Can a stochastic cusp catastrophe model explain housing market crashes?

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Abstract

Similar to the patterns of stock market prices, housing prices also exhibit temporary bubbles and bursts. One possible explanation for such abrupt changes is the catastrophe model. However, due to the deterministic nature of catastrophe theory, applications to social science are rare. It remains a question whether the catastrophe model can be used to explain and predict the dynamics of housing markets. Our paper fits a stochastic cusp catastrophe model to empirical housing market data in different countries for the first time. Two estimation approaches are discussed – Cobb's Method and Euler Discretization. The analysis shows that Euler Discretization provides better short-run predictions while Cobb's better describes the long term invariant density. Moreover, the results using Euler Discretization suggest that the dynamics of housing markets could be explained and predicted by a multiple equilibria cusp catastrophe model. In particular, this paper yields important insights on interest rate policy regarding the stability of economic system.

Keywords: stochastic catastrophe, housing market, instability, multiple equilibria
JEL Classification: C13, C53

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1 Introduction

The collapse of the U.S. housing bubble in 2007 was followed by an enormous worldwide financial crisis. This tragedy has raised great concerns of housing bubbles among financial regulators and researchers. Like stock market bubbles, housing bubbles can be identified through rapid increases in housing prices before they crash. Figure 1 illustrates the bust phase of housing price cycles surrounding banking crises from 1899 to 2008 using real housing prices (Reinhart and Rogoff, 2009). The historical average of the declines from peak to trough is 35.5 percent. A number of countries with major housing crashes are included. For instance, Finland, Colombia, the Philippines and Hong kong have experienced the most severe real housing prices crashes in the past 25 years. Their crashes amounted to 50 to 60 percent from peak to trough. Notably, the duration of housing price declines has been quite long lived, averaging roughly 6 years. After the housing market crash of Japan in 1992, real housing prices declined for consecutive 17 years. In particular, housing price declines are even longer lived than equity price declines. The average historical downturn phase in equity prices lasts 3.4 years, still less than half of the downturn phase in housing prices (Reinhart and Rogoff, 2009). The International Monetary Fund (IMF) recorded that housing price busts lasted nearly twice as long and led to output losses that are twice as large as asset price bursts (IMF World Economic Outlook, 2003). Moreover, financial crises and recessions are often preceded by housing market crashes (Reinhart and Rogoff, 2009). The credit crisis and global financial crisis in 2008 are convincing examples. After housing prices declined in the latter half of 2007, the secondary mortgage market collapsed. A complex chain reaction almost brought down the whole world's financial system. Furthermore, in some papers, housing market bubbles are considered as leading indicators of financial instability and crises (Davis and Heathcote, 2005). For the above reasons, a good understanding of the instability in housing market is crucial.

Housing market models have been studied extensively in literature. Unfortunately, most of the available research in macroeconmics is mainly based on state-of-the art dynamic stochastic general equilibrium (DSGE) models which are based on fundamentals. However, these traditional models are insufficient to explain the observed booms and busts in housing prices. A series of papers by Shiller have argued that the changes in economic fundamentals such as population growth, construction costs, interest rates

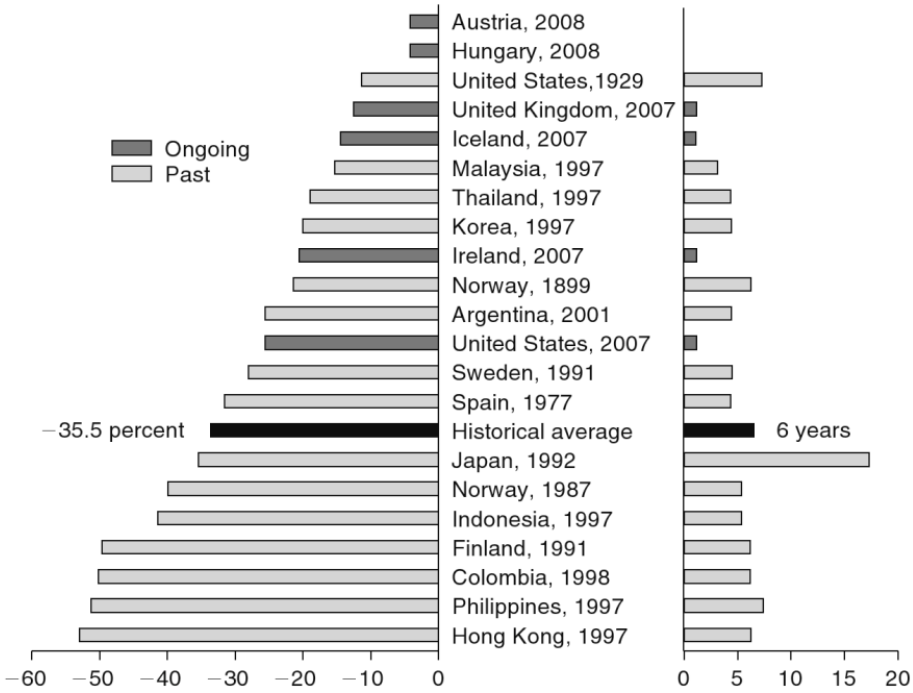


Figure 1: The bust phase of housing price cycles surrounding banking crises. Left panel: peak-to-trough price declines; Right panel: years duration of downturn. Source: Reinhart and Rogoff, 2009

or real rents did not match up with the observed house price fluctuations (Case and Shiller, 2003; Shiller, 2007, 2008, 2012, 2015). Davis and Heathcote (2005) also suggested that DSGE models with housing consumption and production were unable to capture the instability of house price.

During the last decades, a number of theories were proposed that are based on a multiple equilibria approach. Unlike traditional DSGE models under the general equilibrium assumption, they recognise economic systems as complex systems with multiple equilibria. Along these times, a theoretical approach on heterogenous agents models (HAMs) has been introduced to the housing market. It was inspired by the work on heterogenous agent-based financial market models (see Brock and Hommes,1997, 1998 and comprehensive survey in Hommes, 2013). For instance, Kouwenberg and Zwinkels (2011) developed and estimated a HAM model for the U.S. housing market. Their estimated model produced boom and bust price cycles endogenously, which were induced by boundedly rational behaviour of investors. Dieci and Westerhoff (2012, 2013) in-

vestigated the speculative behaviour in housing markets using a heterogenous agent approach. Their examples illustrated a variety of situations that can display irregular endogenous dynamics with long lasting, significant price swings around the fundamental price, like in many actual markets. In a recent paper Bolt *et al.* (2014) established and estimated a HAM model for eight different countries. They found evidence of heterogenous expectations from empirical data and identified temporary house price bubbles for different countries.

Although HAMs have proven to be successful theoretical tools to capture temporary deviations from market equilibrium, a method which allows statistical time series analysis is still lacking. In this paper, we fill this gap by using statistical analysis to capture the crash phenomena in real time series. Forecasting market crashes based on previously observed time series is also possible. In this paper, we are attempting to fill this blank. Catastrophe theory has been suggested to be a good candidate (Zeeman, 1974). It captures the instability in many nonlinear dynamical systems and has proven to be an extremely successful tool to investigate the qualitative properties in a wide range of different complex systems ranging from physics and engineering to biology, psychology and sociology. Its applications involve urban and regional systems (Wilson, 1981), quantum morphogenesis (Aerts *et al.*, 2003), the stability of black holes (Tamaki *et al.*, 2003), the size of bee societies (Poston and Stewart, 2012), the cognitive development of children (Van der Maas and Molenaar, 1992), sudden transitions in attitudes (Van der Maas *et al.*, 2003) and so on. In all these applications, behaviour of the observed system shows sudden and discontinuous changes or phase transitions as a result of a small change in control variables. Catastrophe theory offers a mathematical basis for the number and the type of critical points for the classification of nonlinear dynamical systems. Since the economic system has been recognised as a complex system, displaying quick transitions such as market crashes, catastrophe theory might be a good candidate to explain its extreme fluctuations. Zeeman (1974) already proposed that some of the unstable behaviour of stock exchanges could be explained by a model based on catastrophe theory. A similar model can also be applied to currencies, property markets, or any market that admits speculators. The proposed HAM model in financial market was also motivated by some behavioural finance elements in Zeeman's work. Barunik and Vosvrda (2009) and Barunik and Kukacka (2015) tested Zeeman's idea and fitted a stochastic cusp catastrophe model to stock market data. They provided an important shift in application of

catastrophe theory to stock markets. Their examples showed that stock market crashes were better explained by cusp catastrophe theory than other models.

In our paper, following the idea of Zeeman (1974), we fit a stochastic cusp catastrophe model to housing market data for the first time. The aim of this paper is twofold. Firstly, we discuss the estimation method of fitting stochastic cusp catastrophe model. Two estimation approaches are studied: Cobb's method and Euler Discretization. Secondly, we apply catastrophe behaviour to the housing markets and study its policy implications further.

There are two main contributions in this paper. Firstly, we study two estimation methods: Cobb's method and Euler Discretization. We show that Euler Discretization gives better forecasting ability than Cobb's Method. Secondly, we explain the underlying mechanisms of the instability of housing markets by using a cusp catastrophe model . A critical transition can be distinguished from ordinary fluctuations. We also unfold the underlying link between interest rates and systematic fluctuations in housing markets.

Our analysis sheds some light on the application of catastrophe theory to time series data in social science. We fit a stochastic cusp catastrophe model to housing markets in six different countries. Our results show how the equilibria of the system are changing depending on the interest rate. This scenario can be used to explain several housing price bubbles and crashes in empirical data, such as UK 1978, 1980, 1990, NL 1978, 1990, and the depression of SE after 1990. The policy implication of this paper is that policy makers should commit an interest rate policy which prevents the system from getting too close to the cusp curve that may induce a systemic market crash. To achieve this, the cusp catastrophe fit could provide a reasonable guidebook. This is one of the most appealing contributions of our paper.

This paper is organised as follows. We first introduce, in Section 2, the cusp catastrophe theory and its application to housing markets. Subsequently, we discuss the empirical methods of Cobb and Euler Discretization, estimation variables and empirical data in Sections 3 and 4. The results of Euler Discretization are presented and discussed in Section 5. This paper ends with a summary and conclusion in Section 6.

2 Catastrophe Theory

Catastrophe theory has been first proposed by the French mathematician René Thom (1972). Before his work, most models only described phenomena with smooth and continuous changes. However, the world is full of sudden transformations and unpredictable divergences. The proposed catastrophe theory has shed some lights on "a law of nature". Zeeman cooperated with Thom and proposed catastrophe's applications in the fields of economics, psychology, sociology, political studies, and others (Zeeman, 1974, 1977). In particular, he proposed the application of cusp catastrophe model to stock markets and qualitatively described the bull and bear markets as a result of interaction between two main types of investors: fundamentalists and chartists (Zeeman, 1974). This work contains a number of important behavioural finance elements, which later led to research on HAM models. However, the biggest difficulty in application of catastrophe theory arises from the fact that it stems from deterministic systems, while most scientific investigations allow for random noise. In order to apply it directly to behavioural science in which random influences are common, we need a bridge to lead catastrophe theory from deterministic to stochastic systems. Loren Cobb was the first to address this challenge. He proposed a stochastic version of catastrophe theory based on Itô stochastic differential equation (Cobb, 1980). Later the development of statistical methods made catastrophe theory very useful and applicable empirically on real data. Unfortunately, this maximum likelihood estimation (MLE) method is not invariant under nonlinear diffeomorphic transformations. Much of the topological generality of catastrophe theory was lost in the statistical portion of this theory (Cobb and Watson, 1980). Hartelman (1997) improved Cobb's method by taking into account the Itô transformation rule. They showed that this model remained invariant under smooth 1-to-1 transformations and can be estimated by a straightforward time series analysis based on level crossings (Wagenmakers *et al.*, 2005). This model has been applied to psychology to model transitions in attitudes successfully (Van der Maas *et al.*, 2003). Barunik and Vosvrda (2009) and Barunik and Kukacka (2015) fitted it to stock market data and showed that stochastic cusp catastrophe model explained the crash of stock market much better than other models. Housing market crashes are often prior to financial crises and recessions. Housing bubbles are considered as leading indicators of financial instability and crises. There are no other examples of understanding hous-

ing markets by using catastrophe theory. Therefore, we will use catastrophe theory to explain the dynamical behaviour in housing markets. In what follows, a basic understanding of catastrophe theory is discussed.

Catastrophe theory provides a mathematical basis for systems involving discontinuous and divergent phenomena. In particular, it is effective in those systems where gradually changing forces lead to abrupt changes in behaviour. The nonlinear dynamics of the system under study follows, in the noise-less case

$$dy_t = \frac{-dV(y_t; c)}{dy_t} dt, \tag{1}$$

where y_t represents the state of system. It implies that the studied system changes in response to a change in $V(y_t; c)$; $V(y_t; c)$ is a potential function which is determined by control parameter c, and c determines the specific structure of the system and can consist of one or multiple variables. The system is in equilibrium when the spatial derivative of the potential function equals 0, i.e. $-dV(y_t; c)/dy_t = 0$. The equilibrium corresponds either to a maximum or a minimum of potential function $V(y_t; c)$ with respect to y. When $V(y_t; c)$ takes a minimum, the equilibrium points are stable. The system will always return to it after a small perturbation with respect to system's state; Following the same idea, the equilibrium points are unstable equilibria if the potential function $V(y_t; c)$ takes a maximum. Even a small perturbation will drive the system away from these equilibrium states and move towards a stable equilibrium. The Hessian matrix has eigenvalues equal to 0 in these equilibria, at which a system can give rise to unexpected bifurcations when the control variables are changed. Therefore, catastrophe theory can be employed in systems which can be driven toward an equilibrium state, such as a gradient dynamical systems with critical points.

2.1 Cusp Catastrophe

One of the extraordinary findings of catastrophe theory is that it proposed the behaviour of deterministic dynamical systems around the critical points of potential function $V(y_t; c)$. It proposed that this behaviour can be characterised by a set of seven canonical forms with no more than four control variables and one or two canonical state variables (Thom,1972; Zeeman, 1976; Gilmore,1993). In behavioural sciences, the most commonly used canonical form is the so-called cusp catastrophe. In terms of a

normalised variable z_t, it describes the sudden, discontinuous transitions in equilibria states as a result of continuous changes in two normal form control parameters α and β:

$$-V(z_t;\alpha,\beta) = -\frac{1}{4}z_t^4 + \frac{1}{2}\beta z_t^2 + \alpha z_t. \tag{2}$$

The equilibria can be obtained by solving the cubic equation:

$$-\frac{\partial V(z;\alpha,\beta)}{\partial z} = -z_t^3 + \beta z_t + \alpha = 0, \tag{3}$$

in which the derivative of potential function $V(z;\alpha,\beta)$ equals 0. For descriptive purpose, a statistic so-called Cardan's discriminant is proposed to distinguish the case of three solutions from the case of one solution (Cobb, 1981). This Cardan's discriminant is defined as:

$$\delta = 27\alpha^2 - 4\beta^3. \tag{4}$$

When $\delta < 0$, there are three solutions, while when $\delta > 0$, there is only one solution. When δ exactly equals 0, there are three solutions and two of them have the same value. Figure 2 gives a visual description of the cusp catastrophe model in an example of a housing market. It illustrates a cusp equilibrium surface living in a three dimensional space. The folded surface with a fold cusp represents the equilibrium surface of system. The floor is a two dimentional control plane which is determined by a set of control parameters, α and β. The height predicts the value of the system's state with respect to control parameters. In the middle of the graph, there are two sheets representing the behaviour of system, and they are connected by a middle sheet making a continuous surface. The difference between the middle sheet and the other two sheets is that the middle sheet represents the least probable state of the system. The curve defining the edges of the fold cusp projected onto the control plane showing an across hatched cusp shaped region. The cusp which marks its boundary is called the bifurcation set (Zeeman, 1974, 1976), for which Cardan's discriminant $\delta = 0$. When $\delta > 0$, the system has only one stable equilibrium state. There is only one predicted state value. However, within the cusp fold, where $\delta < 0$, the surface predicts two possible stable state values instead of one. This implies that in a system with noise the state variable is bimodal inside the bifurcation area. In addition, this middle surface predicts that certain state values, such as unstable equilibrium states, should not occur frequently. It "anti-predicts" an intermediate value for these values of the control variables (Grasman *et*

al., 2009). Moreover, the system might get into a hysteresis loop by jumping between these two possible state values. The jump from the top sheet to the bottom sheet of the behaviour surface occurs at a different value of the variable than the jump from the bottom to the top sheet does.

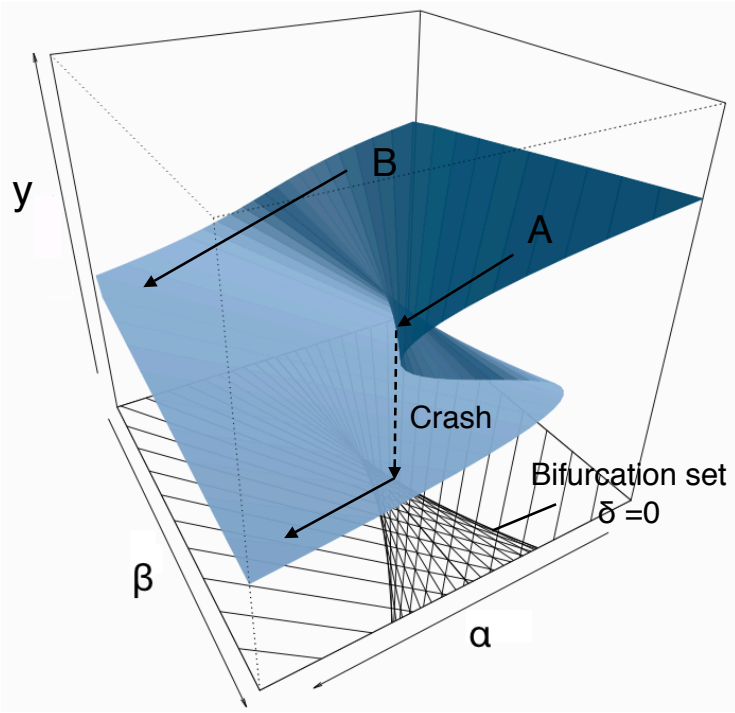


Figure 2: Cusp catastrophe model of housing market

2.2 Stochastic Cusp Catastrophe

Although Zeeman has proposed that catastrophe theory could be applied to multiple disciplines (Zeeman, 1977), a practical empirical investigation requires a model that allows stochastic shocks. In order to address this issue and to build a bridge between catastrophe theory and real scientific data, several stochastic formations of catastrophe theory which allow empirical investigations have been proposed (Oliva *et al.*, 1987; Guastello, 1988; Alexander *et al.*, 1992; Lange *et al.*, 2000). Of all the methods, the method of Cobb and Watson (1980) is arguably the most appealing. They proposed to combine the deterministic catastrophe theory with stochastic systems theory by using Itô stochastic differential equation (SDE). It leads to the definitions of stochastic equilibrium state

and stochastic bifurcation that are compatible with their deterministic counterparts, in such a way that it establishes a link between the potential functions of deterministic catastrophe systems and the stationary probability density functions of stochastic processes.

Assuming that the (canonical) variable z_t is still governed by the potential function of Eq. (2), and that there is a driving noise term with variance σ_z^2 per time unit, the dynamics can be written in terms of the SDE:

$$\begin{aligned} \mathrm{d}z_t &= -\left.\frac{\partial V(z;\alpha,\beta)}{\partial z}\right|_{z=z_t} + \sigma_z \mathrm{d}W_t. \\ &= -z_t^3 + \beta z_t + \alpha + \sigma_z \mathrm{d}W_t. \end{aligned} \tag{5}$$

The deterministic term $-\partial V(z_t;\alpha,\beta)/\partial z_t$ is the drift function, σ_z is the diffusion parameter, and W_t is a Wiener process. z_t is the dependent variable. α and β are the "canonical variates" which are the smooth transformations of actually measured independent control variables $x_1, \ldots, x_n$.

In this stochastic context, on the one hand, if $\alpha = 0$, its density function is symmetric. The sign of α decides whether it is left or right skewed. Thus α is called "asymmetry factor" and determines direction of the skew of the density. On the other hand, as β changes from negative to positive, the density is changing from unimodal to bimodal. β is the "bifurcation factor" and determines the number of modes of the density.

2.3 Cusp catastrophe behaviour of housing market

The best way to understand the nature of models derived from cusp catastrophe is to illustrate it by examples. Zeeman has considered some popular example applications of catastrophe model to different disciplines, such as ecology, physics and psycology (Zeeman, 1974). His work was the first attempt to explain unstable behaviour of stock market using catastrophe model. Some behavioural finance elements in his work motivated HAM models, in which the instability of market is expectations-driven. The housing market has several common characteristics with the stock market. Their dynamical behaviours are connected in some ways. The booms and burst cycle of housing prices was also proved to be partly driven by heterogenous expectations of agents (Bolt *et al.*, 2014). Several theoretical models have been shown to perform well in the analysis of housing markets and stock markets. For instance, HAMs are successful examples

to capture the instability in both the stock market and the housing market (Brock and Hommes, 1997, 1998; Hommes, 2013; Kouwenberg and Zwinkels, 2010; Dieci and Westerhoff, 2012, 2013; Bolt *et al.*, 2014). Since Zeeman has proposed that catastrophe model could be used to explain stock market, it could also be useful in the housing market.

An example of a cusp catastrophe model in housing market is illustrated visually in Figure 2. The smooth folded surface with three levels of sheets represents the equilibria of the system. A set of control parameters α and β forms a two-dimensional control plane. The z-axis of the 3-dimensional space y represents the state variable, such as housing prices.

The catastrophe behaviours are now observable whenever the set of control parameters moves all the way across the cusp equilibrium surface. Each point on the top and bottom sheets of this surface gives equilibrium of system. If the point is on the top sheet and follows the path A on the control surface, the corresponding path moves to left on the top sheet until it reaches the fold curve; the top sheet then vanishes, and the path must suddenly jump to the bottom sheet. A small change in control parameters can produce a sudden large change in the state of the system. Alternatively, the path B on the control surface outside of the cusp bifurcation exhibits the behaviour of an ordinary market. Its corresponding path moves to the bottom sheet slowly and smoothly, without catastrophes.

The mechanism of housing market crashes can now be understood. We assume that the equilibrium of a housing market with rising prices is on the top sheet of the behaviour surface. The housing market with falling prices has its equilibrium on the bottom sheet of the equilibrium surface. A crash can then be induced by any event that changes the control parameters enough to push the behaviour point over the fold curve, fall off the "cliff" and jump to the bottom sheet. In particular, if the equilibrium is in the bifurcation set and very close to the cusp curve, even a small perturbation can induce large market collapse. Similar, the "negative" crashes could induced by an upward jump from the bottom sheet to the top sheet. In particular, the equilibrium of the system could also transition from one stable equilibrium to another without passing through the cusp curve. This can be used to explain the slow recovery from a crash in housing markets. A recovery is affected by the slow feedback from the behaviour of housing market on the control parameters. It does not pass through the catastrophe cusp, but slowly and smoothly follow the reversal of the path B. The system is then set

for another cycle of boom and bust. Therefore, the duration of housing price declines was quite long lived and the subsequent recovery from a crash was exceptional slow.

3 Estimation Methods

In the estimation, α and β are approximated by using a first order Taylor expansion

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_v x_v \\ \beta &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_v x_v \end{aligned} \tag{6}$$

Then $\alpha_0, \alpha_1..., \alpha_v$ and $\beta_0, \beta_1..., \beta_v$ are the parameters to be estimated when fitting the catastrophe model to empirical data. $x_1, x_2..., x_v$ are the independent variables.

3.1 Cobb's Method

The most common estimation method that allows quantitative comparison of catastrophe models with empirical data is that proposed by Cobb (Cobb, 1978, 1980, 1981; Cobb and Watson, 1980; Cobb *et al.*, 1983). He established a stochastic cusp catastrophe by simply introducing a stochastic Gaussian white noise, with the SDE as Eq.(5).

In this approach, maximum likelihood estimation is applied to the cusp probability density function (PDF). To simplify the estimation procedure, instead of using the conditional PDF $f(z_t|z_{t-1}, z_{t-2}, ...; \alpha, \beta)$, Cobb considered the invariant PDF which is given by solving the corresponding Fokker-Planck equation. As time passes, $t \to \infty$, the conditional PDF $f(z_t|z_{t-1}, z_{t-2}, ...; \alpha, \beta)$ converges to a stationary and time invariant form $f(z_t|\alpha, \beta)$. With linear transformations of dependent variable $z_t = (y_t - \lambda)/c$, the distribution of system's states on any moment in time is expressed as

$$f(y) = \psi \exp\left[\tilde{\alpha}\left(\frac{y-\lambda}{c}\right) + \frac{1}{2}\tilde{\beta}\left(\frac{y-\lambda}{c}\right)^2 - \frac{1}{4}\left(\frac{y-\lambda}{c}\right)^4\right], \tag{7}$$

where $\tilde{\alpha} = \left(\frac{\sigma_z^2}{2}\right)^{-\frac{3}{4}} \alpha$ and $\tilde{\beta} = \left(\frac{\sigma_z^2}{2}\right)^{-\frac{1}{2}} \beta$, $c = r\left(\frac{\sigma_z^2}{2}\right)^{\frac{1}{4}}$, and ψ is normalisation constant so this PDF's integral over the entire range equals 1. For a derivation, see Appendix B.2. The location parameter λ and scale parameter c are kept constant.

Consequently, with the invariant PDF as theoretical PDF in the estimation, and under the assumption that the diffusion function is constant, σ_z = constant, the deterministic and stochastic potential function can be linked with each other. The stochastic stable and unstable equilibrium of the potential function are associated with modes and anti-modes of the invariant PDF respectively. The stochastic bifurcations correspond to the changes in the number and type of the modes of the invariant PDF. A qualitative change in the potential function is identical to a qualitative change in the PDF with respect to the change of control parameters. For instance, the PDF changes from unimodal to bimodal as the bifurcation parameter β changes from negative to positive. As a result, by estimating the distribution of invariant PDF, any point on the cusp equilibrium surface can be estimated according to their closest maximum mode of PDF. The inaccessible middle sheet of equilibrium surface is reflected in the middle of the bimodal that a low probability mode is between two high probability modes.

Based on Cobb's statistical catastrophe theory, a series of works by Hartelman (1997), Hartelman *et al.* (1998), and Grasman *et al.* (2009) implemented and extended Cobb's estimation method. They presented robust and practical computer programs which made it easy to fit cusp catastrophe models on empirical data in a statistical way. In these approaches, Cobb's method is combined with the subspace fitting method of Oliva *et al.* (1987). The state of system is measured as a first approximation of

$$z = w_0 + w_1 y_1 + w_2 y_2 + ... + w_v y_v. \tag{8}$$

z is the smooth transformation of the actual state variable of system. $w_0, w_1, ..., w_v$ are the first order coefficient of a polynomial approximation to the smooth transformation. $y_1, y_2..., y_v$ is a set of measured dependent variables. Because independent variables α and β are also approximated by the first approximation (see Equation (6)), fitting the cusp model to empirical data is then reduced to estimate the parameters of $w_0, w_1, ..., w_v$; $\alpha_0, \alpha_1..., \alpha_v$ and $\beta_0, \beta_1..., \beta_v$.

Although the Cobb's method has been proved to be a successful tool in several multiple equilibria systems, we should note that it requires a system with dynamics change much more quickly than the feedback reactions of control parameters, due to its nature of time independent. This method gives a good fit of the overall invariant density for the cross-sectional dataset, or for those systems with quickly changing dynamics. For time series dataset or for systems with slow process, particularly when the scale of state

of system (z-axis in Figure 2) can not be separated from the scale of control parameters (x-axis and y-axis in Figure 2), the forecasting ability of Cobb's method may be not sufficient.

3.2 Estimations of Euler Discretization

In order to estimate the stochastic differential equation (SDE) of cusp catastrophe with changing time, we consider another numerical method - Euler Discretization, which is the numerical method which seeks to approximate the SDE at discrete times.

Time t is subdivided into intervals of length Δt, so that $t_n = n\Delta t$. Then we could approximate the solution at those times t_n. Because $y = \lambda + rz$ is a scaled and/or translated variable, in terms of $y_t = \lambda + rz_t$ the SDE becomes (see Appendix B.3)

$$\frac{1}{r}\mathrm{d}y_t = -\left.\frac{\partial V(z;\alpha,\beta)}{\partial z}\right|_{z=\frac{y_t-\lambda}{r}} + \sigma_z \mathrm{d}W_t, \tag{9}$$

Euler Discretization gives an approximate equation which predicts a future value of state of system y in terms of past value:

$$\begin{aligned} y_{t+\Delta_t} &= y_t - \left.\frac{\partial V(\tilde{y}_t;\alpha,\beta)}{\partial \tilde{y}_t}\right|_{\tilde{y}_t=\frac{y_t-\lambda}{r}} r\Delta t + r\sigma_W \epsilon_t \sqrt{\Delta t} + h.o.t. \\ &\approx y_t + \left(\alpha + \beta\left(\frac{y_t-\lambda}{r}\right) - \left(\frac{y_t-\lambda}{r}\right)^3\right) r\Delta t + r\sigma_z\sqrt{\Delta t}\varepsilon_{t+\Delta_t}, \end{aligned} \tag{10}$$

where $\varepsilon_{t+\Delta t} \sim N(0,1)$.

The distribution of state of system at any moment in time can now be approximated as:

$$f_{y_{t+\Delta t}}(y|y_t) \approx \psi \exp\left\{\frac{-\left[y - \left(y_t - r\Delta t\left(\left(\frac{y_t-\lambda}{r}\right)^3 - \beta\left(\frac{y_t-\lambda}{r}\right) - \alpha\right)\right)\right]^2}{2\sigma_z^2 r^2 \Delta t}\right\} \tag{11}$$

ψ is normalisation constant so the PDF's integral over the entire range equals 1. The location parameter λ and scale parameter c are kept constant. In this paper λ is assumed to be equal to 0 since there is a fundamental equilibrium at 0 (and the mean of

the price fluctuations around this is assumed to be zero). The estimated parameters are $\lambda, c; \alpha_0, \alpha_1, ..., \alpha_v$ and $\beta_0, \beta_1, ..., \beta_v$. The conditional PDF is considered in this approach while invariant PDF is used in Cobb's method. Therefore, by using one-step-ahead forecast, any moment on the equilibrium surface is able to be predicted with changing time. To compare with Cobb's method, the estimation method used here is also Nonlinear Least Squares (NLS). Brillinger (2007) used another approach to estimate a potential function based on a linear model.

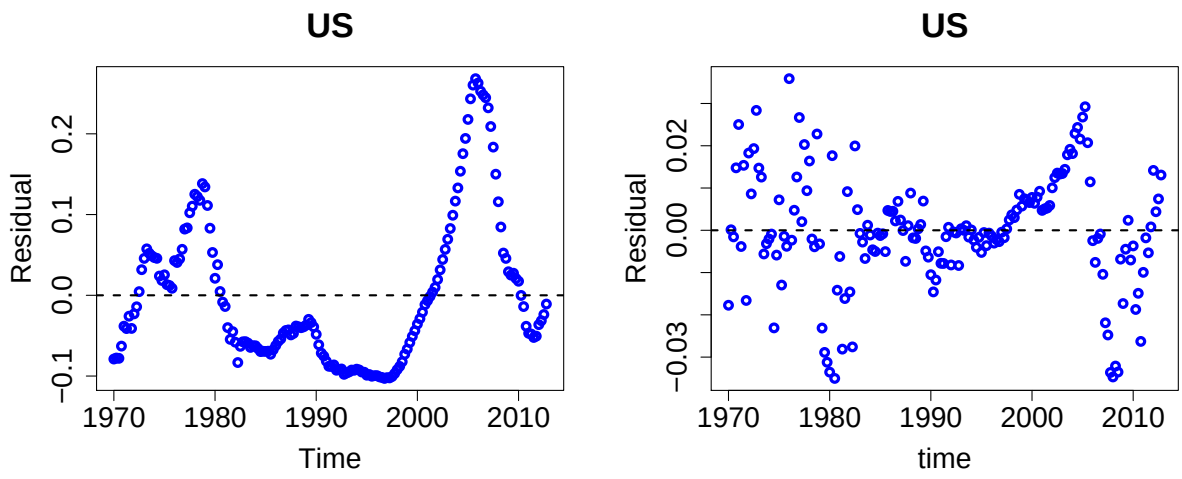
We should note that the one-step-ahead forecast is over-parameterised in the sense that the forecasts the model produces are independent of the value we choose for Δt. This might have been expected, since changing the value of Δt simply corresponds to a change of time units, which should not affect the forecasts. We therefore set $\Delta t = 1$ throughout (i.e. we define one time unit to correspond to one quarter, the time interval between consecutive observations in our data set). We do emphasize that the choice of the time scale does affect the estimated numerical values of α, β and r. Doubling Δt can be compensated in the point forecast by multiplying r and β by $\sqrt{2}$ and dividing α by $\sqrt{2}$. The term $\sigma_z^2 r^2 \Delta t$ in the denominator can be considered a single parameter (the forecast variance over a period of one month), which is independent of the choice made for the time unit.

3.3 Cobb's Method v.s. Euler Discretization

Cobb's method aims to give a good fit of cusp catastrophe based on the overall invariant density of system. This method requires a system with separated measure scales and quickly changing dynamics. As a matter of fact, it is time independent and would give better fit on cross-sectional dataset rather than time series data. On the contrary, Euler Discretization considers the one-step-ahead forecast by estimating the conditional density of system in time. The assumption of time independent in Cobb's method is relaxed. Intuitively, Euler Discretization should make considerable improvement to Cobb's Method regarding the forecasting ability in a time series framework.

To compare the forecasting ability of these two estimation approaches, the most simple and straightforward way is to examine their residuals. Figure 3 shows the plots of residuals against time in the example of US by using Cobb's Method and Euler Discretization respectively. In Figure 3 (a), we observe strongly correlated residuals with

clear patterns and obvious deviations from randomness. Moreover, the values of the residuals are big, i.e. bigger than 0.1. This is due to the fact that the predicted values in Cobb's method are estimated based on the closest maximum mode of invariant PDF rather than by past information. Although it may still give a good fit with respect to invariant density distribution, when it comes to the evaluation of the forecasting ability, our analysis shows it is insignificant. Nevertheless, as shown in Figure 3 (b), these residuals are randomly distributed and are small, i.e. smaller than 0.04 in absolute value. This suggests that Euler Discretization gives much better predictions than Cobb's method. The residuals in the examples of many other countries are observed under the same patterns, which are shown in Figure 13 and Figure 14 in Appendix B.1.



(a) Cobb's Method

(b) Euler Discretization

Figure 3: Plots of residuals against index by using Cobb's Method and Euler Discretization in the example of US.

Hartelman (1997) and Grasman *et al.* (2009) proposed to use the AIC and BIC to assess the model fit. The AIC and BIC in our examples are presented in Table 1. It can be seen that the AIC and BIC by using Euler Discretization are much smaller than using Cobb's method, which suggests a better model fit of Euler Discretization. It further proves that the forecasting ability of Euler Discretization is more promising than Cobb's Method with respect to housing prices.

The possible explanations are following. Firstly, although the estimation method of

Table 1: AIC and BIC of Cobb's method and Euler Discretization in different countries.

		US	JP	UK	NL	SE	BE
Cobb	AIC	497.955	491.587	438.482	357.619	361.697	379.087
	BIC	510.568	504.200	451.096	370.232	373.258	391.049
Euler	AIC	-983.611	-901.021	-715.484	-721.421	-592.250	-782.153
	BIC	-971.021	-888.431	-702.894	-708.831	-580.719	-770.191

Cobb has been proved to be a successful tool to fit cusp catastrophe in several areas, most of these examples considered cross-sectional datasets. Rather than observing with changing time, these examples focused on a cross section of time and aimed to approximate the overall stationary density of the system. However, the dataset in our examples are housing price deviations which are time series data. From this point of view, the one-step-ahead forecast in Euler Discretization should be considered as a better candidate than Cobb's method. Secondly, when fitting time series data, Cobb's method presumes separated time scales; it requires that the state of system changes much more quickly compared with the feedback reactions of control parameters. However, housing markets involve very long boom and burst cycles, the noise is big and the speed of the process is slow. Furthermore, because we are going to discuss the forecasting ability of cusp catastrophe model on the housing market, Euler Discretization is considered as a better candidate than Cobb's method regarding forecasting ability. For the above reasons, in what follows, we will analyse the cusp catastrophe behaviour of housing markets using Euler discretisation.

4 Estimation Variables

In this paper, we are going to observe the housing markets of six different countries: the United states (US), the United Kingdom (UK), the Netherlands (NL), Japan (JP), Sweden (SE) and Belgium (BE). The variables in the estimation of the cusp catastrophe model consist of state variables and control variables.

4.1 State Variables

The state variable is required to be able to describe the unstable behaviour of housing markets. Bolt *et al.* (2014) estimated a HAM model for housing markets in different countries and observed bifurcations driven by policy parameter out of relative deviations of housing price from fundamental price. Following their success, we also use the relative deviation of housing price from the estimated fundamental price as state variable, which is denoted by

$$y_t = \frac{p_t}{p_t^*} - 1 \approx lnp_t - lnp_t^*. \tag{12}$$

The fluctuations of the housing price around the fundamental price (based on expected future rental prices) are described by a model in which agents choose between either buying or renting house. In this model, agents make their decisions at time t based on the expected excess return on investing in housing relative to renting during the period between time t and $t+1$. The fundamental price is assumed as the price that would prevail under rational expectations about the conditional mean of excess return. In equilibrium, the annual cost of home ownership must equal the housing rent adjusted for risk. The supply of the market is the stock of housing. The demand of agents is determined by maximising one-period ahead expected excess returns adjusted for risk. By solving the market clearing condition for price p_t (See Appendix A for detailed calculation), we have the price equation

$$p_t = \frac{1}{1+r+\alpha}\mathbb{E}_{h,t}\left[p_{t+1} + (1+r^{rf})Q_t\right], \tag{13}$$

where Q_t denotes the price for renting one unit of housing in the period between times t and $t+1$. Because rents are typically payed up-front at time t, the rent at time t in terms of currency at time $(t+1)$ should be inflated by a factor $(1+r^{rf})$, where r^{rf} is the risk free mortgage rate. Therefore the cost of renting between t and $t+1$ is given by $(1+r^{rf})Q_t$ in terms of currency at time $(t+1)$. r is the sum of the risk free mortgate rate and the maintenance/tax rate. α is interpreted as a risk premium of buying a house over renting a house. In this model, α is assumed to be constant in order to keep the model tractable. Taking into account the risk premium α in the fundamental price will provide an equilibrium fundamental price from which the market price will deviate by an amount which averages out to 0 in long time series.

The fundamental process underlying the model is assumed to follow a geometric Brownian motion with drift (Boswijk *et al.*, 2007):

$$logQ_{t+1} = \mu + logQ_t + v_{t+1}, \quad \{v_t\} \overset{i.i.d.}{\sim} N(0, \sigma_v^2). \tag{14}$$

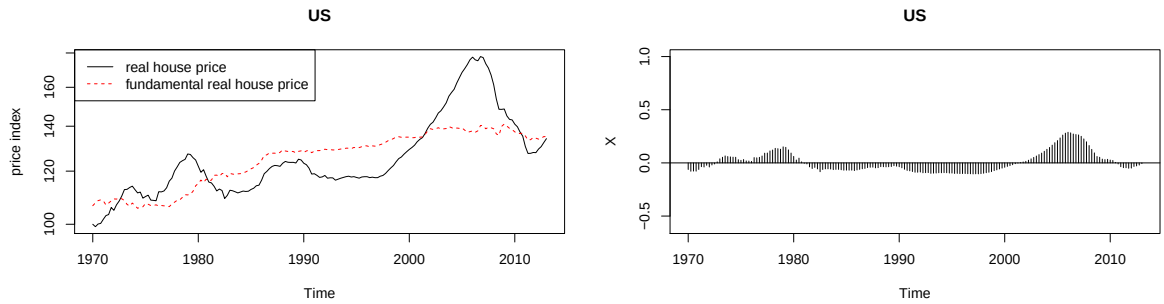
When $g = e^{\mu + 1/2\sigma_v^2} - 1$ and $\epsilon_{t+1} = e^{v_{t+1} - 1/2\sigma_v^2}$, one obtains:

$$\frac{Q_{t+1}}{Q_t} = (1+g)\varepsilon_{t+1}, \tag{15}$$

such that $\mathbb{E}_t(\varepsilon_{t+1}) = 1$. Therefore, by applying the law of iterated expectations and imposing the transversality condition, the fundamental price at time t is as:

$$p_t^* = \mathbb{E}_t \left[\sum_{i=0}^{\infty} \frac{(1+r^{rf})Q_{t+i}}{(1+r+\alpha)^{i+t}} \right] = \frac{1+r^{rf}}{r+\alpha-g} Q_t, \quad r+\alpha > g. \tag{16}$$

It shows that the fundamental price of housing is directly proportional to the actual rent level. Figure 4 shows an example of house price and fundamental price in US from 1970 to 2013. Figure 4(a) presents the housing price index p_t with the corresponding estimated fundamental values p_t^*. The price deviations of $p_t - p_t^*$ is shown in Figure 4(b), which is also the state variable y in our case. The plots of the examples of many other countries are shown in Appendix A in Figure 12.



(a) p_t and p_t^*

(b) $y_t = p_t - p_t^*$

Figure 4: The example of housing price and fundamental price in US from 1970 to 2013. (Source: Bolt *et al.*, 2014). (a)The housing price indices p_t with the corresponding estimated fundamental values p_t^*; (b) The log-difference between the two $p_t - p_t^*$.

4.2 Control Variables

We start with a simple model and discuss only one control variable. One of the advantages is that it allows us to observe critical transitions from time series. This could help us to understand the real market crashes in housing system.

One of the parameters which has the greatest influences on the deviations of housing prices from fundamentals is argued to be the mortgage rate. Several papers have pointed out that monetary policy, especially interest rate policy has great impact on housing prices (Bernanke and Gertler, 1995; Shiller, 2006; Muellbauer and Murphy, 2008; Taylor, 2007, 2009; Crowe *et al.*, 2013; Shi *et al.*, 2014). Moreover, empirical evidence of Bolt *et al.* (2014) suggested that several bifurcations of price equilibria may occur driven by interest rates. Therefore, as an important policy parameter, the interest rate is chosen as our control variable.

4.3 Data Description

The analysed housing markets are the US, UK, NL, JP, SE and BE. In order to observe the critical transitions from nonlinear dynamics of the housing market, we require time series as long as possible to contain as much information as possible. The investigated time window ranges from 1970 to 2013 [1]. This contains several well-known housing market crashes, such as those of the United States (2007), Japan (1992), Sweden (1991), the United Kingdom (2007) and the on-going bubbles in many countries.

Quarterly nominal and real house prices for each country are obtained from the housing dataset in the Organisation for Economic Co-operation and Development (OECD). The nominal house price is indexed using 2005 as base year. The real house price index is derived by deflating with the private final consumption expenditure deflator, which is available from the OECD Economic Outlook 89 database. The price-to-rent ratio is defined as the nominal house price index divided by the rent component of the consumer price index, made available by the OECD. The interest rate is indicated by 10-year government bonds yields, downloaded from the OECD iLibrary for US, UK, NL, BE. Because OECD iLibrary does not have the interest rate dating back to 1970, for JP and SE, we downloaded them from Datastream.

[1] Time window for Sweden (SE) is from 1Q1980 to 1Q2013, and for Belgium is from 2Q1976 to 1Q2013, based on the availability in the datasets.

5 Results and Discussions

The differential equation of cusp catastrophe model is estimated by using Euler Discretization. As described in Section 4, the state variable is the relative deviation of the housing price from the fundamental price. For the control parameters, we consider two variants: estimation with constant control parameters, and estimation with the control variable interest rate governing the control parameters α and β.

5.1 Constant Control Parameters

As a benchmark, we fit the cusp catastrophe model to housing market data given a constant control variable. Thus the control parameters α and β are constant and are defined by

$$\begin{aligned} \alpha &= \alpha_0 \\ \beta &= \beta_0 \end{aligned} \tag{17}$$

To describe the stability of system statistically, we investigate Cardan's discriminant $\delta = 27\alpha^2 - 4\beta^3$. $\delta = 0$ indicates the boundary of bifurcation set. When $\delta < 0$, the state of system is in cusp bifurcation region and unstable. The model predicts two possible state values instead of one; if $\delta > 0$, the state of the system is outside of bifurcation region and stable.

Table 2 shows the estimated parameters and their corresponding standard errors in all six countries. The parameters λ and σ scale the observed state variable. λ is assumed to be equal to 0 since there is a fundamental equilibrium at 0. Bayesian information criterion (BIC) and Akaike information criterion (AIC) indicate the fitness of model (Hartelman, 1997; Grasman *et al.*, 2009). The standard errors of transformed parameters of Cardan's discriminant δ are obtained by the delta method.

Delta method takes the variance of the Taylor series approximation of a function as standard error. Let G be the transformation function and X be the consistent estimator. X converges in probability to its mean vector U. Let $\bigtriangledown G(X)$ be the gradient of $G(X)$. The first two terms of the Taylor expansion are then an approximation for $G(X)$,

$$G(X) \approx G(U) + \bigtriangledown G(U)^T \cdot (X - U), \tag{18}$$

which implies that the variance of $G(X)$ is approximately

$$Var(G(x)) \approx \bigtriangledown G(X)^T \cdot Cov(X) \cdot \bigtriangledown G(X) \tag{19}$$

where $Cov(X)$ is the variance-covariance matrix of X. $Var(G(x))$ is thus the standard error of Cardan's discriminant δ.

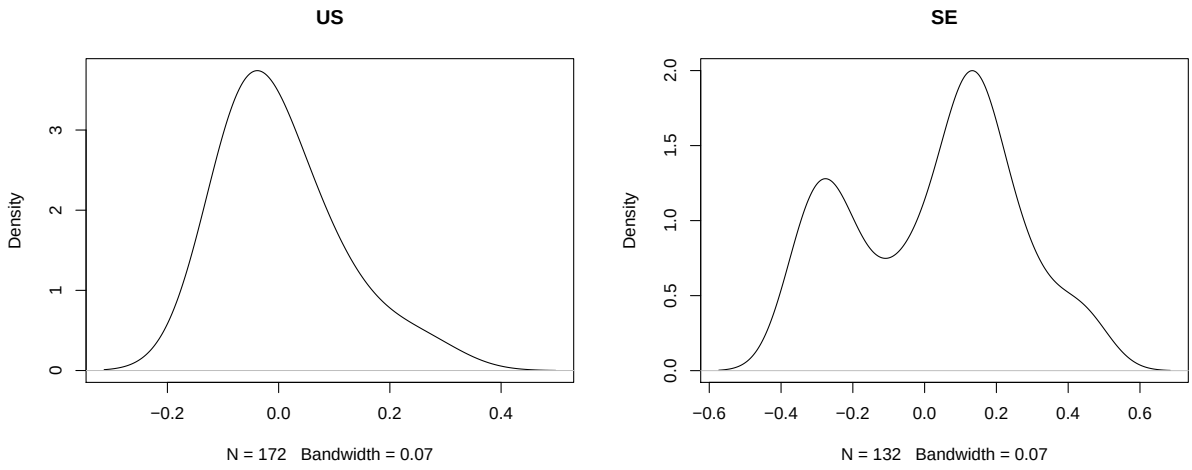
Table 2: Estimation results with respect to constant α and β in different countries. "Stability" shows local stability of the system. The values in the brackets are standard errors.

	US	JP	UK	NL	SE	BE
λ	0.000	0.000	0.000	0.000	0.000	0.000
σ	5.298	$3.420^{\cdot}$	3.056^{*}	2.982^{*}	$2.405^{\cdot}$	2.641^{***}
$(s.e.)$	(2.858)	(2.041)	(1.261)	(1.255)	(1.348e+00)	(0.373)
α	$7.704e-05$	-0.0004	$1.781e-03$	$1.448e-03$	$2.583e-05$	0.003^{**}
$(s.e.)$	(5.314e−04)	(0.0005)	(0.001)	(0.001)	(9.484e−04)	(0.0010)
β	$-9.632e-03$	0.014	$1.376e-02$	$1.353e-02$	$1.131e-02$	0.048^{***}
$(s.e.)$	(1.795e−02)	(0.012)	(0.018)	(0.018)	(2.489e−02)	(0.011)
δ	$3.735e-06$	$-5.528e-06$	$7.526e-05$	$4.673e-05$	$-5.773e-06$	-0.0001
$(s.e.)$	(1.842e−05)	(2.368e−05)	(0.0001)	(8.298e−05)	(3.799e−05)	(0.0002)
AIC	−983.611	−901.021	−715.484	−721.421	−592.250	−782.153
BIC	−971.021	−888.431	−702.894	−708.831	−580.719	−770.191
Loc.stab.	stable	unstable	stable	stable	unstable	unstable

$***$ significant at 0.1% level, $**$ significant at 1% level, $*$ significant at 5% level, $\cdot$ significant at 10% level.

The local stability of the system, which is indicated by the sign of δ, is also presented in Table 2. Using US and SE as examples, Figure 5 shows the kernel density of the price deviations in stable and unstable systems. The US shows an unimodal distribution while the SE has a bimodal distribution.

Figure 6 gives a visual illustration of the estimated location of the behaviour points (equilibria) of different countries on the projection of cusp equilibrium surface in Figure 2. The cusp shaped shaded area is the control plane which is determined by the sets of control parameters α and β. As shown in the legend, each symbol and colour indicates the estimated parameters of equilibrium of one country. The ellipse around it with the same colour corresponds to its 95% probability region. Consistent with the estimation results in Table 2, countries of US, NL and UK are in stable regions, while JP, SE and



(a) $\delta > 0$ (b) $\delta < 0$

Figure 5: The examples of Kernel density estimate on price deviations when $\delta > 0$ and $\delta < 0$.

BE are unstable and inside the grey bifurcation region. With confidence level of 95%, the confidence regions include the points representing the "true" values of behaviour points.

Notably, by combining Figure 6 with the cusp equilibrium surface in Figure 2, we are able to forecast the dynamical behaviour of the housing market by observing where its behaviour point is located on the cusp equilibrium surface. For instance, country SE is inside the bifurcation set where the surface predicts two possible state values instead of one. With the change of control parameters, its behaviour point may follow the path A in Figure 2 and move close to fold curve. A tiny perturbation on the control variables would induce it to fall off the cusp "cliff" and jump to a different equilibrium suddenly. The US is in the "normal" situation and outside of the cusp bifurcation. If the control variable α changes, its state point would follow the path B in Figure 2. Its transition between equilibria is slowly and smoothly without experiencing cusp catastrophes. However, it does not mean that no critical transition could occur. When control variable β increases, it would move into the unstable bifurcation set and experiences a critical transition. We shall observe similar situations for NL and UK. Although they are in stable regions, as long as their control parameters change in certain directions, they

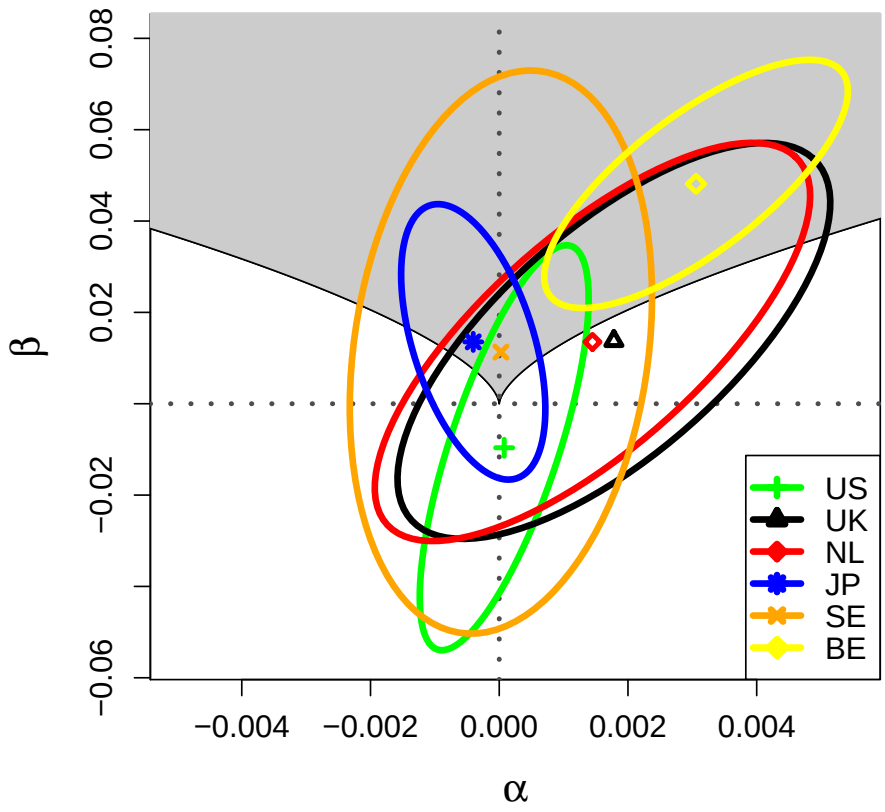


Figure 6: The estimated behaviour points of different countries in different regions of the control plane with respect to constant α and β. The cusp shaped shaded area is the control plane. The ellipses correspond to confidence regions with confidence level of 95%.

may move into bifurcation set and experience possible critical transitions. They are particularly dangerous when they are too close to the bifurcation border. Even a small perturbation may induce a critical transition. Therefore, the changes of corresponding control variables perform an important role in the dynamic behaviour of the systems. By monitoring and controlling these changes, we are able to influence or even prevent the instability of housing markets.

5.2 Interest Rate as Control Variable

In what follows, we study the situation when the interest rate is used as a control variable. Following the estimation methods in Section 3, the control parameters α and β are now defined by

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 x, \\ \beta &= \beta_0 + \beta_1 x, \end{aligned} \tag{20}$$

where x is the control variable - interest rate. It is indicated by the 10-year government bonds yields. $\sigma, \alpha_0, \alpha_1, \beta_0, \beta_1$ are the parameters to be estimated. Table 3 shows the estimation results in different countries. Because we are interested in predicting the changes of equilibria and investigating critical transitions, although some parameters are not very significant, we will later show that the model fits well with respect to the changes of equilibria. Like the example of constant control parameters, we conduct a plot of the control variables with the estimated behaviours for different countries in Figure 7 to provide a visual illustration.

Table 3: Estimated parameters of $\sigma, \alpha_0, \alpha_1, \beta_0, \beta_1$ when using interest rate as the control variable. λ is constant. The values in the brackets are standard errors.

	US	JP	UK	NL	SE	BE
λ	0.000	0.000	0.000	0.000	0.000	0.000
σ	2.097	4.482	2.773^{**}	7.495	1.485^{***}	2.533^{***}
$(s.e.)$	(1.985)	(4.639)	(0.972)	(21.405)	(0.286)	(0.342)
α_0	0.003	-0.002	0.013^{*}	0.005	0.020^{***}	0.015^{***}
$(s.e.)$	(0.004)	(0.002)	(0.006)	(0.014)	(0.006)	(0.003)
α_1	-0.0005	0.0004	-0.001^{*}	-0.0005	-0.002^{***}	-0.002^{***}
$(s.e.)$	(0.0006)	(0.0005)	(0.0005)	(0.002)	(0.0007)	(0.0003)
β_0	0.0123	-0.0002	-0.032	-0.094^{*}	0.025	0.051^{*}
$(s.e.)$	(0.046)	(0.016)	(0.034)	(0.042)	(0.030)	(0.020)
β_1	-0.004	-0.001	0.004	0.013^{**}	0.0009	-0.006^{*}
$(s.e.)$	(0.006)	(0.003)	(0.004)	(0.005)	(0.003)	(0.002)
AIC	-988.770	-912.718	-732.828	-729.114	-630.110	-831.070
BIC	-969.885	-893.832	-713.943	-710.229	-612.813	-813.128

$***$ significant at 0.1% level, $**$ significant at 1% level, $*$ significant at 5% level, $\cdot$ significant at 10% level.

As shown in Figure 7, in different countries, their equilibria changing with time trace

different paths on the control plane. These tracks are distinguished by different symbols and colours. By comparing with the benchmark in Figure 6, we shall see that the interest rate as a control variable has great impact on the instability of housing markets. For the housing markets of NL and UK, although their equilibrium points locate in the stable regions under constant control parameters (Figure 6), the changes of interest rate induce them move into the unstable bifurcation region in certain times (Figure 7). On the contrary, the equilibrium points of SE, BE and JP are now able to move outside of the bifurcation region. As for the US which remains in stable regions, its equilibrium point also changes with interest rate.

5.2.1 Equilibrium

Figure 8 shows the predicted equilibrium values and the time series of housing price deviations from fundamentals in different countries. It allows us to investigate the equilibria of the system, and also provides information on how the state of system transits from one equilibrium to another.

The black line in Figure 8 represents the time series of housing price deviations on fundamentals. The black dotted line indicates the baseline of 0. All six countries exhibit long-lasting periods of fluctuations of price deviations around 0. It can be observed that housing prices have been increasing rapidly since the mid-1990s and have peaked around 2008 in the US and NL. After that they have dropped considerably for those countries. The JP housing prices peaked earlier around 1990, and subsequently declined to levels below the baseline 0. For the UK, SE and BE, they exhibited peaks around 1990 and 2008. After that, housing prices dropped significantly.

The red and blue lines indicate the estimated equilibria when control parameters are constant (Section 5.1), while the scatter plots show equilibria with interest rate as control variables. The red lines or scatters represent stable equilibria on the upper or bottom sheet of cusp equilibrium surface, while the blue lines or scatters imply unstable equilibria on the middle sheet of cusp equilibrium surface.

Although some predicted equilibria do not completely fit the position of real data in the plots, we should note that our objective is to forecast the systematically changes of equilibria. It can be observed that the cusp model forecasts the changes of equilibrium in real housing data well. When the housing price deviation increases or decreases,

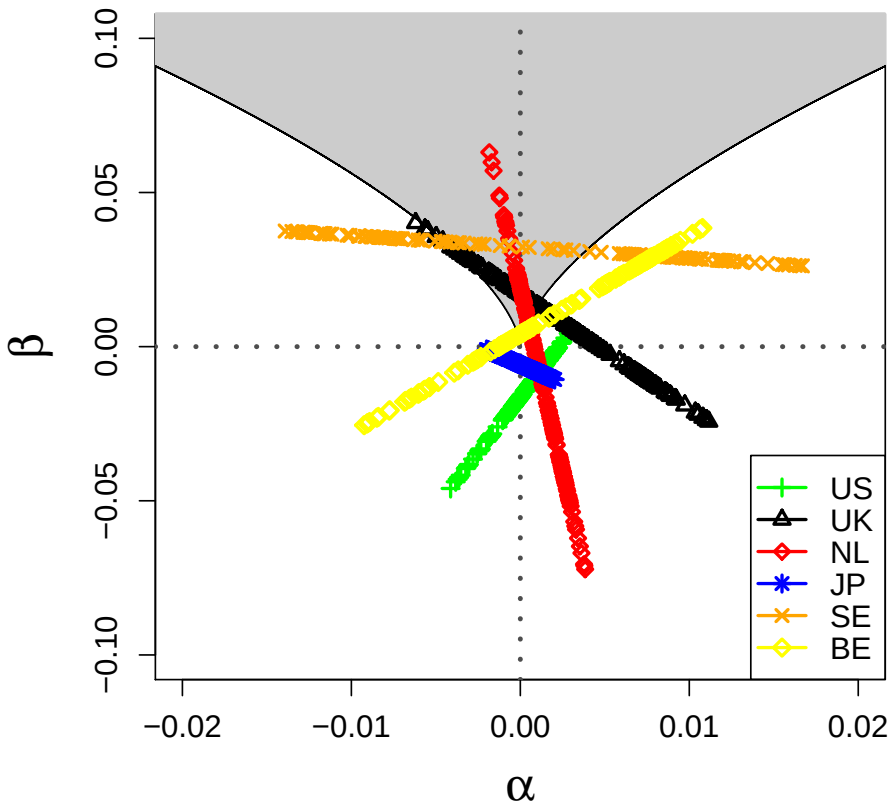


Figure 7: The estimated behaviour points of different countries in different regions of the control plane. The control variable is interest rate. The cusp shaped shaded area is the control plane. Each symbol and colour indicate the estimated behaviour of one country.

the cusp model forecasts its equilibrium changes in the same direction. For instance, in the example of SE around mid-1990s, the equilibrium presented a transition from a lower equilibrium to an upper equilibrium. Around the same time, the corresponding housing price data experienced a rapid drop and a gradual increase. Therefore, the cusp model performs well regarding the endogenous changes.

5.2.2 Critical Transition

Figure 9 presents the time series of price deviations, interest rates and the cube root of Cardan's discriminant δ in different countries. Under the monitoring of monetary policy, interest rates in all countries exhibited long lasting fluctuations regarding different economic situations. In general, for JP, SE and NL, interest rates were dropping in the observed period. For the US, UK and BE, interest rates peaked around 1980. After that, it was followed by persistent droppings.

Because δ is small and close to 0, transformation to its cube root allows us to visually catch more details on its value around 0. We shall observe its up and down oscillations around 0 in different countries. When $\delta > 0$, the system is in a quiet and stable regime; When $\delta < 0$, it is in unstable bifurcation area. As shown in Figure 9, the time series of δ is consistent with the tracks of different countries in Figure 7. A negative value of δ is observed in the examples of the NL, SE, UK and BE, but not in the US and JP.

We are particularly interested in the corresponding dynamics when δ crosses the 0 baseline from negative to positive, which implies that the state of system moves from the unstable bifurcation region to stable region. This is observed during most of the housing price bubble and burst cycles in the UK, NL, SE and BE. When crossing the bifurcation boundary from inside to outside, a systemic change of system may occur through a "critical transition", with respect to the directions of the changes of the control variable - the interest rate. The corresponding cusp behaviour is that the system's state 'falls' off the cusp curve and jumps from one equilibrium to another.

A fluctuation in time series can be a normal oscillation in a single equilibrium or a systematic change via critical transition between two equilibria. To distinguish between the two, a study of the change of equilibria with respect to the control variables is useful. Figure 10 illustrates how the time series of the control variable - the interest rate, corresponded with the bifurcation band in the housing market historically. The grey band indicates the range of interest rate values with respect to $\delta < 0$, which implies the unstable bifurcation region with three equilibria - two stable and one unstable. Because a critical transition happens when the state of system jumps off the cusp curve and ends in a different equilibrium, it can be distinguished when the interest rate cross through the bifurcation band.

By observing the relationship between interest rates and the bifurcation region, dis-

played across time in Figure 10, we are able to study the underlying mechanisms of fluctuations in housing markets. For instance, in the example of SE, the state of housing system has fallen into the bifurcation region twice during 1990 and 2000. This corresponds with the bottom of the downturn of the housing price index. After falling into the bifurcation region for the first time, the state of the system quickly came out of it and went back to the previous stable equilibrium due to a rise of the interest rate. A year after that, the interest rate dropped and the system was brought down to the multiple equilibria bifurcation region for the second time. For this time, it did not end in the previous equilibrium but transitioned to a new stable equilibrium. This transition induced the retrieval of the housing market. Since then, the SE housing price was continuously increasing. The UK and BE show similar bifurcation bands in their interest rates and exhibited similar systematic fluctuations as SE. They had experienced several critical transitions between two equilibria before mid-1990s, which were consistent with the fluctuations in real housing price. After that, the system of UK came back to its previous stable equilibrium. These behaviours corresponded with the recovery of housing market in these two countries. In particular, the housing system of NL has the widest bifurcation band of all. It remained in a multiple equilibria region for almost a decade between 1973 to 1984, while the corresponding housing price was experiencing a bubble and burst cycle. Nevertheless, its equilibrium never went across the bifurcation band and always came back to its previous stable equilibrium. For the US and JP, there is no bifurcation band. In our analysis, their housing systems remained in a single equilibrium.

To analyse the impact of interest rate on the equilibrium of housing system further, Figure 11 presents the bifurcations showing the predicted equilibrium as a function of the interest rate r in different countries. Red scatter represents the stable equilibrium on the upper or bottom sheet of cusp equilibrium surface. Blue scatter represents the unstable equilibrium which lies on the middle sheet.

The results unfold the underlying bifurcations in different housing systems. For the UK, SE and BE, the observed bifurcations are so-called saddle-node bifurcations. With interest rate as a control parameter, the cusp model exhibits three equilibria for a certain range of interest rate. It also shows that as interest rate increases further, the housing system becomes stable again in a new equilibrium. The bifurcation scenario for the example of NL is different. As the interest rate increases, the cusp model exhibits one

stable equilibrium. After passing a critical thresholds when interest rate around 7.5, a new saddle-node bifurcation occurs and creates two new equilibria, one stable and one unstable. The system has been in the multiple equilibria regime from then on. For the US and JP, there are no bifurcations going on during the analysed period.

5.2.3 Policy Implication

The housing bubble and bursts cycles were followed by financial crisis which could create enormous tragedies. They have raised great concerns of the instability of housing price among policy makers. How can a policy maker stabilise the housing price and prevent market instability? As an essential factor in the monetary policy, interest rate has been pointed out to have great influence on the instability of housing market (Bernanke and Gertler, 1995; Shiller, 2006; Muellbauer and Murphy, 2008; Taylor, 2007, 2009; Crowe *et al.*, 2013; Shi *et al.*, 2014). Our study once again shed lights on the importance of interest rates. Moreover, we unfold the underlying link between interest rate and systematic fluctuations in housing market. The dynamic of housing system shows cusp catastrophe behaviour with interest rate as control parameter. It exhibits critical transitions between multiple equilibrium states as a result of the changes of interest rate. It is also dangerous when the system is in the unstable bifurcation region, or gets too close to the cusp curve. Even a small perturbation could induce significant fluctuations. This scenario can be used to explain the majority of housing bubbles and bursts in the data, such as in UK 1978, 1980,1990, NL 1978, BE 1987,1990, and the depression of SE after 1990.

A general lesson for policy makers to be drawn from these examples is that cusp catastrophe may yield important insights on policies that can cause global instability in economics. As we argued in our analysis, policy makers should monitor the instability of economic systems, be alerted when the system approaches a bifurcation, in particular be aware of critical transitions which could lead to significant market bubbles or sudden market collapse. The examples in this paper suggest that interest rate policy plays an important role in keeping the stability of economic system. By performing an appropriate interest rate policy, policy makers are able to prevent endogenous market crashes. There is no empirical evidence to show whether a high or a low interest rate should be beneficial to the economy. " You can never be too rich or too thin". Taking

the SE housing market as an example, as seen in Figure 11, if we set the interest rate too high, i.e. between 7 and 9 percent, the market might collapse. However, when the interest rate is set too low, bubbles may arise. Our method gives us an overall picture about the multiple equilibria of the system and how the equilibria change with interest rate policy. It could provide policy makers a reasonable guide to conduct a proper interest rate policy that keeps the economy in a healthy state. It could also help to deal with markets with bubbles and to establish a post-crisis policy on the recession following a housing market collapse.

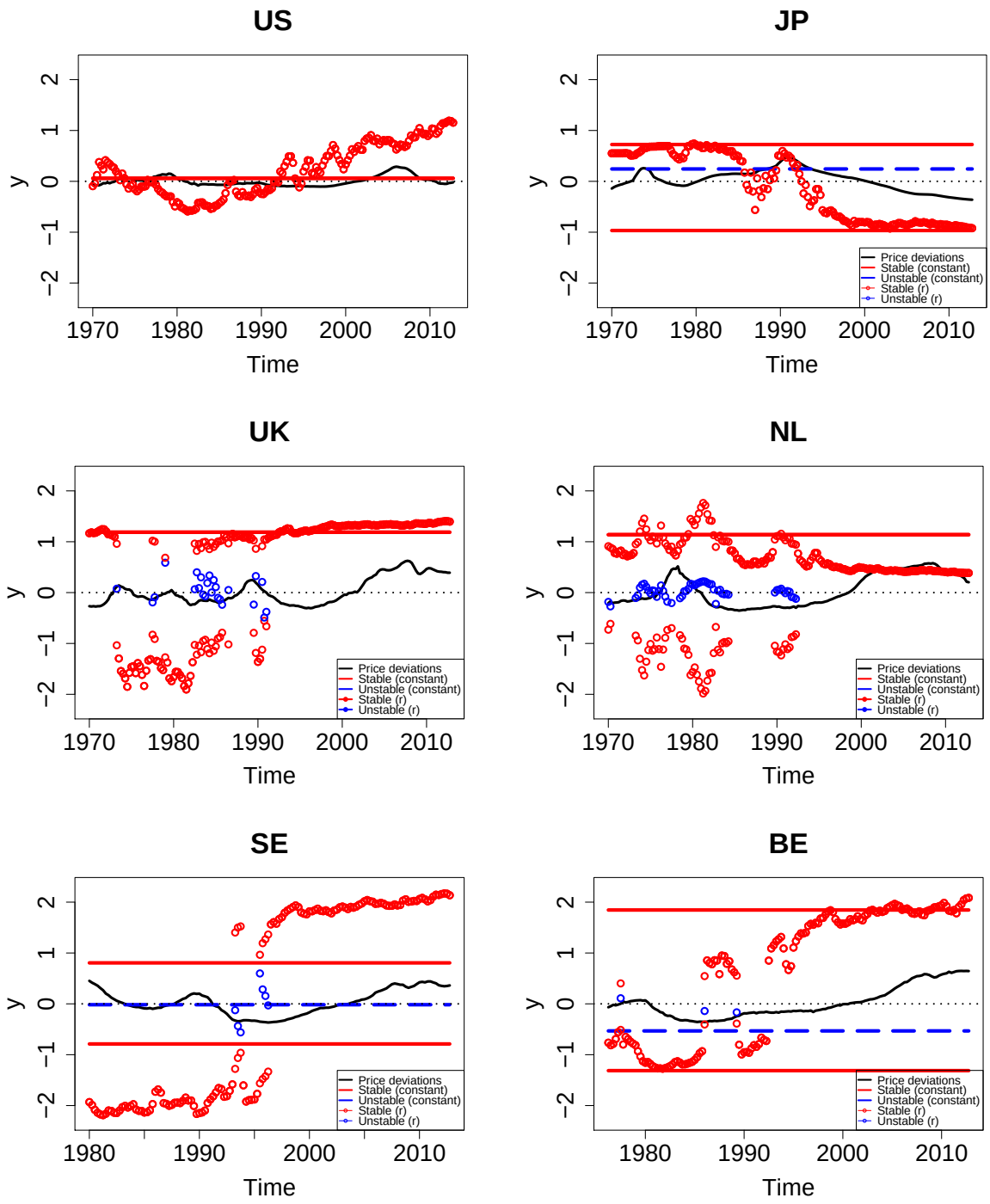


Figure 8: Time series of housing price deviations and predicted equilibria in different countries. Black line represents the time series of housing price deviations. Scatter plots indicate estimated equilibria when control variables are interest rate. Red and blue lines indicate the estimated equilibria when control parameters are constant.

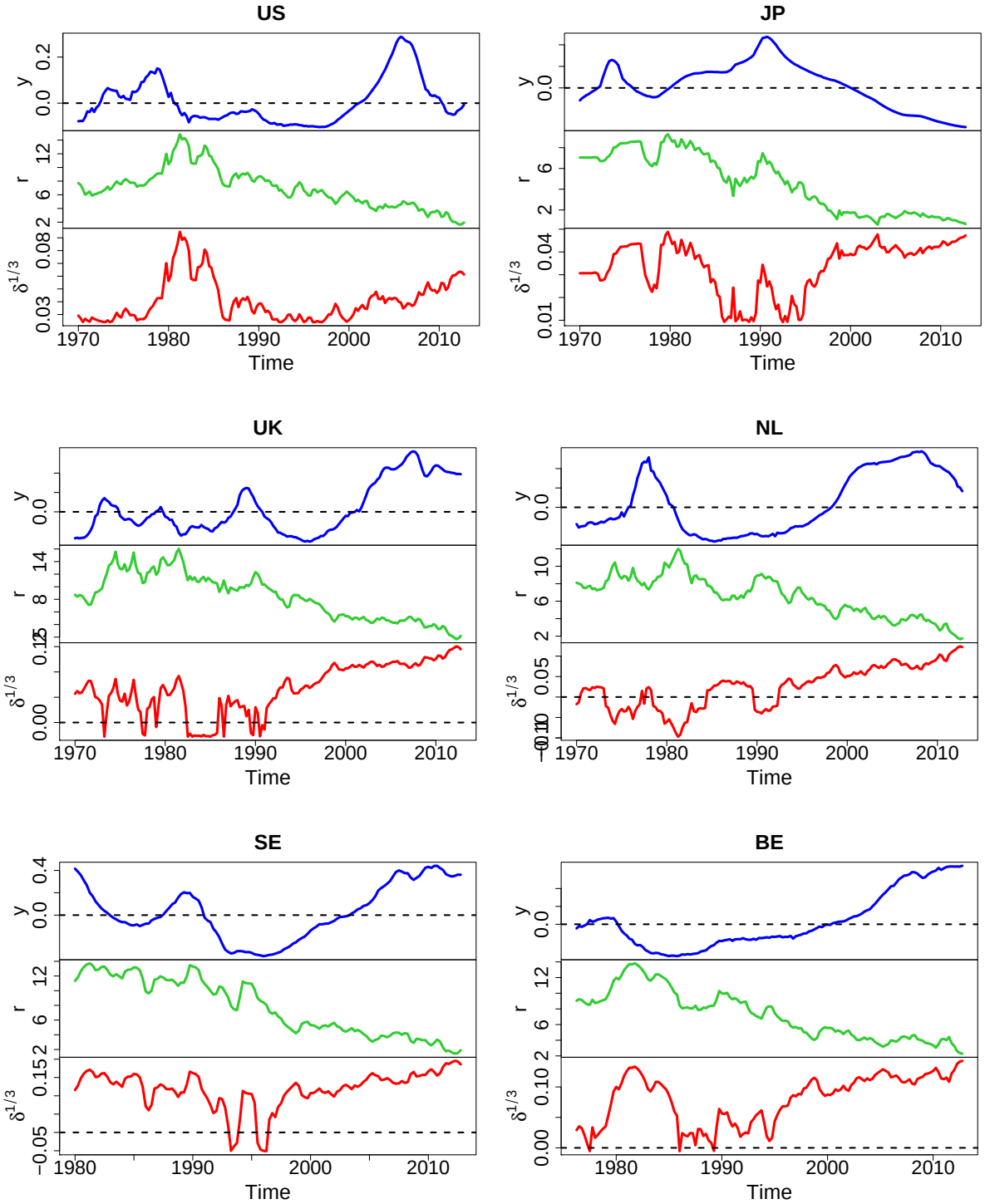


Figure 9: Time series of state variable (top panels), interest rate (middle panels) and cube root of Cardan's discriminant δ (bottom panels) in different countries. The control variable is interest rate. Dashed line is baseline of 0.

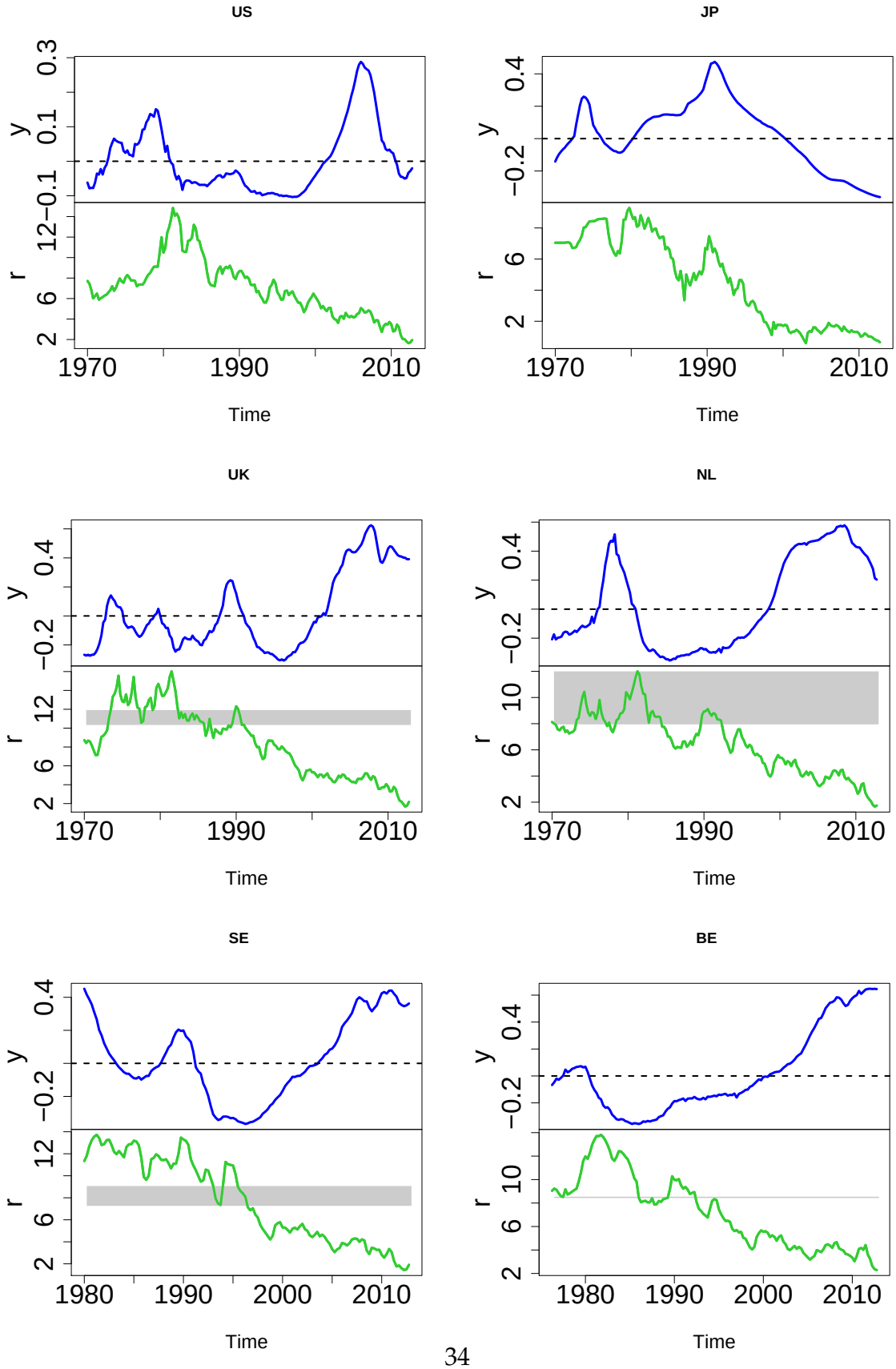


Figure 10: Time series of housing price deviations, and interest rate with grey bifurcation band.

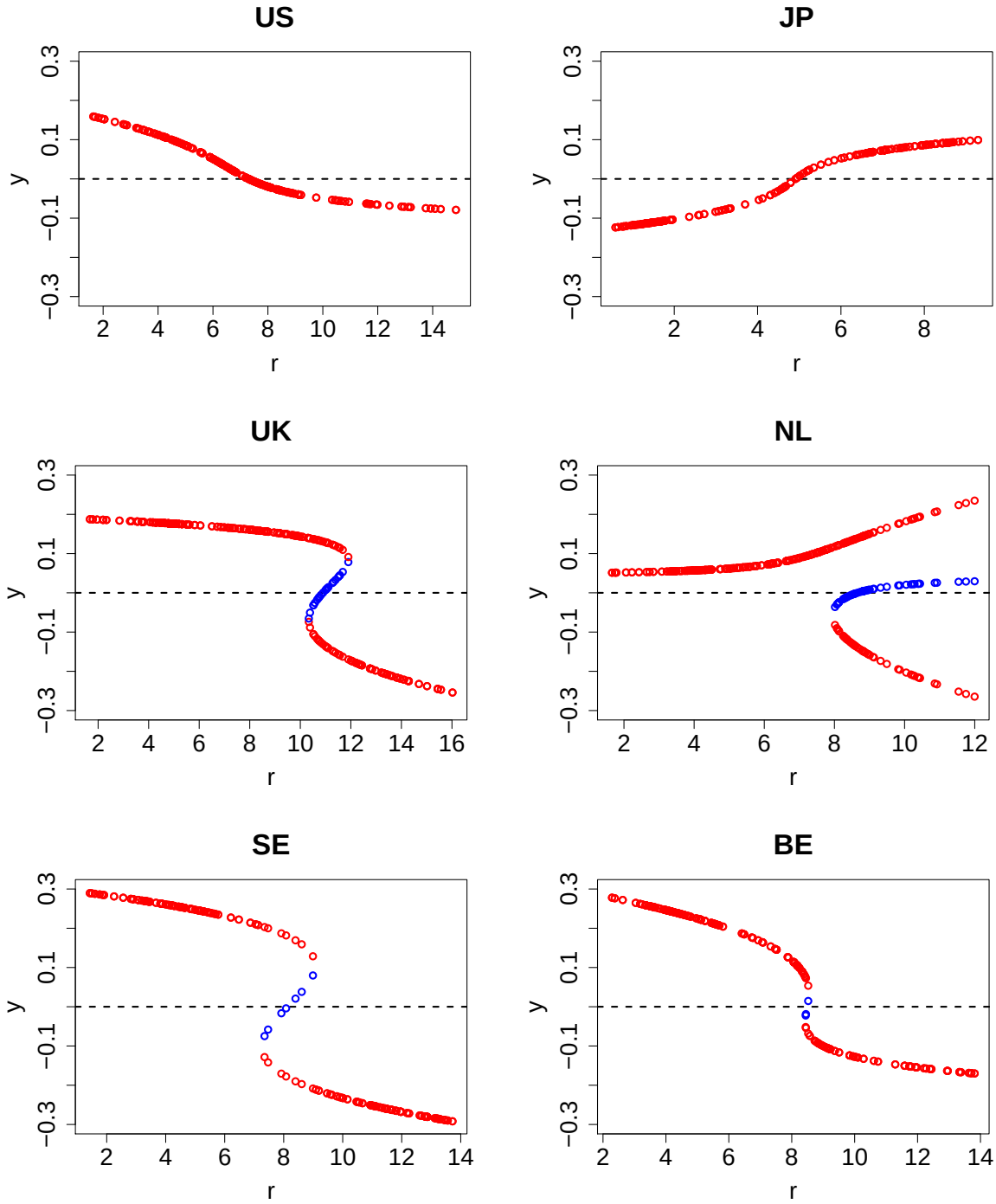


Figure 11: Bifurcations showing the predicted equilibrium as a function of the interest rate r in different countries. Red scatter represents the stable equilibrium (up or bottom sheet). Blue scatter represents the unstable equilibrium (middle sheet)

6 Concluding Remarks

This paper attempts to find out whether instability of housing market can be explained and predicted by catastrophe theory. A stochastic cusp catastrophe model was fitted to empirical housing market data for the first time. Using housing price deviations and quarterly data on long term government interest rates, we estimated the model for six different countries: United States (US), United Kingdom (UK), Netherlands (NL), Japan (JP), Sweden (SE) and Belgium (BE).

Two estimation approaches are discussed - Cobb's Method and Euler Discretization. The analysis shows that Cobb's Method requires a system for which the state variables change fast compared to the control parameters. It performs well when modeling the overall invariant density of state variables. However, when it comes to forecasting, Euler Discretization always gives better predictions. In this paper, because we are using time series data and our objective is to study the forecasting ability of cusp catastrophe model, Euler Discretization is employed in the later sections.

The estimation results obtained using Euler Discretization are discussed in the later part of the paper. We find that the dynamics of the housing market can be explained by cusp catastrophe behaviour. Under constant control parameters, the housing systems of US, UK and NL are in a normal stable regimes while the housing systems of SE, JP and BE are in unstable bifurcation regime. Nevertheless, when using interest rate as control variable, the interest rate changes the stability of the systems; those systems' equilibria vary with interest rate. The predicted equilibria give us a general picture on the changes of equilibria with time. Time series of cardan's discriminant δ links the changes of system equilibria and the bubbles and bursts cycles in empirical data. Moreover, by observing the relationship between interest rates and bifurcation bands, we are able to study the underlying mechanisms of fluctuations in housing markets. A critical transition can be distinguished when the interest rate cross through its bifurcation band. The underlying bifurcations can be given by the correlation between predicted equilibria and interest rates.

Our results yield important insights into policies that monitor the instability in economics. A change of the main control parameter, interest rate, may move the economic system closer to the unstable region with multiple equilibria. As a control variable, interest rate plays an important role in keeping the stability of economic system. Policy

makers should prevent the economic system from moving into the multiple equilibria regions, or from getting too close to the cusp curve that may induce critical transitions. The cusp catastrophe theory could provide policy makers with a reasonable guidebook on interest rate policy.

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A Fundamental Price and Price Deviations

In this model, agents are boundedly rational and have different views about the future values of asset prices. At the same time, agents are allowed to switch from one period to the next between a number of available strategies, based on how well they have performed in the recent past. Agents base their decisions at time t on the expected excess return R_{t+1} on investing in housing relative to renting during the period between time t and $t+1$. Let P_t denote the price of one unit of housing at time t. Let the price for renting one unit of housing in the period between times t and $t+1$ be given by Q_t. Since rents are typically payed up-front (at time t), to express the rent at time t in terms of currency at time $t+1$, it should be inflated by a factor $(1+r^{\mathrm{rf}})$, where r^{rf} denotes the risk free mortgage rate. Therefore, the cost of renting in the period between time t and $t+1$, expressed in terms of currency at time $t+1$, is given by $(1+r^{\mathrm{rf}})Q_t$ rather than Q_t. The ex post excess return R_{t+1} on investing in housing during the period between time t and $t+1$ then is given by the sum of the capital gain minus mortgage/maintenance costs and the saving on rent.

$$R_{t+1} = \frac{(P_{t+1} - (1+r_t)P_t) + (1+r^{\mathrm{rf}})Q_t}{P_t} = \frac{P_{t+1} + (1+r_t^{\mathrm{rf}})Q_t}{P_t} - (1+r_t),$$

where $r_t = r_t^{\mathrm{rf}} + \omega_t$ is the sum of the risk-free (mortgage) rate r_t^{rf} and the maintenance/tax rate ω_t.

The demand, $z_{h,t}$, of agents of belief type h is determined by maximizing one-period ahead expected excess returns adjusted for risk:

$$\mathbb{E}_{h,t}\left(R_{t+1}z_{h,t}\right) - \frac{a}{2}\mathrm{Var}_{h,t}\left(R_{t+1}z_{h,t}\right), \tag{21}$$

where a is a measure of risk aversion. For simplicity we assume r_t^{rf} and ω_t to be constant over time: $r_t^{\text{rf}} = r^{\text{rf}}$, $\omega_t = \omega$ (and hence $r_t = r$). Agents are assumed to be homogeneous with respect to their expectations regarding the conditional variance of the excess return, that is, $\text{Var}_{h,t}\left((P_{t+1} + (1+r^{\text{rf}})Q_t)/P_t - (1+r)\right) = V_t$, while they are heterogeneous concerning their expectations of excess return $\mathbb{E}_{h,t}\left((P_{t+1} + (1+r^{\text{rf}})Q_t)/P_t - (1+r)\right)$. Maximizing Eq. (21) leads to the demand for housing:

$$z_{h,t} = \frac{\left(\mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right)/P_t - (1+r)\right)}{aV_t} = \frac{\mathbb{E}_{h,t}(R_{t+1})}{aV_t},$$

The market clearing condition is:

$$\sum \frac{n_{h,t}\left(\mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right)/P_t - (1+r)\right)}{aV_t} = S_t, \tag{22}$$

where S_t is the stock of housing, and $n_{h,t}$ is the fraction of agents in period t that hold expectations of type h.[2] Solving the market clearing condition for the price P_t leads to the following price equation:

$$P_t = \frac{1}{1+r+\alpha} \sum n_{h,t} \mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right), \tag{23}$$

where $\alpha \equiv aV_t \times S_t$ is assumed to be constant in order to keep the model tractable. Agents require a rate of return on housing equal to $r + \alpha$ rather than $r = r^{\text{rf}} + \omega$. Therefore the parameter α can be interpreted as a risk premium of buying a house over renting a house. Treating α as a constant in the model allows for estimating this extra required rate of return, under the assumption that it is a constant.

We next turn to expectations regarding the fundamental price. Following Boswijk *et al.* (2007), we assume that the fundamental process underlying the model, i.e. Q_t, follows a geometric Brownian motion with drift, i.e.

$$\log Q_{t+1} = \mu + \log Q_t + v_{t+1}, \quad \{v_t\} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_v^2),$$

with commonly known parameters μ and σ_v^2, from which one obtains

$$\frac{Q_{t+1}}{Q_t} = (1+g)\varepsilon_{t+1},$$

with $g = e^{\mu + \frac{1}{2}\sigma_v^2} - 1$ and $\varepsilon_{t+1} = e^{v_{t+1} - \frac{1}{2}\sigma_v^2}$, such that $\mathbb{E}_t(\varepsilon_{t+1}) = 1$.

[2] We assume that S is large enough, such that neither type has an incentive to sell short on houses.

We define the fundamental price as the price that would prevail under rational expectations $\mathbb{E}_t(R_{t+1})$ about the conditional mean of R_t, while taking into account the risk premium α. Taking into account the risk premium in the fundamental price is convenient, as it will provide an equilibrium fundamental price from which the market price will deviate by an amount which averages out to zero in long time series. Under rational expectations on the first conditional moment, we can re-write the price Eq. (23) as

$$(1 + r + \alpha)P_t = \mathbb{E}_t \left(P_{t+1} + (1 + r^{\text{rf}})Q_t \right).$$

By applying the law of iterated expectations and imposing the transversality condition, we obtain the fundamental price at time t, denoted by P_t^*.

$$P_t^* = \mathbb{E}_t \left[\sum_{i=0}^{\infty} \frac{(1 + r^{\text{rf}})Q_{t+i}}{(1 + r + \alpha)^{i+1}} \right] = \sum_{i=0}^{\infty} \frac{(1 + g)^i (1 + r^{\text{rf}})Q_t}{(1 + r + \alpha)^{i+1}} = \frac{1 + r^{\text{rf}}}{r + \alpha - g} Q_t, \qquad r + \alpha > g. \tag{24}$$

This shows that the fundamental price of housing is directly proportional to the actual rent level. Figure 12 shows the examples of house price, fundamental price and price deviations $lnp_t - lnp_t^*$ in all the countries from 1970 to 2013.

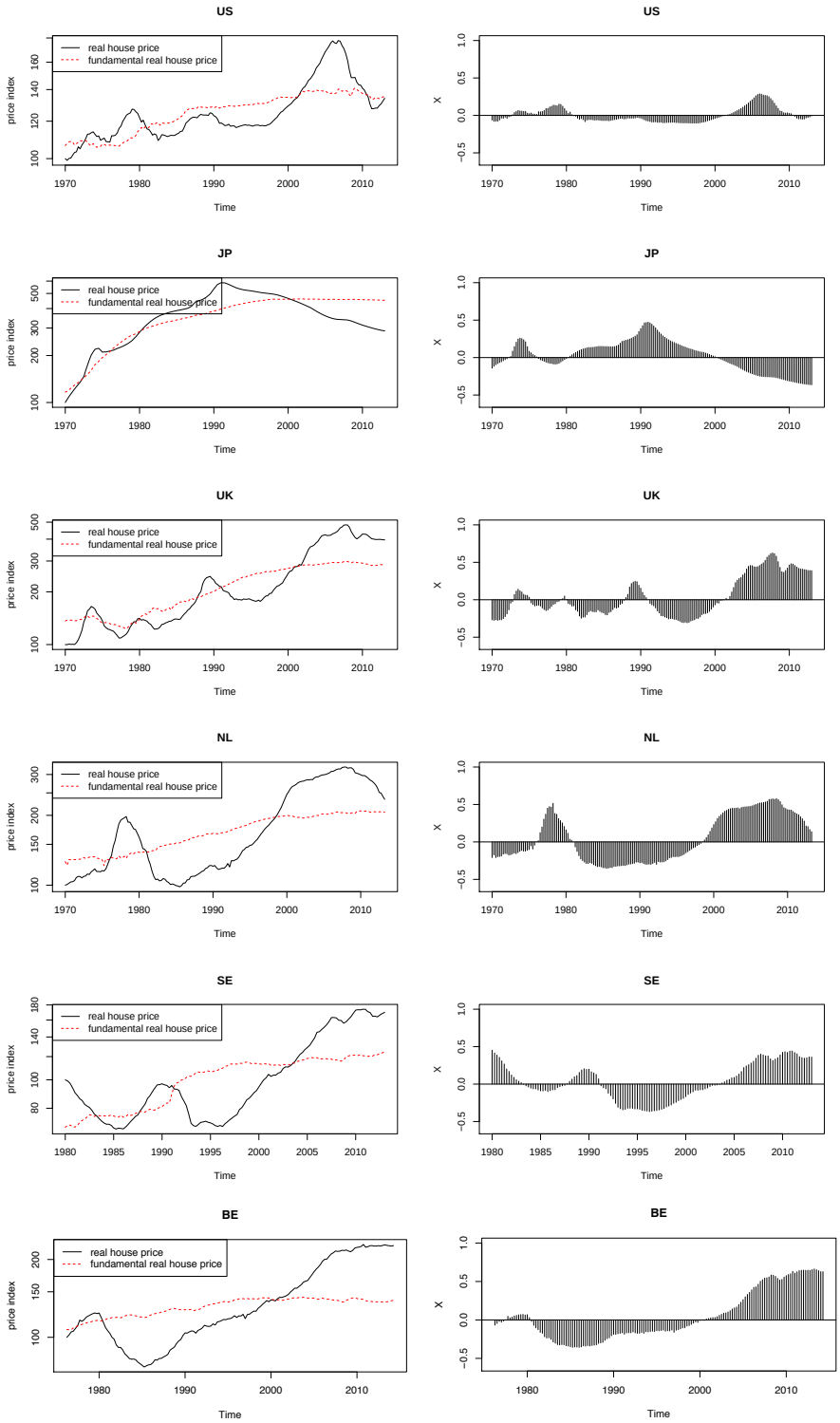

Figure 12: Housing price indices (left, solid lines, 1970Q1=100), estimated fundamental real housing prices (left, dashed lines) and corresponding price deviations X_t (right).

B Cobb's Method v.s. Euler Discretization

B.1 Residuals

B.2 Cobb's Method

For the stochastic differential equation

$$\mathrm{d}y_t = -V'(y_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

with potential $V(y)$, the invariant distribution is the Gibbs distribution with density

$$f(y) = \frac{1}{Z_\sigma} \exp\left(-\frac{2V(y)}{\sigma^2}\right),$$

where Z_σ is a normalization constant (Anderluh and Borovkova, 2008). [With $\epsilon = \sigma^2/2$ this coincides with Cobb (1978)].

For the canonical CUSP potential $-V(y) = \alpha y + \frac{1}{2}\beta y^2 - \frac{1}{4}y^4$, one finds

$$f(y) = \frac{1}{Z_\sigma} \exp\left(-\frac{\alpha y + \frac{1}{2}\beta y^2 - \frac{1}{4}y^4}{\sigma^2/2}\right).$$

Transforming to

$$\tilde{y} = y / \left(\frac{\sigma^2}{2}\right)^{\frac{1}{4}},$$

(leaving aside trivial translations) we obtain for the density of $\tilde{y}$

$$\begin{aligned} f_{\tilde{y}}(\tilde{y}) &\propto \exp\left(\frac{-\alpha\left(\frac{\sigma^2}{2}\right)^{\frac{1}{4}}\tilde{y} - \frac{1}{2}\beta\left(\frac{\sigma^2}{2}\right)^{\frac{1}{2}}\tilde{y}^2 + \frac{1}{4}\left(\frac{\sigma^2}{2}\right)\tilde{y}^4}{\left(\frac{\sigma^2}{2}\right)}\right) \\ &\equiv \exp\left(-\tilde{\alpha}\tilde{y} - \frac{1}{2}\tilde{\beta}\tilde{y}^2 + \frac{1}{4}\tilde{y}^4\right). \end{aligned}$$

This suggests $\tilde{\alpha} = \alpha / \left(\frac{\sigma^2}{2}\right)^{\frac{3}{4}}$ and $\tilde{\beta} = \beta / \left(\frac{\sigma^2}{2}\right)^{\frac{1}{2}}$.

B.3 Euler Discretization

Suppose that the deterministic part of the (canonical) variable z_t is governed by the potential function

$$V(z; \alpha, \beta) = \alpha z + \frac{1}{2}\beta z^2 - \frac{1}{4}z^4,$$

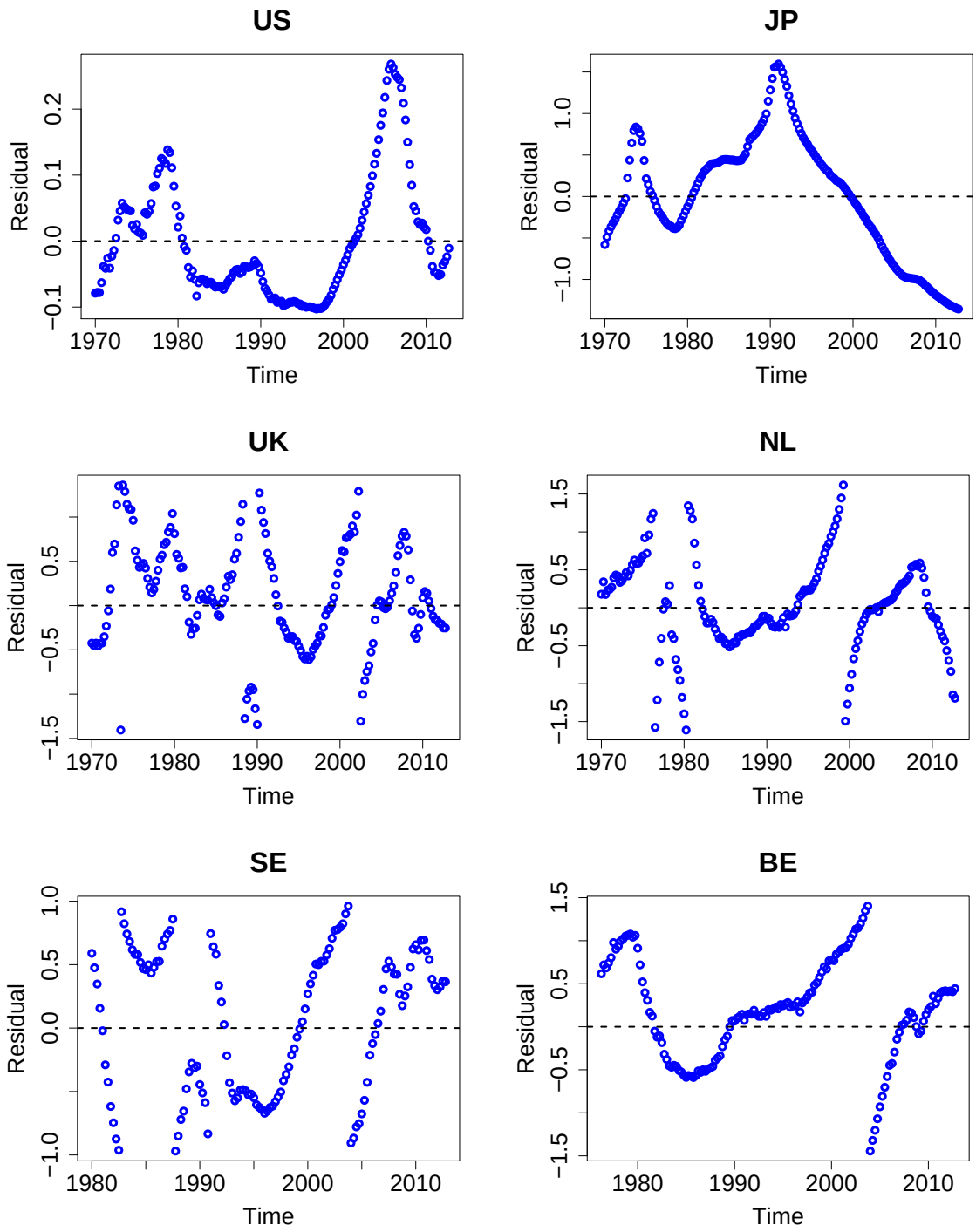

Figure 13: Plots of residuals against index by using Cobb's Method.

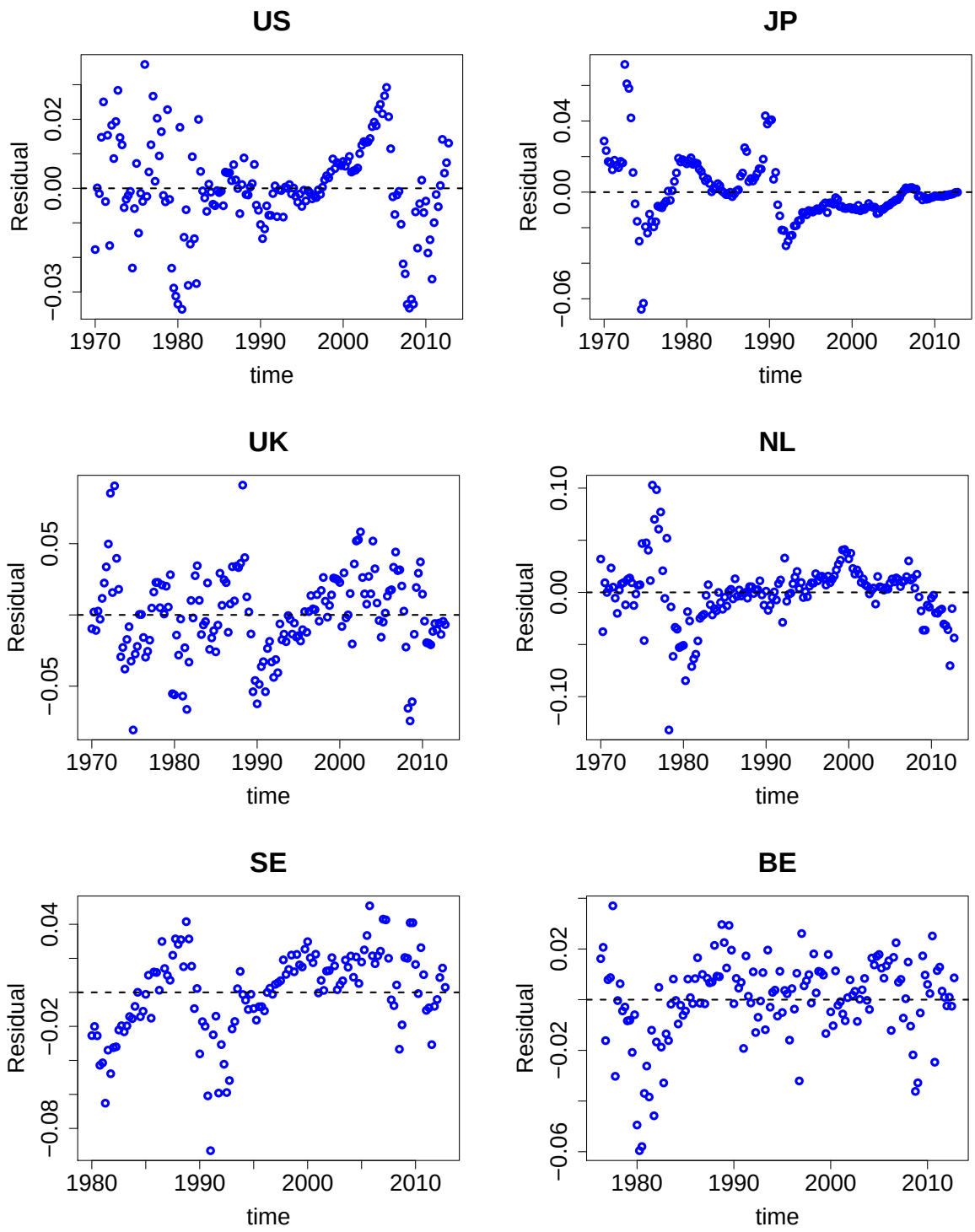

Figure 14: Plots of residuals against index by using Euler Discretization.

and that there is a driving noise term with variance σ_Z^2 per time unit, i.e.

$$\mathrm{d}z_t = -\left.\frac{\partial V(z;\alpha,\beta)}{\partial z}\right|_{z=z_t} + \sigma_z \mathrm{d}W_t.$$

The invariant density of z then is proportional to (Gibbs distribution)

$$f_Z(z) \propto \exp\left[-\frac{2V(z)}{\sigma_z^2}\right] = \exp\left[-\frac{\alpha z + \frac{1}{2}\beta z^2 - \frac{1}{4}z^4}{\sigma_z^2/2}\right].$$

If $y = \lambda + rz$ is a scaled and/or translated variable, then $z = (y-\lambda)/r$, and the density of y is proportional to

$$f_Y(y) \propto \exp\left[-\frac{\alpha\left(\frac{y-\lambda}{r}\right) + \frac{1}{2}\beta\left(\frac{y-\lambda}{r}\right)^2 - \frac{1}{4}\left(\frac{y-\lambda}{r}\right)^4}{\sigma_z^2/2}\right]. \tag{25}$$

The invariant density fitted by CUSP fit is

$$f_Y(y) = \psi \exp\left[\tilde{\alpha}\left(\frac{y-\lambda}{c}\right) + \frac{1}{2}\tilde{\beta}\left(\frac{y-\lambda}{c}\right)^2 - \frac{1}{4}\left(\frac{y-\lambda}{c}\right)^4\right]. \tag{26}$$

Comparing the coefficients of the fourth powers in Eqs (25) and (26), we see that these coincide only if $c = r\left(\frac{\sigma_z^2}{2}\right)^{\frac{1}{4}}$. It can be readily checked that this implies $\tilde{\alpha} = \left(\frac{\sigma_z^2}{2}\right)^{-\frac{3}{4}}\alpha$ and $\tilde{\beta} = \left(\frac{\sigma_z^2}{2}\right)^{-\frac{1}{2}}\beta$.

In terms of $y_t = \lambda + rz_t$ the SDE is

$$\frac{1}{r}\mathrm{d}y_t = -\left.\frac{\partial V(z;\alpha,\beta)}{\partial z}\right|_{z=\frac{y_t-\lambda}{r}} + \sigma_z \mathrm{d}W_t.$$

Euler discretization gives

$$y_{t+\Delta_t} \approx y_t + \left(\alpha + \beta\left(\frac{y_t-\lambda}{r}\right) - \left(\frac{y_t-\lambda}{r}\right)^3\right) r\Delta t + r\sigma_z\sqrt{\Delta t}\varepsilon_{t+\delta_t},$$

where $\varepsilon_{t+\Delta t} \sim N(0,1)$.