On the probability of breakdown in participation games

Pim Heijnen*

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Abstract

In this paper I analyze a participation game i.e. a public good game where contributions to the public good are binary (people either participate or not participate). Although variants of this game have been studied extensively, most previous work takes the benefit of provision of the public good to be independent of the number of players that contribute and show that the probability of breakdown, i.e. the probability that no one participates, is increasing in group size. Here this assumption is dropped. I show when the probability of breakdown is decreasing in group size and also present sufficient conditions under which the probability of breakdown is increasing in group size. Moreover I show that for large groups this probability is non-negligible and exceeding e^{-1} in the limit and that the expected number of participants is less than one. Also two economic examples, concerning R&D and debt overhang, are discussed.

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1 Introduction

Kitty Genovese was murdered in New York City in 1964 while 38 neighbors looked on without calling the police. This story is consistent with evidence from both social psychology and game theory that suggests that people are less likely to help someone in need if they are surrounded by more other people (Dixit and Skeath, 1999, pp. 388–392; other references are Rasmusen, 2001, pp.77–79 and Harrington, 2001). At least from the perspective of game theory, this seems puzzling. Although we expect that freeriding decreases the probability that each individual acts (under the assumption that seeking help is costly and people prefer rescue), it is by no means certain that the probability, that the group as a whole fails to act (i.e. what I call the probability of breakdown), also increases. Assuming constant cost and benefit of rescuing, the probability of breakdown always increases: Nash equilibrium logic ensures that the probability, that any of the *other* people fails to act, stays constant. A direct consequence is that the probability of breakdown is increasing in group size. In general, however, this probability will only decline, if the freeriding effect is strong enough. The question then is, if

*CeNDEF, Department of Quantitive Economics, University of Amsterdam, Roeterstraat 11, 1018WB, Amsterdam, the Netherlands, e-mail: `p.heijnen@uva.nl`. The proof of Proposition 3 owes much to the useful comments and suggestions of Bert Schoonbeek. The author would like to thank Peter Kooreman, Linda Toolsema, Marco Haan, Allard van der Made, José-Luis Moraga-González, Florian Wagener, an anonymous referee and the editor for helpful comments and discussions. Financial support by the Netherlands Organization for Scientific Research (NWO) is gratefully acknowledged.

it is possible for large groups to have a lower probability of breakdown and when this might occur.

To answer this question we model this as a general public good game with binary contributions. This means that contributing to the public good can be interpreted as participating in a program or joining a group to do a certain amount of volunteering. I refer to this as a participation game (cf. Anderson and Engers, 2005). I identify which payoff structures[1] can be considered as characteristic for participation games and then proceed to analyze these games. The contributions of this paper are the following. First, I characterize all Nash-equilibria. While there are many Nash-equilibria, the paper focuses on the symmetric strategy Nash-equilibrium. Second, I show that there are public good games which have the property that the probability of breakdown decreases as the group becomes larger. Hence, large public good games are not neccesarily more inefficient than smaller games. In particular, sufficiently strong positive externalities seem to be required for this to occur. However, finally, I show that there are mild sufficient conditions that guarantee the probability of breakdown to increase. This implies that in most situations, the public good problem worsens as the group size increases.

Anderson and Engers (2005) present a different generalization: they examine the effect of different thresholds and of positive and negative externalities of participating. A threshold is the minimal number of contributions that are needed for the benefits of the public good (per contributor) to exceed the cost of participation. By positive and negative externalities I mean that if more of the other people participate, then the incentive to participate respectively increases and decreases. This is basically a monotonicity requirements on the payoff function. They examine the different kind of equilibria that can occur and some comparative statics such as the effects of increasing the payoff of participating. I impose no monotonicity restrictions on the payoff function, but only examine a threshold of one. It will be argued that in some economic application the monotonicity requirement is unreasonable.

The defining characteristic of participation games is that it is optimal to participate if and only if no one else participates. The following discussion is an incomplete list of articles that employ participation-like games and it serves to show the breadth of possible applications and extensions. Palfrey and Rosenthal (1983) offer one of the first systematic analyses of binary public good games. They consider the following variant: there are two groups of individuals. The group with the most individual contributions wins. A real-life example would be a presidential election. Given the behavior of the other group, there is an incentive to free ride. Haan and Kooreman (2003) show that in this game even if one group has an arbitrarily large potential majority (if everyone in their group will vote), it will not always win. This is due to the fact there is a subgame where members of the majority group have to decide whether they are going to contribute to the public good (i.e. voting to help their group win). Mukhopadhyay (2003) extends the basic model with incomplete information. Mukhopadhyay examines the Condorcet jury theorem which states that a larger jury makes a more accurate verdict. Mukhopadhyay argues that if paying attention in court is costly (i.e. boring) and every juror wants to make an accurate verdict (for which they have to pay attention), then in a large jury there is an increasingly large probability that no one will pay attention. The Condorcet jury theorem does not hold. This conclusion is not altered when the jurors receive a signal, whether the defendant is likely to be guilty, prior to the court case.Although variants of this game have been studied extensively, previous work takes the benefit of provision of

[1]By payoff structure I mean the utility a player receives depending on how many other players participate and whether the player himself participates.

the public good to be independent of the number of players that contribute. While the model presented here cannot be seen as a direct generalization of most of the papers mentioned in the previous paragraphs, at heart the paradoxical results they all report are due to the presence of a participation (sub)game.

The paper is organized as follows: in Section 2 I introduce the participation game, derive all its equilibria and show that the probability of participating for each player is decreasing in group size. Section 3 discusses the probability of breakdown. In this Section I show when this probability is increasing and give sufficient conditions. Section 4 gives two economic examples concerning R&D and debt overhang (based on Kaneko and Prokop, 1993). Section 5 concludes.

2 The participation game

There are $n+1$ players.[2] Denote the group of players with $N = \{1, \ldots, n+1\}$. Each player $i \in N$ can choose to participate ($a_i = 1$) or not participate ($a_i = 0$). For player i, define $A_i = \sum_{j \neq i} a_j \in \{0, \ldots, n\}$ as the total number of other players who participate. The payoff of player i depends only on a_i and A_i; players do not have preferences about *who* participates. The game is anonymous. Denote the payoff function by $\pi : \{0,1\} \times \{0, \ldots, n\} \to \mathbb{R}$.

Assumption 1. *Participation is costly if at least one other player participates: for each player i, for all $A_i \in \{1, \ldots, n\}$ we have $\pi(1, A_i) < \pi(0, A_i)$.*

Assumption 2. *If nobody participates, the best-response is to participate: for each player i we have $\pi(0,0) < \pi(1,0)$.*

I refer to the game defined above as the participation game. As an example of a participation game, consider the model presented in Harrington (2001), where a group of $n+1$ people stand around a lake and someone is drowning in this lake. Each person has the opportunity to rescue this person. Each person receives a payoff of b if the drowner is rescued, but the person who actually rescues the person incurs a cost c. The payoff function for each player i is:

$$\begin{aligned} \pi(0,0) &= 0, \\ \pi(1,0) &= b - c, \\ \pi(0, A_i) &= b, \qquad \text{for all } A_i > 0, \\ \pi(1, A_i) &= b - c, \quad \text{for all } A_i > 0, \end{aligned}$$

where $b, c > 0$ and $b > c$. I will refer to this game as Harrington's model.

Proceeding with the general participation game, it is easy to see that there is no symmetric equilibrium in pure strategies in this game. To see this, suppose that everyone else participates. Then by Assumption 1 it is optimal to not participate. Suppose that no one else participates. Then by Assumption 2 it is optimal to participate. Hence neither of the pure strategies is a best response against itself. However, there is a symmetric mixed equilibrium that can be derived as follows. Suppose that n people participate with probability p. The $(n+1)$-th

[2] In a mixed equilibrium the players are indifferent given the actions of the other players. It is convenient to set the number of other players equal to n.

person should be indifferent between participating and not participating. The expected payoff of participating is:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \pi(1,k), \tag{1}$$

and the expected payoff of not participating is:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \pi(0,k). \tag{2}$$

In equilibrium, these two payoffs should be equal:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \theta_k = 0, \tag{3}$$

where $\theta_k \equiv \pi(0,k) - \pi(1,k)$. This parameter signifies the incentive to not participate if k people are already participating. From Assumptions 1 and 2, it follows that $\theta_0 < 0$ and $\theta_k > 0$ (for $k > 0$). If everyone participates with probability equal to a root of (3) in the unit interval, then this would be a symmetric Nash equilibrium. Define the function

$$f_m(p) = \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} \theta_k, \tag{4}$$

where $0 < m \leq n$. Let p_m denote a root of $f_m(\cdot)$. Before we continue the examination of the properties of the symmetric equilibrium, I first present the following proposition that characterizes all equilibria.

Proposition 1. *There are $2^{n+1} - 1$ equilibria. In each of these equilibria, each member of a set $M \subseteq N$ (with the exception of the empty set) participates with probability $p > 0$ and each member of $N \backslash M$ does not participate. Moreover:*

1. *If $\#M = 1$, then $p = 1$ (equilibrium in pure strategies).*
2. *If $\#M > 1$, then p is the unique root of $f_{\#M-1}(\cdot)$, $0 < p < 1$.*

Outline of the proof. First, I show that there are no symmetric pure strategy equilibria, but there are $(n+1)$ asymmetric pure strategy equilibria, in which one player participates and the other players do not. I proceed by showing that there exists a symmetric mixed-strategy equilibrium. Next it is shown that if a group of $(n+1)$ players play a symmetric equilibrium, then if a $(n+2)$-th player joins but does not participate, then this is an equilibrium for the game with $(n+2)$ players. Therefore, there are also equilibrium in which a subset of m players participate with a certain probability while the remaining $(n+1-m)$ players never participate. The symmetric mixed-strategy equilibrium is unique because the root of $f_n(\cdot)$, that characterizes the symmetric equilibrium, is unique. The same holds for the root of the function $f_m(\cdot)$, that characterizes the equilibria in which a subset of m players participate, and therefore each of these equilibria is also unique. Hence there are exactly $2^{n+1}-1$ equilibria (which is the number of subsets of N minus one). Now all that has to be shown is that there are no other equilibria. If there is any other equilibrium, then there has to be an asymmetric strictly mixed-strategy equilibrium. It turns out that these do not exist. See Appendix A for the details. ■

Corollary 1. *There is a unique symmetric mixed equilibrium, where everybody participates with probability p_n, the unique root of the function $f_n(\cdot)$.*

From now on my focus will be on the symmetric equilibrium. This equilibrium has the following property:

Proposition 2. *In the symmetric equilibrium of the participation game we have $p_{n+1} < p_n$: the probability of participation declines if group size increases.*

Outline of the proof. It is shown that $f_{n+1}(p_n) > 0$ and $f_{n+1}(0) < 0$. Since p_{n+1} is the unique root of $f_{n+1}(\cdot)$, it must be in the interval $(0, p_n)$. See Appendix B for the details. ■

This proposition shows the free-riding problem that is inherent to public good games.

3 The probability that no one participates

Let $S_n \equiv (1-p_n)^{n+1}$. This is the probability that no one participates. I am interested in how S_n depends on n. Returning to Harrington's model, where the benefit to each player if the public good is provided is b and the cost of provision of the public good is c (both benefit and cost do not depend on how many people participate) and the probability of breakdown is always increasing. To find the Nash-equilibrium observe that in the equilibrium the expected payoff of participating, $b-c$, is equal to the expected payoff of not participating, $b[1-(1-p)^n])$, where the latter expression is the probability that at least one other person participates. Equating these two expressions, after some manipulation, yields

$$(1-p_n)^n = 1 - \frac{c}{b}. \tag{5}$$

Now we see that $S_n = \left(1 - \frac{c}{b}\right)(1-p_n)$. Since p_n is decreasing in n, it follows that S_n is increasing.

Now note that in general if $S_{n+1} > S_n$, then:

$$(1-p_{n+1})^{n+2} > (1-p_n)^{n+1} \Leftrightarrow p_{n+1} < 1 - (1-p_n)^{\frac{n+1}{n+2}} \approx p_n, \tag{6}$$

where the approximation holds if n large. This can be seen as an indication that it might be difficult to find examples where $S_{n+1} \leq S_n$ for n large. However, one can construct examples in which S_n is decreasing. For instance, consider the following game: $\theta_0 = -1$, and $\theta_k = \alpha\beta^k$ for all $k \geq 1$. Assume that $\alpha > 0$ and $0 < \beta < 1$. It can be shown that (see Appendix D):

$$S_n = \left[\frac{\beta}{\sqrt[n]{(1+\alpha)/\alpha} - (1-\beta)}\right]^{n+1}. \tag{7}$$

Take $\beta = \frac{1}{25}$ and $\alpha = \frac{25}{4}$. Numerical calculations show that $S_1 = 0.0400$, $S_2 = 0.0399$, $S_3 = 0.0378$, $S_4 = 0.0359$. So, $S_4 < S_3 < S_2 < S_1$. In this example, θ_2, θ_3 etcetera are very close to zero and, hence, $\pi(0,k) \approx \pi(1,k)$ for $k \geq 2$: that is, players are basically indifferent between participating and not participating. In this, what I will call the counterexample, there are strong positive externalities: the incentive to not participate, although present, becomes very small as n increases. The counterexample shows that large groups can exhibit a larger degree of cooperation than small groups.

The following two propositions present sufficient conditions under which the probability of breakdown does increase. The first condition, presented in Proposition 3, is weak in the sense that it only depends on the ratio of θ_0 and θ_1. The second condition, presented in Proposition 4, states that monotonicity of the payoffs, i.e. $\theta_1 < \theta_2 < \ldots$, is also a sufficient condition. The counterexample implies that these condition are not neccesary.

Proposition 3. *If $|\theta_0| < |\theta_1|$, then*

(i) $S_{n+1} > S_n$ *for all* $n \in \mathbb{N}$,

and, moreover,

(ii) $S_\infty = \lim_{n\to\infty} S_n \geq e^{\frac{\theta_0}{\theta_1}}$,

(iii) the expected number of participants is less than one.

Outline of proof. These results follows from the fact that not only is $p_{n+1} < p_n$, the difference between p_{n+1} and p_n cannot be arbitrarily small for given n. Using an upper bound for p_{n+1} (given p_n), a lower bound for S_{n+1} (given S_n) is obtained. Subsequently the lower bound is used to proof the results. See Appendix C for the details. ■

It is interesting to notice that if $|\theta_0| < |\theta_1|$, then the values of $\theta_2, \ldots, \theta_n$ are completely irrelevant: the probability of breakdown will increase. The intuition behind this is the following: If the number of players increases, then the probability to participate in general will be small, i.e. from Proposition 3 *(iii)* we know that $p_n < 1/n$. Hence for the individual decision maker, it is likely that either no other players participate or one other player participates. Unless the incentives to participate if more than one other player participates are high, the player can ignore the probability that more than one other player participates. In this case only θ_0 and θ_1 matter. What if the incentives to participate if more than one other player participates are high? The scope for this limited since high incentives would mean a low θ_2, but $\theta_2 > 0$. If $\theta_1 > \theta_0$, then the incentives cannot be raised high enough to make S_n go down.

Proposition 3 shows that, under a fairly weak condition only dependent on the ratio of θ_0 and θ_1, the probability of breakdown in binary public good games is increasing in group size. Hence, this paradoxical result (not only will people participate less, the group as a whole will participate less) appears very robust to different payoff specifications. Moreover S_∞ is always bigger than $\mathrm{e}^{-1} \approx 0.3679$. So, in all participation games satisfying the condition $|\theta_0| < |\theta_1|$, the probability of breakdown is increasing and always more than 37% in the limit. The Proposition also shows that on average less than one person participates and thus the public good is underprovided (more than one being socially optimal).[3]

As an example of how to use Proposition 3 consider the following extension of Harrington's model. Suppose that being recognized as a rescuer has merits of its own. If there are k rescuers, then each obtains an extra payoff of H/k. Now, $\theta_0 = -[b + H - c]$ and $\theta_1 = c - H/2$, where I assume that $H < 2c$ and $b + H - c > 0$. The condition $|\theta_0| < |\theta_1|$ corresponds to $H < \frac{4}{3}c - \frac{2}{3}b$. So, if the hero bonus is sufficiently small, then the probability that no one rescues the drowner is still increasing in group size. Notice that for $n > 2$ the equilibrium strategy will in general not be solvable without resorting to numerical methods.

For the second sufficient condition I will focus on monotonically increasing payoff functions. This corresponds to negative externalities. The purpose of this is twofold: (1) it will

[3]The socially optimal outcome maximizes $\pi(0,k)(n+1-k) + \pi(1,k)k$ with respect to k.

yield another sufficient condition for the probability of breakdown to be increasing and (2) the sufficient condition of Proposition 3 does not always yield 'nice' conditions as the application in Section 4.2 shows whereas the condition for a negative externality is easier to interpret.

Suppose we have a game with $n+1$ players such that $\theta_0 = -1$ (just a normalization) and, for all $k = 1, \ldots, n$, $\theta_{k+1} > \theta_k$. If the θ_k's are increasing, then $\pi(0,k) - \pi(1,k)$ is increasing in k. As more players participate the incentive to not participate becomes stronger; this is the case of a negative externality (cf. Anderson and Engers (2005) who focus solely on such monotonic payoff functions).

Construct now an auxiliary game with parameters $\theta_0^C = -1$ and for all $k > 0$, $\theta_{k+1}^C = \theta_k^C$ such that this game with $n+1$ players has the same symmetric equilibrium as the game above with increasing θ_k's. Note that for every n such an auxiliary game can be uniquely constructed. Observe that for the auxiliary game it is known that the probability of breakdown is increasing in n (since it is a game similar to the one analyzed at the beginning of Section 3). Denote the equilibrium of the auxiliary game by p_n^C. By construction $p_n^C = p_n$, where p_n is the probability of participation in the game with a negative externality. If we can show that $p_{n+1} < p_{n+1}^C$, then $(1-p_{n+1})^{n+2} > (1-p_{n+1}^C)^{n+2} > (1-p_n^C)^{n+1} = (1-p_n)^{n+1}$. Then the probability of breakdown in a game where, for all $k > 0$, $\theta_{k+1} > \theta_k$, is increasing in group size.

I will now show that $p_{n+1} < p_{n+1}^C$. Examining the function $f_n(p)$, I observe that it is a weighted average of the θ_k's. The most weight is given to $k = \lfloor (n+1)p \rfloor$ since in a binomial distribution this is the most likely outcome.[4] An equilibrium is then a p which shifts the weights in such a way that the weighted average is zero. In the auxiliary game, there exists an integer $m \in \{1, \ldots, n\}$ such that for $0 < k < m$ we have $\theta_k^C > \theta_k$ and for $k \geq m$ we have $\theta_k^C \leq \theta_k$. This implies that $\theta_{n+1}^C < \theta_{n+1}$. So, in determining the equilibrium in the original game p_{n+1} has to be smaller than p_{n+1}^C to compensate.

The following proposition is thus obtained.

Proposition 4. *If, for all $k > 0$, $\theta_{k+1} > \theta_k$, then S_n is increasing in n.*

Proposition 4 shows that if the parameters θ_k are increasing, then this is also a sufficient condition for the probability of breakdown to be increasing. Increasing θ_k's can be interpreted as negative externalities. As the counterexample in Section 3 shows, if the parameters θ_k decrease sufficiently fast, then the probability of breakdown can decrease. Hence, sufficiently strong positive externalities in the payoff (*and* $|\theta_0| > |\theta_1|$) might mitigate the paradox.

4 Applications

4.1 R&D and teamwork

An economic application is teamwork. Suppose that the R&D-department of a company consists of $n+1$ employees. Each employee has to decide whether to exert effort or not. The effort is interpreted as contributing to the research project. If an employee exerts effort, then some innovation is created and the employee bears a cost c. The employee values this innovation, but the value only depends on the number of employees that have exerted effort (including the employee himself). Denote this value by $V(k)$, where k is the number of participants, and assume, without loss of generality, that $V(0) = 0$. The payoff function, using the notation of

[4] The truncation function is denoted by $\lfloor \cdot \rfloor$.

Section 2, is $\pi(a_i, A_i) = V(a_i + A_i) - a_i c$. Notice that

$$\theta_k = V(k) - V(k+1) + c. \tag{8}$$

To fit into my framework, the following restrictions are needed:

$$\theta_0 = -V(1) + c < 0 \tag{9}$$
$$\theta_k = V(k) - V(k+1) + c > 0 \text{ for all } k \geq 1 \tag{10}$$

If we assume that $V(\cdot)$ is concave, then the θ_k's are increasing in k, i.e.

$$\frac{\partial \theta_k}{\partial k} = V'(k) - V'(k+1) > 0. \tag{11}$$

Under concavity of $V(\cdot)$, the probability that the patent is not discovered is increasing in the number of co-workers. Concavity can be interpreted in the following manner: the effort of each employee increases the value of the innovation, but the additional value diminishes. This lack of 'synergy' does not imply that team work is necessarily unwanted. In a cooperative equilibrium, that maximizes $nV(k) - ck$ with respect to k, the optimal number of employees that exert effort can be larger than one. However, if a company puts together a research team, then some additional value of working together is expected. One could imagine that $V(\cdot)$ is S-shaped and teamwork would only create additional value if the number of contributors is below a certain number.[5] In a natural manner, this yields non-monotonic θ_k's and the result from Proposition 3 is useful.

The condition $|\theta_0| < |\theta_1|$ yields (from Proposition 3):

$$V(2) < 2c. \tag{12}$$

A possible interpretation of (12) is the following: If you could put in twice the effort, then this is not profitable. While you may benefit if others exert effort, it does not pay to do the extra effort yourself. This ensures that the probability that the patent is not discovered is increasing in the number of co-workers. This example also shows that in not every application the assumption of monotonicity is innocuous.

4.2 International debt overhang

Another application, due to Kaneko and Prokop (1993), is debt overhang. Kaneko and Prokop (1993, p.2) define it as follows:

> The term 'debt overhang' expresses a situation where a sovereign country has borrowed money from foreign banks and has not succeeded in fulfilling the scheduled repayments for some period. The existence of the debt overhang is a serious problem for the debtor country, which keeps the country in a bad economic situation.

I discuss a symmetric version of their model. In this game there are $n+1$ creditors and one country that is in debt. The debtor country is treated as part of the environment and not as an active player.

[5] A function $V(k)$ is S-shaped if there exists $\bar{k} \in \mathbb{R}$ such that it is convex for all $k < \bar{k}$ and concave for all $k > \bar{k}$.

Each creditor holds a debt of size D. I assume that total outstanding debt is constant, i.e. $(n+1)D = \bar{D}$. If the number of creditors increases, then they also hold a smaller share of total debt. This assumption is needed to ensure that below the parameters θ_k do not depend on n.

The creditors can sell off their debt at a secondary market price. The market price is given by the pricing function $P : \mathbb{R}^+ \to \mathbb{R}^+$ whose domain is total outstanding debt. For instance, if a creditor has an outstanding debt of D and total outstanding debt is $\bar{D}$, then it can sell of this debt at $DP(\bar{D})$.[6] The larger the debt (and the higher the probability of default), the lower the price of buying back the debt.

I follow Kaneko and Prokop (1993) and suppose that each creditor can either sell its debt now at a price $P(\bar{D})$ or wait for one period and then sell it. If the creditor postpones, then its debt will rise to βD, where $\beta > 1$ is one plus the interest rate. However, the present value of the debt will stay D. The price of buying back debt in the next period is $P(\beta(n+1-k)D)$, where k is the number of other creditors that have not postponed. Each creditor has two actions: sell or postpone.

For the moment, and contrary to Kaneko and Prokop (1993), assume that $P(\cdot)$ is a decreasing and differentiable function. Since $P(\cdot)$ is a decreasing function, we have that:

$$DP(\bar{D}) > DP(\beta\bar{D}), \tag{13}$$

where the present value of debt stays constant at D, but the price of buying back debt does change. From (13) it follows that it is optimal for the creditors to sell if none of the other creditors have sold. I will show that this game can be considered as a participation game. In terms of the participation game selling is equal to participating and (13) is equal to Assumption 2. In order to qualify as a participation game, it must also be optimal to postpone if one other creditor sells:

$$DP(\beta(\bar{D} - D)) > DP(\bar{D}), \tag{14}$$

which boils down to $\beta < \frac{n+1}{n}$. This is equal to Assumption 1 if either the number of creditors is small or the interest rate is not too large. From Corollary 1, it follows that there is a unique symmetric equilibrium in this game.

Since the present value of debt, D, is constant for a creditor, the creditor is just interested in the price at which the debt is sold off. Hence, the payoff function is can be simplified to this price. The parameters of the model are defined as follows:

$$\theta_k = P(\beta(\bar{D} - kD)) - P(\bar{D}). \tag{15}$$

If $\beta < \frac{n+1}{n}$ and $P(\cdot)$ is decreasing, then $\theta_0 < 0$ and $\theta_k > 0$ for all $k \geq 1$ ensuring that we have a participation game. Notice that:

$$\frac{\partial \theta_k}{\partial k} = -\beta DP'(\beta(\bar{D} - kD)) > 0, \tag{16}$$

and thus θ_k is increasing in k. Hence, it follows directly from Proposition 4, that an increase in the number of creditors implies that the probability that no one sells is increasing. Therefore having more creditors has a detrimental effect on debt overhang. This result is, unlike the results of Kaneko and Prokop (1993), valid for a small number of creditors.

[6] This price is often far below 1; in the case of Bolivia in 1985 the price of buying back one dollar of debt was 5 cent (cf. Kaneko and Prokop, 1993, p.2).

Kaneko and Prokop (1993) do not assume that $P(\cdot)$ is decreasing: they merely assume that $P(0) \geq P(\bar{D}) > P(\beta\bar{D})$. In this case, Proposition 3 can still be used to formulate sufficient conditions for the probability that no one sells to increase. First, to have $\theta_k > 0$, for all $k \geq 1$, we need:

$$P(\beta(\bar{D} - kD)) > P(\bar{D}). \tag{17}$$

The condition $|\theta_0| < |\theta_1|$ (from Proposition 3) yields

$$2P(\bar{D}) < P(\beta\bar{D}) - P(\beta(\bar{D} - D)). \tag{18}$$

If (17) and (18) both hold, then the probability that no one sells increases in the number of creditors. Remark that the sufficient condition of Proposition 3 does not yield easily interpretable conditions. The reasonable assumption of a decreasing pricing function, however, implies negative externalities and gives the desired result immediately.

5 Conclusion

This paper has investigated if it is possible in a participation game for large groups to have a lower probability probability of breakdown. The counterexample showed that this is possible, but it is not easy to find these example as strong positive externalities seem to be required for this phenomenon to occur. However, I presented two (mild) sufficient conditions under which the probability of breakdown turned out to be increasing. Previous work by Anderson and Engers (2005) already showed how negative externalities will lead to a higher probability of breakdown, but I presented another condition only dependent on the incentive to participate if no or one other player participates. In the example about R&D I argued that assuming negative externalities is not always as innocuous as it seems. This paper only gives sufficient conditions. Clearly, necessary conditions or a full characterization of the set of parameters, for which the probability of breakdown is decreasing, are the next step.

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A Proof of Proposition 1

It is convenient to present the following Lemmata first.

Lemma 1. *There exists no symmetric pure strategy equilibrium.*

Proof. Suppose all players participate: $a_i = 1$ and $A_i = n$ for all i. Each player then receives a payoff $\pi(1, n)$. However, by Assumption 1 the best-response is to not participate since $\pi(0, n) > \pi(1, n)$. Suppose all players do not participate: $a_i = 0$ and $A_i = 0$ for all i. However, by Assumption 2 the best-response is to participate. ■

Lemma 2. *There are $(n + 1)$ non-symmetric pure strategy equilibria.*

Proof. Let $A_i \geq 1$ be the number of other people who participate. Those who participate have a payoff $\pi(1, A_i) < \pi(0, A_i)$ (Assumption 1). Clearly, if $A_i \geq 1$, this cannot be an equilibrium. This does not hold for $A_i = 0$, since $\pi(1, 0) > \pi(0, 0)$ (Assumption 2). Only one player participates is an equilibrium. There are $(n + 1)$ of these equilibria. ■

Lemma 3. *There exists a unique symmetric mixed-strategy equilibrium where everyone participates with probability $p \in (0, 1)$.*

Proof. We know that p must be a real positive root of $f_n(\cdot)$ defined in (4). First, it has to be shown that it exists. Second, it has to be shown that it is unique.

It is straightforward to show that there is at least one root on the unit interval. Note that by Assumption 1 and 2:

$$f_n(0) = \pi(0, 0) - \pi(1, 0) = \theta_0 < 0 \tag{19}$$

$$f_n(1) = \pi(0, n) - \pi(1, n) = \theta_n > 0, \tag{20}$$

and remark that $f_n(\cdot)$ is a continuous function. This also implies that generically the number of roots is odd.

To proof unicity, notice that, for $p \neq 1$, $f_n(\cdot)$ can be rewritten as:

$$f_n(p) = (1 - p)^n \sum_{k=0}^{n} \binom{n}{k} r^k \theta_k, \tag{21}$$

where $r \equiv p/(1-p)$. This allows us to look at the roots of $\sum_{k=0}^{n} \binom{n}{k} r^k \theta_k$. By using Descartes' sign rule (see Atkinson, 1989, p. 95), we now easily obtain that the root is unique. ■

In determining all asymmetric equilibria, it is useful to think of a person joining the group. Suppose that a group of $n+1$ players has already decided to participate with the equilibrium probability of p_n. If an extra person would join this group, what would this person do? The next Lemma shows that this person becomes a bystander: someone who does not participate.

Lemma 4 (The bystander)**.** *The following is an equilibrium: in a group of size $n + 2$, let $n+1$ players participate with probability p_n and the remaining player (the bystander) will not participate.*

Proof. For the players who participate this is obviously an equilibrium strategy. For the bystander, the benefit of not participating over participating is given by:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}p_n^k(1-p_n)^{n-k}\theta_k = p_n^{n+1}\theta_{n+1}+\sum_{k=0}^{n}\binom{n}{k}p_n^k(1-p_n)^{n-k}\theta_k\times\left[\frac{(n+1)(1-p_n)}{n+1-k}\right]. \quad (22)$$

Note that *(i)* the first term on the RHS is positive and *(ii)* the second term on the RHS is also positive. In order to see the latter recall that $\sum_{k=0}^{n}\binom{n}{k}p_n^k(1-p_n)^{n-k}\theta_k = 0$. Observe that this sum consists of one negative term at $k = 0$ followed by exclusively positive terms. The new weighing factor $(n+1)/(n+1-k)$ puts more emphasis on these positive terms since it increases as k increases. Hence, the bystander will not participate. ■

Lemma 5. *Any non-empty subset of the group of players can play the symmetric mixed equilibrium for a subset of that size with the rest of the group being bystanders.*

Proof. Follows from Lemma 4. ■

Lemma 6. *There exists no non-symmetric strictly mixed equilibrium.*

Proof. Suppose that each member of the group in an arbitrary mixed equilibrium participates with probability p_i, where $i = 1, \ldots, n+1$, and $p_i \neq p_j$ for some i, j. Suppose the focus is on a subset of the group without members $n_1, \ldots, n_m$. Call this subset $M \equiv N\backslash\{n_1, \ldots, n_m\}$. The probability that k players in this subset participate is:

$$\sum_{K\subseteq M, \#K=k}\left\{\left(\prod_{j\in K}p_j\right)\left(\prod_{\ell\notin K}(1-p_\ell)\right)\right\}. \quad (23)$$

This expression is obtained by summing over all subsets of M of size k and calculating the probability that for each of these subsets the k people in this subset participate. Let $\Pr(k|n_1, \ldots, n_m)$ denote this probability. If $k < 0$ or $k > \#N - m$, then $\Pr(k|n_1, \ldots, n_m)$ is defined to be zero. Notice that $\Pr(k|i) = \Pr(k|i,j)\times(1-p_j) + \Pr(k-1|i,j)\times p_j$.

For any $i, j \in N$, in a strictly mixed equilibrium the following should hold:

$$\sum_{k=0}^{n}\Pr(k|i)\theta_k = 0 \quad (24)$$

$$\sum_{k=0}^{n}\Pr(k|j)\theta_k = 0. \quad (25)$$

Both player i and j should be indifferent between participating and not participating. This is equivalent with:

$$\sum_{k=0}^{n}[\Pr(k|i,j)\times(1-p_j) + \Pr(k-1|i,j)\times p_j]\theta_k = 0 \quad (26)$$

$$\sum_{k=0}^{n}[\Pr(k|i,j)\times(1-p_i) + \Pr(k-1|i,j)\times p_i]\theta_k = 0, \quad (27)$$

which implies that in equilibrium:

$$p_i = p_j = \frac{\sum \Pr(k|i,j)\theta_k}{\sum \Pr(k|i,j)\theta_k - \sum \Pr(k-1|i,j)\theta_k}. \tag{28}$$

Since $p_i = p_j$ holds for any i and any j, this contradicts are earlier statement that $p_i \neq p_j$ for some i, j. Hence there are no non-symmetric strictly mixed equilibria. ■

Proposition 1 follows easily from Lemmata 1–6. Observe that for each subset of N (with the exception of the empty set), there is an equilibrium where the members of the subset participate with the same probability and the other players do nothing. The number of subsets is 2^{n+1}. Moreover, these probabilities are the unique root of $f_m(\cdot)$, where m the number of players that participate. So there are exactly $2^{n+1} - 1$ equilibria.

B Proof of Proposition 2

Note that $f_{n+1}(p)$ can be rewritten as:

$$f_{n+1}(p) = p^{n+1} \times \theta_{n+1} + (1-p) \times \sum_{k=0}^{n} \frac{n+1}{n+1-k} \binom{n}{k} p^k (1-p)^{n-k} \theta_k. \tag{29}$$

To see that $f_{n+1}(p_n) > 0$, note that:

1. The first term on the RHS of (29) is positive,
2. The second term on the RHS of (29) is also positive in p_n. It is known that $\sum_{k=0}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k} \theta_k = 0$. Observe that this sum consists of one negative term at $k = 0$ followed by exclusively positive terms. The new weighing factor $(n+1)/(n+1-k)$ puts more emphasis on the latter terms since it increases as k increases.

It is known from Lemma 3 that $f_{n+1}(\cdot)$ has a unique positive real root at p_{n+1}. Since p_{n+1} is the unique root of $f_{n+1}(\cdot)$, $f_{n+1}(0) < 0$ and $f_{n+1}(p_n) > 0$, p_{n+1} must be in the interval $(0, p_n)$.

C Proof of Proposition 3

The symmetric Nash-equilibrium is the root of $f_n(\cdot)$. Remark that if $p \neq 1$ we can write

$$f_n(p) = (1-p)^n \sum_{k=0}^{n} \binom{n}{k} r^k \theta_k, \tag{30}$$

where $r = p/(1-p)$. Note that r is a monotonic transformation from $[0,1)$ to $[0,\infty)$. Since the root of $f_n(\cdot)$ is never equal to 1, it is possible to focus on the roots of:

$$g_n(r) = \sum_{k=0}^{n} \binom{n}{k} r^k \theta_k. \tag{31}$$

Let r_n denote the unique positive real root of $g_n(\cdot)$. Note that $p_n = \frac{r_n}{1+r_n}$. The following Lemmata, that are needed for the proof of the Proposition, are presented in terms of r_n.

Lemma 7. *For every $n \in \mathbb{N}$, we have $r_{n+1} < \frac{n}{n+1} r_n$.*

Proof. If $g_{n+1}(nr_n/(n+1)) > 0$, then $r_{n+1} < \frac{n}{n+1} r_n$. This follows from $g_{n+1}(0) < 0$ and $g_{n+1}(\cdot)$ continuous. It has to be shown that $g_{n+1}(nr_n/(n+1)) > 0$ given that $g_n(r_n) = 0$. Let $r \equiv r_n$ and $s \equiv \frac{n}{n+1} r$ to simplify notation. Observe that:

$$\binom{n}{0}\theta_0 + \binom{n}{1} r\theta_1 = \binom{n}{0}\theta_0 + \frac{n+1}{n}\binom{n}{1}\left(\frac{n}{n+1} r\right)\theta_1 \tag{32}$$

$$= \binom{n+1}{0}\theta_0 + \binom{n+1}{1} s\theta_1 \tag{33}$$

and, since $g_n(r) = 0$ that,

$$\binom{n}{0}\theta_0 + \binom{n}{1} r\theta_1 = -\binom{n}{2} r^2\theta_2 - \ldots - \binom{n}{n} r^n\theta_n. \tag{34}$$

Combining these two results gives:

$$g_{n+1}\left(\frac{n}{n+1} r\right) > 0 \tag{35}$$

$$\Leftrightarrow -\binom{n}{2} r^2\theta_2 - \ldots - \binom{n}{n} r^n\theta_n + \binom{n+1}{2} s^2\theta_2 + \ldots + \binom{n+1}{n} s^n\theta_n + s^{n+1}\theta_{n+1} > 0 \tag{36}$$

$$\begin{aligned} \Leftrightarrow -\binom{n}{2} r^2\theta_2 - \ldots - \binom{n}{n} r^n\theta_n + \frac{n+1}{n-1}\binom{n}{2} s^2\theta_2 \\ + \ldots + \frac{n+1}{1}\binom{n}{n} s^n\theta_n + s^{n+1}\theta_{n+1} > 0 \end{aligned} \tag{37}$$

$$\Leftrightarrow \sum_{i=2}^{n} \left\{ \binom{n}{i}\theta_i \left[\frac{n+1}{n+1-i} \times \left(\frac{n}{n+1}\right)^i - 1 \right] r \right\} + s^{n+1}\theta_{n+1} > 0. \tag{38}$$

Since $s_{n+1}\theta_{n+1} > 0$, it suffices to show that the following is true for every n and for all $i = 2, \ldots, n$:

$$\frac{n+1}{n+1-i} \times \left(\frac{n}{n+1}\right)^i - 1 \geq 0. \tag{39}$$

Rearrange this to obtain:

$$\frac{n^i}{(n+1-i)(n+1)^{i-1}} - \frac{(n+1-i)(n+1)^{i-1}}{(n+1-i)(n+1)^{i-1}} = \frac{n^i - (n+1-i)(n+1)^{i-1}}{(n+1-i)(n+1)^{i-1}} \geq 0, \tag{40}$$

which is equivalent with:

$$n^i - (n+1-i)(n+1)^{i-1} \geq 0. \tag{41}$$

The rest of the proof is by induction. Let $i = 2$. Now (41) boils down to $n^2 - (n-1)(n+1) = n^2 - n^2 + 1 = 1 \geq 0$. Next, assume that (41) is true for an $i > 2$ and show that it also is true for $i+1$. Assume that $n^i - (n+1-i)(n+1)^{i-1} \geq 0$ is true. It has to be shown that $n^{i+1} - (n-i)(n+1)^i \geq 0$ is true. Rewrite this as:

$$n^{i+1} - (n-i)(n+1)^i = n[n^i - (n+1-i)(n+1)^{i-1}] + i(n+1)^{i-1} > 0, \tag{42}$$

since $n^i - (n+1-i)(n+1)^{i-1} \geq 0$ by assumption and since $i(n+1)^{i-1} > 0$. ■

Lemma 8. *If $r_{n+1} < \frac{n}{n+1} r_n$ and $r_1 < 1$, then for all n we have $(1+r_n)^{n+1} > (1+r_{n+1})^{n+2}$.*

Proof. It is sufficient to show that this results holds for the limit case of $r_{n+1} = \frac{n}{n+1} r_n$. This defines a recursive relation and therefore r_n can be rewritten as an explicit function of n, i.e. $r_n = \frac{r_1}{n}$. It has to be shown that:

$$\left(1+\frac{r_1}{n}\right)^{n+1} > \left(1+\frac{r_1}{n+1}\right)^{n+2}. \tag{43}$$

Let $T_n \equiv \left(1+\frac{r_1}{n}\right)^{n+1}$. Note that $\lim_{n\to\infty} T_n = \mathrm{e}^{r_1}$ since $\mathrm{e}^x = \left(1+\frac{x}{n}\right)^n$. Using the logarithmic derivative rule, it follows that:

$$\frac{\mathrm{d}T_n}{\mathrm{d}n} = \left[\log\left(1+\frac{r_1}{n}\right) - \frac{n+1}{n+r_1}\frac{r_1}{n}\right]\left(1+\frac{r_1}{n}\right)^{n+1}, \tag{44}$$

which has to be strictly negative. Equivalently:

$$\log\left(1+\frac{r_1}{n}\right) < \frac{n+1}{n+r_1}\frac{r_1}{n}. \tag{45}$$

Since $r_1/n < 1$ by assumption, we can use the following power function (Apostol, 1967: pp. 388–391) to replace $\log\left(1+\frac{r_1}{n}\right)$ with:

$$\frac{r_1}{n} + \sum_{k=1}^{\infty}\left[-\frac{1}{2k}\left(\frac{r_1}{n}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{n}\right)^{2k+1}\right]. \tag{46}$$

It is easily checked that $\frac{r_1}{n} < \frac{n+1}{n+r_1}\frac{r_1}{n}$ if $r_1 < 1$. Remark further that:

$$-\frac{1}{2k}\left(\frac{r_1}{n}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{n}\right)^{2k+1} < 0 \tag{47}$$

$$\Leftrightarrow -\frac{1}{2k} + \frac{1}{2k+1} \times \frac{r_1}{n} < 0 \tag{48}$$

$$\Leftrightarrow \frac{r_1}{n} < \frac{2k+1}{2k}, \tag{49}$$

which is true if $r_1 < 1$. These two facts imply that

$$\log\left(1+\frac{r_1}{n}\right) = \frac{r_1}{n} + \sum_{k=1}^{\infty}\left[-\frac{1}{2k}\left(\frac{r_1}{n}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{n}\right)^{2k+1}\right] < \frac{r_1}{n} < \frac{n+1}{n+r_1}\frac{r_1}{n}. \tag{50}$$

Therefore, T_n is decreasing in n if $r_1 < 1$, which concludes the proof. ■

Now we are able to provide the proof of Proposition 3.

Proof of part *(i)* We have to show that:

$$(1-p_n)^{n+1} < (1-p_{n+1})^{n+2} \tag{51}$$

$$\Leftrightarrow \left(\frac{1}{1+r_n}\right)^{n+1} < \left(\frac{1}{1+r_{n+1}}\right)^{n+2} \tag{52}$$

$$\Leftrightarrow (1+r_n)^{n+1} > (1+r_{n+1})^{n+2}. \tag{53}$$

Now if we solve (3) for $n = 1$, then we see that the condition that $|\theta_0| < |\theta_1|$ is equivalent to $p_1 < \frac{1}{2}$. Since $p_1 < \frac{1}{2}$ implies that $r_1 < 1$ and, from Lemma 7, we know that $r_{n+1} < \frac{n}{n+1} r_n$, the result follows from Lemma 8.

Proof of part *(ii)* Let $\hat{r}_n \equiv \frac{r_1}{n}$. Note that $r_n < \hat{r}_n$ and $(1+r_n)^{n+1} < (1+\hat{r}_n)^{n+1}$. Taking the limit we obtain:

$$S_\infty = \lim_{n\to\infty}(1+r_n)^{n+1} < \mathrm{e}^{r_1}. \tag{54}$$

Since $\lim_{n\to\infty}(1+r_n)^{n+1} = \left[\lim_{n\to\infty}(1-p_n)^{n+1}\right]^{-1}$ and substituting $r_1 = p_1/(1-p_1)$ leads to:

$$\lim_{n\to\infty}(1-p_n)^{n+1} > \mathrm{e}^{-p_1/(1-p_1)}, \tag{55}$$

where the RHS is equal to $\mathrm{e}^{\theta_0/\theta_1}$.

Proof of part *(iii)* The expected number of participants is the probability of participation, p_n, times group size, $n+1$ (since the number of participants follows a binomial distribution). We have to proof that $(n+1)p_n < 1$. Substituting $p_n = r_n/(1+r_n)$ yields:

$$r_n < \frac{1}{n}. \tag{56}$$

It is known that $r_1 < 1$ and $r_{n+1} < \frac{n}{n+1}r_n$. Therefore $r_n < \frac{1}{n}r_1 < \frac{1}{n}$.

D Derivation of the probability of breakdown in the counterexample

The probability of participation is implicitly defined by:

$$\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\theta_k = 0, \tag{57}$$

which in this case may be rewritten as:

$$-(1-p)^n + \alpha\sum_{k=1}^{n}\binom{n}{k}(\beta p)^k(1-p)^{n-k} = 0. \tag{58}$$

If $\beta < 1$, then some straightforward algebra yields:

$$-(1-p)^n + \alpha[\beta p + (1-p)]^n \times \sum_{k=1}^{n}\binom{n}{k}\left(\frac{\beta p}{\beta p + (1-p)}\right)^k\left(\frac{1-p}{\beta p + (1-p)}\right)^{n-k} = 0. \tag{59}$$

Now the part behind the summand is a proper binomial again and represents 'the probability of at least one success'. Exploiting this fact, we obtain:

$$-(1-p)^n + \alpha[\beta p + (1-p)]^n\left(1 - \left(\frac{1-p}{\beta p + (1-p)}\right)^n\right), \tag{60}$$

which is equivalent to:

$$-(1-p)^n + \alpha[\beta p + (1-p)]^n - \alpha(1-p)^n = 0, \tag{61}$$

which is easily solved for p. This yields:

$$p_n = \frac{\sqrt[n]{(1+\alpha)/\alpha} - 1}{\sqrt[n]{(1+\alpha)/\alpha} - (1-\beta)}. \tag{62}$$

Applying $S_n = (1-p_n)^{n+1}$ gives the desired result.