Endogenous Beveridge cycles and the volatility of unemployment

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November 7, 2013

Abstract

This paper aims to explain the magnitude and cyclical structure of the fluctuations in unemployment and vacancies. Adding demand externalities to an otherwise standard search and matching model reduces the need for exogenous shocks in explaining unemployment fluctuations. Under plausible parameter values, the equilibrium dynamics include a stable limit cycle that resembles the empirically observed counterclockwise cycles around the Beveridge curve. Quantitatively, these endogenous 'Beveridge cycles' can explain half of the volatility and almost all persistence of unemployment without any exogenous forces, avoiding the amplification and propagation problems of the standard model.

Keywords: Labor market search; Endogenous cycles; Unemployment and vacancies volatility.
JEL Classification: E24; E32; J63; J64

1   Introduction

The dynamic relation between vacancies and unemployment is characterized by counter-clockwise cycles in the *unemployment rate, vacancy rate*-plane. Pissarides (2000, p. 36) considers these counterclockwise cycles to be a stylized fact of business cycles, although shifts in the natural rate of unemployment can contaminate the picture over long time frames. Removing these trends with an HP-filter results in the cycles in figure 1 for the United States.
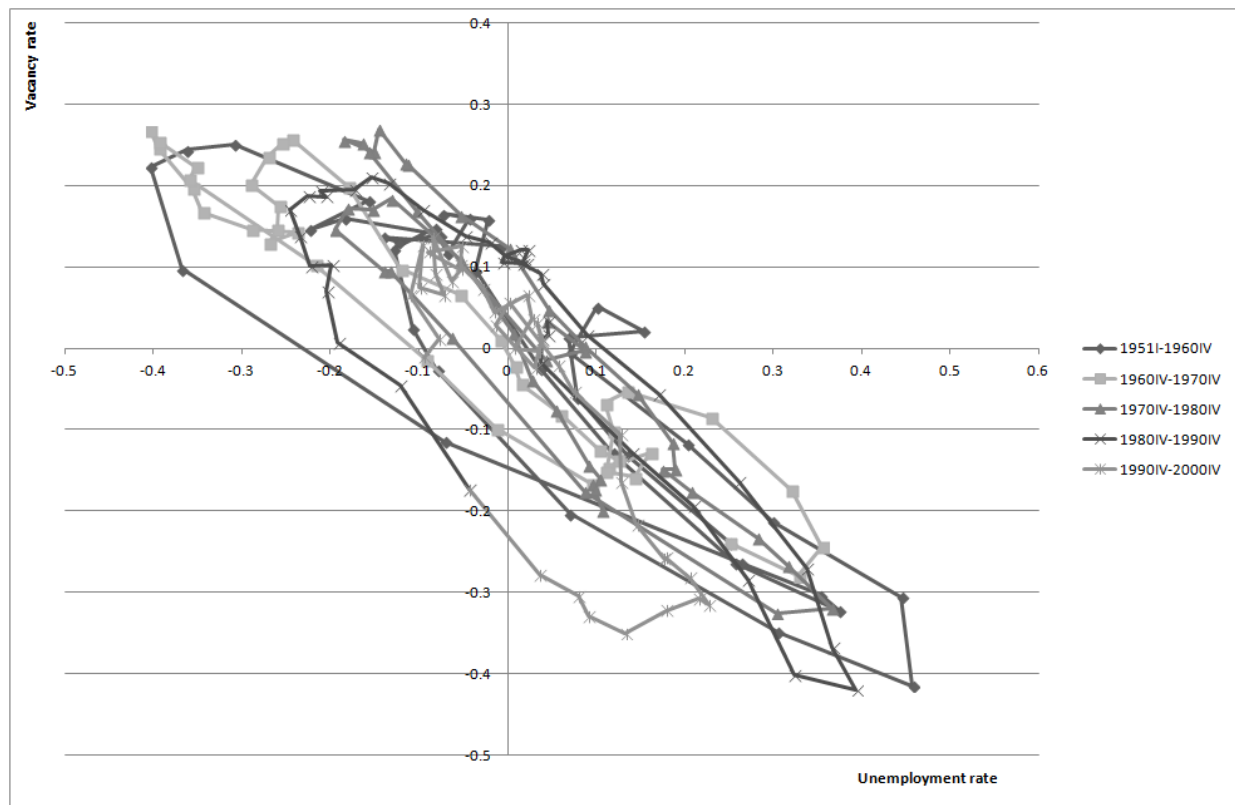

Figure 1: Cyclical component of the Beveridge curve for the United States, 1951-2000.
Notes: The trend is an HP-filter with smoothing parameter 10^5. Data are quarterly averages of monthly series. Vacancy data are based on help-wanted advertising and constructed by Valletta (2005). The unemployment data are the seasonnally adjusted series from the CPS.

The current paper explains these cycles with a limit cycle driven by demand externalities, based on Mortensen (1999). Feedback from aggregate employment to the (expected) revenue per worker can give rise to endogenous rational expectations cycles in vacancies and unemployment, which I call Beveridge cycles. I investigate whether these Beveridge cycles can match the empirically observed standard deviation and autocorrelation of unemployment. Calibrating the cycles to the median duration of the business cycle, 50 percent of the volatility in unemployment can be explained without any exogenous shocks. Since one-third to one-half of unemployment volatility is driven by fluctuations in the job separation rate, the rate

at which employed workers flow into unemployment and that is constant in this model, 50 percent amounts to almost all volatility in unemployment that is driven by job creation. In addition, the simulated autocorrelation of unemployment is close to the observed correlation.

Within the neo-Keynesian tradition, Hansen (1970) and Bowden (1980) explain the counterclockwise direction by exploiting the idea that vacancies respond faster to favorable demand shifts than employment. In the search and matching literature, Pissarides (1985) and Blanchard and Diamond (1989) use the same idea of sluggishness of employment to shocks in output and capital prices, and in aggregate activity respectively. However, all of these explanations of the counterclockwise cycles rely on exogenous shocks, and moreover, lack cyclical responses. Shimer (2005) shows that the amplification and propagation of the canonical search and matching model are too weak to generate the observed fluctuations in unemployment and vacancies in response to realistic exogenous shocks in productivity and/or the separation rate. As a result, a large literature has arisen that adjusted some of the parameters or assumptions within the search and matching framework to increase the amplification, in order to solve this 'Shimer puzzle'.[1] The lack of propagation is generally not overcome by these solutions. Fujita and Ramey (2007) show that cyclical responses can be generated by the introduction of sunk costs of vacancy creation. However, spreading out the impact of a shock in such a way results in a counterfactually high cross correlation between labor market variables and productivity across time.

Alternatively, cyclical responses can result from the self-reinforcing effect of feedback from aggregate employment to the (expected) revenue per worker. This approach is not subject to problems of amplification and propagation, simply because it doesn't involve exogenous shocks. The only article within the endogenous business cycle literature I am aware of that explains the counterclockwise cycles along the Beveridge curve, is Mortensen (1999). I rewrite his model in labor market terms and add a positive value of leisure to investigate whether it can quantitatively explain the labor market facts of the business cycle, while featuring counterclockwise cycles in unemployment and vacancies. I am not aware of any other paper that calibrates a deterministic model to quantify its performance in explaining these characteristics.[2]

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[1]For instance, Hall (2005) investigates the effect of wage rigidity, and Hall and Milgrom (2008) study the effect of an alternating offer bargaining model. Hagedorn and Manovskii (2008) just argue that the outside option of the worker should be calibrated differently, while Pissarides (2009) argues that the solution to the Shimer puzzle should neither be found in wage stickiness nor in the outside option, but in the introduction of fixed matching costs. See also Mortensen and Nagypal (2007) for a discussion of several assumptions in this literature and how a particular combination of them can account for two-thirds of the volatility in the vacancy-unemployment ratio.

[2]There are other papers that quantify the performance of a labor market model with strategic complementarities and multiple equilibria, but those that I know all rely on sunspot shocks to induce switches between different equilibrium paths. This idea is developed in Howitt and McAfee (1992) and Fanizza (1996), and Kaplan and Menzio (2012) perform a calibration.

Several interpretations can be given to the feedback from aggregate employment to the (expected) revenue per worker. In Mortensen (1999), per period productivity of a worker-employer match exhibits increasing returns, which are accounted for by external economies across sectors, maintaining the competitive nature of the goods market. He motivates these externalities by referring to Hall (1988) and Caballero and Lyons (1989). Mortensen also shows that the externalities can result in multiple equilibria, and that these can be Pareto-ranked. As a result, a coordination problem exists. Mortensen (1999, p. 890) furthermore suggests an interpretation based on Diamond (1982). The latter assumes that producers need to exchange their product for it to have value, and that the likelihood that they meet a trading partner is increasing in the total number of producers. As a result, the expected value of output is increasing in employment.[3] Another formulation of specialization in production relative to consumption that results in endogenous rational expectations cycles is the model by Drazen (1988), who exploits spillovers across a labor market with search frictions and a monopolistic goods market.[4] A recent version of such spillovers can be found in Kaplan and Menzio (2012).

Together, these cases capture the three examples of Cooper and John (1988) that can result in strategic complementarities: production technology, matching technology, and agents' demands. In a reduced form, the three interpretations of the feedback are equivalent, since the (expected) value of the product is always an increasing function of aggregate employment. Since high employment results in a higher revenue per worker, it drives vacancy creation of employers, which via the matching process results in more employment, leading to new vacancies, and so on. If an increase in employment results in more vacancy creation via the demand externalities than the number of vacancies that is necessary for the original increase in employment via the matching process, employers overshoot the steady state level of vacancies. However, with more and more vacancies it takes a long time to fill an individual vacancy. Foreseeing an end to high employment and high revenue per worker, employers decrease their costly recruitment activity. At some point, fewer matches are made than jobs are destroyed, and with an increase in unemployment, revenue per worker actually falls. The above process in reverse completes the Beveridge cycle.

The next section presents the model. In section 3 I analyze its steady states, and I continue with the characterization of the Beveridge cycle in section 4. Section 5 summarizes the data, explains the parameter choices, and shows that the calibrated Beveridge cycle can

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[3]Diamond and Fudenberg (1989) already showed that such increasing returns in a matching function for the goods market can result in endogenous rational expectations cycles in reservation costs and employment. However, they have no explicit vacancies in their model.

[4]This model does contain vacancies, and is closest to Mortensen (1999) with a cycle in firm match value and unemployment. However, he assumes an equal fixed number of firms and workers, which equates the number of vacancies and unemployed and thus rules out cycles in the two.

account for most of the hiring driven volatility of unemployment and its autocorrelation.

2 A model of unemployment and tightness

This section presents the model of Mortensen (1999) in terms of labor market tightness and unemployment. I add a positive value of leisure to this model, which is important for its empirical performance. The model is a Pissarides (2000) equilibrium search and matching model with variable search intensity and feedback from employment to productivity, but without (aggregate) uncertainty. It consists of risk-neutral representative agents of two types: employers and workers. Each type is homogeneous with respect to skills and preferences. Matches of workers and employers produce a single consumption good, and the (expected) value of this product is increasing in aggregate employment. First, I present the law of motion for unemployment.

2.1 Dynamics of unemployment

Workers can be either employed or unemployed. The labor force is normalized to one, so that the total number of employed workers, n, is equal to one minus the unemployment rate u. If a worker is unemployed, he can search for a job with a certain intensity s. The discouraged worker effect - unemployed workers stop looking for a job if the prospect of finding one is very bad - is modeled by s, because the labor force is fixed while search intensity will increase with labor market tightness. The representative employer can open vacancies. The total number of unemployed workers that finds a job in a certain period is given by the common constant returns Cobb-Douglas matching function $m(v, su) = m_0 v^\eta (su)^{1-\eta}$, with $0 < \eta < 1$ and $m_0 > 0$. Inputs of this function are aggregate recruiting activity represented by the vacancy rate v (scaled by the labor force), and aggregate search effort given by the unemployment rate times the search intensity of the unemployed. Define labor market tightness θ as the ratio of the inputs of the matching function: $\theta = \frac{v}{su}$. An individual unemployed worker finds work at the Poisson rate $\frac{m_0 v^\eta (su)^{1-\eta}}{u} = sm_0\theta^\eta \equiv s\lambda(\theta)$. Similarly, individual vacancies are filled at a rate $\frac{m_0 v^\eta (su)^{1-\eta}}{v} = \frac{\lambda(\theta)}{\theta}$. $\lambda(\theta)$ is increasing and concave. Jobs are destroyed at an exogenous rate $\delta \in (0, 1]$. These assumptions give the change in unemployment in terms of unemployment and labor market tightness

$$\dot{u} = \delta(1 - u) - su\lambda(\theta). \tag{1}$$

The next subsection describes the asset values of different labor market statuses for both workers and employers.

2.2 Values of different labor market statuses

Employers face a constant cost of opening a vacancy per unit of time, denoted by $k > 0$. Unemployed workers encounter periodical search costs $c(s)$, increasing and strictly convex in the intensity of search, with $c(0) = c'(0) = 0$. I add an exogenous positive gross value of leisure z to the original model, so that the net flow value of leisure of the unemployed is $z - c(s)$.[5] This parameter is assumed to be independent of labor market conditions and captures the combination of the unemployment benefit, the stigma of unemployment, the value of home production and the pure value of leisure that comes with unemployment. Note that the gross value of leisure does not affect the search costs, which seems a reasonable assumption for risk-neutral agents. This assumption highlights the effect that the value of leisure has on vacancy creation rather than labor supply.

All agents have utility that is linear in consumption, and discount the future with the same constant and exogenous rate r. These assumptions result in the following asset price equations for the value of holding a vacancy V for an employer

$$rV = -k + \frac{\lambda(\theta)}{\theta}(J^e - V) + \dot{V}, \tag{2}$$

and of being unemployed U for a worker

$$rU = z - c(s) + s\lambda(\theta)(W^e - U) + \dot{U}, \tag{3}$$

where J^e and W^e are the expected present values of a match to a representative employer and worker respectively.

The stock of vacancies follows from a free entry condition. The competitive behavior of the employers ensures that for interior solutions the value of a vacancy in equilibrium is always zero. However, the value of a vacancy can be negative if employers cannot reduce vacancies because they did not open any at all, so that $V \leq 0$ with $V = 0$ if $\theta > 0$. Using the asset price equation in (2), for all $\theta > 0$ the expected value of a filled vacancy can then be expressed in terms of the costs and benefits of a vacancy, where the latter depends on labor market tightness,

$$J^e = \frac{k\theta}{\lambda(\theta)}. \tag{4}$$

At $\theta = 0$, J^e may be negative, but $\frac{k\theta}{\lambda(\theta)}$ is always bigger than or equal to zero. In the remainder of this section I focus on equilibria that have a positive level of labor market tightness.

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[5]The original model is thus a special case in which $z = 0$. Alternatively, one could say that I only modify Mortensen's functional form of the cost function, from $c(0) = 0$ to $c(0) = -z$.

The demand externalities are modeled by an individual production function that is increasing in the aggregate level of employment. The expected periodical flow benefit of a match can thus be denoted by $\phi(1-u)$, with $\phi'(1-u) > 0$. The per period flows to an employer are then the expected value of output $\phi(1-u)$ minus the wage w. This wage is the per period flow income to the worker. The asset price equations of a job J and W, to an employer and a worker respectively, are then

$$rJ = \phi(1-u) - w - \delta(J - V) + \dot{J}, \tag{5}$$

and

$$rW = w - \delta(W - U) + \dot{W}.$$

With rational expectations, the representative employer is correct in her expectation of the value of a filled vacancy, and thus the number of vacancies opened is actually maximizing the expected utility of the employer. Using the free entry condition (4) correspondingly, and acknowledging that $V = 0$ for all interior equilibria, (5) can be rearranged to a law of motion of the value of a job in w, u and θ only

$$\dot{J} = (r + \delta)\frac{k\theta}{\lambda(\theta)} - \phi(1-u) + w. \tag{6}$$

2.3 Bargaining, wages, and search intensity

The wage is determined by Nash bargaining over the surplus of a match $p = J + W - V - U$, with worker's bargaining power equal to $\beta \in (0,1)$ and separation (U, V) as threat point. This assumption about the distribution implies that the worker's rent is equal to his share of the surplus and thus that

$$W - U = \frac{\beta}{1-\beta}(J - V), \text{ and } \dot{W} - \dot{U} = \frac{\beta}{1-\beta}(\dot{J} - \dot{V}),$$

where the latter follows because wages are continuously renegotiated.[6]

With rational expectations and Nash bargaining, and the condition that for interior equilibria $V = 0$ and thus also $\dot{V} = 0$, the asset price equations can be simplified to yield

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[6]Pissarides (2009) shows that the crucial assumption for job creation is that wages of *new* matches are given by this rule. How rents in ongoing jobs are split is inconsequential for job creation, and thus for the dynamics in this model. Coles and Wright (1998) have shown that outside a steady state, Nash bargaining no longer necessarily corresponds to the outcome of strategic bargaining with appropriately defined threat points as the time between offers goes to zero, as it would in a stationary environment (Binmore, Rubinstein, and Wolinsky, 1986). Consequently, the division rule in this paper should not be interpreted as the outcome of strategic bargaining, but as an axiomatic solution.

the wage w

$$w = \beta\phi(1 - u) + \beta sk\theta + (1 - \beta)\left[z - c(s)\right]. \tag{7}$$

The search intensity s in this wage equation is optimally chosen by the unemployed worker, knowing that the surplus of a match will be shared by Nash bargaining. From (3), the worker's net expected income from search activity g is given by $s\lambda(\theta)(W^e - U) - c(s)$. With the expression for J^e in (4), net expected income from search activity can be expressed in terms of labor market tightness

$$g(\theta) = \max_{s}\left[\frac{\beta}{1 - \beta}sk\theta - c(s)\right]. \tag{8}$$

This expression can be recognized as an element of the wage. Using (8), the equation for the wage in (7) can be simplified to

$$w = g(\theta) + z + \beta\left[\phi(1 - u) - g(\theta) - z\right]. \tag{9}$$

As usual, the wage is a linear combination of the net expected income from search and the gross value of leisure - together the outside option of the worker - and the expected value of output of the match.

The worker must balance the benefits of search with the costs. Given an increasing and strictly convex search cost function with $c'(0) = 0$, an optimal intensity exists and is unique. Moreover, since the benefits of search increase in labor market tightness, while the cost function is independent of tightness, the optimal intensity increases in tightness. It is zero if tightness is zero, since without vacancies there is no payoff to search. Following Mortensen in choosing a constant elasticity specification for the increasing and strictly convex cost function, so that $c(s) = c_0 s^\gamma$ with $\gamma > 1$, I have an explicit function for $s(\theta)$[7,8]

$$s(\theta) = \left(\frac{\beta}{1 - \beta}\frac{k}{c_0\gamma}\theta\right)^{\frac{1}{\gamma - 1}}.$$

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[7]Note that s, just as u, is part of the definition of θ, but that aggregate tightness is given for the representative unemployed worker. Consequently, $s(\theta)$ can be expressed in terms of θ.

[8]This expression also allows for elimination of s in the expression of net expected income from search activity (8):

$$g(\theta) = \left(\frac{\beta}{1 - \beta}k\theta\right)^{\frac{\gamma}{\gamma - 1}}\left(\frac{1}{c_0\gamma}\right)^{\frac{1}{\gamma - 1}}\left[1 - \frac{1}{\gamma}\right].$$

Note that $g(\theta)$ is nonnegative given that $\gamma > 1$, as it should be. Secondly, $g(\theta)$ is increasing in θ, since it increases the likelihood of a match for a worker, and is zero if there are no vacancies.

2.4 Dynamics of tightness

In this subsection, I complete the model by deriving the law of motion for labor market tightness. Taking the time derivative of (4) yields a differential equation for the expected value of a job which holds for all interior solutions

$$\dot{J}^e = \frac{k\dot{\theta}\lambda(\theta) - k\theta\lambda'(\theta)\dot{\theta}}{\lambda(\theta)^2} = \frac{k\dot{\theta}}{\lambda(\theta)}(1 - \frac{\theta\lambda'(\theta)}{\lambda(\theta)}). \tag{10}$$

Since the matching function is Cobb-Douglas, the elasticity $\frac{\theta\lambda'(\theta)}{\lambda(\theta)}$ is simply η. Therefore, equation 10 implies the following law of motion for labor market tightness

$$\dot{\theta} = \dot{J}^e \frac{\lambda(\theta)}{k(1-\eta)}.$$

By opening or closing vacancies, employers translate changes in expectations about the surplus of a filled vacancy in changes in labor market tightness.

Substituting the actual law of motion of J in (6) for the expected law of motion, and using the wage from (9), yields the following second differential equation in θ and u, valid for all interior solutions

$$\dot{\theta} = (r+\delta)\frac{\theta}{1-\eta} + (1-\beta)\frac{\lambda(\theta)}{k(1-\eta)}\left[g(\theta) + z - \phi(1-u)\right]. \tag{11}$$

Together with the law of motion for unemployment in (1), (11) describes rational expectations equilibria. High employment expectations make employers open vacancies immediately, because revenue per worker is expected to be higher in the future, but then hiring will be more costly too. Since more vacancies bring about higher employment, expectations are self-fulfilling. The next section shows that these self-fulfilling expectations may result in multiple equilibria. For an equilibrium, however, the laws of motion are restricted to a certain state space. Unemployment can never be negative or exceed one, and labor market tightness can never be negative either. An equilibrium path starts at the given initial unemployment rate, and satisfies the transversality condition that tightness goes to zero if time goes to infinity, since then no surplus of a filled vacancy can be expected anymore.[9]

In the presence of search frictions, an equilibrium is not necessarily efficient. Positive externalities of search and recruiting activity occur for the trading partners, for whom matching is *more* likely because of the availability of more (effective) trading partners. Negative externalities of search and recruiting activity occur for searchers of the same type, for whom matching is *less* likely because of increased congestion for trading partners. Both

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[9]See Mortensen (1999, p. 897) for a more detailed and formal characterization of an equilibrium.

externalities only cancel once the net private returns from search and recruiting activity equal the net social returns. As can easily be checked, the familiar Hosios (1990) condition characterizes the efficient sharing rule in this model. Only if the bargaining power of employers $1 - \beta$ is equal to the elasticity of the matching function η, search intensity s and labor market tightness θ (by employers' adjustment of vacancies) is efficient. Note that the Hosios condition only concerns the search externalities, not the demand externalities. In fact, Mortensen (1999) shows that the multiple equilibria presented in the next section can be Pareto-ranked.

3 Steady state unemployment and tightness

This section studies the existence, uniqueness, and stability of the steady states of the dynamic system in labor market tightness θ and unemployment u, as given by the differential equations in (11) and (1). I focus on the case with a positive gross value of leisure z, which is new relative to Mortensen (1999) and which is both plausible and important for the empirical performance of the model. Knowledge of the steady states and their stability helps to understand the Beveridge cycle that ultimately explains the data.

3.1 A no-trade steady state

Setting (1) equal to zero results in steady state unemployment in terms of labor market tightness, the $\dot{u} = 0$-locus or employment nullcline:

$$u = \frac{\delta}{\delta + s(\theta)\lambda(\theta)} = \frac{\delta}{\delta + m_0 \left(\frac{\beta}{1-\beta} \frac{k}{c_0 \gamma} \right)^{\frac{1}{\gamma-1}} \theta^{\eta + \frac{1}{\gamma-1}}}, \tag{12}$$

where the last equality uses the Cobb-Douglas matching function and the optimal search intensity derived from the constant elasticity cost of search function. One can see that all workers are unemployed if tightness is zero, and that unemployment decreases in tightness via the job finding rate.

Specifying a production function, the tightness nullcline can also be expressed as unemployment in terms of tightness. Following Mortensen (1999) in choosing a constant elasticity function, production per match will be given by $\phi_0 (1-u)^\alpha$, where α represents the elasticity of the external effects. For all interior equilibria, the tightness nullcline can then be written as

$$u = 1 - \left[(r+\delta) \frac{k\theta}{\phi_0 (1-\beta)\lambda(\theta)} + \frac{g(\theta) + z}{\phi_0} \right]^{\frac{1}{\alpha}}. \tag{13}$$

Again unemployment decreases in tightness: at a lower level of unemployment, the expected

value of output will be higher, and therefore in equilibrium employers open more vacancies. At some level of unemployment, the nullcline crosses the $\theta = 0$-axis, and employers do not want to open any vacancies. From (13) with $\theta = 0$ we can see that this happens at

$$u_0 = 1 - (z/\phi_0)^{1/\alpha}. \tag{14}$$

This u_0 is the upper limit on the unemployment level, and thus the lower limit on the expected value of output, to have any vacancy creation at all. For smaller values of leisure, wages are lower. As a result, there is more surplus of a match, and employers open more vacancies. Due to the complementary slackness condition, the tightness nullcline thus consists of two sections: the combinations of u and θ as given by (13), and a linear section with $\theta = 0$ and $u_0 \leq u \leq 1$. Figure 2 shows the typical shapes of the unemployment and tightness nullclines, and indicates the location of u_0 for a positive gross value of leisure.[10]
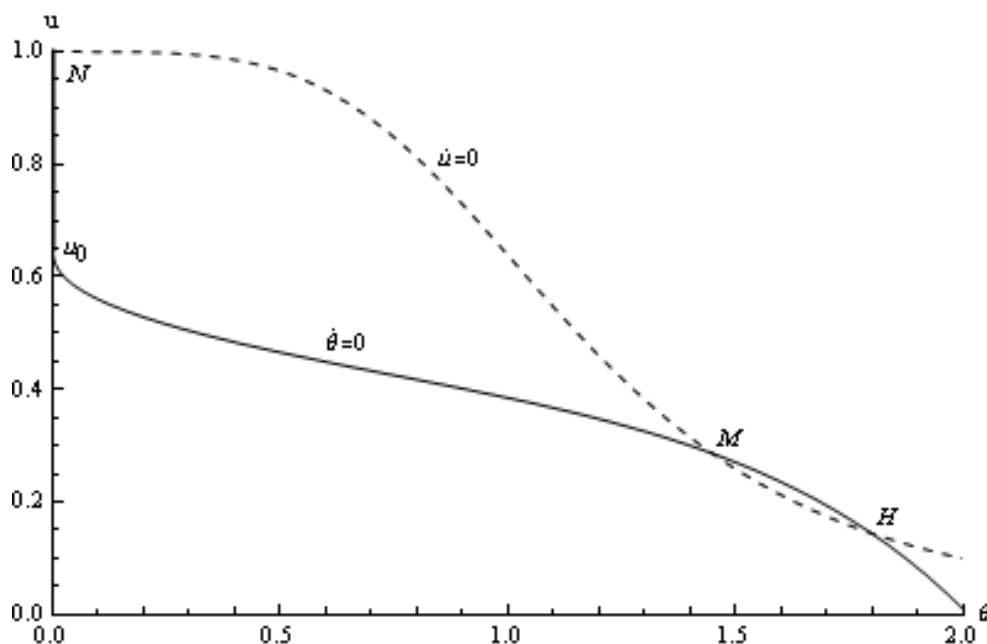

Figure 2: Nullclines of unemployment (dashed) and labor market tightness, resulting in the three steady states N, M, and H.
Notes: Except for $z > 0$, parameters are as much as possible from Mortensen (1999). They are $k = 0.3$, $\delta = 0.15$, $r = 0.01$, $\eta = 0.6$, $\gamma = 1.29$, $c_0 = 0.3$, $\phi_0 = 0.12$, $\alpha = 0.3$, $\beta = 0.29$, $m_0 = 4.3$ and $z = 0.71\phi_0$.

A steady state occurs if the two downward-sloping nullclines intersect. It is not generally possible to give an explicit solution for a steady state. However, for $z = 0$ it can be seen that

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[10]Since unemployment can be expressed more easily as a function of tightness than the reverse, I plot unemployment on the vertical axis and tightness on the horizontal. Once I later plot vacancies to unemployment, unemployment will be on the horizontal axis as is common in the literature. With these conventions, counterclockwise cycles in unemployment and vacancies thus correspond to clockwise cycles in unemployment and tightness.

(13) and (12) intersect at $\theta = 0$ and $u = 1$, so that a no-trade steady state exists for $z = 0$, as in Mortensen (1999). For $z > 0$, u_0 will be smaller than 1, but the no-trade equilibrium will still exist, due to the linear section of the tightness nullcline on the $\theta = 0$-axis that intersects with (12) at $\theta = 0$ and $u = 1$. The example of figure 2 results in three steady states: the no-trade steady state N at zero vacancies and zero employment, a steady state M with a relatively low but positive employment and labor market tightness, and a steady state H with high employment and tightness. The existence of steady states in the positive quadrant (positive steady states for short) is discussed in the next subsection.

3.2   Two positive steady states

In lemma 1 below I state that the unemployment nullcline can have two different shapes, depending on the elasticities of the matching function η and search cost function γ. The proof is in appendix A. Since unemployment decreases for a higher tightness via the job finding rate, but never becomes zero unless tightness becomes infinite, the end of the nullcline is always convex. It is possible for the entire nullcline to be convex, but the nullcline has a logistic shape if the cost function of search intensity is not too convex, so that search intensity varies considerably with tightness.

Lemma 1. *For $\eta + \frac{1}{\gamma - 1} \leq 1$ the unemployment nullcline is convex, and for $\eta + \frac{1}{\gamma - 1} > 1$ the nullcline has a logistic shape.*

Mortensen (1999) refers to Burdett, Kiefer, Mortensen, and Neumann (1984) to argue that $\gamma < 2$. Since $0 < \eta < 1$, the empirically relevant unemployment nullcline is logistic, and this is the case that is depicted in figure 2.

For a given $z > 0$ (and $\alpha > 0$), there are only steady states in the positive quadrant if the scale parameter of the revenue per worker ϕ_0 is large enough. If not, the unemployment nullcline will lie entirely above the tightness nullcline, and the no-trade steady state is the only steady state.[11] As soon as the scale parameter of the revenue per worker becomes large enough, the two nullclines touch and a saddle-node bifurcation occurs. In a bifurcation, the qualitative properties of a dynamic system change as the result of change in a parameter, ϕ_0 in this case.[12] In a saddle-node bifurcation two steady states emerge: a saddlepoint and a node or focus. Focusing on the empirically relevant case in which $\alpha \leq 1$, proposition 1 states that if any positive steady states exist, as long as $z > 0$ is sufficiently big, there are exactly two of them. The proof is in appendix B.

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[11]If $z = 0$, a steady state always exists in the positive quadrant if $\eta + \frac{1}{\gamma - 1} < 1$, independent of any other parameters.

[12]Alternatively, for a given ϕ_0, z, α, the efficiency of matching parameter m_0 must be large enough, or vacancy costs k low enough. In section 5 I will subsequently vary α and z to their respective saddle-node bifurcation values, given the other parameters. See e.g. Kuznetsov (1998) for more on bifurcation theory.

Proposition 1. *Assume ϕ_0 is large enough to guarantee the existence of a steady state in the positive orthant. Then if $\alpha \leq 1$, a sufficient condition for the existence of exactly two positive steady states is $z > \phi_0 \left(1 - \frac{\delta}{\delta + (r+\delta)(1-\eta)\eta(\gamma-1)}\right)^{\alpha}$.*

The sufficient condition of proposition 1 for the existence of exactly two positive steady states is satisfied by my final calibration as presented in subsection 5.3, but is by no means necessary. For all parameter values that I experimented with in my calibration, the interior tightness nullcline looks inverse tangential for a positive value of leisure, as in figure 2. Moreover, for $z > 0$ the tightness nullcline enters the positive orthant below the unemployment nullcline. Given that the unemployment nullcline is convex or logistic from lemma 1, and treating the tightness nullcline as inverse tangential or concave on the relevant segment, a saddle-node bifurcation occurs only once, resulting in exactly two positive steady states. The no-trade steady state always continues to exist. The case of the existence of positive steady states is the relevant one to explain empirically observed levels of unemployment, and has therefore been depicted in figure 2. In the next subsection I study the stability of the positive steady states.

3.3 Stability of steady states

Equation 1 shows that unemployment is always decreasing in unemployment and tightness. Since revenue per worker is increasing in employment, (11) shows that tightness is globally increasing in unemployment. However, it is not clear how tightness globally responds to tightness itself. For that reason, I only study the stability of tightness on the tightness nullcline. Equalizing (11) to zero gives $(r+\delta)\frac{\eta}{1-\eta} = (1-\beta)\frac{\lambda(\theta)}{\theta}\frac{\eta}{k(1-\eta)}[\phi(1-u) - g(\theta) - z]$. With the Cobb-Douglas matching function, $\lambda'(\theta) = \eta\frac{\lambda(\theta)}{\theta}$. In a positive steady state, the Jacobian matrix is then

$$\left(\begin{array}{cc} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial u} \\ \frac{\partial \dot{u}}{\partial \theta} & \frac{\partial \dot{u}}{\partial u} \end{array}\right)\Bigg|_{\dot{\theta}=0} = \left(\begin{array}{cc} r + \delta + s(\theta)\lambda(\theta)\frac{\beta}{(1-\eta)} & (1-\beta)\frac{\lambda(\theta)}{k(1-\eta)}\phi'(1-u) \\ -s'(\theta)\lambda(\theta)u - s(\theta)\lambda'(\theta)u & -\delta - s(\theta)\lambda(\theta) \end{array}\right). \tag{15}$$

Since the upper left entry is positive, tightness is increasing in itself on the $\dot{\theta} = 0$-locus. We know that one of the steady states is a saddlepoint, and the other is an antisaddle (node or focus). Only a focus has the oscillating dynamics that can result in endogenous fluctuations. The Jacobian matrix allows me to distinguish which steady state is the saddlepoint, and which is the antisaddle. Proposition 2 therefore states a necessary condition for oscillatory dynamics. The proof is in appendix C.

Proposition 2. *A steady state in the positive quadrant is an antisaddle if and only if the unemployment nullcline crosses the tightness nullcline from above.*

As can be seen in figure 2, steady state M is the intersection of the unemployment nullcline crossing the tightness nullcline from above. As a result, only this steady state can feature surrounding oscillatory dynamics, and only if its eigenvalues are complex. Steady state H is characterized by saddlepath dynamics, just as the no-trade steady state.

By calculating the trace of the Jacobian matrix in (15), one can determine whether the dynamics locally converge to the antisaddle (the trace is negative and the steady state is a sink), diverge from it (positive trace; source), or neither (zero trace; center). In the latter case the antisaddle is a focus, but the same applies more generally if the trace is small relative to the determinant.[13] The trace is given by the sum of the diagonal elements of the Jacobian matrix, and is thus

$$\frac{\partial \dot{\theta}}{\partial \theta} + \frac{\partial \dot{u}}{\partial u} = r + s(\theta)\lambda(\theta)\left(\frac{\beta}{(1-\eta)} - 1\right). \tag{16}$$

The trace and the corresponding stability of the steady states is crucial for understanding the dynamics of the model. Equation 16 shows that if the Hosios condition is satisfied, thus if $\beta = 1 - \eta$, the trace is only r. Consequently, if the discount rate r is zero and the sharing rule is efficient, the antisaddle will be a center. If $r \geq 0$ and $\beta \geq 1 - \eta$ with at least one of the inequalities strict, the trace is positive, so that the node or focus is unstable. However, even with $r > 0$, the trace can become zero and negative if β is smaller than efficient, turning the steady state stable. Figure 3 shows the trace of both positive steady states, as a function of the workers' bargaining power β. The trace is r at $\beta = 1 - \eta$, and for both steady states there exists a smaller than efficient workers' bargaining power that makes the trace zero (since tightness is positive).

Mortensen (1999) uses cases distinguished by the workers' bargaining power to characterize the dynamics of his model. In the next section I show when a Beveridge cycle exists, using that Mortensen's system and the one presented here are topologically equivalent, so that his results transfer to this paper, and mine to his.

4 Dynamic equilibria

In this section I present necessary conditions for a Beveridge cycle, and show how it emerges through a series of bifurcations. Rather than characterizing all topological possibilities of my model, I exploit that it is very similar to that of Mortensen (1999). I first prove equivalence between the dynamic system of Mortensen and the one presented here, so that I can subsequently focus on the dynamic equilibria that are useful for my calibration.

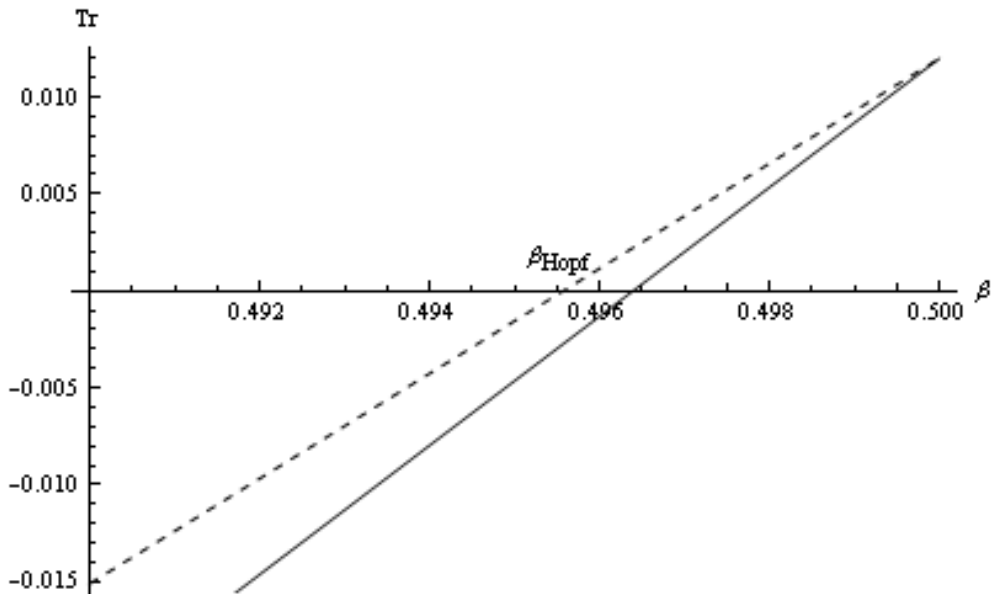[13]The eigenvalues are complex if $Tr^2 < 4Det$

Figure 3: The trace of antisaddle M (dashed) and saddlepoint H for different values of β.

Notes: Other parameters at their final calibration values as presented in subsection 5.3. In particular, $r = 0.012$ and $\eta = 0.5$, so that efficient bargaining corresponds to $\beta = 0.5$. Conditional on the job finding rate in steady state, the trace is invariant to changes in parameters different from these three.

4.1 Smooth equivalence

Mortensen (1999) models a system of differential equations in the surplus of a match p and employment n. To assess the empirical performance of this model, I have modeled it in labor market terms. However, apart from the introduction of a positive value of leisure in my model, the two systems are topologically the same. Proposition 3 states that they are diffeomorphic, or smoothly equivalent, for all interior equilibria. In this case, the two systems can be seen as essentially the same system written in different coordinates, retaining the same eigenvalues of the corresponding equilibria and the same periods of the corresponding limit cycles (Kuznetsov, 1998, p. 41). It is proven by the recognition that there is a smooth one-to-one correspondence between employment and unemployment, and surplus and tightness respectively. The proof is in appendix D.

Proposition 3. *The dynamical system in unemployment u and labor market tightness θ without a positive value of leisure ($z = 0$) and Mortensen (1999)'s dynamical system in employment n and surplus p are smoothly equivalent for all interior equilibria.*

Since the systems are smoothly equivalent, all dynamic properties carry over from one system to the other.[14] Consequently, since Mortensen (1999) finds a clockwise limit cycle in

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[14]If and only if n and p reach a steady state, u and θ do as well; if and only if n and p are on a stable arm, u and θ are as well; and if and only if n and p close an orbit, u and θ do as well. Moreover, if and only if $p = 0$, $\theta = 0$; if and only if p satisfies its transversality condition, θ does as well; and if and only if $n \in [0, 1]$, the same holds for u.

n and p under certain conditions, there must be a counterclockwise limit cycle in u and θ under the same conditions. To clarify the calibration, I present these conditions in the next subsection. Of course one must realize that Mortensen's results concern the model without a parameter for the gross value of leisure. I point out the differences along the way.

4.2 A Beveridge cycle

In this subsection I present necessary conditions for the existence of a Beveridge cycle. I define a Beveridge cycle as a limit cycle in labor market tightness and unemployment, which results in enduring endogenous fluctuations in vacancies and unemployment.[15] Restricting my analysis to the empirically relevant case of positive discounting r and a positive value of leisure z, necessary conditions for a Beveridge cycle are

- a smaller than efficient worker's bargaining power, i.e. $\beta < 1 - \eta$,

- a sufficiently high scale parameter for the expected revenue per worker ϕ_0, given the other parameters.

The first condition is necessary for the trace of the Jacobian matrix in M to become zero. In this case the antisaddle is a focus, the eigenvalues are purely imaginary, and a Hopf bifurcation occurs. In a Hopf bifurcation periodic equilibria emerge or disappear, depending on the direction of the parameter change. Keeping r and η fixed, worker's bargaining power β is the bifurcation parameter. Since a Hopf bifurcation in a hyperbolic system guarantees the existence of a limit cycle, figure 3 shows that a limit cycle in unemployment and labor market tightness exists for a workers' bargaining power on one side but close to the β_{Hopf} that makes the trace in M zero. Because search intensity increases with labor market tightness, a rise in tightness is not the result of a fall in search intensity but the consequence of an increase in vacancies, giving rise to enduring endogenous fluctuations in vacancies and unemployment.[16]

The second condition is sufficient for the existence of multiple steady states in the positive quadrant for $z > 0$, including a steady state in which the unemployment nullcline crosses

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[15]That is, the fluctuations are not due to fluctuations in unemployment and search intensity only.

[16]Under positive discounting, a smaller than efficient workers' bargaining power is not only necessary for limit cycles resulting from a Hopf bifurcation, but for closed orbits in general. If $r > 0$, the trace will always be positive if $\beta \geq 1 - \eta$. Mortensen (1999) argues by Bendixson's criterion that under these conditions no limit cycles can occur. Bendixson's criterion (see e.g. Guckenheimer and Holmes (1983, p. 44)) states that if on a simply connected state space the trace is not identically zero and does not change sign, a dynamical system has no closed orbits lying entirely in the state space. In Mortensen's system it is possible to compute the Jacobian matrix for every point in the state space, not only on the tightness nullcline. He shows that under positive discounting, the condition for a trace that is not identically zero and that does not change sign in the entire state space is simply $\beta \geq 1 - \eta$. Because of proposition 3 and because $z > 0$ does not affect the trace in Mortensen's system, the same condition has to hold in my model. As a result, the dynamical system in (1) and (11) has no closed orbits under positive discounting if $\beta \geq 1 - \eta$.

the tightness nullcline from above, as is necessary by proposition 2. This proposition implies that unemployment must decrease more from an increase in labor market tightness than the decrease in unemployment (and thus the increase in the revenue per worker) that is necessary to motivate such an increase in tightness. Tightness must thus result in many matches, and employers must react strongly to profit opportunities. This interpretation of proposition 2 helps to explain the Beveridge cycle.

Imagine a recovery from a trough, in which tightness increases and unemployment decreases via the matching process. Unemployment decreases more from an increase in labor market tightness than the decrease in unemployment (and thus the increase in the revenue per worker) that is necessary to motivate such an increase in tightness. As a result, employers want to open even more vacancies. As soon as employers open more vacancies, it becomes more attractive for workers to search, which makes it more attractive for employers to open vacancies, and so on and so forth. Employers overshoot the steady state tightness for the alluringly high revenue per worker at high aggregate employment levels. However, with so many vacancies and such a low unemployment rate it takes a long time to fill an individual vacancy. Consequently, while unemployment still decreases but employers foresee an end to the boom, they do not want to spend valuable resources on vacancies that are hard to fill. Expecting higher unemployment in the future and thus smaller benefits of a filled vacancy, employers reduce labor market tightness. With tightness decreasing, workers decrease their search intensity, and the benefits of opening a vacancy for an employer decrease even more. As a result, at some point fewer matches are made than jobs are destroyed, and unemployment increases. Higher unemployment feeds back to a lower revenue per worker, so that employers overshoot the steady state level of tightness in the trough as well. However, with so few vacancies, a single vacancy is filled very fast. Consequently, while unemployment still increases but employers foresee an end to the trough, they are willing to spend some resources on vacancies that will be filled very soon and pay off at the higher employment levels of the future. As a result, search intensity recovers, and vacancies are filled even easier. Job matching takes over from job destruction again, and unemployment decreases, completing the cycle. In the next subsection, I show that the Beveridge cycle is stable and exists for a workers' bargaining power in between the Hopf and a saddle-loop bifurcation.

4.3   A saddle-loop bifurcation

A saddle-loop bifurcation occurs when the stable and the unstable manifolds of a steady state with saddle-path stability connect to form a homoclinic orbit. In Hamiltonian systems homoclinic orbits are a common phenomenon, but in systems with hyperbolic steady states

the existence of homoclinic orbits is not robust to small perturbations of the parameters. The Andronov-Leontovich theorem (see e.g. Kuznetsov (1998, p. 200)) states that given a hyperbolic saddlepoint and a bifurcation parameter, a limit cycle bifurcates on one side of the homoclinic orbit.

Mortensen (1999) shows that his model is characterized by Hamiltonian dynamics if the discount rate r is zero and the sharing rule is efficient. As a result, a homoclinic orbit generically exists. By proposition 3, the same is true for my model under the same conditions and with $z = 0$. Moreover, for $z > 0$ the Hamiltonian function continues to exist for interior solutions.[17] However, Hamiltonian dynamics are an event of measure zero and never exist for a more realistic positive discount rate. Using Melnikov perturbation, Mortensen shows that also for a small distortion of the conditions - a positive r combined with a smaller than efficient β - the existence of a homoclinic orbit can still be proven. Proposition 4 states that a homoclinic orbit can continue to exist under positive discounting and for a positive value of leisure as a perturbation of the Hamiltonian system. The proof is in appendix E.

Proposition 4. *There exist parameter values such that, also for $r > 0$ and $z > 0$, a homoclinic orbit exists.*

As a corollary, the Andronov-Leontovich theorem establishes the existence of a limit cycle for an $r > 0$ and a workers' bargaining power β that is even smaller than the β_{SL} for which the saddleloop bifurcation occurs, which is already smaller than the efficient value. From figure 3 we can infer that this limit cycle is stable. Contained by the stable and unstable manifolds of H for β's smaller than β_{SL}, the limit cycle cannot disappear in any other way than in the Hopf bifurcation. Consequently, the limit cycle exists for β's in the range $(\beta_{Hopf}, \beta_{SL})$, and it is stable because the trace at M is positive for β's slightly larger than β_{Hopf}. It is larger for larger β's until it coincides with the homoclinic orbit, the biggest possible closed orbit.

The bifurcations and the limit cycle in between are illustrated in figure 4. These phase diagrams zoom in on the positive steady states at the parameter values of the final calibration as presented in subsection 5.3 (except, of course, for β), and the nullclines are dashed. Starting off from a small β in figure 4(a), the trace of the antisaddle is still negative, so that the steady state is stable. For a large set of initial conditions for unemployment, the antisaddle will be the attractor. At the supercritical Hopf bifurcation, the limit cycle inherits the stability of the antisaddle, which becomes unstable. Figure 4(b) shows that this limit cycle (depicted in the center) still attracts from outside its own orbit as well, although convergence may take a long time. However, at the saddle-loop bifurcation of figure 4(c), the basin of attraction outside the closed loop - the homoclinic orbit - has disappeared. For

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[17]Mortensen's function becomes simply: $H(p, n) = \int_0^n \phi(x)dx + (1 - n)[g(p) + z] - \delta pn$.

an infinitesimally larger β as in figure 4(d) no closed orbits exist any longer. The antisaddle remains unstable, but now the orbits originating from its neighborhood will no longer be bounded, except for the stable manifolds of the saddlepoints.

(a) $\beta = 0.45$

(b) $\beta = 0.4956$

(c) $\beta \approx \beta_{SL} \approx 0.4957$
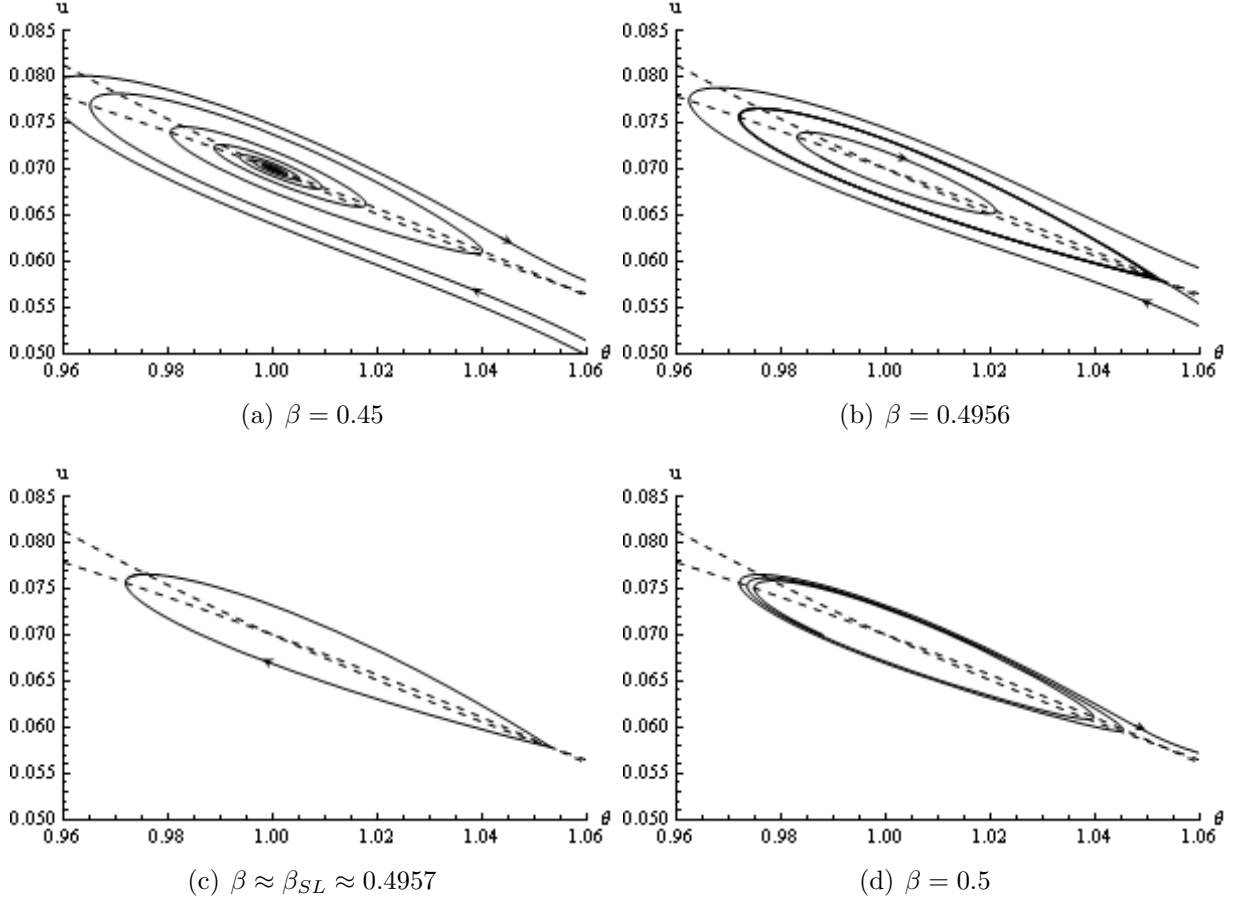
(d) $\beta = 0.5$

Figure 4: (a) Stable steady state; (b) Stable limit cycle (inner circle); (c) Homoclinic orbit of the saddle-loop bifurcation; (d) No closed orbits at efficient bargaining. $\beta_{Hopf} \approx 0.49557$.
Notes: All orbits are actual simulated paths, but panel (b) is only an approximation of the actual homoclinic orbit. Nullclines are dashed, intersecting twice. k and c_0 used to normalize steady state M tightness to 1. Except for β, other parameter values from final calibration as presented in subsection 5.3.

I take the dynamics of the limit cycle to be the relevant dynamics to explain the actual data, because of the evidence on the counterclockwise cycle as presented in figures **??** and 1, and its status as a stylized fact. Moreover, the stability of the Beveridge cycle further supports its plausibility as a data-generating process, although its basin of attraction may be small. For these reasons, I pay only very limited attention to other equilibria in the subsequent sections. Since for the Beveridge cycle to exist β must be smaller than efficient, I will assume this market failure in my calibration. This choice is based on the observed counterclockwise cycles rather than evidence on the bargaining power itself. In addition, Mortensen (1999) shows that saddlepoint H Pareto-dominates all other equilibria, so that my focus on the Beveridge cycle assumes yet another coordination failure.

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Figures 3 and 4 shows that the set of β's that give rise to a Beveridge cycle has a positive but small measure. Although the values of β that do result in a Beveridge cycle seem not implausible as a value for the worker's bargaining power, my proposed data-generating process is not very robust to changes in β. On the other hand, the range for which orbits oscillate is much bigger, containing almost all β's smaller than β_{SL}.[18] Indeed, there exists a large set of β's smaller than β_{Hopf} for which the dynamics oscillate around the antisaddle, eventually settling down in it. Especially if β is smaller than but close to β_{Hopf}, it may take a very long time to reach the steady state. In such a case, many business cycles could be explained with one exogenous shock in fundamentals or beliefs. Consequently, these kind of spirals could also be the data-generating process of the counterclockwise cycles in unemployment and vacancies, but I will focus on the limit cycles. The next section presents the numerical part of this paper.

5   Calibrating the Beveridge cycle

In this section I calibrate the Beveridge cycle. First I summarize the data on unemployment, vacancies and the duration of the business cycle. I subsequently explain my parameter choices, and show the calibration results. Finally, I discuss the robustness of the calibration.

5.1   Data on the business cycle

I take information on the volatility and persistence of unemployment and other statistics from Shimer (2005), who reported summary statistics of quarterly US data from 1951 to 2003. The standard deviation and autocorrelation of unemployment u are presented in table 1, together with those of other relevant variables in the model and their cross-correlations. These variables are vacancies v, the ratio of vacancies to unemployment v/u, the job finding rate f, the job separation rate δ, and labor productivity y.[19]

My model of the Beveridge cycle has a constant job separation rate. However, since the reported standard deviation of the job separation rate in table 1 is not zero, in reality some of the variance of unemployment is driven by job separations rather than fluctuations in hiring. Indeed, Pissarides (2009, p. 1344) argues that one-third to one-half of the volatility in unemployment is driven by fluctuations in the inflow into unemployment. As a result, a model with constant job separation rate should not explain all volatility in unemployment.

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[18]Only for small β's the eigenvalues stop to be complex.

[19]Statistics on vacancies are based on the help-wanted advertising index (HWI), so that an average is uninformative. On the other hand, vacancies are likely to be underreported anyway. The index offers a more reliable picture of their cyclical properties, as correspondence with the shorter JOLTS data confirms (Shimer, 2005, p. 29).

		u	v	v/u	f	δ	y
Standard deviation		0.190	0.202	0.382	0.118	0.075	0.020
Quarterly autocorrelation		0.936	0.940	0.941	0.908	0.733	0.878
	u	1	-0.894	-0.971	-0.949	0.709	-0.408
	v		1	0.975	0.897	-0.684	0.364
Correlation matrix	v/u			1	0.948	-0.715	0.396
	h				1	-0.574	0.396
	δ					1	-0.524
	y						1

Table 1: Summary statistics of table 1 of Shimer (2005), quarterly US data, 1951-2003.
Notes: f stands for the job finding rate, y for the revenue per worker. Standard deviations and correlation coefficients in logs as deviations from an HP trend with smoothing parameter 10^5.

Other relevant statistics are the average unemployment, job finding, and separation rates, which are 0.0567, 1.355, and 0.102 respectively. Remarkably, the mean unemployment rate over time lies substantially lower than the steady state value u^{SS} consistent with the observed job finding and separation rates: $u^{SS} = 0.102/(0.102 + 1.355) = 0.0700$. Also in my model, the steady state and mean unemployment rates need not coincide. From the perspective of this model, the economy spends more time on the lower than on the upper part of the Beveridge cycle, or negative deviations exceed positive deviations.

Finally, the duration of the business cycle provides another natural statistic for an endogenous cycle. One can argue that only for an AR(1)-process the autocorrelation coefficient is the single most informative statistic for persistence. Unlike the autocorrelation coefficient, the duration of the business cycle is invariant to logarithmic transformations and HP-filtering of the (simulated) data, so that it is a robust calibration target. Figure 5 shows the duration of the business cycles falling entirely in the sample period from 1951 to 2003. The duration seems skewed to the right, and depending on whether the cycle is measured from peak to peak or from trough to trough, the median cycle last 20.5 or 19.3 quarters respectively. The average is 23.8 quarters irrespective of measurement from peak or trough. The next subsection presents the parameter choices.

5.2   Parameter choices

The parameters α, z, and β are chosen to target a particular steady state unemployment rate, relative value of leisure, and duration of the business cycle. The remaining parameters are used to normalize steady state tightness, or chosen exogenously. More specifically, I follow Mortensen (1999, p. 909-910) in choosing γ equal to 1.29, which is based on empirical evidence in Burdett, Kiefer, Mortensen, and Neumann (1984). I set the elasticity of the
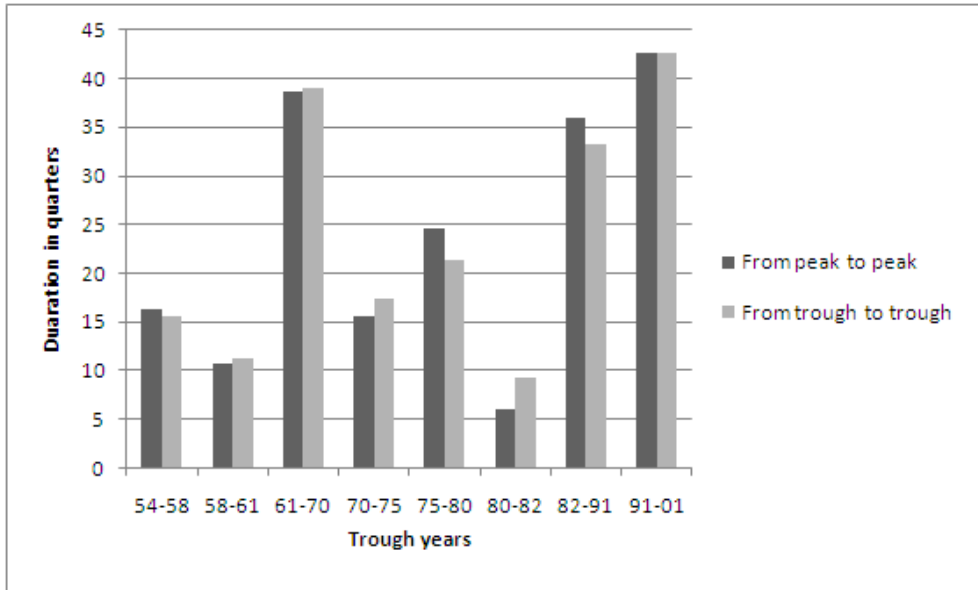
Figure 5: Duration in quarters of all completed NBER business cycles between 1951 and 2003.
Source: http://www.nber.org/cycles.html, collected at May 27, 2013.

matching function η to 0.5. Parameter choices for the elasticity of the matching function have varied considerably across the literature, ranging from 0.28 (Shimer, 2005) to 0.6 (Blanchard and Diamond, 1989). I follow the choice of Hall and Milgrom (2008), which is consistent with the evidence on the United States in Petrongolo and Pissarides (2001). Finally, I set the constant separation rate δ equal to its empirical quarterly average of 0.102, fix the quarterly discount rate r at a common 0.012, and normalize the scale parameter for output ϕ to 1.

The model allows for two more normalizations, because the level of the ratio of vacancies to unemployment $\frac{v}{u}$ and the value of search intensity s are intrinsically meaningless.[20] Consequently, recruiting and search parameters k and c_0 can be chosen to target $\frac{v}{u}$ and s respectively, and thus labor market tightness θ, in steady state M. Following Shimer (2005), I normalize the steady state ratio of vacancies to unemployment to 1, just as search intensity, so that steady state tightness $\theta \equiv \frac{v}{su}$ is 1 as well. Since the job finding rate is $s(\theta)m_0\theta^\eta$, these choices force matching efficiency m_0 to the observed quarterly rate of 1.355. The exogenously fixed parameters are summarized in table 2.

Not only steady state tightness, but also steady state unemployment can be a calibration

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[20]Doubling the cost of vacancies k and multiplying the scale parameter of the matching function m_0 by 2^η, halves θ by reducing the number of vacancies opened, but leaves the search intensity, job finding rate, and unemployment rates unchanged. This is not only true in the nullclines, but also globally, as can be checked in the laws of motion of (1) and (11). Similarly, doubling the scale parameter of the search cost function c_0, changes search intensity s by a factor $2^{-\frac{1}{\gamma}}$ and thus θ by $2^{\frac{1}{\gamma}}$, while leaving search costs $c(s)$ unchanged. Combined with a multiplication of matching efficiency m_0 by a factor of $2^{\frac{1-\eta}{\gamma}}$, however, the job finding and unemployment rates are unaffected.

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Parameter		Value	Source
Discount rate	r	0.012	Standard
Scale revenue per worker	ϕ_0	1	Normalization
Separation rate	δ	0.102	Shimer (2005)
Scale matching function	m_0	1.355	Shimer (2005)
Elasticity matching function	η	0.5	Petrongolo and Pissarides (2001)
Elasticity search cost	γ	1.29	Burdett, Kiefer, Mortensen, and Neumann (1984)

Table 2: Exogenously fixed parameters.

target. I use the elasticity of the externalities α and the gross value of leisure z to match the location of steady state M to the 7 percent steady state unemployment rate consistent with the observed average job finding and separation rates. This calibration target requires a high gross value of leisure or substantial demand externalities. As argued in section 3, an increase in the gross value of leisure z shifts the tightness nullcline down to $u_0 = 1 - (z/\phi)^{1/\alpha}$ at $\theta = 0$. This means that the revenue per worker (which increases in the employment rate) must be higher to have any vacancy creation at all. Since such an increase leaves the unemployment nullcline unaffected, a higher z lowers the unemployment rate in steady state M as long as it continues to exist. The magnitude of this coordinating effect is mediated by the size of the externalities α.

As discussed in the introduction, the reduced form of the feedback from employment to (expected) revenue per worker is mathematically equivalent across the three cases of Cooper and John (1988): matching technology, production technology, and agents' demands. Concerning matching technology, I am not aware of any empirical studies on thick-market externalities in the goods market. Secondly, increasing returns in production have been studied more often, but are equally controversial.[21] Finally, there is not much empirical evidence of demand spillovers between the labor and goods market, except for an interesting study on shopping externalities by Kaplan and Menzio (2012). Based on the idea that unemployed people spend less *and* force down markups, their calibration implicitly shows a decrease of 0.72 percent in real labor productivity as the result of unemployment rising from 5 to 9 percent. With my constant elasticity functional form - $\phi(1-u)^\alpha$ - such a small decrease in labor productivity corresponds to significant externalities: $\alpha = 0.169$.

Authors explaining the volatility of unemployment vary considerably in their parameter

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[21]For instance, Caballero and Lyons (1989) estimate aggregate returns to scale of 1.3, so that Mortensen sets α equal to 0.3. However, the methodology and corresponding high estimates of this earlier literature have been questioned by Basu and Fernald (1995) and Burnside (1996), who are unable to reject constant returns to scale. Responding to this critique, Harrison (2001) allows for sector-specific external effects. While she rejects increasing returns in the consumption goods sector, her two-standard-error confidence intervals (across different specifications) of externalities in the investment goods sector suggest values between 0.021 and 0.172.

choices for the flow benefit of leisure.[22] In contrast to these calibrated models, my model includes both a gross value of leisure z and variable search costs $c(s)$, together making up the net value of leisure. On top of that, while productivity in other models fluctuates around an exogenous average of 1, productivity in my model is endogenous. For that reason, the relevant calibration target of my model is not z, but the net value of leisure relative to output

$$\zeta = \frac{z - c(s)}{\phi(1 - u)} = \frac{z - c_0 \left(\frac{\beta}{1 - \beta} \frac{k}{c_0 \gamma} \theta \right)^{\frac{\gamma}{\gamma - 1}}}{\phi_0 (1 - u)^\alpha}.$$

In matching the steady state unemployment rate to 7 percent, I do not allow the relative value of leisure to exceed the 0.955 of Hagedorn and Manovskii (2008). Moreover, I also present an alternative static calibration in which I target a relative value of leisure of 0.71. The size of the externalities α is subsequently used as to obtain an unemployment rate of 7 percent at steady state M. For the sake of illustrating the trade-off between α and ζ only, I use the steady state at the saddle-node bifurcation and set β at β_{Hopf}.[23] Since β_{Hopf} results in the lowest steady state unemployment rate consistent with a limit cycle, I make these choices to illustrate the smallest required externalities for a Beveridge cycle around a 7 percent unemployment rate.[24]

In case one allows for $\zeta = 0.955$, this static calibration results in required externalities of $\alpha = 0.151$ to explain a 7 percent steady state unemployment. Given the evidence above, this magnitude of the external effect seems quite reasonable. In case I require $\zeta = 0.71$, the saddle-node bifurcation occurs at 7 percent unemployment for $\alpha = 0.973$. Such a value seems outside a plausible range for the externalities. As a last alternative, one could also exploit the normalized labor force of my model, and argue that not unemployment, but non-employment should be the calibration target. CPS data show an average employment rate of 60 percent, implying a job finding rate of 0.153.[25] Matching a non-employment rate

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[22]Shimer (2005) started assumes a relatively low 0.4. This value encourages Hagedorn and Manovskii (2008) to argue that Shimer's lack of amplification is due to his calibration, not to the model. Their own calibration strategy results in a relatively high value of 0.955. Hall and Milgrom (2008), followed by Pissarides (2009), argue that a flow benefit of 0.955 is too high. Hall and Milgrom themselves come up with a calibration that gives a value equal to 0.71, which is close to the 0.73 of Mortensen and Nagypal (2007). They argue that this value is much more realistic because of its small elastic labor supply. In the calibration of Hagedorn and Manovskii, unemployment would be very responsive to changes in the value of leisure, such as a policy change in the unemployment benefit.

[23]At the saddle-node bifurcation the nullclines just touch and the two positive steady states overlap. This single steady state will not have surrounding oscillatory dynamics, but a steady state with such dynamics emerges for an infinitesimally bigger value for bifurcation parameter α, keeping other parameters fixed. Similarly, at the Hopf bifurcation no limit cycle exists, but it does for an infinitesimally larger β. Given the parameter values in table 2 $\beta_{Hopf} \approx 0.4956$.

[24]A high β gives rise to a higher search intensity but fewer vacancies, each having a second order effect on the other, so that the net effect on unemployment is not immediately clear. Within the limit cycle range, however, numerical analysis shows that the unemployment rate monotonically increases in β.

[25]As a result of this lower rate, also β_{Hopf} decreases to approximately 0.46. The other parameters remain

of 40 percent, while keeping the value of leisure ζ at 0.71, only requires modest externalities: $\alpha = 0.127$. Table 3 presents the trade-off between high externalities and a high relative value of leisure in explaining a 7 percent unemployment rate. It shows the calibration targets u and ζ, the required α and z, and also the vacancy and search cost parameters k and c_0 required to normalize the steady state search intensity and tightness to 1.[26]

Calibration		High leisure	High externalities	Non-employment
St.st. unemployment	u^{SS}	0.07	0.07	**0.40**
St.st. net value of leisure	ζ^{SS}	**0.955**	0.71	0.71
Elasticity externalities	α	0.151	**0.973**	0.127
Gross value of leisure	z	0.974	0.841	0.746
Vacancy cost	k	0.039	0.235	0.122
Scale search cost	c_0	0.029	0.179	0.081

Table 3: Parameters to target a given unemployment rate u and relative net value of leisure ζ at the saddle-node bifurcation steady state.
Notes: Trade-off unaffected by scale parameter of search costs c_0 and costs of vacancy k, but chosen to equalize search intensity s and labor market tightness θ to 1 in steady state. $\beta = 0.495572$ and other parameters from table 2, except for last column where $m_0 = 0.153$ and $\beta = 0.460784$ (m_0 and β chosen at job finding rate and Hopf bifurcation respectively).

In my subsequent calibration I only continue with a 7 percent unemployment rate and a 0.955 relative value of leisure as calibration targets for steady state M. For a slightly bigger α than at the saddle-node bifurcation, two positive steady states exist, and the endogenous Beveridge cycle surrounding M can be used to explain the volatility and persistence of unemployment. The bifurcation analysis of the last section makes clear that the bargaining power of workers β is important for the existence of the Beveridge cycle. I choose the β that is closest to efficient bargaining while still giving rise to a limit cycle. The limit cycle of this β approaches the homoclinic orbit, the largest closed orbit possible taking the other parameters as given.[27] By moving β further away from efficient bargaining in the direction of the Hopf bifurcation, the limit cycle can be made arbitrarily small, but then the Beveridge cycle can explain little volatility.

Increasing the elasticity of the externalities α drives the two positive steady states apart, while M can be kept at its unemployment calibration target by matching efficiency parameter m_0. I choose α such that the duration of the Beveridge cycle, given the other parameters, corresponds to the median duration of the NBER business cycle over the sample period. For that reason I choose α to target a Beveridge cycle duration of 20 quarters. Together with the parameters in table 2, the calibration strategy described here results in

the same.

[26]If other parameters (such as large externalities) make opening vacancies and searching for jobs very attractive, the normalization requires bigger search and vacancy costs.

[27]Computationally, I find the largest (in machine precision) β resulting in a limit cycle that still has a basin of attraction outside its orbit.

the parameters of table 4. Note that $\alpha = 0.185$, close to the $\alpha = 0.169$ that is implicit in Kaplan and Menzio (2012). The next subsection presents the results of my calibration.

Parameter		Value	Calibration target	
Vacancy cost	k	0.0386158	St.st. vacancy ratio	$v/u = 1$
Scale search cost	c_0	0.0294244	St.st. search intensity	$s = 1$
Gross value of leisure	z	0.9716870	Relative net value of leisure	$\zeta = 0.955$
Workers' bargaining power	β	0.4957014	Saddle-loop bifurcation	β_{SL}
Elasticity externalities	α	0.185	Duration business cycle	20 quarters

Table 4: Calibrated parameters, with their respective calibration targets.

5.3 Calibration results

In this subsection I present the results of a calibration of the Beveridge cycle. I draw 212 quarterly observations from the simulated Beveridge cycle, and compute time series for unemployment, vacancies, the vacancy-unemployment ratio, the job finding rate, and revenue per worker. More specifically, for each variable I sample 20 time series for 20 different starting points all around the cycle. I report the average statistics over the cycle, although differences are small. As for the original data, I take logs and deviations from an HP trend with smoothing parameter 10^5. Table 5 is the simulated counterpart to table 1, presenting the standard deviation, autocorrelation and cross-correlations of the endogenous variables of the model.

		u	v	v/u	f	y
Standard deviation		0.095	0.037	0.119	0.105	0.001
Quarterly autocorr.		0.915	0.856	0.909	0.909	0.909
	u	1	-0.544	-0.966	-0.966	-0.999
	v		1	0.742	0.742	0.544
Correlation matrix	v/u			1	1.000	0.965
	f				1	0.965
	y					1

Table 5: Summary statistics of the calibrated Beveridge cycle over 212 quarters.
Notes: f stands for the job finding rate, y for revenue per worker. Statistics are averages from samples across the 20 quarter cycle. All variables are in logs as deviations from an HP trend with smoothing parameter 10^5.

Table 5 shows that the calibrated Beveridge cycle is able to account for half of the standard deviation of unemployment, of which the total is 0.190. Since one-third to one-half of the volatility in unemployment is driven by fluctuations in the job separation rate that is constant in this model, the Beveridge cycle can explain at least three quarters of the hiring

driven volatility, and may even explain all of it. In addition, the simulated autocorrelation is 0.915, close to the empirical estimate of 0.936. The autocorrelation coefficients of all other endogenous variables are also similar to their observed counterparts.

The calibrated Beveridge cycle fails to explain sufficient volatility in vacancies and in the ratio of vacancies to unemployment. However, the standard deviation of the job finding rate is close to the observed standard deviation. Most importantly, the 0.001 standard deviation of the expected revenue per worker is smaller than the actual 0.020, so that the Shimer puzzle is not reintroduced in disguise: I don't need a high volatility in my endogenously changing revenue per worker to explain high volatility in unemployment, whereas the standard search and matching model requires large shocks in productivity to have sufficient volatility in unemployment. Moreover, the volatility of revenue per worker is so low that the volatility of vacancies could be increased by introducing productivity shocks.

Unemployment and the expected revenue per worker are almost perfectly correlated in my model by construction, so that the simulated correlation coefficient overstates the actual negative correlation between these two variables. In addition, my calibration understates the correlation between unemployment and vacancies. As a result, vacancies are not as closely associated with their ratio to unemployment and the job finding rate as in the data. On the other hand, this lack of correlation breaks the extremely tight counterfactual link between vacancies and revenue per worker in the standard search and matching model of Shimer (2005). The high counterfactual correlation with revenue per worker survives to a much larger extent for the ratio of vacancies to unemployment and the job finding rate, because of the choice to model the demand externalities as a function of unemployment. However, the assumed almost perfect correlation between unemployment and the expected revenue per worker can be broken by additional productivity shocks as well.

For analytical convenience I have modeled a system in unemployment and labor market tightness, rather than in vacancies. Because search intensity increases with labor market tightness, a rise in tightness is not the result of a fall in search intensity but the consequence of an increase in vacancies, giving rise to enduring cycles in vacancies and unemployment. To show graphically that my model explains the counterclockwise cycles in the *unemployment,vacancy rate*-plane, I present figure 6. The figure contains 20 simulated quarterly observations that complete a cycle, connected by straight lines. It is qualitatively similar to the observed cycles in figure 1, and rotates counterclockwise. Figure 7 shows a simulated time series of unemployment over 212 quarters, connected by straight lines. The time series is very regular, much more so than actual data. However, exogenous shocks in fundamentals or beliefs can cause variations in amplitude and period of the cycle, without altering its driving mechanism. The next subsection discusses robustness issues.
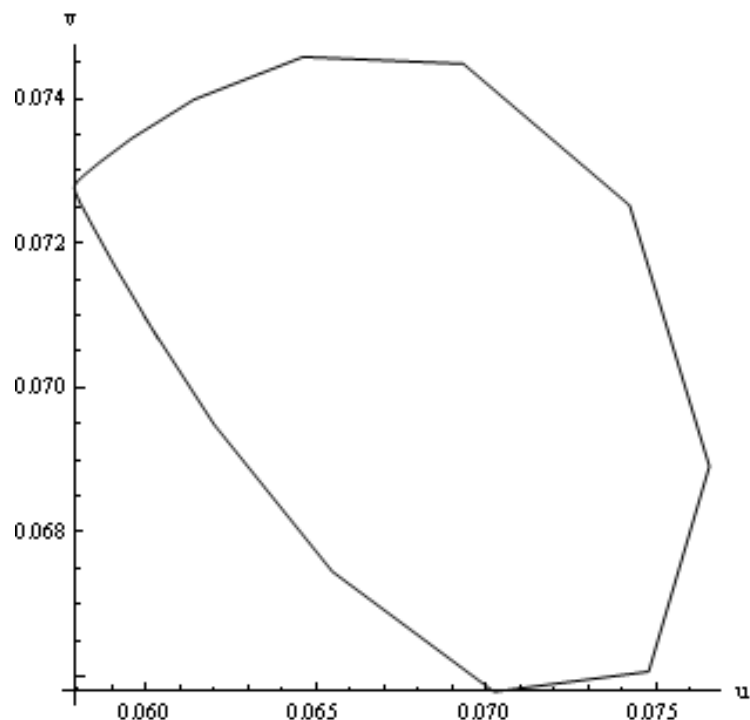
Figure 6: A simulated counterclockwise cycle in unemployment and vacancies.
Notes: 20 quarterly datapoints from the calibrated Beveridge cycle, connected by straight lines.

5.4 Robustness and the unemployment benefit

I claim that the calibrated Beveridge is able to explain almost all hiring-driven volatility of unemployment, unlike the standard search and matching model as calibrated by Shimer (2005). However, Hagedorn and Manovskii (2008) argue that the low value of leisure in Shimer's calibration is the reason for his lack of amplification. Since I use Hagedorn and Manovskii's high net relative value of leisure to explain the *level* of unemployment, I stress that my results on the *volatility* of unemployment are not driven by this value of leisure. To show this, I also adopt the calibration targets of column 3 of table 3. I explain 40 percent nonemployment and a 0.71 net relative value of leisure, keeping the elasticity of the externalities equal to the previous calibration so that $\alpha = 0.185$. Adopting the workers' bargaining power approaching the homoclinic orbit ($\beta \approx 0.4641$), gives a standard deviation of unemployment of 0.125, two-thirds of the observed volatility. I conclude that the value of leisure does not drive my results on the volatility of unemployment.[28]

As another robustness exercise I calibrate the elasticity of the externalities to the average

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[28]The volatility of unemployment is mainly driven by the elasticity of the externalities. The fact that the standard deviation of unemployment *increases* for a lower net relative value of leisure, keeping the elasticity of the externalities fixed, is because in this case a lower α is required for the saddle-node bifurcation, as can be seen in table 3. Consequently, the same $\alpha = 0.185$ is larger relative to the smallest required externalities and drives the two steady states further apart, allowing for a larger volatility.
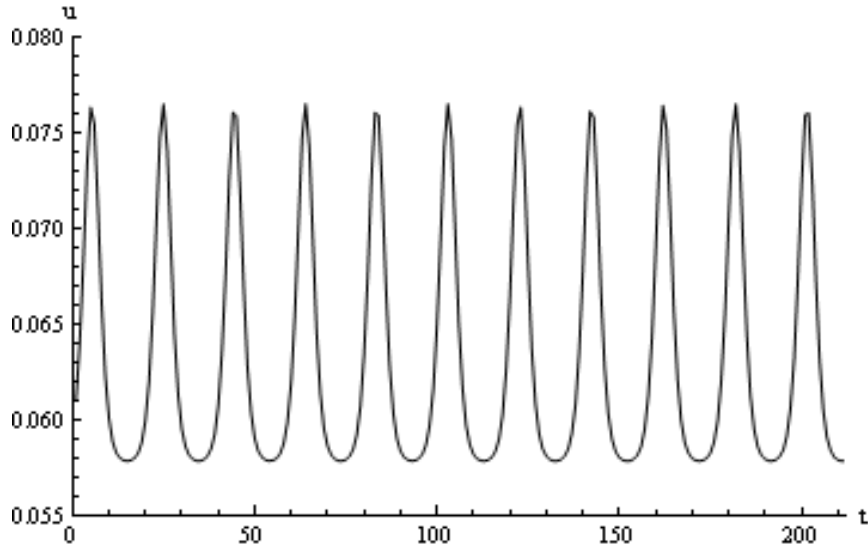
Figure 7: Simulated time series of unemployment.
Notes: 212 quarterly datapoints from the calibrated Beveridge cycle, connected by straight lines.

duration of the business cycle, instead of the median. With 23 quarters as a calibration target and worker's bargaining power approaching its new saddle-loop bifurcation value, $\alpha = 0.177$ and $\beta \approx 0.49567$. Not surprisingly, the 23-quarter Beveridge cycle results in a larger autocorrelation coefficient than the 20-quarter cycle, and smaller elasticities result in a smaller standard deviation. The full statistics of the 23-quarters cycle are presented in table 6.

		u	v	v/u	f	y
Standard deviation		0.074	0.026	0.092	0.083	0.001
Quarterly autocorr.		0.932	0.889	0.928	0.928	0.929
	u	1	-0.590	-0.973	-0.973	-1.
	v		1	0.760	0.760	0.590
Correlation matrix	v/u			1	1.	0.973
	f				1	0.973
	y					1

Table 6: Summary statistics of the 23-quarter Beveridge cycle.
Notes: f stands for the job finding rate, y for revenue per worker. Statistics are averages from 212-quarters samples across the 23 quarter cycle. All variables are in logs as deviations from an HP trend with smoothing parameter 10^5.

Finally, I have shown that a higher gross value of leisure z lowers the steady state M unemployment rate. A high gross value of leisure serves like a cue that communicates to all labor market participants that all possible positive steady states require a high expected revenue per worker and thus a low unemployment rate. A lower revenue per worker is not sufficient anymore to pay the wages that rise with the value of leisure. Consequently,

employers expect that revenue per worker will be high in the future, and open vacancies accordingly. Via the matching process, more vacancies result in lower unemployment, and thus the expectation of a high revenue per worker is realized. A higher unemployment benefit - a component of the gross value of leisure - can therefore shift coordination to a low unemployment equilibrium, which might seem counterintuitive. I show that an increase in the unemployment benefit which reduces unemployment is not necessarily counterfactual.

Empirical evidence on the effect of unemployment benefits must take general equilibrium effects into account, since in my model the unemployment benefit reduces unemployment via vacancy creation. Microstudies in which the unemployment benefit of only a fraction of the population of unemployed workers is varied, and in which subsequently individual labor market outcomes are compared, are therefore not suited to reject my model. For the same reason, and because natural experiments are rare, Costain and Reiter (2008) perform a panel-data study across countries. They conclude that a higher unemployment benefit increases unemployment. However, the explanatory variable in their regressions is not the unemployment benefit, but the replacement ratio. I will show that in my model the replacement ratio moves in the opposite direction as the unemployment benefit. As a result, my model is consistent with the data of Costain and Reiter (2008) and similar studies, but not with their conclusion.

Remember that the relative value of leisure ζ differs from the gross value of leisure z, and is given by (5.2). This expression already shows that the net relative value of output may move in the opposite direction as the gross value of leisure, if the gross value of leisure reduces unemployment to such an extent that the denominator increases more than the numerator. Similarly, the replacement ratio z/w with wage w given by (9) may decrease after an increase in z if w increases relatively more. Table 7 shows by a numerical example that this is exactly what happens. The table shows values of steady state M for two different values of z: at its final calibration value of subsection 5.3, and at its (higher) saddle-node bifurcation value.[29] It shows that an increase in z can increase labor market tightness θ (and as a result decrease unemployment u and thus increase revenue per worker y) to such an extent that the net relative value of leisure ζ and the replacement ratio z/w decrease.

z	θ	u	y	ζ	z/w
0.9717	1	0.0700	0.9867	0.9550	0.9881
0.9718	1.026	0.0636	0.9879	0.9523	0.9870

Table 7: Comparative steady states M for different values of leisure z.
Notes: z at its final calibration value as presented in subsection 5.3, and its saddle-node bifurcation value respectively. All other parameters as in calibration.

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[29]Now z is used as bifurcation parameter, keeping α, ϕ_0, m_0, k and other parameters fixed.

The qualitative message that the replacement ratio moves in the opposite direction as the unemployment benefit for steady state M and its surrounding dynamics is robust to all parameterizations experimented with - including those consistent with a net relative value of leisure equal to 71%. The replacement ratio is an endogenous variable, because the numerator z determines the unemployment rate u and thus denominator w. Costain and Reiter (2008) address the possibility that the unemployment benefit might vary in response to the unemployment rate, and for that reason instrument labor market policies by their lags. However, the intrinsic endogeneity of using the replacement ratio as explanatory variable is not overcome in this way. Moreover, the estimated coefficients of Costain and Reiter (2008) decrease once they control for the international business cycle by including time dummies or the mean output gap (across countries). Consequently, their results might at least partially be driven by the denominator of the replacement ratio, which endogenously varies over the business cycle. The cross-country evidence on the effect of the unemployment benefit is therefore not necessarily in contradiction with the predictions of this model.[30] Realize, however, that the model doesn't allow for moral hazard effects on the search intensity of the unemployed. Via this channel the unemployment benefit might have a positive impact on the unemployment rate.

6   Conclusion

Mortensen (1999) presents a parsimonious model to show that multiple Pareto-ranked equilibria can coexist, and that different expectations can self-fulfillingly result in each of the equilibria. The equilibrium dynamics include a stable limit cycle - the Beveridge cycle - that I calibrate to the median duration of the business cycle. For plausible parameter values, the calibrated Beveridge cycle can account for most of the hiring-driven volatility and almost all persistence of unemployment. In addition, this Beveridge cycle looks qualitatively similar to the counterclockwise cycles in the *unemployment, vacancy rate*-plane.

The calibration requires a substantial value of leisure to explain a Beveridge cycle around a sufficiently low unemployment rate. However, it cannot be stressed enough that a substantial value of leisure does not drive my results on the volatility of unemployment, as it does in Hagedorn and Manovskii (2008). A limitation of this study is that the range of values for the workers' bargaining power that results in a limit cycle is small. Although my calibration focuses on this purely deterministic Beveridge cycle, for a much bigger set of

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[30]In my model, the immediate (non-steady state) effect of the unemployment benefit is affected by the economy's location on the cycle, while the timing of a policy shock is unlikely to be exogenous. In addition, the immediate effect of the unemployment benefit depends on the beliefs about the effects of the policy, which can be correlated to fundamental shocks. These possibilities have to be addressed in empirical studies on the effects of the unemployment benefit.

bargaining power parameter values a single shock can result in counterclockwise fluctuations that are able to explain many business cycles but eventually settle down in a steady state.

The model of this paper ignores other relevant sources of unemployment volatility. Most importantly, job destruction is constant. Shocks to this and other parameters provide a natural complement to the endogenous mechanism of this paper. On top of that, the indeterminacy of equilibrium allows for belief shocks, which directly result in another level of labor market tightness by the opening or closing of vacancies by employers and the adjusted search intensity of the unemployed. While additional exogenous shocks can result in more irregular time series than those generated by my calibration, they also provide additional degrees of freedom.

My calibration shows that exogenous shocks are not required to explain almost all persistence of unemployment, and most of the volatility that is driven by job creation. Rather than providing another solution to the amplification and propagation problems of the standard search and matching model, I present the quantitative performance of a deterministic model in explaining the volatility and persistence of unemployment over the business cycle. Understanding endogenous unemployment fluctuations can help in designing policies that efficiently reduce the level or volatility of unemployment, and guide future empirical studies on the effect of the unemployment benefit.

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Appendices

A Proof of lemma 1

Proof. Define $\kappa \equiv \eta + \frac{1}{\gamma-1}$. Differentiate (12) twice with respect to θ, to obtain

$$\frac{d^2u}{d\theta^2} = \frac{\delta m_0 \kappa \left(\frac{\beta}{1-\beta}\frac{k}{c_0\gamma}\right)^{\frac{1}{\gamma-1}} \theta^{\kappa-2} \left[\delta(1-\kappa) + (1+\kappa)m_0 \left(\frac{\beta}{1-\beta}\frac{k}{c_0\gamma}\right)^{\frac{1}{\gamma-1}} \theta^\kappa\right]}{\left[\delta + m_0 \left(\frac{\beta}{1-\beta}\frac{k}{c_0\gamma}\right)^{\frac{1}{\gamma-1}} \theta^\kappa\right]^3}.$$

For $\kappa \leq 1$ the second derivative is positive for all $\theta > 0$, so that the unemployment nullcline is convex. For $\kappa > 1$, the second derivative can be positive or negative, depending on θ. More specifically, for $\kappa > 1$ there exists a unique inflection point at the positive labor market tightness given by

$$\theta^* = \left[\frac{\delta(\kappa-1)}{(1+\kappa)m_0 \left(\frac{\beta}{1-\beta}\frac{k}{c_0\gamma}\right)^{\frac{1}{\gamma-1}}}\right]^{\frac{1}{\kappa}}.$$

Consequently, for $\kappa > 1$ the unemployment nullcline is concave for $0 < \theta < \theta^*$, and convex for all $\theta > \theta^*$, so that as a whole it has a logistic shape. $\qquad\square$

B Proof of proposition 1

Proof. The second derivative with respect to θ of the tightness nullcline in (13) is

$$\frac{d^2u}{d\theta^2} = -\frac{1}{\alpha}\left[\frac{(r+\delta)k\theta}{\phi_0(1-\beta)\lambda(\theta)} + \frac{g(\theta)+z}{\phi_0}\right]^{\frac{1-\alpha}{\alpha}}\left[\frac{k\beta s(\theta)}{(\gamma-1)\phi_0(1-\beta)\theta} - \frac{(r+\delta)k(1-\eta)\eta}{\phi_0(1-\beta)\lambda(\theta)\theta}\right]$$
$$-\frac{1-\alpha}{\alpha^2}\left[\frac{(r+\delta)k\theta}{\phi_0(1-\beta)\lambda(\theta)} + \frac{g(\theta)+z}{\phi_0}\right]^{\frac{1}{\alpha}-2}\left[\frac{(r+\delta)k(1-\eta)}{\phi_0(1-\beta)\lambda(\theta)} + \frac{k\beta s(\theta)}{\phi_0(1-\beta)}\right]^2.$$

One can see that for $\alpha \leq 1$, the tightness nullcline is concave for sure on the segment of the nullcline for which

$$s(\theta)\lambda(\theta) > (r+\delta)(1-\eta)\eta(\gamma-1).$$

Define ξ as the job finding rate equal to $(r+\delta)(1-\eta)\eta(\gamma-1)$. Given that the unemployment nullcline is convex or logistic by lemma 1, if any positive steady state exist, generically exactly two positive steady states exist if the tightness nullcline lies below the unemployment nullcline for any potential non-concave segment of it. In that case, the concave segment

of the tightness nullcline intersects at most twice with the unemployment nullcline. A sufficient condition for any non-concave segment of the tightness nullcline to lie below the unemployment nullcline is the maximum unemployment rate giving rise to any vacancy creation (u_0 as given by (14)) to be lower than the unemployment rate consistent with the job finding rate ξ. Consequently, assuming the existence of a steady state in the positive orthant, for $\alpha \leq 1$ generically exactly two steady states exist if

$$z > \phi_0 \left(1 - \frac{\delta}{\delta + \xi} \right)^\alpha.$$

$\square$

C Proof of proposition 2

Proof. We know from the Jacobian matrix in (15) that $\frac{\partial \dot{\theta}}{\partial \theta} > 0$, $\frac{\partial \dot{\theta}}{\partial u} > 0$, $\frac{\partial \dot{u}}{\partial \theta} < 0$, and $\frac{\partial \dot{u}}{\partial u} < 0$ in steady state. Depending on whether the product of the diagonal elements $\frac{\partial \dot{\theta}}{\partial \theta} \frac{\partial \dot{u}}{\partial u}$ or the product of the cross-diagonal elements $\frac{\partial \dot{\theta}}{\partial u} \frac{\partial \dot{u}}{\partial \theta}$ is more negative, the determinant is negative or positive respectively. If and only if the determinant of the Jacobian matrix at a steady state is negative, it has eigenvalues of different signs and thus saddle path dynamics (see e.g. Kuznetsov (1998, p. 49). If and only if the determinant is positive, the eigenvalues are of equal sign and the steady state is an antisaddle.

The slopes of the nullclines in any of the positive steady states are given by $\frac{du}{d\theta}|_{\dot{u}=0} = -\frac{\partial \dot{u}}{\partial \theta} / \frac{\partial \dot{u}}{\partial u}$ and $\frac{du}{d\theta}|_{\dot{\theta}=0} = -\frac{\partial \dot{\theta}}{\partial \theta} / \frac{\partial \dot{\theta}}{\partial u}$. Now we see that $\frac{\partial \dot{\theta}}{\partial \theta} \frac{\partial \dot{u}}{\partial u} > \frac{\partial \dot{\theta}}{\partial u} \frac{\partial \dot{u}}{\partial \theta}$ if and only if $\frac{du}{d\theta}|_{\dot{u}=0}$ is steeper than $\frac{du}{d\theta}|_{\dot{\theta}=0}$ in the steady state under study. Since both terms are negative by the sign restrictions, this corresponds to the unemployment nullcline crossing the tightness nullcline from above. $\square$

D Proof of proposition 3

Following the definition of Kuznetsov (1998, p. 42), two smooth systems $\dot{x} = \mu(x)$, $x \in \mathbb{R}^n$ and $\dot{y} = \nu(y)$, $y \in \mathbb{R}^n$ are not only topologically equivalent, but also smoothly equivalent if (1) an invertible map $f : \mathbb{R}^n \to \mathbb{R}^n$ exists such that $y = f(x)$, if (2) this map is smooth together with its inverse, and if (3) f can be used to change coordinates such that holds that $\mu(x) = M^{-1}(x)\nu(f(x))$, where $M(x) = \frac{df(x)}{dx}$ is the Jacobian matrix of $f(x)$ at x. As a result, f is not only a homeomorphism, but also a diffeomorphism.

Proof. My dynamic system in u and θ with $z = 0$ is given for all interior equilibria by the

two smooth differential equations in (11) and (1), for convenience reprinted below without z

$$\dot{\theta} = (r + \delta)\frac{\theta}{1 - \eta} + (1 - \beta)\frac{\lambda(\theta)}{k(1 - \eta)}\left[g(\theta) - \phi(1 - u)\right],$$
$$\dot{u} = \delta(1 - u) - s(\theta)u\lambda(\theta),$$

with $\theta \in (0, \infty)$ and $u \in [0, 1]$. The dynamic system of Mortensen (1999) is for all interior equilibria given by the following two smooth differential equations

$$\dot{p} = (r + \delta)p + g(p) - \phi(n),$$
$$\dot{n} = h(p)(1 - n) - \delta n,$$

with $p \in (0, \infty)$ and $n \in [0, 1]$, and where $h(p) = s(\theta)\lambda(\theta)$, and $g(p) = g(\theta)$ for the invertible map defined by

$$p = \frac{k\theta}{(1 - \beta)\lambda(\theta)} = \frac{k}{(1 - \beta)m_0}\theta^{1-\eta}, \tag{17}$$
$$n = 1 - u. \tag{18}$$

Recognizing that rational Nash bargaining implies $J^e = (1 - \beta)p$, the first equation follows from the free-entry condition in (4), while the second is true by definition. Both equations are smooth together with their inverses, so that they satisfy the second requirement as well. The Jacobian matrix of this diffeomorphism is given by

$$M(x) = \begin{pmatrix} \frac{k(1-\eta)}{(1-\beta)\lambda(\theta)} & 0 \\ 0 & -1 \end{pmatrix}.$$

If we apply the map in (17) and (18), then indeed

$$\begin{pmatrix} (r + \delta)\frac{\theta}{1-\eta} + (1 - \beta)\frac{\lambda(\theta)}{k(1-\eta)}\left[g(\theta) - \phi(1 - u)\right] \\ \delta(1 - u) - s(\theta)u\lambda(\theta) \end{pmatrix} = \begin{pmatrix} \frac{(1-\beta)\lambda(\theta)}{k(1-\eta)} & 0 \\ 0 & -1 \end{pmatrix} \times$$
$$\begin{pmatrix} (r + \delta)\frac{k\theta}{(1-\beta)\lambda(\theta)} + g(\theta) - \phi(1 - u), \\ s(\theta)u\lambda(\theta) - \delta(1 - u) \end{pmatrix},$$

so that the two systems also satisfy the last of the three requirements. As a result, they are smoothly equivalent as long as $p > 0$, or equivalently $\theta > 0$. $\qquad\square$

E Proof of proposition 4

Proof. To show that parameters exist such that the system with $r > 0$ continues to have a homoclinic orbit for $z > 0$, it is sufficient to show that Melnikov's method can still be applied. The argument in Mortensen (1999) based on the distortion of the efficient bargaining and no discounting assumptions, using the trace at the steady state, remains largely valid, since z is no argument in the trace. However, even if the existence of the saddlepoint in the positive quadrant is not affected, as in my calibration, the area of integration is. The issue is whether by the introduction of a value of leisure part of the homoclinic orbit falls outside the positive quadrant.

Whether the homoclinic orbit remains an equilibrium for $z > 0$, $r > 0$, and $\beta < 1 - \eta$ can be investigated in the associated Hamiltonian system where $r = 0$ and $\beta = 1 - \eta$, for which all equilibrium paths are level curves. For this reason, combinations of u and θ on the homoclinic orbit have the same value of the Hamiltonian as the saddlepoint H. By the same arguments as in Mortensen (1999, p. 900), the antisaddle M continues to be a local minimum in the system with $z > 0$. Moving along the continuum of surrounding closed orbits, the homoclinic orbit must have a higher value, which can be computed in the saddlepoint H. I can subsequently compare this value to the value of the Hamiltonian at u_0 of (14), the intersection of the tightness nullcline with the unemployment axis where $\theta = 0$.[31] If and only if this value is bigger than the value at the saddlepoint H, the homoclinic orbit continues to lie within the positive quadrant.[32] This argument is true because of the ordering of the level curves, and because only below the tightness nullcline the orbit would be able to escape to a negative tightness, so that u_0 would always fall within such an 'escaping' homoclinic orbit. Since I have no analytical expressions for the positive steady states, I computed the value of the Hamiltonian function numerically. For all parameters experimented with in the calibration, but maintaining $r = 0$ and $\beta = 1 - \eta$, this test verifies the existence of a homoclinic orbit as a perturbation of the Hamiltonian system with $z > 0$. $\qquad\square$

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[31]Normalizing ϕ_0 to 1, in Mortensen's system the Hamiltonian in this point is simply: $z - \frac{\alpha}{\alpha+1} z^{\frac{\alpha+1}{\alpha}}$.

[32]If both values are equal, the orbit is a heteroclinic orbit, connecting the two saddlepoints.