{\exp(\beta(\pi_t^R - C))}{\exp(\beta(\pi_t^R - C)) + \exp(\beta\, \pi_t^N)}, \tag{2.12}$$

and the fraction of agents choosing the naive predictor in period t is

$$n_t^N = 1 - n_t^R. \tag{2.13}$$

A crucial feature of this evolutionary predictor selection is that agents are boundedly rational, in the sense that most agents use the predictor that has the highest fitness. Indeed, from (2.12-2.13) we have for instance that $n_t^R > n_t^N$ whenever $\pi_t^R - C > \pi_t^N$, although the optimal predictor is not chosen with probability one. The parameter β is called the *intensity of choice*; it measures how fast producers switch between the two prediction strategies. Let us briefly discuss the two extreme cases $\beta = 0$ and $\beta = \infty$. For

[3] The case where the performance measure is realized net profit in the most recent past period, leads to a two-dimensional dynamic system. The more general case, with a weighted sum of past net realized profits as the fitness measure, leads to higher dimensional systems, which are not as analytically tractable as the two-dimensional case. In this more general higher dimensional case however, numerical simulations suggest similar dynamic behaviour.

$\beta = 0$, both fractions are fixed over time and equal to 1/2. The other extreme $\beta = \infty$, corresponds to the *neoclassical limit* in which agents are unboundedly rational, and *all* producers choose the optimal predictor in each period. Hence, the higher the intensity of choice the more rational, in the sense of evolutionary fitness, agents are in choosing their prediction strategies. The neoclassical limit $\beta = \infty$ will play an important role in what follows.

The timing of predictor selection in (2.12) is important. In (2.9) the old fractions n^R_{t-1} and n^N_{t-1} determine the new equilibrium price p_t. This new equilibrium price p_t is used in the fitness measures (2.10) and (2.11) for predictor choice and the new fractions n^R_t and n^N_t are updated according to (2.12) and (2.13). These new fractions are then used in determining the next equilibrium price p_{t+1}, etc.. Equilibrium prices and fractions thus co-evolve over time.

It will be convenient to define the difference m_t of the two fractions:

$$m_t \equiv n^R_t - n^N_t, \tag{2.14}$$

so $m_t = -1$ corresponds to all producers being naive, whereas $m_t = +1$ means that all producers prefer the rational expectations predictor. The evolution of the equilibrium price, p_t, and the difference of fractions, m_t, is then summarized by the following two-dimensional, non-linear dynamical system

$$p_t = \frac{a - n^N_{t-1} s p_{t-1}}{d + n^R_{t-1} s} = \frac{2a - (1 - m_{t-1}) s p_{t-1}}{2d + (1 + m_{t-1}) s}, \tag{2.15}$$

$$m_t = \tanh\left(\frac{\beta}{2}(\pi^R_t - \pi^N_t - C)\right). \tag{2.16}$$

BH97a called the coupling (2.15-2.16) between the equilibrium price dynamics and adaptive predictor selection an *Adaptive Rational Equilibrium Dynamics (ARED)* model.

For a linear supply curve, using (2.10) and (2.11) the difference in realized profits of rational and naive agents can be simplified to

$$\pi^R_t - \pi^N_t = \frac{s}{2}(p_t - p_{t-1})^2, \tag{2.17}$$

that is, the difference in realized profits is proportional to the squared prediction error of the naive forecast. In the case of a linear demand and supply the ARED thus becomes

$$p_t = \frac{2a - (1 - m_{t-1}) s p_{t-1}}{2d + (1 + m_{t-1}) s} \tag{2.18}$$

$$m_t = \tanh\left(\frac{\beta}{2}\left[\frac{s}{2}(p_t - p_{t-1})^2 - C\right]\right). \tag{2.19}$$

The reader may easily verify that the model has a unique steady state $(p^*, m^*) = (a/(d+s), \tanh(-\beta C/2))$. Notice that $p^* = a/(d+s)$ is exactly the price where demand and supply intersect. It will be convenient to rewrite (2.18-2.19) in *d*eviations from the steady state price, $x_t = p_t - p^*$, yielding[4]

$$x_t = \frac{-(1-m_{t-1})sx_{t-1}}{2d+(1+m_{t-1})s} \tag{2.20}$$

$$m_t = \tanh(\frac{\beta}{2}\,[\,\frac{s}{2}(x_t - x_{t-1})^2 - C\,]). \tag{2.21}$$

In the sequel we use the shorthand notation $(x_t, m_t) = F_\beta(x_{t-1}, m_{t-1})$ for the ARED-model (2.20-2.21). We are especially interested in the dynamics when the 'degree of rationality', that is, the intensity of choice, β to switch forecasting strategies, becomes high.

2.2 Local (in)stability of the steady state

We will now discuss the dynamical behaviour of prices and fractions in the cobweb model with rational versus naive expectations, starting with the stability conditions for the steady state. Recall that the model has a unique steady state $(p^*, m^*) = (a/(d+s), \tanh(-\beta C/2))$, where $p^* = a/(d+s)$ is the price where demand and supply intersect. Notice that for $C = 0$, i.e. when there are no costs for rational expectations, $m^* = 0$, so that at the steady state the fractions of the two types are equal. In contrast, for positive information costs for rational expectations, i.e. for $C > 0$ we have $m^* < 0$, so that at the steady state most agents employ the naive forecasting rule. This makes sense, because at the steady state both forecasting rules yield exactly the same forecast, and most agents then prefer the cheap, naive forecast.
The stability properties of the steady state are determined by the derivatives of supply and demand at the steady state price p^*. A straightforward computation shows that the eigenvalues of the Jacobian matrix of (2.18-2.19) evaluated at the steady state are $\lambda_1 = 0$, and

$$\lambda_2 = \frac{(1-m^*)\,S'(p^*)}{2\,D'(p^*) - (1+m^*)\,S'(p^*)} = \frac{-(1-m^*)\,s}{2\,d + (1+m^*)\,s} < 0. \tag{2.22}$$

Since m^* is less than or equal to one in absolute value, the value of the second eigenvalue lies between $S'(p^*)/D'(p^*) = -s/d$ and 0. Hence, if the familiar cobweb stability condition $|S'(p^*)/D'(p^*)| = |s/d| < 1$ is satisfied, implying that the model is stable under naive expectations, then in the cobweb model with rational versus naive expectations and linear demand and supply, the steady state is globally stable, for all β. Prices then always converge to p^*, and the difference of fractions converges to m^*. To allow for the possibility of an unstable steady state and endogenous price fluctuations in the evolutionary ARED-model, from now on we assume the following.

Assumption U. *The market is locally unstable when all producers are naive, that is,* $S'(p^*)/D'(p^*) = -s/d < -1$.

[4] Notice that (2.20) is equivalent to fixing $a = 0$ in (2.18), so that $p^* = 0$. In fact, we are just choosing the steady state price as the origin.

The stability properties of the steady state in the evolutionary ARED-model are summarized as follows.

Proposition 1. *Under assumption U, the evolutionary ARED-model satisfies:*

(*i*) *when information costs are zero ($C = 0$), the steady state is globally stable for all β,*

(*ii*) *when information costs are strictly positive ($C > 0$), there exists a critical value β_1 such that the steady state is (globally) stable for $0 \le \beta < \beta_1$ and unstable for $\beta > \beta_1$. At $\beta = \beta_1$ the second eigenvalue satisfies $\lambda_2 = -1$, and F_β in (2.18-2.19) exhibits a period doubling bifurcation.*

For $C = 0$, the steady state difference in fractions $m^* = 0$ and the eigenvalue in (2.22) satisfies $-1 < \lambda_2 < 0$, implying that the steady state is locally stable. Global stability follows by observing that, for $C = 0$ we must have $m_t \ge 0$, for all $t \ge 1$, and then using (2.20) x_t must converge to 0, or equivalently, prices always converge to their steady state value. The second part of the proposition follows by observing that the eigenvalue $\lambda_2 = -1$ when the steady state difference in fractions $m^* = \bar{m} = -d/s$. Assumption U implies that $-1 < \bar{m} = -d/s < 0$. As the intensity of choice β increases from 0 to $+\infty$ the steady state difference in fractions m^* decreases from 0 to -1 and, for some critical value of $\beta = \beta_1$ we have $m^* = \bar{m}$ and an eigenvalue $\lambda_2 = -1$ and the second part of the proposition follows.

2.3 A rational route to randomness

According to Proposition 1, for positive information costs C, the steady state $(p^*, m^*) = (p^*, \tanh(-\beta C/2))$ becomes unstable as the intensity of choice β increases. In this subsection we investigate the dynamics for large values of the intensity of choice. It will be useful however, to consider the neoclassical limit, that is, the case $\beta = \infty$ first.

For $\beta = \infty$ and $C > 0$, the steady state difference in fractions $m^* = tanh(-\beta C/2) = -1$, that is, at the steady state all agents are naive. Furthermore, for $\beta = \infty$ in each period *a*ll agents choose the optimal predictor, that is, in each period $t \ge 1$ either all agents are rational or all agents are naive. In fact, for $\beta = +\infty$ the switching between forecasting strategies (2.19) simplifies to

$$m_t = \begin{cases} +1 & \text{if} \quad \pi_t^R - \pi_t^N = \dfrac{s}{2}(p_t - p_{t-1})^2 > C, \\ -1 & \text{if} \quad \pi_t^R - \pi_t^N = \dfrac{s}{2}(p_t - p_{t-1})^2, \le C. \end{cases} \tag{2.23}$$

Stated differently, as long as the squared prediction error from naive expectations is sufficiently small compared to the per period information costs for rational expectations, i.e. as long as $(p_t - p_{t-1})^2 \le 2C/s$, all agents employ the simple, cheap forecasting strategy. As long as all agents are naive, the price dynamics is governed by $p_t = D^{-1}(S(p_{t-1}))$, a linear unstable oscillation around the steady state price p^*, and prices diverge from their steady state value oscillating with increasing amplitude. The squared forecasting error from naive expectations will increase, and at some point must exceed the critical level $2C/s$, and all agents will then switch to rational expectations. When all producers become rational in period t, next periods price $p_{t+1} = p^*$ and the price immediately jumps back to the steady state price. These simple observations prove the following proposition:

Proposition 2. *For an infinite intensity of choice $\beta = \infty$ and positive information cost $C > 0$, all time paths in the ARED system (2.18-2.19) converge to the steady state $S = (p^*, -1)$, even under assumption U when the steady state is a locally unstable saddle point.*

Now suppose we add a small amount of noise to the neoclassical limit system, by adding a small random shock (e.g. a demand shock) in each period to the equilibrium pricing equation (2.18). Almost the same story as above applies, except that when all agents switch to rational expectations the system will not be driven exactly onto the steady state, but only close to the steady state. With prices close to the steady state value, all agents will then switch back to the cheap, naive forecasting rule and prices will start to oscillate and diverge, and the story repeats. The noisy neoclassical limit is thus characterized by an irregular switching between an unstable phase in which all agents are naive and prices diverge from the steady state, and a stable phase in which all agents become rational and prices return close to the steady state.

BH97a have shown that the same behavior arises in the deterministic, noise free case for a high, but finite, intensity of choice β. In fact, for high values of the intensity of choice, the dynamical behavior becomes chaotic with prices and fractions moving on a strange attractor. Figure 1 shows an example of a strange attractor, with corresponding time series of prices p_t and fractions n_t^R of rational producers. Numerical simulations suggest that for (almost) all initial states (p_0, m_0) the orbit converges to this strange attractor. Its intricate geometric shape explains why it is called a strange attractor.

Figure 2 illustrates the *r*ational route to randomness, that is, the bifurcation route from a stable steady state for low values of the intensity of choice to the complicated dynamical behavior for high values of the intensity of choice. The primary bifurcation is a period doubling bifurcation for $\beta \approx 0.77$, in which the steady state becomes unstable and a stable 2-cycle is created. As the intensity of choice increase more bifurcations occur and the dynamical behavior becomes more and more complicated. The Lyapunov exponent plot in Figure 2 shows that for large values of the intensity of choice the largest Lyapunov exponent becomes positive and therefore the dynamics becomes chaotic.

For a high intensity of choice price fluctuations are characterized by an irregular switching between a stable phase, with prices close to the steady state, and an unstable phase with fluctuating prices, as illustrated in Figure 1. There is a strikingly simple economic intuition explaining this switching behavior when the intensity of choice is large. Suppose we take an initial state close to the (locally unstable) steady state. Most agents will use the cheap, naive forecasting rule, because it does not pay to buy a costly, sophisticated forecasting rule that yields an almost identical forecast. With most agents using the cheap, naive predictor prices diverge from the steady state, start fluctuating, and net realized profits from the naive predictor decrease. At some point, it becomes profitable to buy the rational expectations forecast, and when the intensity of choice to switch pre-

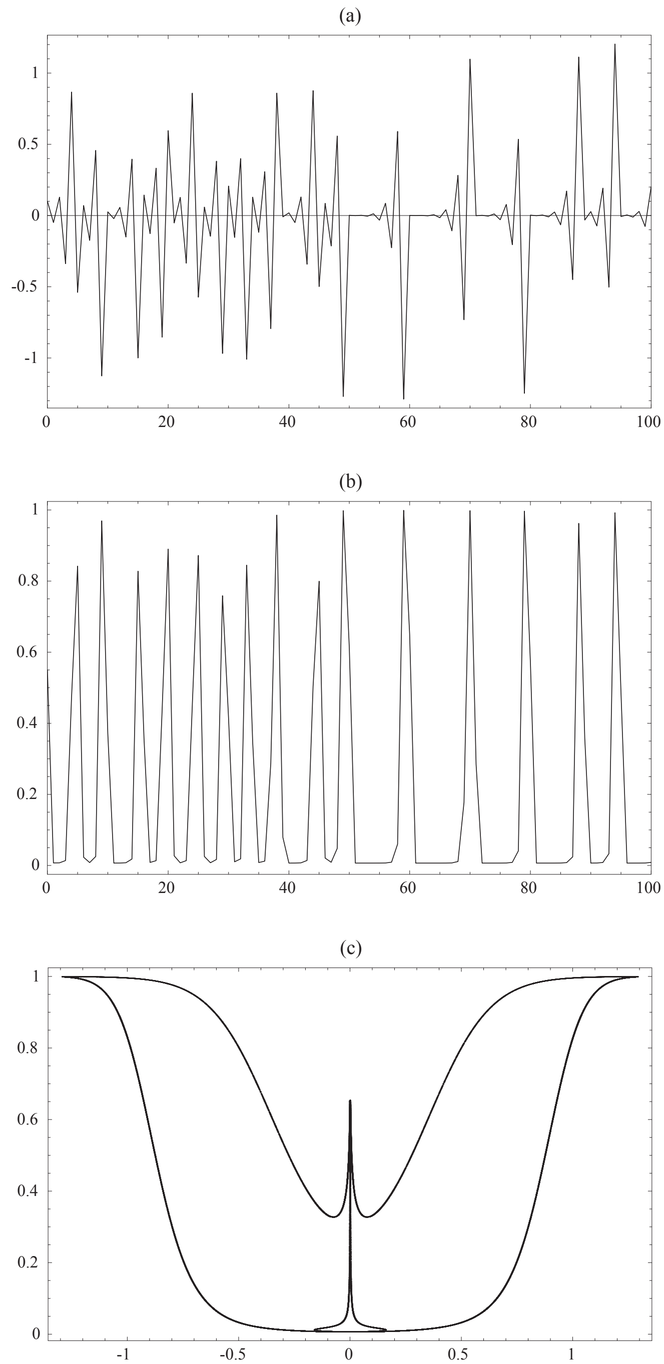


Figure 1. Chaotic time series of deviations x_t from the steady state price (top pannel) and fractions n_t^R of rational agents (middle pannel) and corresponding strange attractor in the (x, n^R)-phase space (bottom pannel). Parameters are: $\beta = 5$, $a = 0$, $d = 0.5$, $s = 1.35$ and $C = 1$.

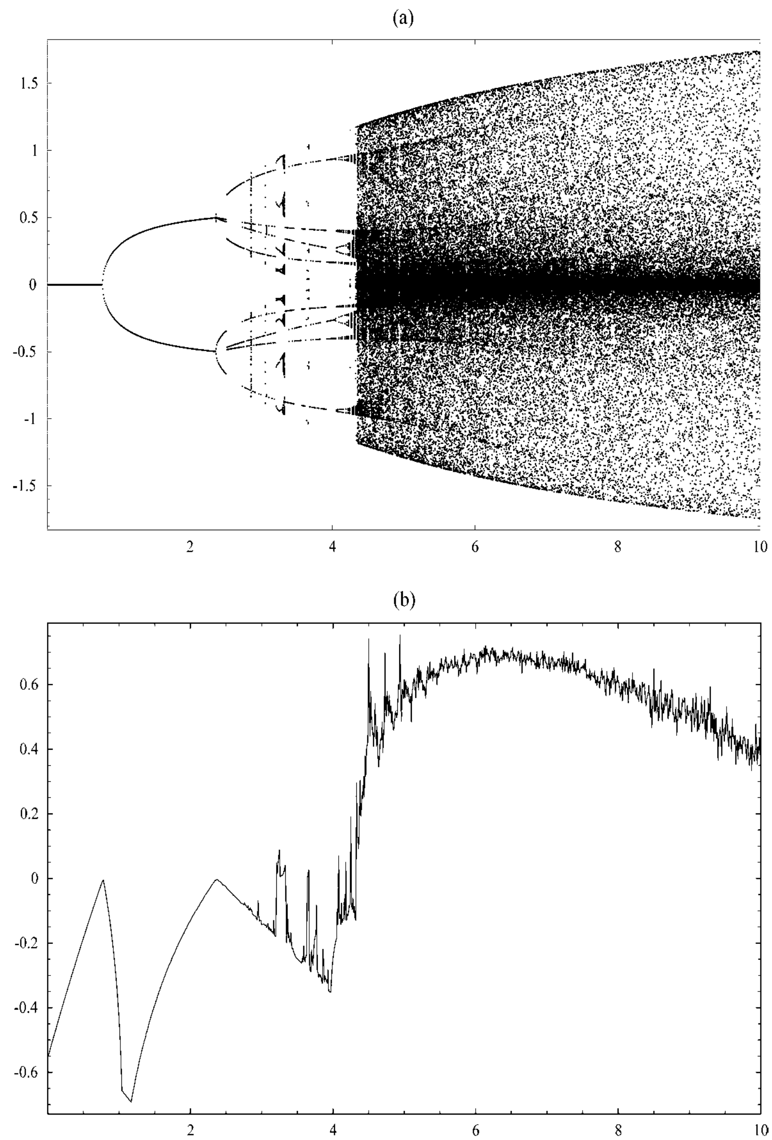


Figure 2. A rational route to randomness. The bifurcation diagram (top panel) shows a bifurcation route from a stable steady state for small values of the intensity of choice β to chaotic price fluctuations, with positive largest Lyapunov exponent (bottom panel) for high values of the intensity of choice. Parameters are $a = 0$, $d = 0.5$, $s = 1.35$ and $C = 1$, $0 \leq \beta \leq 10$.

dictors is high, most agents will then switch to rational expectations. As a result, prices are driven back close to the steady state, and the story then repeats. Irregular, chaotic price fluctuations thus result from a (boundedly) rational choice between cheap 'free riding' and costly sophisticated prediction. In fact, the above economic mechanism already suggests that for a large intensity of choice, the ARED-cobweb model will be close to a so-called homoclinic orbit associated to the unstable, saddle point steady state.

2.4 Homoclinic points and the unstable manifold of the steady state

A key feature of chaotic dynamical behavior in two- and higher dimensional systems is the existence of so-called *homoclinic points*. This concept was introduced already by Poincaré (1890), in his prize winning essay on the stability of the three-body system. Let us briefly discuss this important notion.

Recall that after the primary bifurcation in the ARED-model, the steady state S loses its stability and becomes a saddle point. In deviations x from the steady state price p^*, the steady $S = (0, m^*) = (0, tanh(-\beta C/2))$. The *stable manifold* and the *unstable manifold* of the steady state are defined as

$$W^s(S) = \left\{ (x, m) \mid \lim_{n\to\infty} F_\beta^n(x, m) = S \right\},$$

$$W^u(S) = \left\{ (x, m) \mid \lim_{n\to-\infty} F_\beta^n(x, m) = S \right\}.$$

A *transversal homoclinic point* $Q \neq S$, associated to the saddle S, is an intersection point of the stable and unstable manifold of S. It was already pointed out by Poincaré that the existence of a homoclinic intersection implies that the geometric structure of both the stable and unstable manifold is quite complicated, and the system exhibits some form of sensitive dependence on initial conditions. It is now well-known that a system having a homoclinic point is in fact chaotic. See Palis and Takens (1993) for an extensive mathematical treatment.

The unstable manifold of the steady state plays a crucial role for understanding the global characteristics of the evolutionary dynamics. Figure 3 illustrates the geometric shape of the unstable manifold of the steady state for diffent values of the intensity of choice β. Using (2.20-2.21), the reader may easily verify that all points $(0, m)$ are mapped exactly onto the steady state $(0, m^*)$ in the next period. This implies that the steady state S has an eigenvalue 0 and the stable manifold of the steady state S must contain the vertical line segment $p = p^*$, or in deviations, the vertical segment $x = 0$. For $\beta > \beta_1$, the steady state is locally unstable and has a second eigenvalue $\lambda_2 < -1$. Therefore, the unstable manifold has two different branches, each branch spiralling around one of the two points of the (un)stable period 2 orbit, as illustrated in Figure 3. Moreover, when the intensity of choice becomes large, each branch of the unstable manifold moves closer to the vertical line segment $x = 0$ of the stable manifold. For β large, the ARED-system is thus close to having a homoclinic orbit.

The geometric explanation for the dynamic complexity of the ARED-dynamics, based upon the shape of the unstable manifold of the steady state, bears a close similarity to the economic mechanism underlying complicated price fluctuations. On the one hand, for high values of the intensity of choice the system is driven towards the steady state

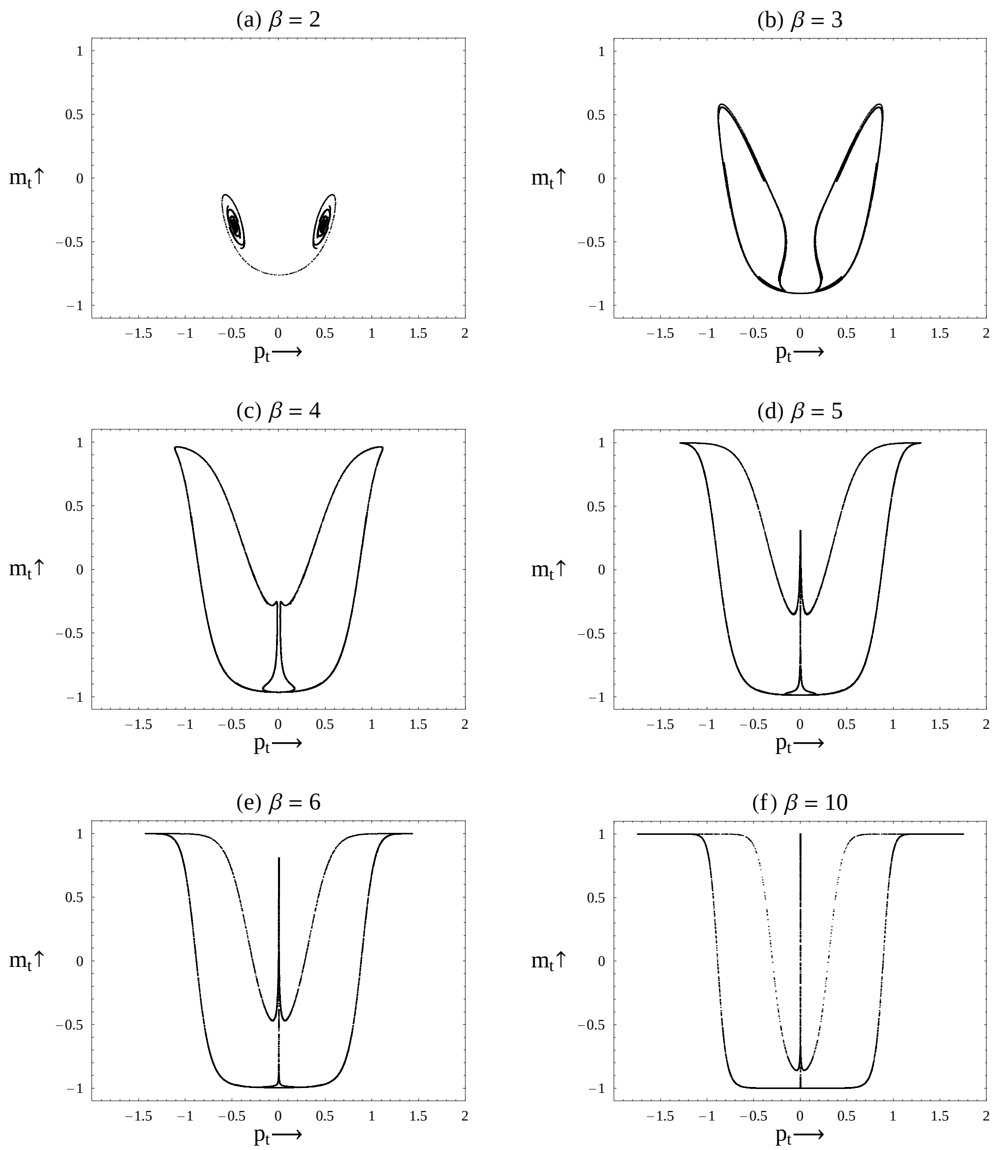


Figure 3. The unstable manifold of the steady state, for different values of the intensity of choice β. The stable manifold contains the vertical line segment $p = p^*$, or equivalently $x = 0$. The two branches of the unstable manifold spiral around the two points of the (un)stable 2-cycle. As the intensity of choice increases, the unstable manifold moves closer to the vertical line segment $x = 0$ contained in the stable manifold. For large β-values the ARED system is therefore close to a homoclinic tangency between the stable and the unstable manifold of the steady state.

by a stabilizing force when most agents become rational. On the other hand, once prices are close to their steady state, due to information costs for rational expectations most agents switch to cheap naive expectations, leading to market instability and diverging prices. Price fluctuations on the strange attractors are thus characterized by an irregular switching between a destabilizing force of cheap free riding and a costly, but stabilizing force of sophisticated prediction.

For a large value of the intensity of choice (corresponding to a high degree of rationality) the ARED system does not settle down to simple (periodic) behavior, but chaotic price fluctuations on a *strange attractor* arise. Applying the mathematical theory of homoclinic bifurcations (see e.g. Palis and Takens (1993)) BH97a have shown that the ARED system exhibits complicated dynamical behavior for a large set of parameter values:

Theorem. *Under assumption U, i.e. when the market is unstable under naive expectations, if information cost C for rational expectations is strictly positive, the ARED-model (2.18-2.19) has strange attractors for a set of β-values of positive Lebesgue measure.*

Adaptive rational equilibrium dynamics is a way of modeling evolutionary competition in a market with heterogeneous traders. The example of the cobweb model with rational versus naive expectations shows that differences in fitness may lead to market instability and endogenous fluctuations. In the next subsection we discuss a financial market applications of the evolutionary framework.

3 An asset pricing model

In this section we discuss a second application of the evolutionary framework proposed in Brock and Hommes (1997a). This application has been coined *Adaptive Belief Systems (ABS)*, and has been introduced in Brock (1997) and Brock and Hommes (1997b,1998), henceforth BH98. An ABS is in fact a standard discounted value asset pricing model derived from mean-variance maximization, extended to the case of *heterogeneous beliefs*.

Agents can either invest in a risk free asset or in a risky asset. The risk free asset is perfectly elastically supplied and pays a fixed rate of return r; the risky asset, for example a large stock or a market index, pays an uncertain dividend. Let p_t be the price per share (ex-dividend) of the risky asset at time t, and let y_t be the stochastic dividend process of the risky asset. Wealth dynamics is given by

$$\mathbf{W}_{t+1} = (1+r)W_t + (\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t)z_t, \tag{3.1}$$

where bold face variables denote random variables at date $t+1$ and z_t denotes the number of shares of the risky asset purchased at date t. Let E_t and V_t denote the conditional expectation and conditional variance based on a publically available information set such as past prices and past dividends. Let E_{ht} and V_{ht} denote the 'beliefs' or forecasts of trader type h about conditional expectation and conditional variance. Agents are assumed to be myopic mean-variance maximizers so that the demand z_{ht} of type h for the risky asset solves

$$Max_{z_t}\{E_{ht}[\mathbf{W}_{t+1}] - \frac{a}{2}V_{ht}[\mathbf{W}_{t+1}]\}, \tag{3.2}$$

where a is the risk aversion parameter. The demand z_{ht} for risky assets by trader type h is then

$$z_{ht} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{aV_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]} = \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{a\sigma^2}, \tag{3.3}$$

where the conditional variance $V_{ht} = \sigma^2$ is assumed to be equal and constant for all types.[5] Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let n_{ht} denote the fraction of type h at date t. Equilibrium of demand and supply yields

$$\sum_{h=1}^{H} n_{ht} \frac{E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t]}{a\sigma^2} = z^s, \tag{3.4}$$

where H is the number of different trader types. BH98 focus on the special case of zero supply of outside shares, i.e. $z^s = 0$, for which the market equilibrium pricing equation becomes[6]

$$(1+r)p_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}]. \tag{3.5}$$

3.1 The EMH benchmark with rational agents

Let us first discuss the EMH-benchmark with rational expectations. In a world where all traders are identical and expectations are *homogeneous* the arbitrage market equilibrium equation (3.5) reduces to

$$(1+r)p_t = E_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1}], \tag{3.6}$$

where E_t denotes the common conditional expectation of all traders at the beginning of period t, based on a publically available information set I_t such as past prices and dividends, i.e. $I_t = \{p_{t-1}, p_{t-2}, ...; y_{t-1}, y_{t-2}, ...\}$. This arbitrage market equilibrium equation (3.6) states that today's price of the risky asset must be equal to the sum of tomorrow's expected price and expected dividend, discounted by the risk free interest rate. It is well known that, using the arbitrage equation (3.6) repeatedly and assuming that the *transversality condition*

$$\lim_{t\to\infty} \frac{E_t[\mathbf{p}_{t+k}]}{(1+r)^k} = 0 \tag{3.7}$$

[5]Gaunersdorfer (2000) investigates the case with time varying beliefs about variances and shows that the results are quite similar to those for constant variance.

[6]Brock (1997) motivates this special case by introducing a risk adjusted dividend $y^{\#}_{t+1} = y_{t+1} - a\sigma^2 z^s$ to obtain the market equilibrium equation (3.5). In general however, the equilibrium equation (3.5) ignores a risk premium $a\sigma^2 z^s$ for investors holding the risky asset. Since dividends and a risk premium affect realized profits and wealth, in general they will affect the fractions n_{ht} of trader type h. The question how exactly the risk premium affects evolutionary competition should be investigated in future work, by taking z^s as a bifurcation parameter. The market equilibrium pricing equation (3.5) in fact represents the case of risk neutral investors.

holds, the price of the risky asset is uniquely determined by

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[\mathbf{y}_{t+k}]}{(1+r)^k}. \tag{3.8}$$

The price p_t^* in (3.8) is called the EMH fundamental rational expectations (RE) price, or the *fundamental price* for short. The fundamental price is completely determined by economic fundamentals and given by the discounted sum of expected future dividends. In general, the properties of the fundamental price p_t^* depend upon the stochastic dividend process y_t. We focus on the case of an IID dividend process y_t, with constant mean $E[y_t] = \bar{y}$. We note however that any other random dividend process y_t may be substituted in what follows[7]. For an IID dividend process y_t with constant mean, the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{3.9}$$

There are two crucial assumptions underlying the derivation of the RE fundamental price. The first is that expectations are *homogeneous*, all traders are *rational* and it is *common knowledge* that all traders are rational. In such an ideal, perfectly rational world the fundamental price can be derived from economic fundamentals. Conditions under which a RE price can be derived can be relaxed, to include for example noise traders or limited heterogeneity of information. In general however, in a world with *heterogeneous* traders having different beliefs or expectations about future prices and dividends, derivation of a RE fundamental price requires perfect knowledge about the beliefs of *all* other traders. In a real market understanding the beliefs and strategies of all other, competing traders is virtually impossible, and therefore in a heterogeneous world derivation of the RE-fundamental price becomes impossible. The second crucial assumption underlying the derivation of the fundamental price is the transversality condition (3.7), requiring that the long run growth rate of prices (and risk adjusted dividends) is smaller than the risk free growth rate r. In fact, in addition to the fundamental solution (3.8) so-called *speculative bubble solutions* of the form

$$p_t = p_t^* + (1+r)^t (p_0 - p_0^*) \tag{3.10}$$

also satisfy the arbitrage equation (3.6). It is important to note that along the speculative bubble solution (3.10), traders have rational expectations. Solutions of the form (3.10) are therefore called *rational bubbles.* These rational bubble solutions are explosive and do *not* satisfy the transversality condition. In a perfectly rational world, traders realize that speculative bubbles cannot last forever and therefore they will never get started and the finite fundamental price p_t^* is uniquely determined. In a perfectly rational world, all traders thus believe that the value of a risky asset equals its fundamental price forever. Changes in asset prices are solely driven by unexpected changes in dividends and random

[7]Brock and Hommes (1997b) for example discuss a non-stationary example, where the dividend process is a geometric random walk .

'news' about economic fundamentals. In a heterogeneous evolutionary world however, the situation will be quite different, and we will see that evolutionary forces may lead to endogenous switching between the fundamental price and the rational self fulfilling bubble solutions.

3.2 Heterogeneous beliefs

In the asset pricing model with heterogeneous beliefs, market equilibrium in (3.5) states that the price p_t of the risky asset equals the discounted value of tomorrow's expected price plus tomorrow's expected dividend, *averaged* over all different trader types. In such a *heterogeneous* world temporary upward or downward bubbles with prices deviating from the fundamental may arise, when the fractions of traders believing in those bubbles is large enough. Once a (temporary) bubble has started, evolutionary forces may reinforce deviations from the benchmark fundamental. We shall now be more precise about traders' expectations (forecasts) about future prices and dividends. It will be convenient to work with

$$x_t = p_t - p_t^*, \tag{3.11}$$

the *deviation* from the fundamental price. We make the following assumptions about the beliefs of trader type h:

B1 $V_{ht}[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t] = V_t[\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - (1+r)p_t] = \sigma^2$, for all h, t.

B2 $E_{ht}[\mathbf{y}_{t+1}] = E_t[\mathbf{y}_{t+1}]$, for all h, t.

B3 All beliefs $E_{ht}[\mathbf{p}_{t+1}]$ are of the form

$$E_{ht}[\mathbf{p}_{t+1}] = E_t[\mathbf{p}^*_{t+1}] + f_h(x_{t-1}, ..., x_{t-L}), \qquad \text{for all} \;\; h, t. \tag{3.12}$$

According to assumption B1 beliefs about conditional variance are equal and constant for all types, as discussed above already. Assumption B2 states that expectations about future dividends $\mathbf{y}_{t+1}$ are the same for all trader types and equal to the conditional expectation. All traders are thus able to derive the fundamental price p_t^* in (3.8) that would prevail in a perfectly rational world. According to assumption B3, traders nevertheless believe that in a heterogeneous world prices may *deviate* from their fundamental value p_t^* by some function f_h depending upon past deviations from the fundamental. Each forecasting rule f_h represents the *model of the market* according to which type h believes that prices will deviate from the commonly shared fundamental price. For example, a forecasting strategy f_h may correspond to a technical trading rule, based upon short run or long run moving averages, of the type used in real markets.

Strictly speaking (3.12) is not a technical trading rule, because it uses the fundamental price in its forecast. Including price forecasting rules depending upon past prices only, not using any information about fundamentals, yields similar results. We will use the short hand notation

$$f_{ht} = f_h(x_{t-1}, ..., x_{t-L}) \tag{3.13}$$

for the forecasting strategy employed by trader type h. An important and convenient consequence of the assumptions B1-B3 concerning traders' beliefs is that the heterogeneous agent market equilibrium equation (3.5) can be reformulated in deviations from the benchmark fundamental. In particular substituting the price forecast (3.12) in the market equilibrium equation (3.5) and using the facts that the fundamental price p_t^* satisfies $(1+r)p_t^* = E_t[p_{t+1}^* + y_{t+1}]$ and the price $p_t = x_t + p_t^*$ yields the equilibrium equation in deviations from the fundamental:

$$(1+r)x_t = \sum_{h=1}^{H} n_{ht} E_{ht}[\mathbf{x}_{t+1}] \equiv \sum_{h=1}^{H} n_{ht} f_{ht}, \tag{3.14}$$

with $f_{ht} = f_h(x_{t-1}, ..., x_{t-L})$. An important reason for our model formulation in terms of deviations from a benchmark fundamental is that in this general setup, the benchmark rational expectations asset pricing model is *nested* as a special case, with all forecasting strategies $f_h \equiv 0$. In this way, the adaptive belief systems can be used in empirical and experimental testing whether asset prices deviate significantly from anyone's favorite benchmark fundamental.

3.3 Evolutionary dynamics

The evolutionary part of the model describes how beliefs are updated over time, that is, how the fractions n_{ht} of trader types in the market equilibrium equation (3.14) evolve over time. Fractions are updated according to an *evolutionary fitness* or performance measure. The fitness measures of all trading strategies are publically available, but subject to noise. Fitness is derived from a random utility model and given by

$$\tilde{U}_{ht} = U_{ht} + \epsilon_{ht}, \tag{3.15}$$

where U_{ht} is the *deterministic part* of the fitness measure and ϵ_{ht} represents noise. Assuming that the noise ϵ_{ht} is IID across $h = 1, ...H$ drawn from a double exponential distribution, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy h is given by the well known *discrete choice model* or 'Gibbs' probabilities[8]

$$n_{ht} = \frac{exp(\beta U_{h,t-1})}{Z_{t-1}}, \qquad Z_{t-1} = \sum_{h=1}^{H} exp(\beta U_{h,t-1}), \tag{3.16}$$

where Z_{t-1} is a normalization factor in order for the fractions n_{ht} to add up to 1. The crucial feature of (3.16) is that the higher the fitness of trading strategy h, the more traders will select strategy h. The parameter β in (3.16) is called the *intensity of choice*, measuring how sensitive the mass of traders is to selecting the optimal prediction strategy. The intensity of choice β is inversely related to the variance of the noise terms ϵ_{ht}. The extreme case $\beta = 0$ corresponds to the case of infinite variance noise, so that differences in fitness cannot be observed and all fractions (3.16) will be fixed over time and equal to $1/H$. The other extreme case $\beta = +\infty$ corresponds to the case without noise, so that the deterministic part of the fitness can be observed perfectly and in each period, *all* traders choose the optimal forecast. An increase in the intensity of choice β represents

[8]See Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993) for extensive discussion of discrete choice models and their applications in economics.

an increase in the degree of rationality w.r.t. evolutionary selection of trading strategies. The timing of the coupling between the market equilibrium equation (3.5) or (3.14) and the evolutionary selection of strategies (3.16) is crucial. The market equilibrium price p_t in (3.5) depends upon the fractions n_{ht}. The notation in (3.16) stresses the fact that these fractions n_{ht} depend upon *past* fitness $U_{h,t-1}$, which in turn depend upon past prices p_{t-1} and dividends y_{t-1} in periods $t-1$ and further in the past as will be seen below. After the equilibrium price p_t has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions $n_{h,t+1}$. These new fractions $n_{h,t+1}$ will then determine a new equilibrium price p_{t+1}, etc.. In the ABS, market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

A natural candidate for evolutionary fitness is accumulated *realized profits*, as given by

$$U_{ht} = (p_t + y_t - Rp_{t-1})\frac{E_{h,t-1}[\mathbf{p}_t + \mathbf{y}_t - Rp_{t-1}]}{a\sigma^2} - C_h + wU_{h,t-1} \tag{3.17}$$

where $R = 1 + r$ is the gross risk free rate of return, C_h represents an average per period *cost* of obtaining forecasting strategy h and $0 \leq w \leq 1$ is a *memory* parameter measuring how fast past realized fitness is discounted for strategy selection. The cost C_h for obtaining forecasting strategy h will be zero for simple, habitual rule of thumb forecasting rules, but may be positive for more sophisticated forecasting strategies. For example, costs for forecasting strategies based upon economic fundamentals may be positive representing investors' effort for information gathering and market research, whereas costs for technical trading rules may be (close to) zero. The first term in (3.17) represents last period's realized profit of type h given by the realized excess return of the risky asset over the risk free asset times the demand for the risky asset by traders of type h. In the extreme case with no memory, i.e. $w = 0$, fitness U_{ht} equals net realized profit in the previous period, whereas in the other extreme case with infinite memory, i.e. $w = 1$, fitness U_{ht} equals total wealth as given by accumulated realized profits over the entire past. In the intermediate case, the weight given to past realized profits decreases exponentially with time.

Fitness can now be rewritten in deviations from the fundamental as

$$U_{ht} = (x_t - Rx_{t-1})(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}) - C_h + wU_{h,t-1}. \tag{3.18}$$

3.4 Forecasting rules

To complete the model we have to specify the class of forecasting rules. Brock and Hommes (1998) have investigated evolutionary competition between *simple linear* forecasting rules with only *one lag*, i.e.

$$f_{ht} = g_h x_{t-1} + b_h. \tag{3.19}$$

It can be argued that, for a forecasting rule to have any impact in real markets, it has to be simple. For a complicated forecasting rule it seems unlikely that enough traders will coordinate on that particular rule so that it affects market equilibrium prices. Although the linear forecasting rule (3.19) is extremely simple, it represent a number of important cases. For example, when both the trend parameter and the bias parameter $g_h = b_h = 0$ the rule reduces to the forecast of *fundamentalists*, i.e.

$$f_{ht} \equiv 0, \tag{3.20}$$

believing that the market price will be equal to the fundamental price p^*, or equivalently that the deviation x from the fundamental will be 0. Other important cases covered by the linear forecasting rule (3.19) are the pure *trend followers*

$$f_{ht} = g_h x_{t-1}, \qquad g_h > 0, \tag{3.21}$$

and the pure *biased belief*

$$f_{ht} = b_h. \tag{3.22}$$

Notice that the simple pure bias forecast (3.22) represents *any* positively or negatively biased forecast of next periods price that traders might have. Instead of these extremely simple habitual rule of thumb forecasting rules, some economists might prefer the rational, *perfect foresight* forecasting rule

$$f_{ht} = x_{t+1}. \tag{3.23}$$

We emphasize however, that the perfect foresight forecasting rule (3.23) assumes perfect knowledge of the heterogeneous market equilibrium equation (3.5), and in particular perfect knowledge about the beliefs of *all* other traders. Although the case with perfect foresight certainly has theoretical appeal, its practical relevance in a complex heterogeneous world should not be overstated since this underlying assumption seems highly unrealistic.[9]

3.5 Simple examples

This section presents simple, but typical examples of ABS, with two, three resp. four competing *linear* forecasting rules (3.19), where the parameter g_h represents a perceived *trend* in prices and the parameter b_h represents a perceived upward or downward *bias*[10].

[9] In the cobweb model with rational versus naive agents of the previous section, the implicitly defined heterogeneous market equilibrium equation (2.8) remains tractable and can be solved explicitly for the unique market equilibrium price (2.9). In general however, with one type of agents having rational expectations or perfect foresight a *temporary equilibrium* model with heterogeneous beliefs such as the asset pricing market equilibrium equation in (3.24) becomes an *implicitly defined* dynamical system with x_t on the LHS and x_{t+1} and e.g. x_{t-1} on the RHS. Typically such implicitly defined evolutionary systems cannot be solved explicitely and often they are not even well-defined.

[10] Brock, Hommes and Wagener (2004) recently introduced the notion of *Large Type Limit (LTL)* to study the model with a large number of different belief types.

The ABS then becomes (in deviations from the fundamental):

$$(1+r)x_t = \sum_{h=1}^{H} n_{ht}(g_h x_{t-1} + b_h) + \epsilon_t \tag{3.24}$$

$$n_{h,t} = \frac{exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} exp(\beta U_{h,t-1})} \tag{3.25}$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2})(\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}) + wU_{h,t-2} - C_h, \tag{3.26}$$

where ϵ_t is a small noise term representing uncertainty about economic fundamentals, e.g. random outside supply of the risky asset. In order to keep the analysis of the dynamical behavior tractable, BH98 have mainly focused on the case where the memory parameter $w = 0$, so that evolutionary fitness is given by last period's realized profit. Here, we review examples with two, three and four forecasting rules stating the most important bifurcation results without; for proofs, the interested reader is referred to Brock and Hommes (1998). A common feature of all examples is that, as the intensity of choice to switch prediction or trading strategies increases, the fundamental steady state becomes locally unstable and non-fundamental steady states, cycles or even chaos arise.

Costly fundamentalists versus trend followers The simplest example of an ABS only has *two* trader types, with forecasting rules

$$f_{1t} = 0 \qquad \text{fundamentalists} \tag{3.27}$$

$$f_{2t} = gx_{t-1}, \qquad g > 0, \qquad \text{trend followers} \tag{3.28}$$

that is, the first type are fundamentalists predicting that the price will equal its fundamental value (or equivalently that the deviation will be zero) and the second type are pure trend followers predicting that prices will rise (or fall) by a constant rate. In this example, the fundamentalists have to pay a fixed per period positive cost C_1 for information gathering; in all other examples discussed below information costs will be set to zero for all trader types.

For small values of the trend parameter, $0 \leq g < 1+r$, the fundamental steady state is always stable. Only for sufficiently high trend parameters, $g > 1+r$, trend followers can *destabilize* the system. For trend parameter, $1+r < g < (1+r)^2$ the dynamic behavior of the evolutionary system depends upon the intensity of choice to switch between the two trading strategies[11]. For low values of the intensity of choice, the fundamental steady state will be stable. As the intensity of choice increases, the fundamental steady state becomes unstable due to a *pitchfork bifurcation* in which two additional non-fundamental steady states $-x^* < 0 < x^*$ are created. The evolutionary ABS may converge to the positive non-fundamental steady state, to the negative non-fundamental steady state, or,

[11] For $g > (1+r)^2$ the system may become *globally unstable* and prices may diverge to infinity. Imposing a stabilizing force, for example by assuming that trend followers condition their rule upon deviations from the fundamental e.g. as in Gaunersdorfer et al. (2000), leads to a bounded system again, possibly with cycles or even chaotic fluctuations.

in the presence of noise, switch back and forth between the high and the low steady state. As the intensity of choice increases further, the two non-fundamental steady states also become unstable due to a Hopf-bifurcation, and limit cycles or even strange attractors can arise around each of the (unstable) non-fundamental steady states, as illustrated in Figure 4. The evolutionary ABS may cycle around the positive non-fundamental steady state, cycle around the negative non-fundamental steady state or, driven by the noise, switch back and forth between cycles around the high and the low steady state.

This example shows that, in the presence of information costs and with zero memory, when the intensity of choice in evolutionary switching is high fundamentalists can *not* drive out pure trend followers and persistent deviations from the fundamental price may occur. Brock and Hommes (1999) show that this result also holds when the memory in the fitness measure increases. In fact, an increase in the memory of the evolutionary fitness leads to bifurcation routes very similar to bifurcation routes due to an increase in the intensity of choice.

Figure 5 illlustrates that the asset pricing model with costly fundemtalists versus cheap trend following exhibits a rational route to randomness, i.e. a bifurcation route to chaos as the intensity of choice to switch strategies increases.

Fundamentalists versus opposite biases In the cobweb model with rational versus naive expectations in the previous section as well as in the two type asset pricing model with fundamentalists versus trend followers rational routes to randomness occur due to information costs for the sophisticated forecasting strategy. The second example of an ABS is an example with *three* trader types without any information costs. The forecasting rules are

$$f_{1t} = 0 \qquad \text{fundamentalists} \tag{3.29}$$

$$f_{2t} = b \qquad b > 0, \qquad \text{positive bias (optimists)} \tag{3.30}$$

$$f_{3t} = -b \qquad -b < 0, \qquad \text{negative bias (pessimists).} \tag{3.31}$$

The first type are fundamentalists again, but as stated above there will be *no* information costs for fundamentalists or other types. The second and third types have a purely *biased* belief, expecting a constant price above respectively below the fundamental price.

For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases the fundamental steady becomes unstable due to a *Hopf* bifurcation and the dynamics of the ABS is characterized by cycles around the unstable steady state. This example shows that, even when there are *no* information costs for fundamentalists, they cannot drive out other trader types with opposite biased beliefs. In the evolutionary ABS with high intensity of choice, fundamentalists and biased traders co-exist with fractions varying over time and prices cycling around the unstable fundamental steady state. Moreover, Brock and Hommes (1998, p.1259, lemma 9) show that as the intensity of choice tends to infinity the ABS converges to a (globally) stable cycle of period 4. Average profits along this 4-cycle are equal for all three trader types. Hence, if the initial wealth is equal for all three types, then in this evolutionary system in the long run accumulated wealth will be equal for all three types. This example suggests

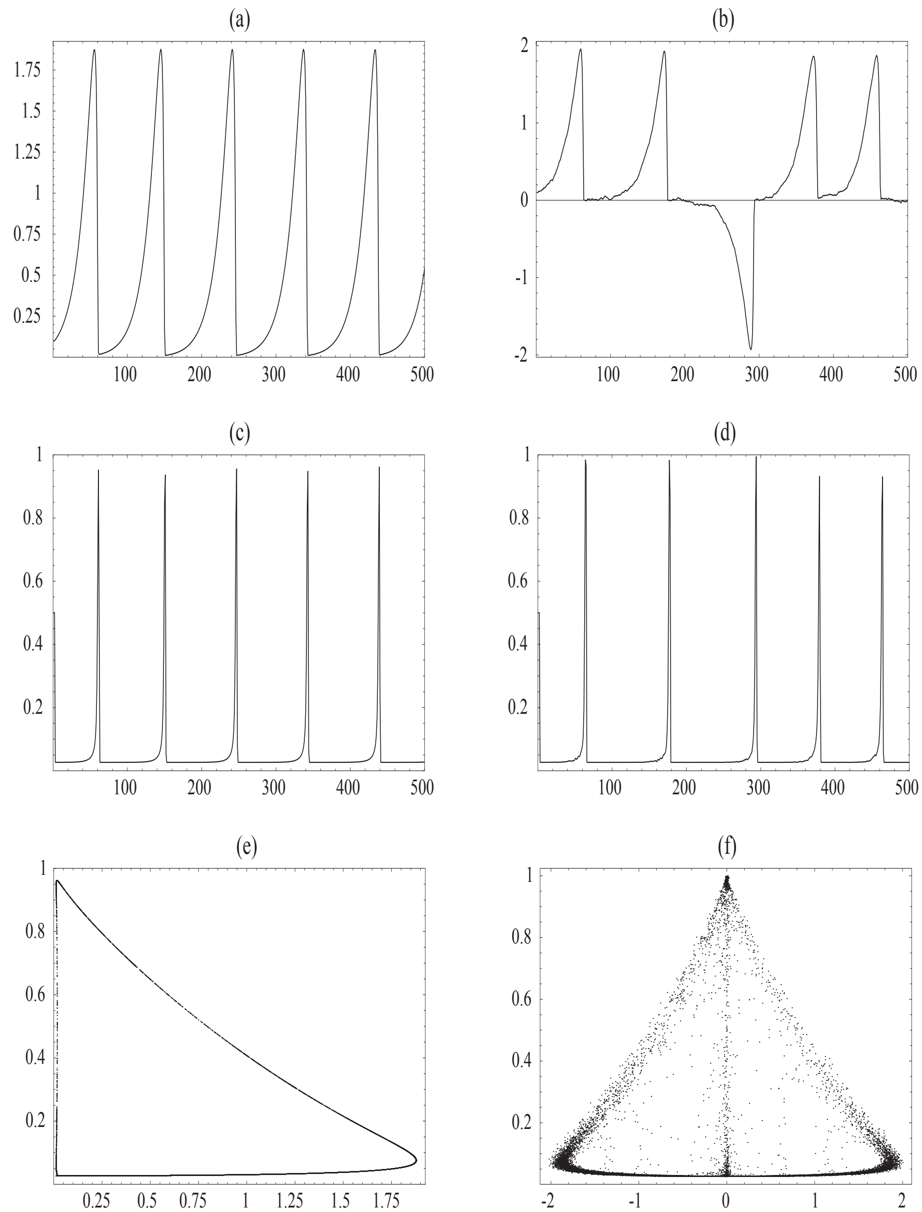


Figure 4. Time series of prices (deviations from the fundamental steady state) and fractions and attractors in the phase space for 2-type model with costly fundamentalists versus trend followers. The left panel shows chaotic dynamics without noise and the right panel illustrates the model buffeted with small noise (SD=0.01 of noise term ϵ_t in (3.24)). Without noise (left panel) the system settles down to the attractor with prices above the fundamental value. In the presence of (small) noise, the system switches back and forth between the two co-existing attractors with prices jumping between above and below fundamental values. Parameters are: $\beta = 3.6$, $g = 1.2$, $R = 1.1$ and $C = 1$.

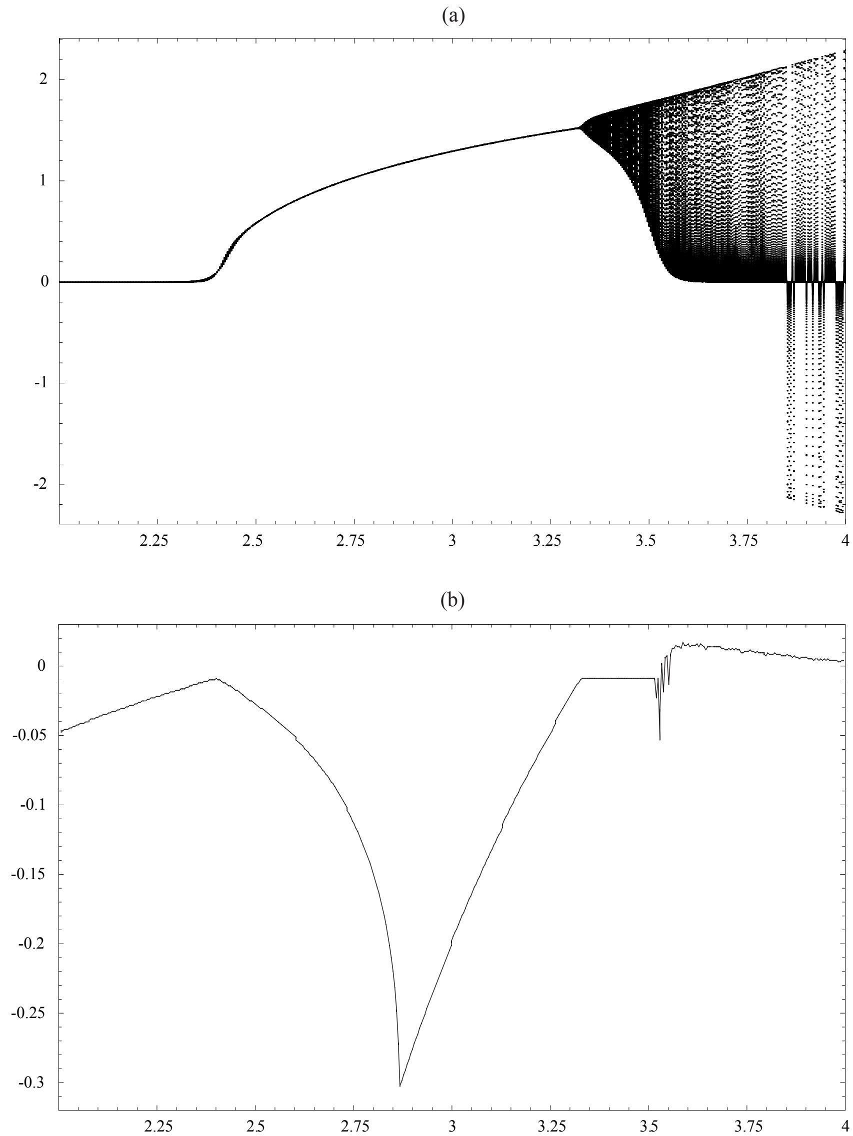
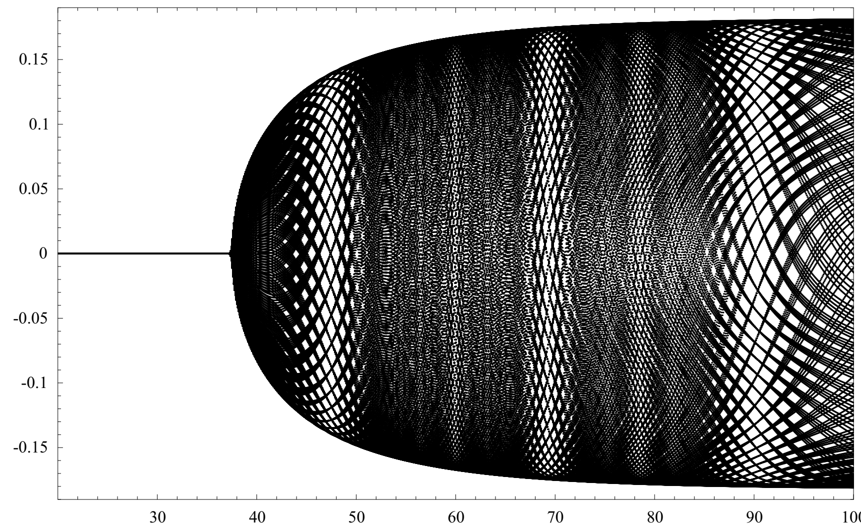


Figure 5. Bifurcation diagram (top) and largest Lyapunov exponent plot (bottom) for 2-type model with costly fundamentalist versus trend followers. In both plots the model is buffeted with very small noise ($SD = 10^{-6}$ for the noise term ϵ_t in (3.24)), to avoid that for large β-values the system gets stuck in the locally unstable steady state. Parameters are: $g = 1.2$, $R = 1.1$, $C = 1$ and $2 \leq \beta \leq 4$. A pitchfork bifurcation of the fundamental steady state, in which two stable non-fundamental steady states are created, occurs for $\beta \approx 2.37$. The non-fundamental steady states become unstable due to a Hopf-bifurcation for $\beta \approx 3.33$, and (quasi-)periodic dynamics arises. For large values of β the largest Lyapunov exponent becomes positive indicating chaotic price dynamics.

(a)



(b)

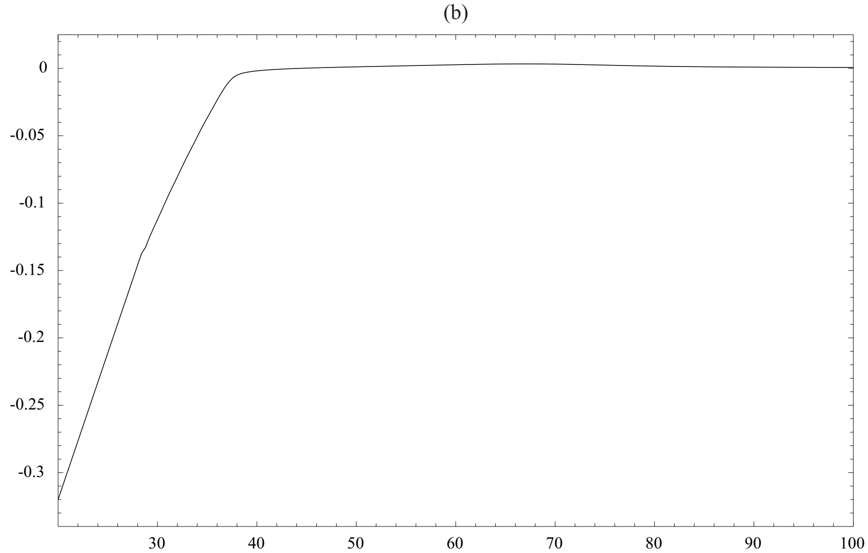


Figure 6. Bifurcation diagram and largest Lyapunov exponent plot for 3-type model. In both plots the model is buffeted with very small noise ($SD = 10^{-6}$ for the noise term ϵ_t in (3.24)), to avoid that for large β-values the system gets stuck in the locally unstable steady state. Belief parameters are: $g_1 = 0$, $b_1 = 0$; $g_2 = 0$, $b_2 = 0.2$ and $g_3 = 0$, $b_3 = -0.2$; other parameters are $r = 0.1$, $20 \leq \beta \leq 100$, $w = 0$ and $C_h = 0$ for all $1 \leq h \leq 3$ The 3-type model with fundamentalists versus opposite biases exhibits a Hopf bifurcation for $\beta \approx 37.4$. For large values of β periodic and quasi-periodic dynamics occurs, but chaos with positive largest Lyapunov exponent does not arise.

that the Friedman argument that smart-fundamental traders will automatically drive out simple habitual rule of speculative traders should be considered with care.

In this example with three trader types, cycles can occur but chaos does not arise. This is illustrated in Figure 6 showing a bifurcation diagram and a plot of the largest Lyapunov exponent. In the three type example with fundamentalists versus opposite biases, even in the presence of (small) noise, price fluctuations will be fairly regular and therefore returns will be predictable. This predictability will disappear however when we combine trend following with biased beliefs.

3.6 Fundamentalists versus trend and bias

The third example of an ABS is an example with *four* trader types, with linear forecasting rules (3.19) with parameters $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$ and $g_4 = 1 + r = 1.01$, $b_4 = 0$. The first type are fundamentalists again, without information costs, and the other three types follow a simple linear forecasting rule with one lag. The dynamical behaviour is illustrated in Figures 7 and 8.

For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases, as in the previous three type example, the fundamental steady becomes unstable due to a *Hopf* bifurcation and a stable invariant circle around the unstable fundamental steady state arises, with periodic or quasi-periodic fluctuations. As the intensity of choice further increases, the invariant circle breaks into a strange attractor with chaotic fluctuations. In the evolutionary ABS fundamentalists and chartists co-exist with fractions varying over time and prices moving chaotically around the unstable fundamental steady state. Figure 8 shows that in this 4-type example with fundamentalists versus trend followers and biased beliefs a rational route to randomness occurs, with positive largest Lyapunov exponents for large values of β.

The (noisy) chaotic price fluctuations are characterized by an irregular switching between phases of close-to-the-EMH-fundamental-price fluctuations, phases of 'optimism' with prices following an upward trend, and phases of 'pessimism', with (small) sudden market crashes, as illustrated in Figure 7. Recall from subsection 3.1 that the asset pricing model with homogeneous beliefs, in addition to the benchmark fundamental price, has rational bubble solutions as in (3.10). One might say that in the ABS prices are characterized by an evolutionary switching between the fundamental value and these temporary speculative bubbles. In the purely deterministic chaotic case, the timing and the direction of the temporary bubbles seem hard to predict. However, once a bubble has started, in the deterministic case, the length of the bubble seems to be predictable in most of the cases. In the presence of noise, as in figure 7 (top right), the timing, the direction and the length of the bubble all seem hard to predict.

In order to investigate this (un)predictability issue further, we employ a so called *nearest neighbor forecasting method* to predict the returns, at lags 1 to 20 for the purely chaotic as well as for several noisy chaotic time series, as illustrated in figure 9.[12] Nearest neighbor forecasting looks for past patterns close to the most recent pattern, and then yields as the prediction the average value following all nearby past patterns. It follows essentially from Takens' embedding theorem that this method yields good forecasts for

[12]I would like to thank Sebastiano Manzan for providing this figure.

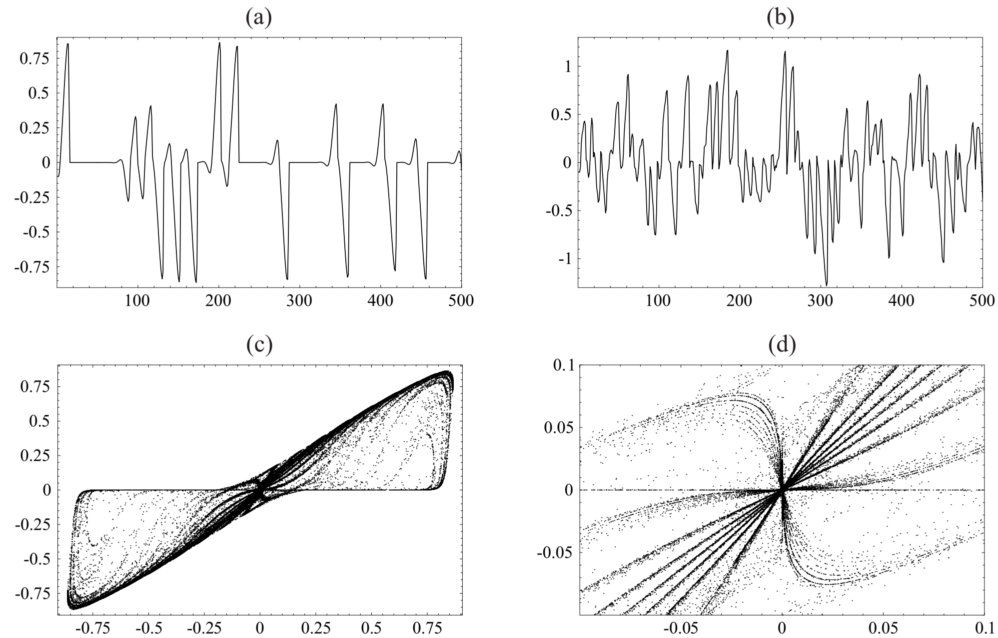


Figure 7. Chaotic (top left) and noisy chaotic (top right) time series of asset prices in adaptive belief system with four trader types. Strange attractor (bottom left) and enlargement of strange attractor (bottom right). Belief parameters are: $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$ and $g_4 = 1 + r = 1.01$, $b_4 = 0$; other parameters are $r = 0.01$, $\beta = 90.5$, $w = 0$ and $C_h = 0$ for all $1 \leq h \leq 4$.

deterministic chaotic systems[13]. Figure 9 shows that as the noise level increases, the forecasting performance of the nearest neighbor method quickly deteriorates. Hence, in our simple nonlinear evolutionary ABS with noise it is hard to make good forecasts of future returns. Our simple nonlinear ABS with small noise thus captures some of the intrinsic unpredictability of asset returns also present in real markets.

This 4-type example shows that when memory is zero, even when there are *no* information costs for fundamentalists, they cannot drive out other simple trader types and fail to stabilize price fluctuations towards its fundamental value. As in the three type case, the opposite biases create cyclic behavior but apparently the additional trend parameters turn these cycles into unpredictable chaotic fluctuations.

[13]See Kantz and Schreiber (1997) for a recent and extensive treatment of nonlinear time series analysis and forecasting techniques.

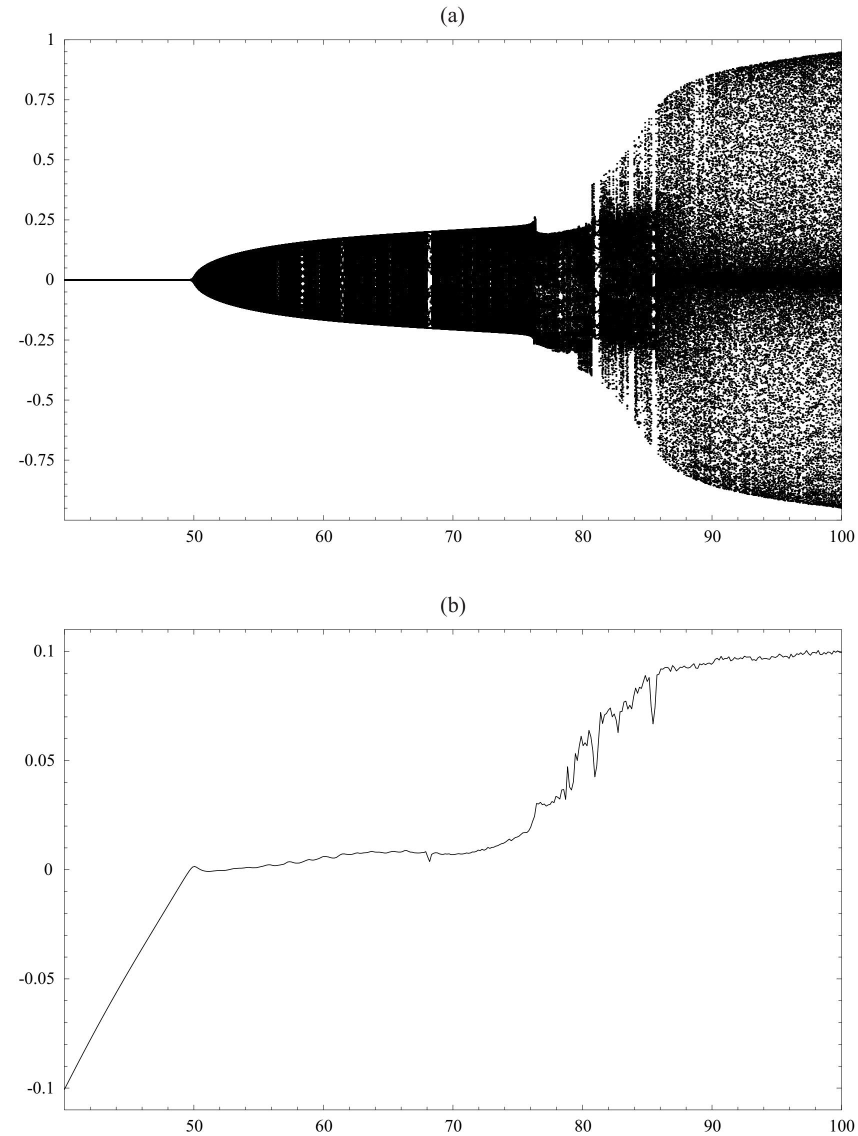


Figure 8. Bifurcation diagram and largest Lyapunov exponent plot for 4-type model. In both plots the model is buffeted with very small noise ($SD = 10^{-6}$ for noise term ϵ_t in (3.24)), to avoid that for large β-values the system gets stuck in the locally unstable steady state. Belief parameters are: $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$ and $g_4 = 1 + r = 1.01$, $b_4 = 0$; other parameters are $r = 0.01$, $\beta = 90.5$, $w = 0$ and $C_h = 0$ for all $1 \leq h \leq 4$. The 4-type model with fundamentalists versus trend followers and biased beliefs exhibits a Hopf bifurcation for $\beta = 50$. A rational route to randomness occurs, with positive largest Lyapunov exponents, when the intensity of choice becomes large.

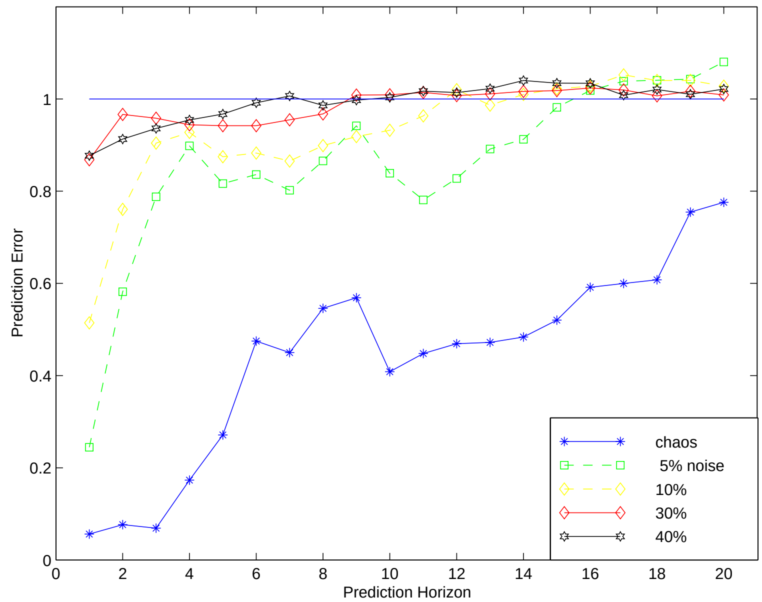


Figure 9. Forecasting errors for nearest neighbor method applied to chaotic returns series as well as noisy chaotic returns series, for different noise levels, in ABS with four trader types. All returns series have close to zero autocorrelations at all lags. The benchmark case of prediction by the mean 0 is represented by the horizontal line at the normalized prediction error 1. Nearest neighbor forecasting applied to the purely deterministic chaotic series leads to much smaller forecasting errors (lowest graph). A noise level of say 10% means that the ratio of the variance of the noise term ϵ_t and the variance of the deterministic price series is 1/10. As the noise level slowly increases, the graphs are shifted upwards. Small dynamic noise thus quickly deteriorates forecasting performance.

4 Concluding remarks

We have reviewed two simple heterogeneous agent models. In both the cobweb model and the asset pricing model we have focussed on the case of linear demand and supply curves. Heterogeneity and evolutionary updating of trading strategies creates an important *nonlinearity* in the model. In particular, when agents are highly sensitive to differences in evolutionary fitness the nonlinear evolutionary switching mechanism causes complicated, unpredictable price fluctuations. In the cobweb model, due to costs for information gathering, perfectly rational agents can not drive out boundely rational agents using simple habitual rule of thumb trading strategies. In the asset pricing model, even without information costs for fundamentalist traders, prices need not converge to the RE fundamental benchmark. Our evolutionary framework thus explains excess volatility and persistent deviations from the fundamental benchmark. More work is needed to investigate how general these phenomena are, for example with respect to introducing more memory in the evolutionary fitness measures.

Is a significant part of changes in stock prices driven by 'Keynesian animal spirits'? For many decades already, this question has lead to heavy debates among economic academics as well as financial practitioners. In the evolutionary adaptive belief systems discussed here, price changes are explained by a *combination* of economic fundamentals and 'market psychology'. Negative economic 'news' (e.g. on inflation or interest rates) may act as a trigger event for a decline in stock prices, which may become reinforced by investors sentiment and evolutionary forces. Price movements are driven by an interaction of fundamentalism and chartism, the two most important trading strategies in financial practice.

Our evolutionary ABS may be seen as, what Sargent (1999) calls an *approximate rational expectations equilibrium.* Traders are boundedly rational and use relatively simple strategies. The class of trading rules is disciplined by evolutionary forces based upon realized profits or wealth. A convenient feature of our theoretical setup is that the benchmark rational expectations model is *nested* as a special case. This feature gives the model flexibility with respect to experimental and empirical testing. It is worthwhile noting that Baak (1999) and Chavas (2000) have run empirical tests for heterogeneity in expectations in agricultural data and indeed find evidence for the presence of boundedly rational traders in the cattle market. It may seem even more natural that heterogeneity and evolutionary switching between different trading strategies play an important role in financial markets. Understanding the role of market psychology seems to be a crucial part of understanding the huge changes in stock prices observed so frequently these days. But much more insight into 'financial psychology' is needed, before 'market sentiment' based policy advice can be given. Theoretical analysis of stylized evolutionary adaptive market systems, as discussed here, and its empirical and experimental testing may contribute in providing such insight.

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