On the Leitmann equivalent problem approach

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20th June 2008

Abstract

The purpose of this note is to show how Leitmann's equivalent problem approach ties in with the classical notions of the Calculus of Variations, and how it can be exploited to give a rapid and elegant approach to Weierstrass' theory of sufficient conditions. Both fixed and free endpoint conditions are considered.

Keywords Calculus of variations, equivalent problems, rectifying coordinates, sufficient conditions

1 Introduction

George Leitmann has introduced a method of transforming a calculus of variations problem into an equivalent, and possibly simpler, form (Leitmann, 1967, 2001). This method consists of performing a coordinate transformation of the state space, followed by a transformation to an equivalent variational problem, in the sense of Carathéodory, by subtracting a null Lagrangian, that is, a total derivative of a function that depends on time and space. Carlson and Leitmann have recently pointed out that the equivalent problem takes a particularly simple form if the coordinate transformation is furnished by a field of extremals (Carlson and Leitmann, 2008a,b); in fact, they obtain the so-called Weierstrass representation formula (see Giaquinta and Hildebrandt, 1996, chapter 6, p. 333) with a particular simple form of the Weierstrass excess function.

The elements of this approach are well-known. Euler himself already noted the covariance of the Euler-Lagrange equation under coordinate transformations (see Euler, 1744, chapter IV), Weierstrass introduced fields (Kneser, 1900; Weierstrass, 1927) and Carathéodory formulated his "royal road" approach using equivalent variational problems (Carathéodory, 1935). It however appears that the natural idea to view the field of extremals as a coordinate transformation is a new contribution to the classical theory.

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†The author would like to thank George Leitmann and Saeed Mohammadian Moghayer for helpful remarks and discussions. This work has been supported by NWO under a MaGW-VIDI grant.

2 Equivalences of variational problems

George Leitmann has introduced a method of transforming a calculus of variations problem into an equivalent, and arguably simpler, form (Leitmann, 1967, 2001), complemented by Carlson's discussion in (Carlson, 2002). I shall discuss his method in the context of minimising a functional

$$J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t. \tag{1}$$

where x is a continuous and piecewise continuously differentiable vector-valued function on the interval $[a, b] \subset \mathbb{R}$, notation $x \in \mathrm{PC}^1([a, b], \mathbb{R}^m)$. That is, there are points $a = t_1 < \cdots < t_n = b$ such that on the intervals (t_i, t_{i+1}), $i = 1, \cdots, n-1$, the derivative $\dot{x}$ exists and is a differentiable functions. The function L is assumed to be at least C^3 on the open set $O \times \mathbb{R}^m \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$.

The case of fixed boundary conditions $x(a) = \alpha$ and $x(b) = \beta$ is investigated first; subsequently, the extension to free boundary conditions is made.

In this note, the derivative of a function L with respect to a variable x is indicated by L_x; that is, $L_x\xi = \sum_{i=1}^m L_{x_i}\xi^i$; a consequence of this notation is that

$$L_{xv}\xi\eta = \sum_{i=1}^m \sum_{j=1}^m L_{x_i v_j}\xi^j\eta^i = \sum_{j=1}^m \sum_{i=1}^m L_{v_j x_i}\eta^i\xi^j = L_{vx}\eta\xi. \tag{2}$$

Attention will be restricted to the case that the integrand L is C^2, and that for fixed values of (t, x), the map $v \mapsto L(t, x, v)$ has a positive definite Hessian matrix $L_{vv}(t, x, v)$.

2.1 Two notions of equivalence. Leitmann's notion generalises the notion of equivalent variational problems introduced by Carathéodory Carathéodory (1935).

Let two points $(a, \alpha), (b, \beta) \in \mathbb{R} \times \mathbb{R}^m$ be given, as well as an open and simply connected set $R \subset (a, b) \times \mathbb{R}^m$ which is such that (a, α) and (b, β) are contained in the closure $\bar{R}$ of R, and such that $\bar{R} \subset O$. Introduce the sets $R_t = \{x \in \mathbb{R}^m \mid (t, x) \in R\}$, and note that R_t is open for every $t \in (a, b)$. Define

$$\mathcal{A} = \Big\{ x \in \mathrm{PC}^1([a, b], \mathbb{R}^m) \;\Big|\; x(t) \in R_t \text{ for all } t \in (a, b),\ x(a) = \alpha,\ x(b) = \beta \Big\}.$$

Moreover, let another open set $R^* \subset (a, b) \times \mathbb{R}^m$ and a diffeomorphism Ξ be given which maps an open set O^* containing R^* diffeomorphically onto O, and which is such that

$$\Xi(t, x) = (t, \xi(t, x)). \tag{3}$$

The conditions on Ξ will be weakened somewhat in section 4. Finally, let α^* and β^* be such that $\xi(a, \alpha^*) = \alpha$ and $\xi(b, \beta^*) = \beta$, and define

$$\mathcal{A}^* = \Big\{ y \in \mathrm{PC}^1([a, b], \mathbb{R}^m) \;\Big|\; y(t) \in R_t^* \text{ for all } t,\ y(a) = \alpha^*,\ y(b) = \beta^* \Big\}.$$

Define the operator $\mathcal{X} : \mathcal{A}^* \to \mathcal{A}$ by setting $(\mathcal{X}y)(t) = \xi(t, y(t))$, and note that $\mathcal{X}$ maps $\mathcal{A}^*$ bijectively onto $\mathcal{A}$.

Definition 2.1
Let Ξ, $\mathcal{A}$ and $\mathcal{A}^*$ be as above. The functionals $J : \mathcal{A} \to \mathbb{R}$ and $J^* : \mathcal{A}^* \to \mathbb{R}$, where

$$J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t \quad \text{and} \quad J^*(y) = \int_a^b L^*(t, y, \dot{y})\,\mathrm{d}t, \tag{4}$$

are *Leitmann equivalent (by Ξ)* if there is a C^1 function $S^* : O^* \to \mathbb{R}$ such that the equation

$$L(t, \xi, \xi_t + \xi_y \dot{y}) = L^*(t, y, \dot{y}) + S^*_t(t, y) + S^*_y(t, y)\dot{y} \tag{5}$$

holds identically in $(t, y, \dot{y}) \in O^* \times \mathbb{R}^m$.

The two functionals are *Carathéodory equivalent*, if equation (5) holds with Ξ being the identity map. □

We have the following well-known theorem (Carathéodory, 1935; Leitmann, 2001; Carlson, 2002); in the context of this note, the proof bears repeating.

Theorem 2.1
If J and J^* are Leitmann equivalent by Ξ, then a minimiser $\bar{y}$ of J^* gives rise to a minimiser $\bar{x}$ of J by setting $\bar{x}(t) = \xi(t, \bar{y}(t))$.

Proof
Let the function $\bar{y}$ minimise J^* over $\mathcal{A}^*$. Note that $\bar{x} = \mathcal{X}\bar{y}$; we want to show that $\bar{x}$ minimises J over $\mathcal{A}$. For this, pick any $x \in \mathcal{A}$, and let $y = \mathcal{X}^{-1}x$. Compute:

$$\begin{aligned} J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t &= \int_a^b \left(L^*(t, y, \dot{y}) + \frac{\mathrm{d}}{\mathrm{d}t} S^*(t, y) \right) \mathrm{d}t \\ &= J^*(y) + S^*(b, \beta^*) - S^*(a, \alpha^*) \geq J^*(\bar{y}) + S^*(b, \beta^*) - S^*(a, \alpha^*) \\ &= \int_a^b \left(L^*(t, \bar{y}, \dot{\bar{y}}) + \frac{\mathrm{d}}{\mathrm{d}t} S^*(t, \bar{y}) \right) \mathrm{d}t = J(\bar{x}). \end{aligned}$$

As x was chosen arbitrarily, it follows that $J(x) \geq J(\bar{x})$ for all $x \in \mathcal{A}$. □

2.2 Equivalence of the equivalences. Given $J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t$ and a continuously differentiable function $\xi(t, x)$ such that $\xi(t, .)$ is a diffeomorphism, introduce

$$\hat{L}(t, y, \dot{y}) = L\Big(t, \xi(t, y), \xi_t(t, y) + \xi_y(t, y)\dot{y}\Big), \tag{6}$$

and consider the corresponding variational problem of minimising $\hat{J}(y) = \int_a^b \hat{L}(t, y, \dot{y})$. Let moreover $J^*(y) = \int_a^b L^*(t, y, \dot{y})\,\mathrm{d}t$. As

$$L(t, x, \dot{x}) = \hat{L}(t, y, \dot{y}) \tag{7}$$

it follows immediately that Leitmann equivalence by Ξ of J and J^* is the same thing as Carathéodory equivalence of $\hat{J}$ and J^*.

3 Simple variational problems

The point of Leitmann's method is that by taking appropriate coordinates, the transformed problem is particularly easy to solve (see Carlson and Leitmann, 2008a). This extends Carathéodory's "royal road" approach to field theory, which I shall sketch briefly.

3.1 The royal road of Carathéodory. In the "royal road" approach, the equivalent problem is required to satisfy the following. For every pair (t, x), there is a vector $v(t, x)$ such that $\dot{x} = v(t, x)$ minimises

$$\dot{x} \mapsto L^*(t, x, \dot{x}). \tag{8}$$

Moreover, the minimum should be equal to 0. If this is the case, it follows immediately that the integral curves of the differential equation

$$\dot{x} = v(t, x) \tag{9}$$

satisfy $J^*(x) = 0$, and that this is indeed the smallest value possible; that is, the integral curves of (9) are absolute minimisers of J.

From the identity

$$L^*(t, x, \dot{x}) = L(t, x, \dot{x}) - S_t(t, x) - S_x(t, x)\dot{x}, \tag{10}$$

the above conditions imply Carathéodory's *fundamental equations*

$$\begin{aligned} L^*(t, x, \dot{x}) &= L(t, x, \dot{x}) - S_t(t, x) - S_x(t, x)\dot{x} = 0, \\ L^*_v(t, x, \dot{x}) &= L_v(t, x, \dot{x}) - S_x(t, x) = 0, \end{aligned}$$

if $\dot{x} = v(t, x)$ minimises $v \mapsto L^*(t, x, v)$. Introducing the costate $p = L_v(t, x, \dot{x})$ and the Hamilton function

$$H(t, x, p) = \max_{\dot{x}} \{p\dot{x} - L(t, x, \dot{x})\}, \tag{11}$$

it follows that the fundamental equations are equivalent to the well-known relations

$$p = S_x(t, x), \qquad S_t + H(t, x, S_x) = 0; \tag{12}$$

the second of these is the Hamilton-Jacobi equation. The point of Carathéodory's approach is that if the Hamilton-Jacobi equation can be solved, then the transformation to the equivalent system is possible, and absolute minimisers are obtained.

3.2 The Tao of Leitmann. Leitmann's approach simplifies the original variational problem by choosing the transformation ξ as the inverse of a rectifying transformation of a field of extremals of the initial minimisation problem. In the Leitmann approach the Hamilton-Jacobi equation is also solved, but only implicitly.

By definition, extremals are solutions of the Euler-Lagrange equation

$$L_x - \frac{\mathrm{d}}{\mathrm{d}t} L_v = 0. \tag{13}$$

As a differential equation of order $2m$, the general solution $x = x(t, c)$ depends on $2m$ parameters, the integration constants $c = (c_1, \cdots, c_{2m})$.

Assume that there is a value $c = \bar{c}$ such that the extremal $\bar{x}(t) = x(t, \bar{c})$ satisfies the boundary conditions $x(a) = \alpha$, $x(b) = \beta$. Let $Y \subset \mathbb{R}^m$ be a simply connected open set. Any map $c : Y \to \mathbb{R}^{2m}$ defines an m-parameter subfamily ξ of extremals

$$\xi = \xi(t, y) = x(t, c(y)). \tag{14}$$

The subfamily ξ *embeds* the extremal x_0 if $c(\bar{y}) = \bar{c}$ for some $\bar{y} \in Y$.

To introduce the notion of a *field* of extremals, define first the covectors

$$\eta = \eta(t,y) = L_v(t, \xi(t,y), \xi_t(t,y)). \tag{15}$$

Technically speaking, the element (ξ, η) is a point in the cotangent bundle $T^*\mathbb{R}^m$; for the purposes of this note $T^*\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$, but in the following I shall be using the bundle notation. Assuming that the vectors $v_i = (\xi_{y_i}, \eta_{y_i})$, $i = 1, \cdots, m$ are linearly independent for all y, then the set

$$\mathcal{F}_t = \left\{ (x,p) \in T^*\mathbb{R}^m \;\middle|\; x = \xi(t,y), \quad p = \eta(t,y), \quad y \in Y \right\} \tag{16}$$

is for every $t \in [a,b]$ an m-dimensional submanifold of $T^*\mathbb{R}^m$. Note that the tangent space to $\mathcal{F}_t$ at a point $(x_0, p_0) = (\xi(t,y_0), \eta(t,y_0))$ is spanned by

$$v_j = \left(\xi_{y_j}(t,y_0), \eta_{y_j}(t,y_0)\right), \qquad j = 1, \cdots, m. \tag{17}$$

If $\omega = \mathrm{d}p \wedge \mathrm{d}x$ is the canonical 2-form (see Arnol'd, 1989), then

$$\omega(v_i, v_j) = \eta_{y_i}\xi_{y_j} - \eta_{y_j}\xi_{y_i}.$$

Carathéodory uses the older "Lagrange bracket" notation $[y_i, y_j]$ instead of $\omega(v_i, v_j)$. Recall that a submanifold $\mathcal{M} \subset T^*\mathbb{R}^m$ is called *Lagrangian*, if $\omega = 0$ on $\mathcal{M}$; that is, $\mathcal{F}_t$ is Lagrangian if all Lagrange brackets vanish identically on $\mathcal{F}_t$. Actually, the vanishing of the Lagrange brackets has only to be verified for a single value of t.

Theorem 3.1
If $\mathcal{F}_{t_0}$ is Lagrangian, then $\mathcal{F}_t$ is Lagrangian for all t.

For completeness sake, the proof of this theorem is given in the appendix.

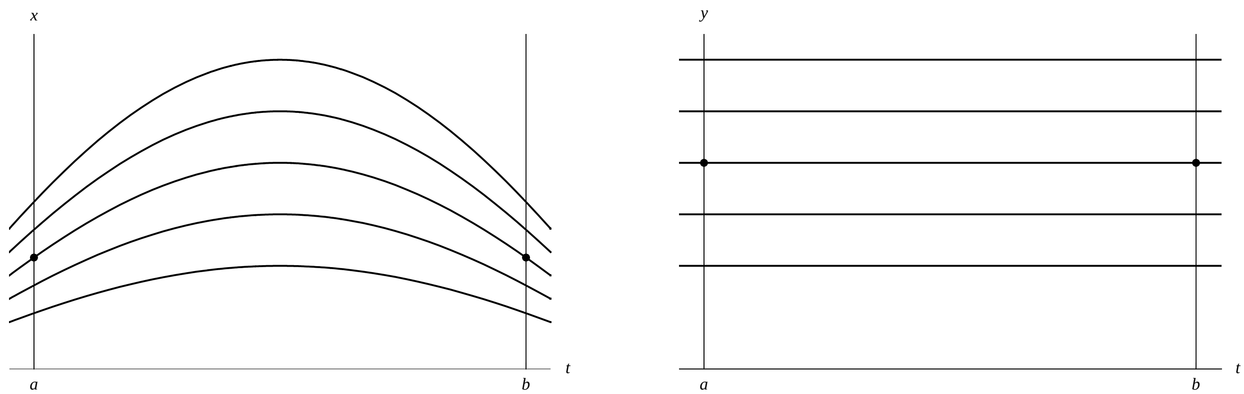
Definition 3.1
Let $\xi : [a,b] \times Y \to \mathbb{R}^m$ be a m-parameter subfamily of extremals. The family ξ defines an *extremal field* of the minimisation problem, if

1. for $t \in (a,b)$, the map $y \mapsto \xi(t,y)$ is a diffeomorphism from Y onto its image; that is $\det \xi_y(t,y) \neq 0$ for all $(t,y) \in (a,b) \times Y$ and $\xi(t,y_1) \neq \xi(t,y_2)$ for any $t \in (a,b)$. In particular, no extremals of the family intersect.
2. The manifold $\mathcal{F}_t$ is Lagrangian for some $t \in [a,b]$; or, equivalently, all Lagrange brackets vanish identically in t.

In this note, a field is called *regular*, if $\xi(t,.)$ is a C^2 diffeomorphism onto its image. □

In the following, typically $R^* \subset (a,b) \times Y$ and $R \subset \Xi((a,b) \times Y)$; that is, the fields of extremals $t \mapsto \xi(t,y)$ with $y \in Y$ *covers* R. Note moreover that the region R^* is simply connected, as it is the diffeomorphic image of the simply connected set R.

If $\xi(t,y)$ is a regular field, then the inverse of $(t,x) = \Xi(t,y)$ is a rectifying transformation of the field; see figure 1, where any extremal $x(t) = \xi(t,y)$ corresponds to a constant value of y. Figure 1 suggests strongly that if to transform the problem a field ξ is used that embeds



(a) Field of extremals (b) Rectified field of extremals

Figure 1: *Taking for the Leitmann transformation the inverse of a rectifying transformation of a field of extremals that embeds an extremal satisfying the boundary conditions, the equivalent Leitmann problem simplifies.*

an extremal satisfying the boundary conditions, then the equivalent variational problem has to have simple solutions.

To make this precise, let

$$\hat{L}(t, y, \dot{y}) = L(t, \xi(t, y), \xi_t(t, y) + \xi_y(t, y)\dot{y}), \tag{18}$$

and expand this expression in the last argument around $\xi_t(t, y)$. Taylor's theorem yields that

$$\hat{L}(t, y, \dot{y}) = L(t, \xi, \xi_t) + L_v(t, \xi, \xi_t)\xi_y\dot{y} + \ell(t, y, \dot{y})\dot{y}^2, \tag{19}$$

where

$$\ell(t, y, \dot{y}) = \frac{1}{2}L_{vv}(t, \xi, \xi_t + \vartheta\xi_y\dot{y})\xi_y^2 \tag{20}$$

for $0 < \vartheta = \vartheta(t, y) < 1$.

Introduce the functions

$$s^0(t, y) = L(t, \xi, \xi_t),$$

and

$$s^i(t, y) = L_v(t, \xi, \xi_t)\xi_{y_i} = \eta\xi_{y_i}, \quad i = 1, \cdots, m.$$

The following observation goes back to Beltrami and Hilbert (Giaquinta and Hildebrandt, 1996, p. 396).

Theorem 3.2
There is a C^2 function $S^*(t, y)$ such that

$$S^*_t = s^0, \qquad S^*_{y_i} = s^i, \quad i = 1, \cdots, m. \tag{21}$$

Proof
This is an integrability statement; since the domain of definition R^* of ξ is simply connected, the function S^* exists if the implied mixed partials of S^* are equal. We have

$$s^0_{y_i} = L_x \xi_{y_i} + L_v \xi_{t y_i}, \qquad s^i_t = \left(\frac{\mathrm{d}}{\mathrm{d}t} L_v\right) \xi_{y_i} + L_v \xi_{y_i t};$$
$$s^i_{y_j} = \eta_{y_i} \xi_{y_j} + \eta \xi_{y_i y_j}.$$

The equation

$$s^0_{y_i} - s^i_t = \left(L_x(t, \xi, \xi_t) - \frac{\mathrm{d}}{\mathrm{d}t} L_v(t, \xi, \xi_t)\right) \xi_{y_i} = 0. \tag{22}$$

holds by virtue of $t \mapsto \xi(t, y)$ satisfying the Euler-Lagrange equation.

Moreover

$$s^i_{y_j} - s^j_{y_i} = \eta_{y_i} \xi_{y_j} - \eta_{y_j} \xi_{y_i} + \eta(\xi_{y_i y_j} - \xi_{y_j y_i}) = 0,$$

by the vanishing of the Lagrange brackets and the equality of mixed partial derivatives. □

Combining this result with equation (19) yields that there is a C^2 function $S^* = S^*(t, y)$ such that

$$L(t, \xi, \xi_t + \xi_y \dot{y}) = S^*_t + S^*_y \dot{y} + \ell(t, y, \dot{y}) \dot{y}^2.$$

Remark that solving the Hamilton-Jacobi equation in the Carathéodory approach reduces to integrating $\dot{S} = L$ along extremals. As in the Leitmann approach the extremals are rectified, this equation is equivalent to $S^*_t(t, y) = \hat{L}(t, y, 0)$.

It is a corollary to theorem 3.2 that if Ξ is taken to be the inverse of a rectifying transformation of a regular field of extremals, then the original variational problem is Leitmann equivalent, by Ξ, to a problem that can be solved by inspection.

Theorem 3.3
Assume that there is a regular field ξ of extremals of $J = \int_a^b L \, \mathrm{d}t$ that covers R, and that the associated map $\Xi(t, y) = (t, \xi(t, y))$ is a diffeomorphism on an open set O^* containing R^*. Then J is Leitmann equivalent by Ξ to

$$J^*(y) = \int_a^b \ell(t, y, \dot{y}) \dot{y}^2 \, \mathrm{d}t. \tag{23}$$

Proof
This is a direct corollary of theorem 3.2. □

Recall the assumption that L is regular, that is, that $v \mapsto L_{vv}(t, x, v)$ is positive definite for all $(t, x) \in R$.

Theorem 3.4
Under the same assumptions as in theorem 3.3, if ξ embeds an extremal $\bar{x}$ in the form $\bar{x}(t) = \xi(t, \bar{y})$, with constant $\bar{y}$, then $\bar{x}$ is the unique minimiser of J in $\mathcal{A}$.

Proof
By inspection from theorem 3.3. □

To summarise: if ξ can be taken as the inverse of a regular rectifying transformation of a field that embeds an extremal satisfying the boundary conditions, the original variational problem is Leitmann equivalent to a problem whose minimum can be determined by inspection.

3.3 Example. As an example, consider the problem to minimise

$$J(x) = \int_{-1}^{1} \left(\dot{x}^2 - x^2\right) \,\mathrm{d}t,$$

subject to the boundary conditions $x(-1) = x(1) = 1$.

The Euler-Lagrange equation for this problem reads as

$$\ddot{x} + x = 0.$$

The general solution to this differential equation is $x(t) = c_1 \cos t + c_2 \sin t$; the single solution $\bar{x}$ that satisfies the boundary conditions is obtained by setting $c_1 = 1/\cos 1$ and $c_2 = 0$.

A field of extremals embedding $\bar{x}$ is given by $\xi(t, y) = y \cos t$. Transforming the functional by $\Xi(t, y) = (t, y \cos t)$ leads to

$$\hat{J}(t, y, \dot{y}) = \int_{-1}^{1} \left(\dot{y}^2 \cos^2 t - y^2 \cos 2t - y\dot{y} \sin 2t\right) \mathrm{d}t,$$

with transformed boundary conditions $y(-1) = y(1) = 1/\cos 1$. The function S reads as

$$S(t, y) = -\frac{y^2}{2} \sin 2t.$$

The functional J is seen to be Leitmann equivalent, and the functional $\hat{J}$ is seen to be Carathéodory equivalent, to

$$J^*(y) = \int_{-1}^{1} \ell(t, y, \dot{y})\dot{y}^2 \,\mathrm{d}t = \int_{-1}^{1} \dot{y}^2 \cos^2 t \,\mathrm{d}t.$$

Note that $\ell > 0$ for $t \in (-1, 1)$. By inspection, we see that the function $\bar{y}(t) = 1/\cos 1$ is the unique minimiser of J^*, and consequently

$$\bar{x}(t) = \frac{\cos t}{\cos 1}$$

is the unique minimiser of J.

4 Boundary conditions

In this section different types of boundary conditions are considered.

4.1 Fixed endpoints. Consider again the fixed endpoint problem to minimise J, where

$$J(x) = \int_a^b L(t, x, \dot{x})\, \mathrm{d}t, \qquad x(a) = \alpha, \quad x(b) = \beta.$$

Recall the idea of a central field: this is a family of extremals that all pass through a single point (a, α). Such a family obviously embeds any extremal satisfying the boundary conditions. Moreover, it has automatically the field property.

Theorem 4.1
Let $Y \subset \mathbb{R}^m$ be an open simply connected set, and let $\xi : (a, b) \times Y \to \mathbb{R}^m$ be an m-parameter family of extremals such that for $t \in (a, b)$, the map $y \mapsto \xi(t, y)$ maps Y diffeomorphically onto its image, and with the property that for all $y \in Y$:

$$\lim_{t \downarrow a} \xi(t, y) = \alpha.$$

Then ξ is a field of extremals; it is called the **central** field around (a, α).

Note that $\Xi : R^* \to R$ cannot be extended diffeomorphically to an open set O^* containing R^*, since $y \mapsto \xi(t, y)$ fails to be one-to-one if $t = a$. Therefore, the definitions of the sets $\mathcal{A}$ and $\mathcal{A}^*$ and the definition of Leitmann equivalence have to be adapted.

The new definitions of $\mathcal{A}$ and $\mathcal{A}^*$ are

$$\mathcal{A} = \Big\{ x \in \mathrm{PC}^1((a, b), \mathbb{R}^m) \;\Big|\; x(t) \in R_t \text{ for all } t \in (a, b), \\ \lim_{t \downarrow a} x(t) = \alpha, \, \lim_{t \uparrow b} x(t) = \beta \Big\},$$

and

$$\mathcal{A}^* = \Big\{ y \in \mathrm{PC}^1((a, b), \mathbb{R}^m) \;\Big|\; y(t) \in R_t^* \text{ for all } t \in (a, b), \\ \lim_{t \downarrow a} \xi(t, y(t)) = \alpha, \, \lim_{t \uparrow b} \xi(t, y(t)) = \beta \Big\}.$$

The sharper version of Leitmann and Carathéodory equivalence is given in the following definition.

Definition 4.1
Let Ξ, $\mathcal{A}$ and $\mathcal{A}^*$ be as introduced in this section. Moreover, let $R_a = \{(a, \alpha)\}$ and $R_b = \{(b, \beta)\}$. The functionals $J : \mathcal{A} \to \mathbb{R}$ and $J^* : \mathcal{A}^* \to \mathbb{R}$, where

$$J(x) = \int_a^b L(t, x, \dot{x})\, \mathrm{d}t \quad \text{and} \quad J^*(y) = \int_a^b L^*(t, y, \dot{y})\, \mathrm{d}t, \tag{24}$$

are *Leitmann equivalent (by* Ξ*)* if there is a continuous function

$$S : R \cup R_a \cup R_b \to \mathbb{R}$$

such that S is C^1 on R and such that if $S^*(t, y) = S(t, \xi(t, y))$, the equation

$$L(t, \xi, \xi_t + \xi_y \dot{y}) = L^*(t, y, \dot{y}) + S_t^*(t, y) + S_y^*(t, y)\dot{y} \tag{25}$$

holds identically in $(t, y, \dot{y}) \in R \times \mathbb{R}^m$.

The two functionals are *Carathéodory equivalent*, if equation (25) holds with Ξ being the identity map. □

Theorem 2.1 still holds if in the proof the quantities $S^*(a, \alpha^*)$ and $S^*(b, \beta^*)$ are replaced by $\lim_{t \downarrow a} S^*(t, y(t)) = S(a, \alpha)$ etc.

Given a central field ξ around (a, α), and using again theorem 3.2, the problem to minimise $J(x)$ is seen to be Leitmann equivalent by ξ to the problem to minimise

$$\hat{J}(y) = S(b, \beta) - S(a, \alpha) + \int_a^b \ell(t, y, \dot{y})\dot{y}^2 \, \mathrm{d}t,$$

subject to the single boundary condition

$$\lim_{t \uparrow b} \xi(t, y(t)) = \beta.$$

In other words, the transformation has changed an optimisation with to fixed boundary conditions into one with only a single fixed boundary condition. Assuming that $y \mapsto \xi(b, y)$ is invertible, and that $\xi(b, \beta^*) = \beta$, the boundary condition can be written as

$$y(b) = \beta^*.$$

Theorem 4.2
Let $J = \int_a^b L(t, x, \dot{x}) \, \mathrm{d}t$ with L regular on R, and let ξ be a regular central field covering R. If $y \mapsto \xi(t, y)$ is a diffeomorphism for $t = b$ and if $\bar{y}$ is such that

$$\xi(b, \bar{y}) = \beta, \tag{26}$$

then the function $\bar{x}(t) = \xi(t, \bar{y})$ is the unique minimiser of J.

Proof
Since ξ is central field, we have for all y that

$$\lim_{t \downarrow a} \xi(t, y) = \alpha \quad \text{and} \quad \lim_{t \downarrow a} \xi_y(t, y) = 0. \tag{27}$$

By equation (21), it follows that $\lim_{t \downarrow a} S^*_y(t, y) = 0$ and that $S^*(a, y) = C$ does not depend on y. It follows that the function $S : R \cup \{(a, \alpha), (b, \beta)\} \to \mathbb{R}$ is a well-defined continuous function if we set $S(t, \xi(t, y)) = S^*(t, y)$ on R^*; in particular $S(a, \alpha) = C$. Invoking theorem 3.2, it is seen that J is Leitmann equivalent to minimising

$$J^*(y) = S(b, \beta) - C + \int_a^b \ell(t, y, \dot{y})\dot{y}^2 \, \mathrm{d}t. \tag{28}$$

As $S(b, \beta) - C$ does not depend on y, it is seen by inspection that J^* is minimised if $\dot{y} = 0$, that is $y(t) = \bar{y}$ for all t. □

4.2 Single free endpoint. Consider next the free endpoint problem to minimise

$$J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t, \qquad x(a) = \alpha, \qquad x(b) \text{ free.}$$

It is obvious how to change the definitions of the sets $\mathcal{A}$ and $\mathcal{A}^*$. In the definition of Leitmann equivalence, the set R_b has now to be taken as $\bar{R} \cap \{b\} \times \mathbb{R}^m$. A simple modification of the proof of theorem 4.2 yields

Theorem 4.3
Let $J = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t$ with L regular on R, and let ξ be a regular central field covering R. Let $S^* = S^*(t, y)$ be the function whose existence has been proved in theorem 3.2.

If $y \mapsto \xi(b, y)$ is a diffeomorphism, and if $\bar{y}$ minimises $y \mapsto S^*(b, y)$, then $\bar{x}(t) = \xi(t, \bar{y})$ minimises J.

Proof
The only difference to the proof of theorem 4.2 is that equation (28) now reads as

$$J^*(y) = S^*(b, y(b)) - S(a, \alpha) + \int_a^b \ell(t, y, \dot{y})\dot{y}^2\,\mathrm{d}t. \tag{29}$$

It is clear that this integral is minimised if $y(t)$ is always equal to a constant $\bar{y}$, implying $\dot{y} = 0$, and that the value of that constant should minimise $y \mapsto S^*(b, y)$. □

Note that since $S_y = L_v$, the familiar necessary transversality condition $L_v = 0$ follows as a corollary.

4.3 Endpoint on a manifold. The extension to the case of a fixed initial point (a, α) and an endpoint located on some manifold is straightforward.

Let

$$J(x) = \int_a^b L(t, x, \dot{x})\,\mathrm{d}t,$$

and let ξ be a central field of L through the point (a, α). Moreover, let Γ be an embedded submanifold of $\mathbb{R} \times \mathbb{R}^m$ that is contained in the boundary of R, and assume that ξ covers an open set U such that $R \cup \Gamma \subset U$. Let $\Gamma^* = \Xi^{-1}(\Gamma)$ be the diffeomorphic image of Γ under the inverse of Ξ.

The problem is to minimise J over

$$\mathcal{A} = \Big\{x \in \mathrm{PC}^1((a,b), \mathbb{R}^m) \;\Big|\; x(t) \in R_t \text{ for all } t \in (a,b), \\ \lim_{t \downarrow a} x(t) = \alpha,\ (b, x(b)) \in \Gamma\,\}.$$

Set also

$$\mathcal{A}^* = \Big\{y \in \mathrm{PC}^1((a,b), \mathbb{R}^m) \;\Big|\; y(t) \in R^*_t \text{ for all } t \in (a,b), \\ \lim_{t \downarrow a} \xi(t, y(t)) = \alpha,\ \Xi(b, y(b)) \in \Gamma \Big\}.$$

Theorem 4.4
Under the given hypotheses, let $S^* = S^*(t,y)$ be the function whose existence has been proved in theorem 3.2.

If $(\bar{t},\bar{y}) \in \Gamma^*$ minimises S^* over Γ, then $\bar{x}(t) = \xi(t,\bar{y})$ defined on $(a,\bar{t})$ minimises J over $\mathcal{A}$.

Proof
In this case equation (28) reads as

$$J^*(y) = S^*(b, y(b)) - S(a,\alpha) + \int_a^b \ell(t,y,\dot{y})\dot{y}^2 \,\mathrm{d}t, \tag{30}$$

where $(b, y(b)) \in \Gamma^*$. It is clear that this integral is minimised if $y(t)$ is always equal to a constant $\bar{y}$, implying $\dot{y} = 0$, and that the value of that constant should minimise $S^*(t,y)$ restricted to Γ^*. □

4.4 Double free endpoint. Finally consider the problem of minimising

$$J(x) = \int_a^b L(t,x,\dot{x})\,\mathrm{d}t,$$

with free endpoint conditions on both sides. The modifications of the definitions of $\mathcal{A}$, $\mathcal{A}^*$, R_a and R_b are left to the reader.

Theorem 4.5
Let $Y \subset \mathbb{R}^m$ be an open simply connected set, and let $\xi : (a,b)\times Y \to \mathbb{R}^m$ be an m-parameter family of extremals such that for $t \in (a,b)$, the map $y \mapsto \xi(t,y)$ maps Y diffeomorphically onto its image, and with the property that for all $y \in Y$:

$$\lim_{t\downarrow a} \eta(t,y) = 0.$$

Then ξ is a field; I shall call it the field that is *transversal* at $t = a$.

Proof
The proof is analogous to that of theorem 4.1, except that instead of $\xi_{y_i} = 0$ here $\eta_{y_i}(a,y) = 0$ for all y and all i. □

Theorem 4.6
Let $J = \int_a^b L(t,x,\dot{x})\,\mathrm{d}t$ with L regular on R, and let ξ be a regular field, transversal at $t = a$, covering R. Let $S^* = S^*(t,y)$ be the function whose existence has been proved in theorem 3.2.

If $y \mapsto \xi(b,y)$ is a diffeomorphism, and if $\bar{y}$ minimises $y \mapsto S^*(b,y)$, then $\bar{x}(t) = \xi(t,\bar{y})$ minimises J.

Proof
The proof is parallel to the proof of theorem 4.3, with some modifications. Since ξ is a transversal field, we have for all y that

$$\lim_{t\downarrow a} S_y(t,y) = L_v = 0. \tag{31}$$

It follows that $S(a,y)=C$ does not depend on y. To minimise J is therefore equivalent to

$$J^*(y) = S(b, y(b)) - C + \int_a^b \ell(t, y, \dot{y})\dot{y}^2 \,\mathrm{d}t. \tag{32}$$

This integral is clearly minimised by a constant function $y(t)=\bar{y}$ such that $\bar{y}$ minimises $y \mapsto S(b,y)$. □

A Proof of theorem 3.1

Proof
Compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega(v_i, v_j) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\eta_{y_i}\xi_{y_j} - \eta_{y_j}\xi_{y_i}\right) \\ &= \eta_{y_i t}\xi_{y_j} + \eta_{y_i}\xi_{y_j t} - \eta_{y_j t}\xi_{y_i} - \eta_{y_j}\xi_{y_i t}.\end{aligned} \tag{33}$$

To evaluate this expression, the quantities η_{y_i} and η_{ty_i} have to be determined. Deriving equation (15) with respect to y_i yields

$$\eta_{y_i} = L_{vx}\xi_{y_i} + L_{vv}\xi_{ty_i}. \tag{34}$$

Since $\xi = \xi(t,y)$ is a family of extremals, the equation

$$L_x(t,\xi,\xi_t) - \frac{\mathrm{d}}{\mathrm{d}t}L_v(t,\xi,\xi_t) = L_x(t,\xi,\xi_t) - \frac{\mathrm{d}}{\mathrm{d}t}\eta(t,y) = 0 \tag{35}$$

holds identically in y. Deriving with respect to y_i and rearranging terms yields

$$\eta_{ty_i} = L_{xx}\xi_{y_i} + L_{xv}\xi_{ty_i}. \tag{36}$$

Substitution of (34) and (36) into (33) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega(v_i, v_j) &= \left(L_{xx}\xi_{y_i} + L_{xv}\xi_{ty_i}\right)\xi_{y_j} + \left(L_{vx}\xi_{y_i} + L_{vv}\xi_{ty_i}\right)\xi_{y_j t} \\ &\quad - \left(L_{xx}\xi_{y_j} + L_{xv}\xi_{ty_j}\right)\xi_{y_i} - \left(L_{vx}\xi_{y_j} + L_{vv}\xi_{ty_j}\right)\xi_{y_i t} \\ &= L_{xv}\xi_{ty_i}\xi_{y_j} + L_{vx}\xi_{y_i}\xi_{y_j t} - L_{xv}\xi_{ty_j}\xi_{y_i} - L_{vx}\xi_{y_j}\xi_{y_i t} = 0.\end{aligned} \tag{37}$$

The last equality holds by virtue of equation (2). □

References

V.I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Springer, Heidelberg, 1989.

C. Carathéodory. *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*. B.G. Teubner, Berlin, 1935.

D.A. Carlson. An observation on two methods of obtaining solutions to variational problems. *Journal of Optimization Theory and Applications*, 114(2):345–361, 2002.

D.A. Carlson and G. Leitmann. Fields of extremals and sufficient conditions for the simplest problem of the calculus of variations. *Journal of Global Optimization*, 40:41–50, 2008a.

D.A. Carlson and G. Leitmann. Fields of extremals and sufficient conditions for the simplest problem of the calculus of variations in n-variables. Preprint, 2008b.

L. Euler. *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti*. Bousquet, Lausannae et Genevae, 1744.

M. Giaquinta and S. Hildebrandt. *Calculus of Variations I*, volume 310 of *Grundlehren der mathematischen Wissenschaften*. Springer, Heidelberg, 1996.

A. Kneser. *Lehrbuch der Variationsrechnung*. Vieweg, Braunschweig, 1900.

G. Leitmann. On a class of direct optimization problems. *Journal of Optimization Theory and Applications*, 108(3):467–481, 2001.

G. Leitmann. A note on absolute extrema of certain integrals. *International Journal of Non-linear Mechanics*, 2:55–59, 1967.

K. Weierstrass. *Vorlesungen über Variationsrechnung*. Akademische Verlagsgesellschaft, Leipzig, 1927. Werke Vol. 7.