Managing the environment and the economy in the presence of hysteresis and irreversibility

Pim Heijnen [*] Florian Wagener [†]

December 24, 2008

Abstract

The shallow lake optimal management problem is one of the simplest ecological-economic interest conflict models for which several qualitatively different long run outcomes are possible. We extend the original model by adding the capital stock of an industry as a second state variable. A government can mitigate the effects of pollution arising from industrial activities by imposing the requirement to abate emissions. Within this framework two scenarios are examined. In the social optimal benchmark, the social planner optimally allocates investment. In the competitive equilibrium, market forces determine the investment in capital, but the social planner can still abate emissions. We show that in the case of irreversibilities catastrophes are avoided in the competitive equilibrium when it is socially optimal to do so. However, in the competitive equilibrium, either the catastrophe is avoided in an inefficient way or the catastrophe is badly managed. In case of hysteresis, catastrophes are almost always avoided. Moreover, the decision to avoid catastrophes does not depend on long-term considerations.

JEL-codes: C61, Q20, Q50
Keywords: shallow lakes, optimal abatement, optimal control, capital stock, irreversibility, hysteresis

1 Introduction

Ecological studies show that a wide variety of ecological systems (lakes, coral reefs, oceans, forests, arid lands) do not respond to gradual change in a smooth way (Scheffer, Carpenter, Foley, Folke, & Walker, 2001). In particular, these systems may be prone to catastrophic shifts, where a small change in external conditions can drastically alter the steady state of the ecosystem. The paradigmatic model for catastrophic shifts in ecosystems is the so-called shallow lake (Scheffer, 1998). In its pristine state, a shallow lake exhibits clear water, rich submerged vegetation and a high number of fish. The alternative steady state is one with turbid water, a high concentration of algae and almost devoid of fish. The external factor that determines the steady state is the inflow of phosphorus. The state of the lake (i.e. its turbidity) seems almost

[*] CeNDEF, Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018WB Amsterdam, the Netherlands, e-mail: p.heijnen@uva.nl. Financial support by the Netherlands Organization for Scientific Research (NWO) is gratefully acknowledged. We thank Tatiana Kiseleva for helpful discussions.

[†] CeNDEF, Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018WB Amsterdam, the Netherlands, e-mail: f.o.o.wagener@uva.nl. Financial support by the Netherlands Organization for Scientific Research (NWO) under a MaGW-VIDI grant is gratefully acknowledged.

irresponsive to the inflow of phosphorus except near a threshold value where, due to biological feedback mechanisms, the turbidity suddenly jumps from low to high. If the lake ecosystem is very fragile, the jump is irreversible. But even in less fragile lakes, the inflow of phosphorus has to be lowered far below the threshold value to make the lake jump back to a clear state.

The problem of optimally managing shallow lakes, introduced by Mäler, Xepapedeas, and de Zeeuw (2003), is important to environmental economics for two reasons. First, it captures a basic tradeoff. The inflow of phosphorus into lakes is mainly due to the use of fertilisers. Therefore more intensive agriculture has a detrimental effect on fishery. The question then is how to optimally manage the lake taking into account the returns to agriculture and fishery, and the dynamics of the lake. In more general terms, the model can be used to discuss the tradeoff between consumption and the state of the environment. The inclusion of this basic tradeoff is a common feature of the models used in environmental economics. Second, and more importantly, by incorporating a positive feedback mechanism, it allows to move beyond simple linear models of the environment. In particular, issues of irreversibility, hysteresis, and multiple basins of attractions can be studied. This has been the focus of a number of recent papers (Brock & Starrett, 2003; Mäler et al., 2003; Wagener, 2003; Kiseleva & Wagener, 2008). In the shallow lake model, the optimal policy depends on the initial state of the lake. In certain cases, if the initial phosphorus level is below a certain threshold, it is optimal to move towards a clean lake. Otherwise, it is optimal to let the lake become more polluted.

One of the drawbacks of the shallow lake management problem is that, by taking the stock of pollution as the only stock variable, it implies a highly stylised model of the economy. In particular, at any moment in time any positive level of consumption can be reached. We therefore extend the shallow lake model by adding industrial capital as a second stock variable, assuming that the inflow of pollution is proportional to the installed base of capital. The installed base of capital puts constraints on both present and future levels of consumption. We think of capital as changing on a slower time scale than the state of the environment; as capital changes slowly, also the inflow of pollution adjusts sluggishly. Even in the case of extremely fast growth, it seems to take more than a generation to change a poor agricultural society in a rich post-industrial society, while the detrimental effects on the state of the environment of this change occur earlier.[1]

We present two extensions of the model of Mäler et al. (2003). The benchmark extension is that of a social planner managing both the industry and the lake. This is then contrasted to a competitive industry, where the externalities of pollution are not properly taken into account. As the capital stock of the industry is polluting the environment, we examine the effect of forcing the industry to clean up part of this pollution. Specifically, the industry is taxed and proceedings from this tax are used to clean a fixed proportion of the pollution. This is different from the situation in Mäler et al. (2003) where the tax was returned lump-sum to society. The tax rate is a fixed constant: this reflects the fact that dynamic taxing rules are very hard to implement, although we investigate one instance where the tax can be adjusted once over the planning horizon.

The model presented in this paper is related to that of Bovenberg and Heijdra

[1]The anecdotal evidence is in favour of this statement. Two prime examples of fast economic growth at unprecedented level, are Japan and China. Even before their transformation from poor to rich was or is completed, they experienced or are experiencing grave environmental problem. E.g. the Minamata poisoning in 1965 in Japan (Timothy, 2001) or the attempts of the Chinese government to improve the air quality in Beijing before the Olympics of 2008

(1998, 2002). The difference is in how we analyse the model. They analyse a non-linear externality problem by considering small deviations from the steady state and performing then a linearization of the model. However in a non-linear system local analysis is in general not sufficient. This is caused by history dependence. Specifically, we focus on the effect of initial level of pollution and government policy, both of which determine which of the steady states is reached in the long-run. Our contribution is in providing a global analysis.

Our results are the following. We show that in the case of irreversibilities catastrophic shifts, or catastrophes for short, are avoided in the competitive equilibrium when it is socially optimal to do so. However in the competitive equilibrium, either the catastrophe is avoided in an inefficient way, or the catastrophe is badly managed. In case of hysteresis, catastrophes are almost always avoided. Moreover, the decision to avoid catastrophes does not depend on long-term considerations. Hence myopia of the decision maker may not matter that much for the long-run outcome.

Some recent papers explore similar issues. Ranjan and Shortle (2007) also add capital dynamics to the shallow lake model of Mäler et al. (2003) and find environmental Kuznets curves. However, they do not examine the case of a competitive industry (with corrective taxation/abatement).

Prieur (forthcoming), extending John and Pecchenino (1994), adds non-linear environmental dynamics to a Diamond-type overlapping generations model. In contrast to the approach of this paper, he only considers the case of a competitive equilibrium. He finds that John and Pecchenino's result, that abatement is sufficient for environmental quality to improve, does not hold when environmental dynamics are non-linear. As far as we know, ours is the first paper that compares the optimal solution to the competitive equilibrium.

Janmaat and Ruijs (2007) argue that in models of harvesting the carrying capacity of the ecosystem depends non-linearly on the harvesting intensity. To model the influence of harvesting intensity, they use shallow lake dynamics. They show that to achieve an efficient outcome taxes need to be initially higher than their efficient levels to avoid moving towards an undesirable steady state. We shall use this insight in our discussion of optimal state-dependent tax rates.

The paper is structured as follows. In Section 2 the model is introduced. We first focus on the case of irreversibilities. Section 3 discusses the social planner benchmark. Section 4 investigates the analogous perfectly competitive economy. In Section 5 the two economies are compared to each other. Section 6 examines the effect of taxes that change over time. In Section 7 we turn our attention to the case of hysteresis. In section 8 we perform a sensitivity analysis. Section 9 concludes.

2 The model

2.1 Shallow lake dynamics

The state of the environment is denoted by $d \geq 0$, the level of phosphorus in the shallow lake. The change in the level of pollution is given by:

$$\dot{d} = \ell - bd + \frac{d^2}{d^2 + 1}, \tag{1}$$

where $b > 0$ denotes the sedimentation rate and $\ell \geq 0$ the loading, that is, the net inflow of pollutants. The nonlinear term on the right hand side of (1) is a very simple model of the biological positive feedback mechanisms working in the lake: as the phosphorus level rises above a threshold value (more accurately a threshold range), the foodweb in the lake breaks down. As a consequence, a significant amount of additional phosphorus is released. Equation (1) gives a reasonable description of the phosphorus dynamics in shallow lakes. We will use it as a broader metaphor for non-linearities in ecosystems. Therefore we will refer to the level of phosphorus as the level of pollution or the level of damage.

The qualitative properties of the shallow lake dynamics depend critically on the "physical" parameter b. If $b \geq 3\sqrt{3}/8 \approx 0.65$, then the dynamics (1) has for every value of ℓ a unique steady state: see figure 1(a). For $\frac{1}{2} < b < 3\sqrt{3}/8$ there are for some values of ℓ several steady states. However, by shutting down the inflow of phosphorus, that is, by setting $\ell = 0$, the lake regenerates completely and reverts to the pollution-free state $d = 0$: this is the reversible case where hysteresis is present, depicted in figure 1(b). Finally, if $0 < b \leq \frac{1}{2}$, there is a critical level d_{cr} given by the first positive zero of the function $d \mapsto -bd + d^2/(d^2+1)$, which is such that if $d \geq d_{\mathrm{cr}}$ at $t = 0$, then even shutting down the inflow of phosphorus completely will not induce the lake to revert to the pollution-free state: this irreversible case is illustrated in figure 1(c).

Note that both in the reversible and the irreversible case, if the inflow of pollutants increases slowly and the state of the system is near the low pollution steady state a catastrophic shift occurs. At the steady state, a slow increase in the pollution level induces an equally slow response in the level of pollution. There is a critical level ℓ_{cr} such that for $\ell > \ell_{\mathrm{cr}}$ the clean steady state disappears only the high pollution steady state remains. Consequently the level of pollution starts to increase until it reaches the polluted steady state. We say that the system has undergone a catastrophic shift, cf. Rinaldi and Scheffer (2000); Thom (1972); Zeeman (1977).

In systems prone to catastrophic shifts, such as the environment, a relatively small decrease in ℓ from just above ℓ_{cr} to just below it, can result in very different long-run outcomes. This often depends on the initial conditions. When managing such systems, this has to be taken into account.

2.2 The standard shallow lake optimal management problem

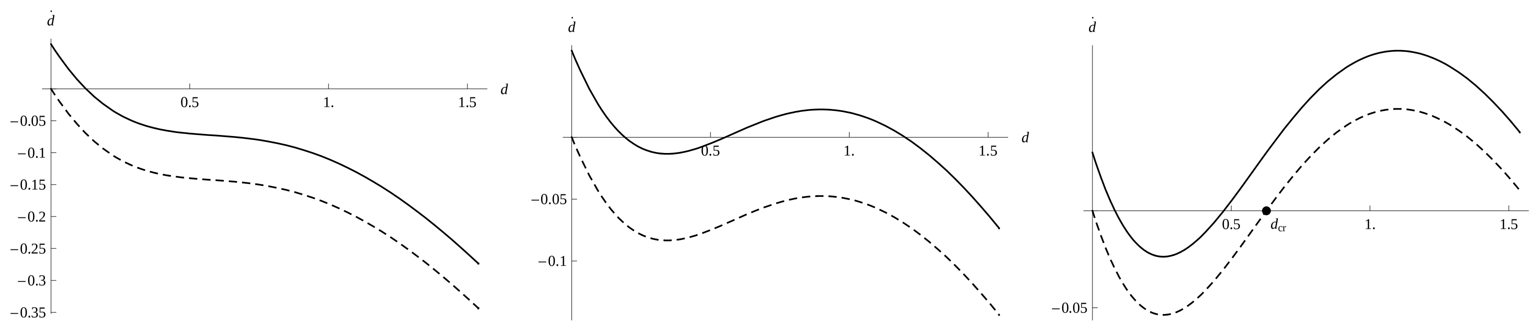
Mäler et al. (2003) introduced the following optimal control problem:

$$\max_{\ell(\cdot)} \int_0^\infty \left[\log \ell(t) - \frac{1}{2} a d(t)^2 \right] \mathrm{e}^{-\rho t} \mathrm{d}t \tag{2}$$

subject to the shallow lake dynamics (1) and the initial condition $d(0) = d_0$. The parameter a determines the relative weight assigned to the environmental damage and ρ is the discount rate. We will refer to this optimisation problem as the standard shallow lake problem. Note that utility is derived both from a higher inflow of pollutants, associated with higher production and hence higher consumption, and a lower level of environmental damage. The optimal inflow of pollutants is determined by this tradeoff.[2]

Depending on the values of the parameters a, b and ρ, one, two or three three steady states of the state-costate system; if there are three, they can be ordered by increasing

[2]For an analysis of this model see also Wagener (2003).



(a) Monotonous case, $b \geq 3\sqrt{3}/8$

(b) Reversible case, $\frac{1}{2} < b < 3\sqrt{3}/8$

(c) Irreversible case, $0 \leq b \leq \frac{1}{2}$

Figure 1: *Change in pollution level as a function of pollution. Solid lines: dynamics with positive loading; dashed lines: dynamics in the absence of loading.*

levels of pollution. The steady states with the lowest and the highest level of pollution are saddles of the state-costate system; the intermediate steady state is a source. An optimal solution will move towards one of the saddles; which one it converges to depends on the values of the parameters and the initial conditions.

In particular, for a large set of parameters there is a threshold state d_*, which is such that if the initial level of pollution satisfies $d(0) > d_*$, the optimal policy moves the system towards the highly polluted steady state. On the other hand, if $d(0) < d_*$, the policy maker lead the system to the low pollution level steady state. Note that in the irreversible case necessarily $d_* \leq d_{\mathrm{cr}}$. If $d(0) = d_*$, there are two possibilities: either there are two optimal solutions originating at d_*; then d_* is called an indifference threshold (or strong Skiba point). Or there is only a single optimal solution, for which $d(t) = d_*$ for all t. Then d_* is a threshold steady state (or weak Skiba point). See Wagener (2003); Kiseleva and Wagener (2008); Grass, Caulkins, Feichtinger, Tragler, and Behrens (2008).

2.3 Adding a capital stock

In this article, we extend the standard shallow lake model by adding a capital stock. Capital produces output and emissions of a pollutant. Output is either consumed, invested in capital or used for pollution abatement, which can be interpreted as investment in environmental capital. Investment determines the growth of capital.

We examine two economies. In the first, a social planner determines the split between consumption and investment; this is taken as the socially desirable benchmark. In the second economy, the production sector is competitive and consumers save to smoothen their consumption paths. In both economies, we allow for the possibility that emissions can be abated. Abatement lowers the inflow of pollution, but the cost of abatement also lower investment in capital. The abatement fraction τ is a policy instrument, and is taken to be constant. One interpretation is that reflects the slowness of political processes. Or a second interpretation would be that the fraction τ is resulting from some international environmental agreement and can therefore not be changed easily.

Let $K(t)$ denote the capital stock available in the economy at time t. We assume that population size and by implication the labour force $L = L(t)$ are constant. Capital and labour are used to produce output $Y = F(K, L)$, where $F(\cdot,\cdot)$ is a production function that is homogenous of the first degree. In per capita terms, we have

$$\frac{Y}{L} = F\left(\frac{K}{L}, 1\right). \tag{3}$$

Define $y = \frac{Y}{L}$ (output per capita), $k = \frac{K}{L}$ (capital per capita) and $f(k) = F(k, 1)$. Then $y = f(k)$, where $f(\cdot)$ is assumed to be increasing and concave. As a parametric form we will use $f(k) = \frac{1}{1-\vartheta}k^{1-\vartheta}$.

We assume that the capital stock produces ηk units of pollution (where η is the conversion factor) of which a fraction τ is removed. The cost of removing one unit of pollution per unit of time is π. A tax of size $\pi\tau\eta k$ is levied to pay for the abatement. We will refer to τ as the level of abatement or as the tax rate (since tax and abatement amount the same in this model). This differs from Mäler et al. (2003), where a pigovian tax whose revenue is returned as a lump-sum transfer, is introduced. We cover the cost of abatement by a lump-sum tax.

The national income identity is

$$y = c + i + \pi\tau\eta k, \tag{4}$$

where c the level of consumption, i investment in capital and $\pi\tau\eta k$ the cost of cleaning a fraction τ of pollution (or investment in environmental capital if you will). We can without loss of generality set η equal to one (see the appendix for details). The growth of capital is governed by the following equation:

$$\dot{k} = i - \delta k, \tag{5}$$

where δ is the rate of depreciation.

There is an infinitely-lived representative agent whose preferences are given by:

$$J = \int_0^\infty \left[\log c(t) - \frac{1}{2}ad(t)^2\right] \mathrm{e}^{-\rho t}\mathrm{d}t. \tag{6}$$

This is also the welfare function for the social planner. The two models, the social planner benchmark and the competitive equilibrium, differ only in how the level of consumption is determined. In the social planner benchmark, consumption is chosen such that it maximises welfare. In the competitive equilibrium, the consumer maximises his utility subject to his budget constraint. The prices and his wage are set in a competitive market. Moreover, the consumer realises that his influence on the state of the environment is infinitesimal and subsequently takes environmental damage as given. Details of both models are given in the appropriate sections.

Finally, we want to be able to control the speed at which capital moves relative to environmental damage. Therefore the ε-parameter is introduced and the evolution of capital is given by:

$$\dot{k} = \varepsilon(i - \delta k) \tag{7}$$

where small (large) values of ε implies that capital moves more slowly (quickly). The appendix gives details of the derivation. The net inflow of pollutants is $(1 - \tau)k \geq 0$, i.e. the part of pollution that is not abated. The dynamics of environmental damage then becomes:

$$\dot{d} = (1 - \tau)k - bd + \frac{d^2}{d^2 + 1}. \tag{8}$$

3 Optimal capital investment policies

3.1 The optimization problem

In this section, the effects of an optimal capital investment policy are analyzed. Recall that capital dynamics are given by:

$$\dot{k} = \varepsilon\Big(i - \delta k\Big), \tag{9}$$

where the investment level i is determined by the requirement that the output y equals investment plus consumption:

$$y = f(k) = i + c + \pi\tau k. \tag{10}$$

Eliminating i, the problem is to determine a consumption path c that maximises

$$J = \int_0^\infty \left[\log c - \frac{1}{2}ad^2\right] \mathrm{e}^{-\rho t}\,\mathrm{d}t, \tag{11}$$

subject to

$$\dot{k} = \varepsilon\Big(f(k) - c - (\delta + \pi\tau)k\Big), \qquad k(0) = k_0, \tag{12}$$

$$\dot{d} = (1-\tau)k - bd + \frac{d^2}{1+d^2}, \qquad d(0) = d_0. \tag{13}$$

To obtain the state-costate equations, form the Pontryagin function[3]

$$P = \log c - \frac{a}{2}d^2 + \varepsilon p\Big(f(k) - c - (\delta + \pi\tau)k\Big) + q\Big((1-\tau)k - bd + \frac{d^2}{1+d^2}\Big). \tag{14}$$

Applying the maximum principle yields for the control

$$0 = \frac{\partial P}{\partial c} = \frac{1}{c} - \varepsilon p; \qquad c = \frac{1}{\varepsilon p}. \tag{15}$$

The Hamilton function can then be written as

$$H = -\log \varepsilon p - 1 - \frac{1}{2}ad^2 + \varepsilon p\Big(f(k) - (\delta + \pi\tau)k\Big) + q\Big((1-\tau)k - bd + \frac{d^2}{1+d^2}\Big). \tag{16}$$

The reduced canonical equations then take the form

$$\dot{k} = \frac{\partial H}{\partial p} = \varepsilon\left(f(k) - \frac{1}{\varepsilon p} - (\delta + \pi\tau)k\right), \tag{17}$$

$$\dot{d} = \frac{\partial H}{\partial q} = (1-\tau)k - bd + \frac{d^2}{1+d^2}, \tag{18}$$

$$\dot{p} = \rho p - \frac{\partial H}{\partial k} = p\Big(\rho + \varepsilon(\delta + \pi\tau - f'(k))\Big) - (1-\tau)q, \tag{19}$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial d} = q\left(\rho + b - \frac{2d}{(d^2+1)^2}\right) + ad. \tag{20}$$

3.2 Investigation of the reduced canonical equations

Before we proceed with the computation of the optimal consumption path, we first want to get some insight in the qualitative features of the dynamical system described by the reduced canonical equations: in particular the number of steady states for different values of the parameters. The main parameters of interest will be the level of abatement (or tax rate) τ and the sedimentation rate b.

Throughout the paper the other parameters will have the following values: $\varepsilon = \frac{1}{100}$, $\vartheta = \frac{3}{4}$, $\pi = 5$, $\delta = \frac{4}{5}$, $\rho = 0.045$ and $a = 1$. By taking ε this small, we ensure that capital moves slowly compared to environmental damage. The other parameters do not affect the qualitative features of the system other than ensuring that the consumer cares enough about the environment and abatement is not prohibitively costly (high a and low π). Otherwise the optimal policy would be to 'ignore' environmental damage, which is neither true nor interesting to study.

Figure 2 shows the bifurcation diagram. We either have one saddle or two saddles, depending mainly on the value of b. If the the lake is irreversible or slightly hysteretic, there

[3] Also called pre-Hamilton or Hamilton function.

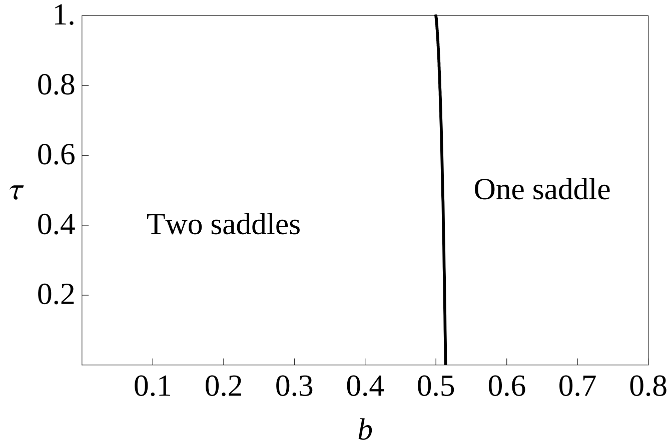


Figure 2: *Bifurcation diagram with respect to b and τ. The solid line indicates a fold bifurcation.*

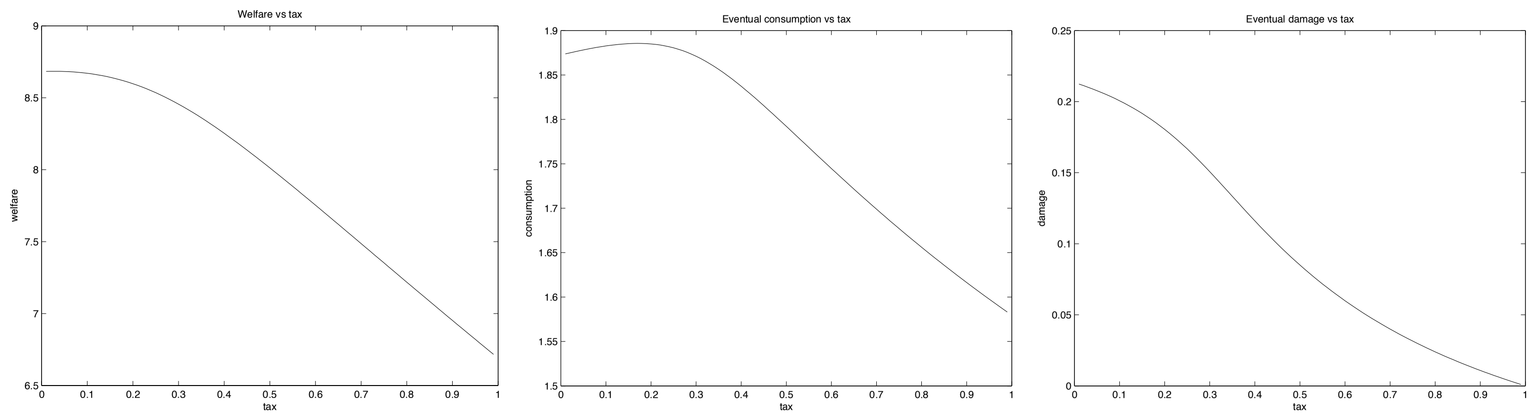
are two saddles. For $b \gtrapprox 0.51$ there is a single saddle. Observe that the value of τ hardly matters: the use of abatement does not seem to influence the number of saddle points greatly. If $b \lessapprox 0.51$, the situation is more complicated. We leave a full bifurcation analysis to future work and concentrate here on an exploration of the properties of the model.

3.3 Results

In this section we fix b equal to 0.45, i.e. we focus on the irreversible case. The critical level of damage d_{cr} for this value of b is 0.627. It implies that we are in the region with three steady states (cf. Figure 2). The initial level of capital will be set to $k_0 = 0.01$. This is much lower than the steady state level of capital. Our focus is therefore on developing economies. We start by examining the effects of changing the level of abatement for a given initial level of damage. Then we proceed by looking at the optimal level of abatement for different initial levels of damage.

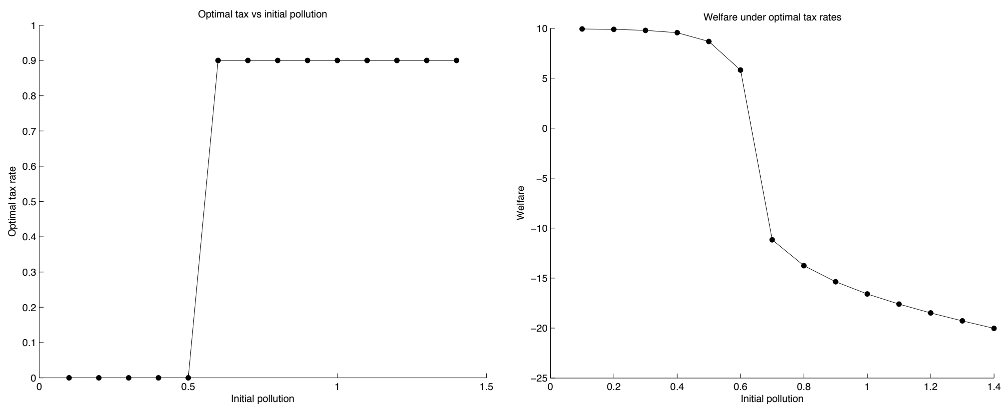
Figure 3 shows the effect of varying the level of abatement if $d_0 = 0.5$. As τ increases, the asymptotic level of damage becomes lower as was to be expected. The effect on the asymptotic level of consumption is more interesting. It seems that a higher level of abatement can be used to increase the asymptotic level of consumption without hurting the environment in the long run. That is, the extra pollution is more than offset by the extra abatement, which is made possible by higher production. However, this effect is small and eventually high abatement will lower consumption. Due to dynamical effects, the total effect on welfare is monotonic. More abatement leads to lower welfare. Hence, the optimal level of abatement is zero when $d_0 = 0.5$.

Since we are interested in history dependence, we will let the initial level of damage d_0 vary between 0.1 and 1.4. We have determined the optimal level of τ by computing the optimised welfare functional J for $\tau = 0, 0.1, \cdots, 1$, and taking the largest. This relatively crude procedure is imposed by computational restrictions. The results for various initial values of environmental damage are given in Figure 4. The social planner will not abate if $d_0 \leq 0.5$. For higher initial values the level of abatement will almost be maximal. This increase in abatement is accompanied by a huge drop in welfare. (Welfare is decreasing in initial damage as



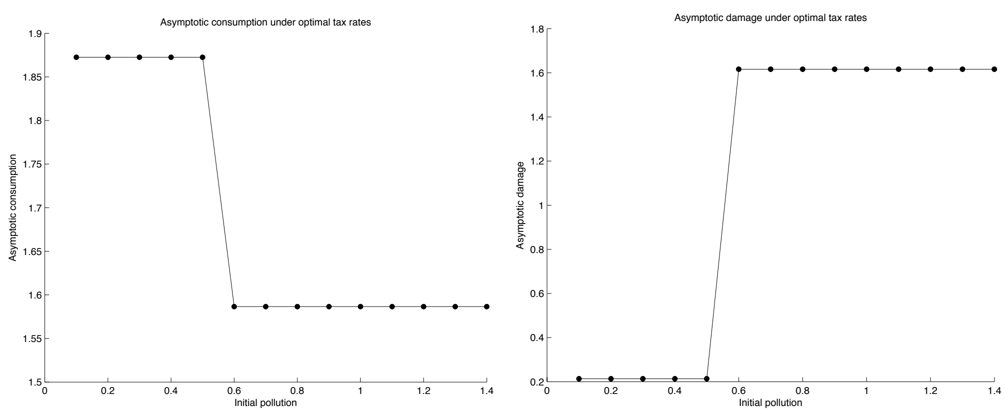
(a) Welfare (b) Consumption (c) Damage

Figure 3: *Welfare, asymptotic optimal consumption and damage as function of the tax rate for* $d_0 = 0.5$.



(a) Optimal tax (b) Welfare

Figure 4: *Optimal tax level as function of initial environmental damage, and the corresponding welfare levels.*



(a) Consumption (b) Damage

Figure 5: *Asymptotic optimal consumption and damage as function of initial environmental damage, and the corresponding welfare levels.*

expected.) Figure 5 shows that at this point, the system shifts from the clean steady state to the dirty steady state. Given the irreversibility of the lake, this is unavoidable, but it happens before the critical level of damage. Avoiding the clean steady state is possible, but too costly. Now the abatement serves a purpose as it is used to reduce the inflow of pollution and make the dirty steady state less dirty. The tradeoff is that it lowers consumption. Observe that the relative decrease in the asymptotic level of consumption is small compared to the relative increase in the asymptotic level of damage (approx. 18% vs. 700%). But if damage is high enough, the social planner is willing to make this tradeoff. Note that abatement serves as a tool to counter the negative effects of the high damage steady state. It is not used to prevent a catastrophe.

4 The competitive equilibrium

4.1 The Ramsey model

Our model for the competitive case is based on the Ramsey model. The derivation is standard (cf. Heijdra and van der Ploeg, 2002, pp. 422–447). Hence we will just summarise the main assumptions. There is a continuum of identical consumers of mass one aggregated into a single representative consumer who supplies one unit of labour for which he receives the prevailing wage rate $w(t)$. Wage is either consumed or put into a bank account. The money in the bank account $b(t)$ evolves in the following way:

$$\dot{b} = rb - c + w, \tag{21}$$

where $r(t)$ is the rate of interest. The discounted value of the consumer's debt has to be finite at any point in time (the No Ponzi Game or transversality condition).

Subject to these constraints, the consumer then maximises

$$\int_0^\infty \left[\log c - \frac{1}{2}ad^2\right] \mathrm{e}^{-\rho t}\mathrm{d}t \tag{22}$$

Note that utility is separable in consumption and environmental damage. Combined with the assumption of a continuum of consumers, each consumers knows that his impact on $d(t)$ is negligible. Hence the consumer acts as if he maximises $\int_0^\infty [\log c]\mathrm{e}^{-\rho t}\mathrm{d}t$. The environment is a pure externality. Of course the social planner problem of Section 3 did take this externality into account.

The time path of consumption is then given by

$$\dot{c} = (r(t) - \rho)c, \tag{23}$$

where $c_0 = c(0)$ is chosen such that the discounted value of life time income equals the discounted value of life time consumption.

The value of $r(t)$ is determined as follows. We assume perfectly competitive and identical firms who produce one output (the consumption good whose price is the numeraire) using capital $k(t)$ and the one unit of labour available. The marginal productivity of capital is then equal to the factor price:

$$f'(k) = r(t) + \delta + \pi\tau, \tag{24}$$

where the factor price equal to the sum of the rental price, depreciation and the cost of cleaning up pollution. We still have that $\dot{k} = i - \delta k$ and $y = c + i + \pi\tau k$. Combining this all yields the following dynamics for consumption and capital:

$$\dot{k} = f(k) - c - (\delta + \pi\tau)k \tag{25}$$

$$\dot{c} = (f'(k) - \rho - \delta - \pi\tau)c \tag{26}$$

Under our assumptions on $f(\cdot)$, this system has a unique saddle. Given k_0, the value of c_0 is obtained by jumping to the stable manifold of this saddle.

The competitive equilibrium of our model than becomes:

$$\dot{d} = (1 - \tau)k - bd + \frac{d^2}{d^2 + 1}, \tag{27}$$

$$\dot{k} = \varepsilon[f(k) - c - (\delta + \pi\tau)k], \tag{28}$$

$$\dot{c} = \varepsilon(f'(k) - \rho - \delta - \pi\tau)c, \tag{29}$$

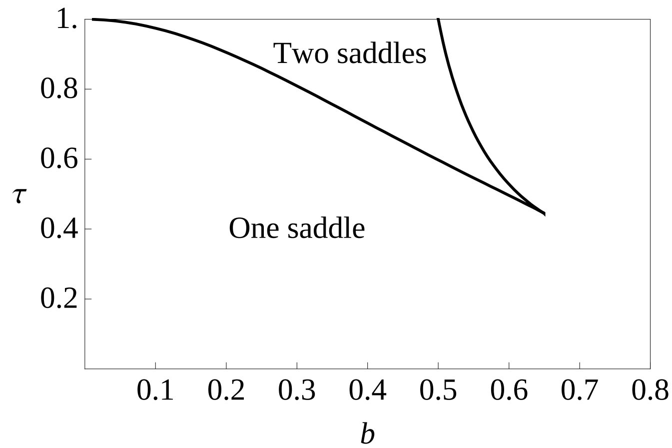


Figure 6: *Bifurcation diagram with respect to b and τ. The solid line indicates a fold bifurcation.*

where $d(0) = d_0$, $k(0) = k_0$ and $c(0)$ is determined as previously discussed. Recall that the ε-parameter determines the relative speed of the capital-consumption subsystem and is derived in the same manner as in Section 3.

4.2 Preliminary analysis of the competitive equilibrium

Observe that the capital and consumption dynamics do not depend on the level of environmental damage; on the other hand the level of capital does influence the change in environmental damage. There is a unique steady state level of capital and consumption, denote this steady state level of capital by $k^*(\tau)$. The steady state level of inflow of pollution is given by $(1-\tau)k^*(\tau)$.

Consequently the steady state level of environmental damage is determined by:

$$0 = (1-\tau)k^*(\tau) - bd + \frac{d^2}{d^2+1} \tag{30}$$

This equation has between one and three roots depending on the value of τ and b. The bifurcation diagram in Figure 6 shows for which values of τ and b this is the case. In case of maximal abatement, there are three steady states if $b < \frac{1}{2}$. For lower values of τ, the number of steady state may decrease to one. The typical phase portrait for the case of three steady state is given in Figure 7. Observe that the social planner's ability to influence the system is mainly to put the system in the basin of attraction of the steady state of choice. If the initial level of damage is high, the social planner may not even have this choice: it can be trapped in the basin of attraction of high pollution steady state. Then the effect of τ is solely to reduce the inflow of pollution and the cost of reducing this inflow is a lower stock of capital and level of consumption.

For $\tau = 1$, the pollution dynamics are the same in the benchmark and the competitive equilibrium since for $\tau = 1$ there is no inflow of pollution. Hence the environment behaves in the same manner. For $b < \frac{1}{2}$, there are two steady states, one with low pollution and one with high pollution. Only the high pollution steady state remains as if $b > \frac{1}{2}$. When we compare Figure 6 to Figure 2, the resemblance becomes less as τ becomes smaller. For small τ the competitive equilibrium always moves to the dirty steady state whereas in the benchmark the clean steady state never disappears.

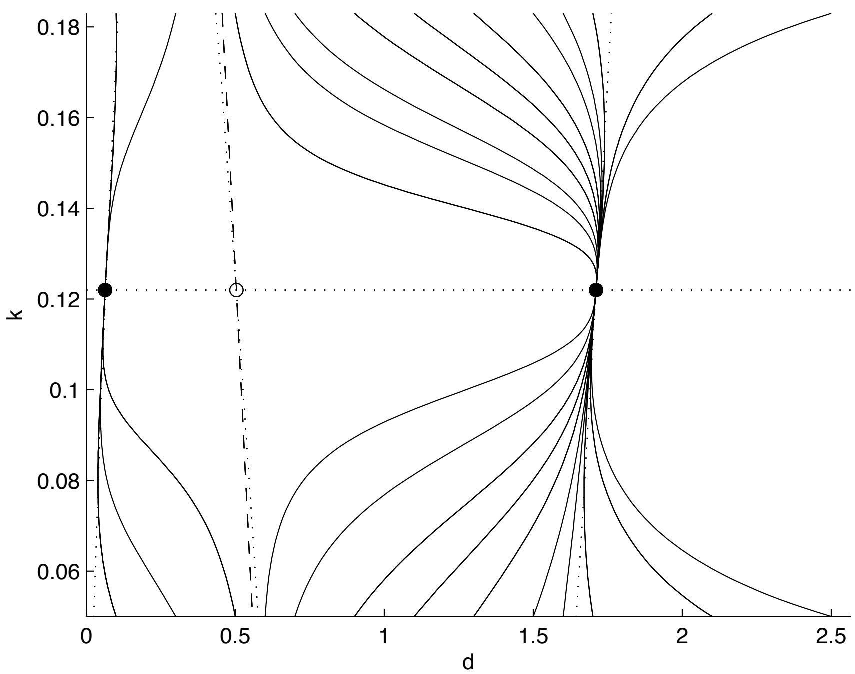


Figure 7: *Typical phase portrait in the competitive case: $b = 0.45$, $\tau = 0.8$. Solid lines indicate trajectories leading to stable steady states. The dashed line is the stable manifold of the saddle separating the two sinks. The dotted lines are the nullclines. Consumption is not shown, but is always proportional to the capital stock.*

4.3 Result

As in the social planner benchmark, we fix b equal to 0.45. Depending on τ, this system will either have three steady states or one (cf. Figure 6). Again the initial level of capital will be set to $k_0 = 0.01$.

First, to see what the effect of abatement is in the competitive equilibrium, we pick $(k_0, d_0) = (0.01, 0.5)$ and vary τ. See Figure 8. We see that τ has a purely monotonic effect. Increasing abatement will lead to lower steady state levels of consumption and damage. For some point in the interval $\tau \in (0.6, 0.7)$ asymptotic damage jumps down, but asymptotic consumption (and capital) adjust smoothly. In the social planner benchmark, the steady state level of consumption was more or less constant, whereas here consumption almost halves over the range of possible abatement levels.

Note that the large change in welfare is before the flipping point where the system jumps from a high steady state level of damage to a low steady state level of damage. Again observe that the flipping point is before the critical level of damage as in the social planner benchmark. The actual change in welfare after the jump is miniscule in comparison, but there is an interior maximum around $\tau = 0.7$.

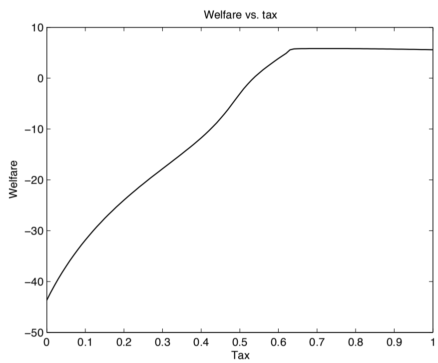
Second, the initial level of damage d_0 will vary between 0.1 and 1.4. We have determined the optimal level of τ by computing the welfare functional J for $\tau = 0, 0.1, \cdots, 1$, and taking the largest. Here the relatively crude procedure is not imposed by computational restrictions, but to make the results comparable to the social planner benchmark. The results for various initial values of environmental damage are given in Figure 9. For increasing levels of initial damage, the level of abatement rises. As we see in Figure 10, the purpose of this increase is to end up in the steady state with low environmental damage. If τ increases above a certain threshold, even the maximum level of abatement will not force the system to the clean steady state. As in the benchmark, in the competitive equilibrium (almost) full abatement for high initial levels of damage is optimal and this is reflected in the asymptotic level of consumption, which is declining in the initial level of damage. Limited control implies that an initially more damaged environment is bad in every possible way.

5 Perfect competition versus the benchmark

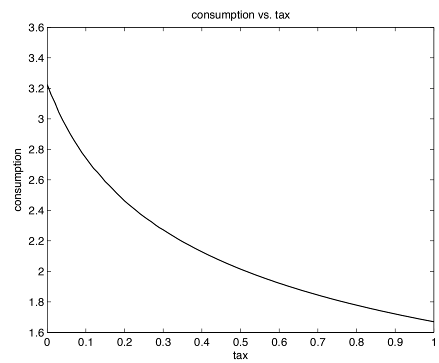
For high initial values of damage, $d_0 > 0.6$, the two cases are roughly similar. In both cases, abatement is high. The asymptotic level of consumption and damage are comparable. Moreover the difference in welfare is relatively small.

The differences occur when the initial value of damage d_0 is lower than 0.6. Here abatement is used to force the system to the clean steady state. Both in the competitive equilibrium and the social planner benchmark, they succeed in doing so. Since in the competitive equilibrium abatement is the only instrument to achieve this, the level of abatement will be high compared to the benchmark. Consequently the difference in welfare is high.

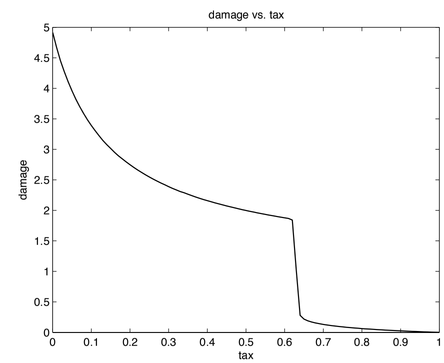
We conclude that in the case of the competitive equilibrium an environmental catastrophe is avoided when it is socially optimal. However large differences in welfare occur when the catastrophe is avoided as the competitive equilibrium is then particularly inefficient.



(a) Welfare

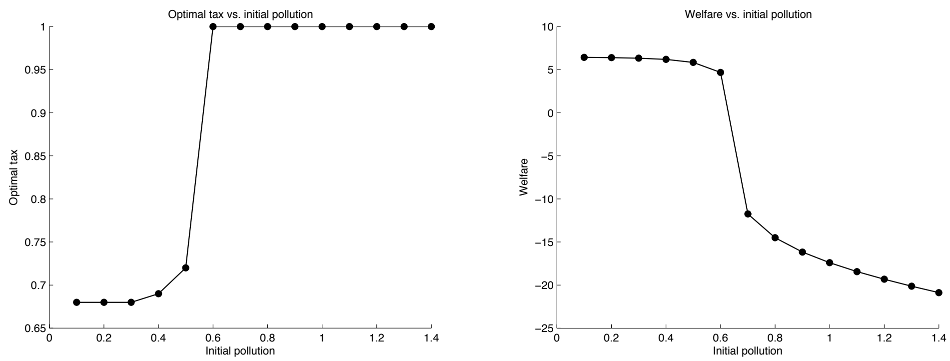


(b) Consumption



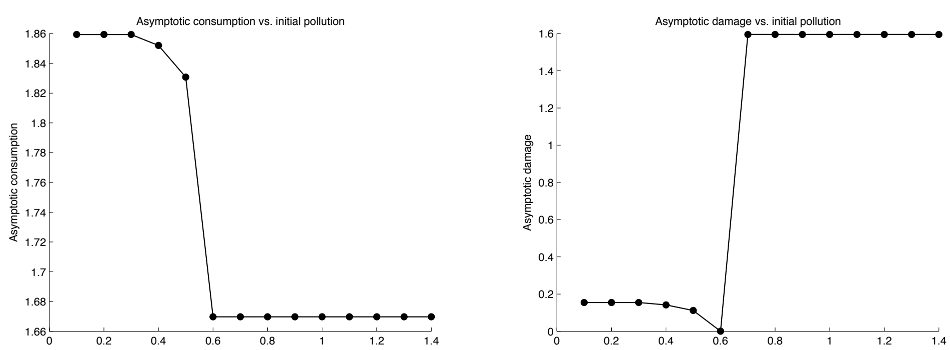
(c) Damage

Figure 8: *Welfare, asymptotic optimal consumption and damage as function of the tax rate for* $d_0 = 0.5$.



(a) Optimal tax (b) Welfare

Figure 9: *Optimal tax level as function of initial environmental damage, and the corresponding welfare levels.*



(a) Consumption (b) Damage

Figure 10: *Asymptotic optimal consumption and damage as function of initial environmental damage, and the corresponding welfare levels.*

6 State-dependent taxation

One could argue that the difference between the benchmark and the competitive equilibrium will disappear if τ can change continuously over time. Due to computational difficulties, we only examine what happens if τ can be switched at one point in time between two levels.

The idea is the following. When abatement is a fixed constant, it has to serve two conflicting purposes. The first is to make sure that the system ends up in the correct steady state, which is the main purpose at the early stage. The second purpose is to abate efficiently in a neighbourhood of the steady state. This is the main purpose at the later stage when the trajectory is actually near the steady state. If we allow the social planner to change the tax rate at some point in the future, then this conflict should be largely resolved.

The abatement scheme will have the following form:

$$\tau(t) = \begin{cases} \tau_1 & \text{if } 0 \leq t < T, \\ \tau_2 & \text{if } t \geq T, \end{cases} \tag{31}$$

where τ_1, τ_2 and T are decision variables. Recall that τ_1 is used to force the system to a clean steady state and τ_2 is an efficient tax near the steady state. We will choose τ_1 and τ_2 in accordance with these roles and independent of the initial state of the system. Hence the only choice variable will be the switching time T. We allow T to be zero (immediate switching) or infinitely high (never switch).

The most straightforward way to force the system to the clean steady state is to pick $\tau_1 = 1$ (maximal abatement). An efficient tax near the steady state should have the property that once we are near the steady state, the social planner does not want to renege on this commitment. Hence τ_2 is such that it maximises instantaneous welfare subject to being in the steady state (cf. Janmaat and Ruijs, 2007 or Mäler et al., 2003 for similar ideas). We will refer to this level of abatement as the static optimum τ^{OS}, and $\tau_2 = \tau^{OS}$. For the values of the parameters we use here $\tau^{OS} \approx 0.66$.

The switching time T is now determined as follows. At time T the social planner should be indifferent between a level of abatement equal to one and τ^{OS}. Otherwise it would be optimal to pre- or postpone the switch to τ^{OS}. This indifference curve will depend on d and k only.

In the long run, two things can happen. The social planner never decides to switch and the environmental damage will be high. Or the social planner does switch to τ^{OS}. In this case environmental damage ends up being low. The indifference curve together with the basins of attractions of these two steady states will characterise the qualitative picture of the dynamics.

Figure 11 shows the indifference curve and the different basins of attractions. The regions marked (i) are the basin of attraction of the clean steady state P if $\tau = \tau^{OS}$. And any trajectory starting in these two region would have converged to P even without a high initial level of abatement. Trajectories starting in region (i) switch immediately.

Region (ii) is more interesting. Trajectories starting in this region are neither in the basin of attraction of P or Q (if $\tau = 1$), but they move away from Q and will at some point intersect with the dashed line. After the switch to $\tau = \tau^{OS}$, they will converge to P. In this region a temporary sacrifice will lead to a good long-term steady state. Trajectories starting in region (iii) are in the basin of attraction of Q and will move toward Q (or another steady state level with high environmental damage) for any level of abatement.

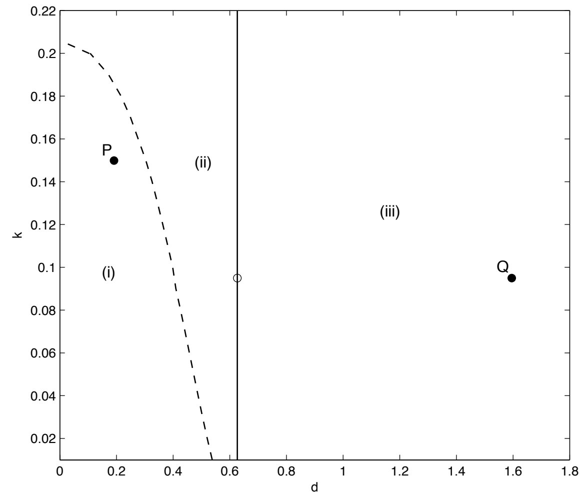


Figure 11: *The optimal switching time. There are two steady-states: P, the clean steady state if $\tau = \tau^{OS}$ and Q, the dirty steady state if $\tau = 1$. Trajectories starting in region (i) will converge to P, in region (ii) they will switch and then converge to P, but they need the initially high level of abatement to converge to P, in region (iii) they will not switch and converge to Q.*

Note that every trajectory that can go to P (as long as T is big enough) will go to P. This can be interpreted that in the case with irreversibilities, any sacrifice in the present is worth the better steady state outcome. This is remarkable considering the relatively high discount rate of 0.045 we have chosen.

7 Hysteresis

So far we have only discussed the case where the dynamics of the environment exhibited irreversibilities (in particular $b = 0.45$). We have seen that if the tax can be changed from $\tau = 1$ to $\tau = \tau^{OS}$, then for every initial state for which it is possible to go to a clean steady state, the system will go to the clean steady state. Because of irreversibilities, it is impossible to reach a clean steady state from some initial levels of damage and capital. With hysteresis it is always possible to go to a steady state with low levels of environmental damage. This suggests that in systems characterised by hysteresis the social planner will almost always go to clean steady states.

In this section, we will set $b = 0.51$. From the bifurcation diagram in Figure 6, we see that there is only one steady state if $\tau = 1$ and this steady state has an steady state damage level of zero. For lower values of τ, including $\tau^{OS} \approx 0.60$, there are three steady states.

Figure 12 shows the optimal fixed abatement as a function of the initial level of damage. There are no major differences with the irreversible case. For $d_0 > 0.8$, it is optimal to set abatement equal to one. This implies that the asymptotic damage level will be zero.

This has major implication for the case where the social planner is allowed to switch from maximal abatement to the optimal static level. Suppose that abatement is at its maximum level for a sufficiently long time. The level of environmental damage will be close to

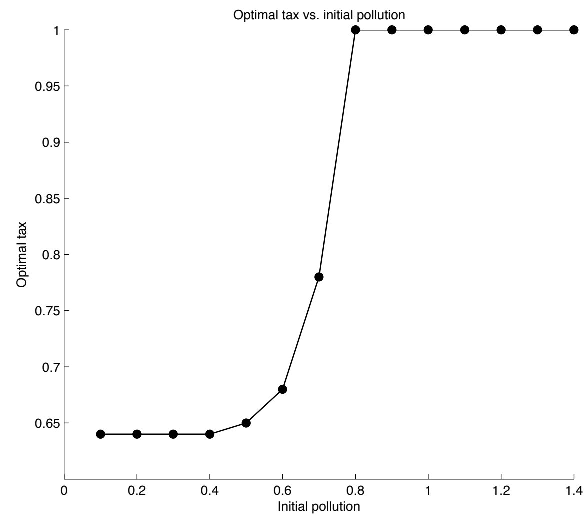


Figure 12: *Optimal tax level as function of initial environmental damage.*

zero and the associated levels of capital and consumption will be low since a substantial amount of the resources will be used to clean up pollution. Computations show that at this point the social planner would like to switch to the optimal static tax. This would lead to higher levels of environmental damage, but it would be compensated by higher levels of consumption.

If at the steady state level associated with $\tau = 1$ the social planner strictly prefers to switch to τ^{OS}, then the social planner will eventually switch to τ^{OS}. Hence for the case with hysteresis the system will always reach the steady state that maximises instantaneous welfare.

Note that this is only true if, for trajectories with high initial levels of damage, the social planner prefers to set $\tau = 1$ above $\tau = \tau^{OS}$. Otherwise the system ends up in the high pollution steady state associated with $\tau = \tau^{OS}$. The argument crucially depends on abatement not being prohibitively expensive.

8 Sensitivity analysis

In this section we will examine the case where the relative weight of the environment is much smaller, but also abatement is much cheaper. We will return to the case of the irreversible case where $b = 0.45$. The parameter values will be $\varepsilon = \frac{1}{100}$, $\vartheta = \frac{3}{4}$, $\pi = 1$, $\delta = \frac{4}{5}$, $\rho = 0.045$ and $a = \frac{1}{20}$. This section has the same structure. Section 3 and we first examine the effect of abatement for a particular initial level of damage. Then we proceed by examining the optimal level of abatement for different initial values of damage.

For the particular initial state $(k_0, d_0) = (0.01, 0.5)$, welfare, asymptotic consumption and asymptotic damage are given as functions of the level of abatement in Figure 13. First, we see that intitially higher levels of abatement lead to higher levels of consumption while the level of environmental damage hardly changes. At some point this can be no longer sustained and both consumption and damage decline steeply. This is explained as follows.

Higher abatement has two effects. First, a higher value of τ implies a higher level of capital is possible without making the environment worse off. This higher level of capital can sustain higher levels of consumption as well. Second, a higher value of τ also implies higher abatement costs which depress the accumulation of capital. This explains why asymptotic consumption first increases when there is more abatement, but also why it has to go down eventually. Abatement creates the possibility to both consume and have a clean environment. In this case, the optimal level of τ is positive and around 0.3.

The results for various initial values of environmental damage are given in Figure 14. We observe a sharp increase of the optimal rate of τ as the initial damage increases up to $d_0 = 0.5$, after which it falls to zero. Welfare drops sharply at the same point.

Figure 15 provides an explanation of these drops in terms of the asymptotic optimal consumption and optimal damage levels. Up to and including $d_0 = 0.5$, imposing abatement has the effect that the optimal investment strategy ends at the clean steady state. However, the optimal level of abatement has to increase sharply. When this is not economically feasible any more, the system will end up at the polluted steady state. For these solutions, the best choice for the level of abatement is $\tau = 0$. Note the small consumption gains as contrasted to the big increase in pollution levels.

Compared to the results in Section 3, two things are noticeable. First, the asymptotic level of consumption is increasing in the initial level of damage. Second, abatement is only used for low levels of initial damage to avoid a catastrophe. If the initial level of damage is high, then the social planner will cease all abatement and hence the asymptotic level of consumption will grow larger, because the social planner does not abate for higher levels of initial damage. It is not clear why the social planner stops abating. Given the willingness to abate for lower levels of initial damage, we could conclude that abatement is relatively cheap for these values of a and π, but then we would also expect full abatement for high levels of initial damage to alleviate the dirty steady state.

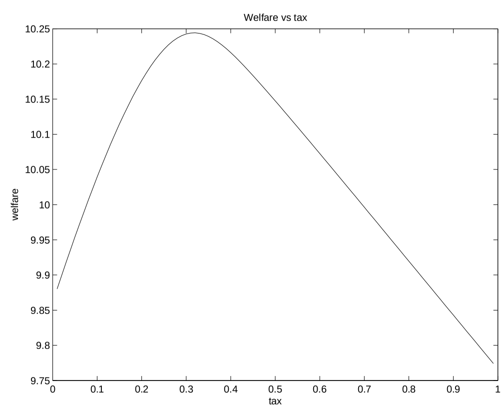
This suspicion is raised further if we turn our attention to the competitive case for these parameter values. Abatement is apparently cheap and it is optimal to fully abate for every initial value of d. Therefore the asymptotic level of consumption is independent of the initial level of damage and there is threshold for which the steady state shift from clean to dirty.

The behaviour in both cases seems to depend on the values of a and π in a non-trivial manner. The relative cost of abatement should be decreasing in a and increasing in π. Although we would expect the relative cost of abatement to go in the same direction in both cases. So, higher levels of abatement in the social planner benchmark should coincide with higher levels of abatement in the competitive case, bit this does not happen.

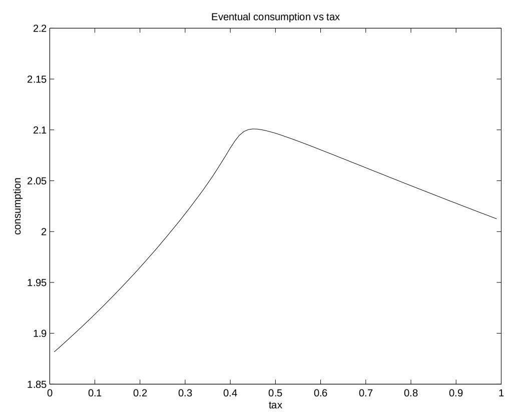
Still, there are similarities to the original simulations. First, disasters are avoided when it optimal to do so according to the benchmark. Second, it seems that the use of abatement in the competitive case (when not used by the social planner in the benchmark) leads to high disparities in welfare. Here disasters are badly managed, in the original simulations they were poorly avoided. In both cases, the competitive equilibrium made use of abatement whereas the benchmark did not.

9 Concluding remarks

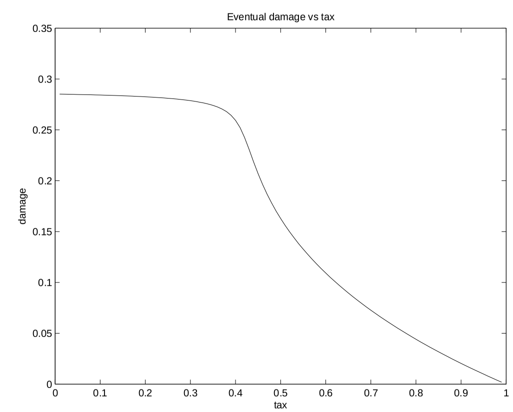
In this paper, we have presented a first foray in adding a (simple) model of the economy to the standard shallow lake model. The social planner benchmark was compared with a competitive economy. In both scenarios, we used the level of abatement as policy variable.



(a) Welfare

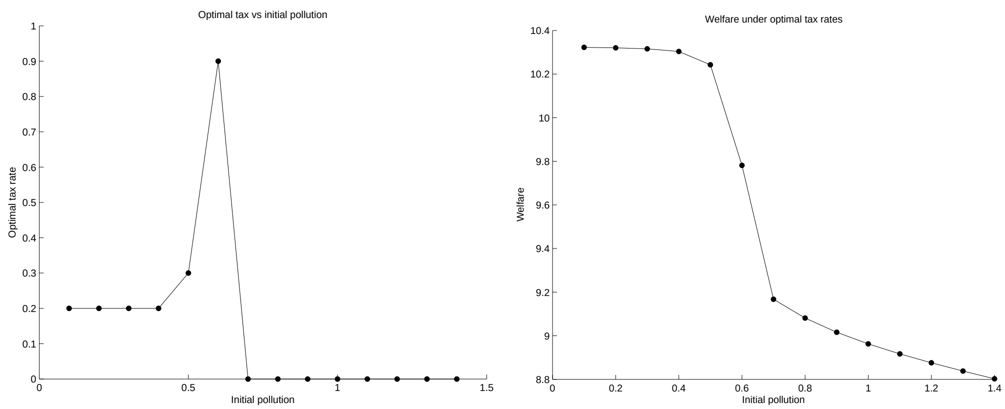


(b) Consumption



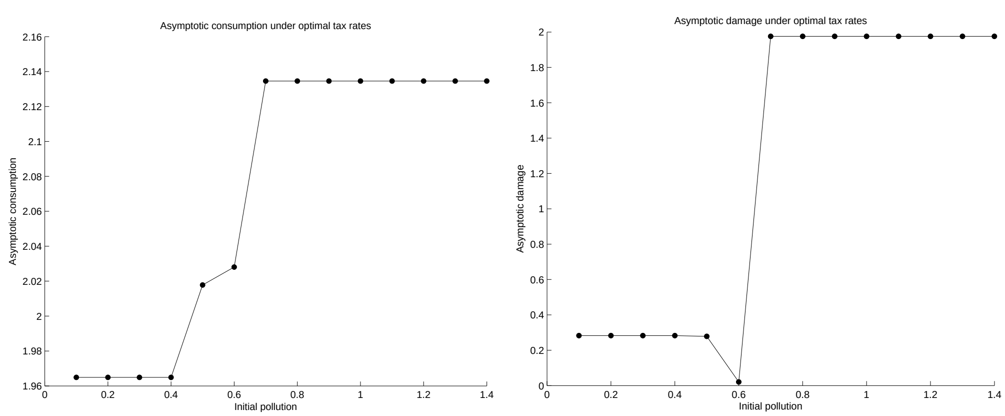
(c) Damage

Figure 13: *Welfare, asymptotic optimal consumption and damage as function of the tax rate for* $d_0 = 0.5$.



(a) Optimal tax (b) Welfare

Figure 14: *Optimal tax level as function of initial environmental damage, and the corresponding welfare levels.*



(a) Consumption (b) Damage

Figure 15: *Asymptotic optimal consumption and damage as function of initial environmental damage, and the corresponding welfare levels.*

In the irreversible case, a fixed level of abatement was sufficient to avoid catastrophe (when in the benchmark the social planner would also have avoided it). This happened for low initial levels of environmental damage. Large disparities in welfare can occur when abatement is used extensively in the competitive case, but not in the socially optimal benchmark. In the competitive equilibrium, either the catastrophe is avoided in an inefficient way or the catastrophe is badly managed.

Next the effect of a simple state-dependent level of abatement on the competitive economy was investigated. The social planner was allowed to switch from an emergency level of abatement (i.e. removing all pollution) to the efficient level of abatement at some point in time. The effect of this was that the system moved to a clean steady state whenever possible.

These insight were used to study the case where the system exhibited hysteresis. It was shown that the state-dependent level of abatement always steered the system towards a clean steady state. The reason behind this is that, for high initials level of pollution, the social planner prefers to abate all pollution. With hysteresis, this implies that eventually environmental damage will be low (and then the planner will change towards the social optimal tax). However, since this is so far into the future, the decision to abate maximally will be mainly influenced by current conditions. Tentatively, we conclude that if the current initial state is bad enough to force the social planner to abate and hysteresis is present, then action will be taken when it is needed.

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A Parameter rescaling

The starting point is the following system:

$$\frac{\partial d}{\partial \tilde{t}} = (1-\tau)\tilde{\eta}k - \tilde{b}d + \tilde{\gamma}\frac{d^2}{d^2+1}, \tag{32}$$
$$\frac{\partial k}{\partial \tilde{t}} = f(k) - c - \tilde{\pi}\tau\tilde{\eta}k - \delta k, \tag{33}$$

where a parameter with a tilde corresponds to the parameters without tilde employed in the main text. Note that we have introduced a parameter $\tilde{\gamma}$, which denotes the strength of the feedback. In the standard formulation of the shallow lake model the parameters are already rescaled such that $\tilde{\gamma}$ is equal to one. Moreover we have substituted $i = f(k) - c - \tilde{\pi}\tau\tilde{\eta}k$ in the last equation.

We will change the time scale. The new time scale t is such that $t = \frac{1}{\epsilon}\tilde{t}$, making the new time scale a factor ϵ slower than the original time scale. This also changes the parameters. The sedimentation rate, measured in units per unit of time, changes by a factor $\frac{1}{\epsilon}$ when measured on the new time scale. In particular:

$$\eta = \frac{\tilde{\eta}}{\epsilon}, \quad b = \frac{\tilde{b}}{\epsilon}, \quad \gamma = \frac{\tilde{\gamma}}{\epsilon} \text{ and } \quad \pi = \epsilon\tilde{\pi}. \tag{34}$$

This leads to the following expressions for $\dot{d}$ and $\dot{k}$:

$$\dot{d} = \frac{\partial d}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = (1-\tau)\eta k - bd + \gamma\frac{d^2}{d^2+1}, \tag{35}$$
$$\dot{k} = \frac{\partial k}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = \epsilon(f(k) - c - \pi\tau\eta k - \delta k). \tag{36}$$

Along similar lines by rescaling t (again) and k we can set $\eta = \gamma = 1$ yielding equations (8) and (7). Rescaling t and k implies that the following parameters have to be rescaled: π, ϵ, δ as well as the units in which consumption c is measured.