

Evolutionary Cournot competition with endogenous technology choice: (in)stability and optimal policy*

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Abstract

We study a dynamic oligopoly market model where quantity setting firms can choose one of two production technologies. We find that boundedly rationality in production (best-reply dynamics) and technology choice (evolutionary selection of better performing technologies) as sources of market dynamics, can generate endogenous instability and complicated dynamics, including chaotic fluctuations and co-existing attractors with fractal basins of attraction. By studying successively more complex versions of our model we analyze these two different sources of instability separately and also investigate their interaction. We find that boundedly rational production decisions amplify technological instability whereas boundedly rational technology decisions do not contribute to the production-driven destabilization of the Nash equilibrium. In any case, whenever the two types of decisions interfere in an endogenously unstable market, fluctuations follow a visibly different pattern compared to the fluctuations of a market with only one source of instability. Finally, we show that an innovation policy that aims to alter the market equilibrium without taking into account off-equilibrium dynamics may, in an intrinsically dynamic world, generate welfare losses by destabilizing a stable equilibrium and/or by raising the amplitude of market fluctuations.

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1 Introduction

Corporate control is usually a complex, dynamic and multidimensional problem. Companies typically adopt a segmented management structure with specialized departments handling production, sales, marketing, HR, R&D, finance, etc. The fact that these departments intensively communicate with each-other indicates that managers clearly know that decisions taken on one dimension interfere with the actions performed by their colleagues in another department. Abstracting away from all fronts of decision-making other than production and R&D, our paper formally studies their effects on market dynamics and equilibria. We introduce a model where firms may take adaptively rational decisions on either production, innovation, or both, and show how these decisions and their interaction in a dynamic environment relate to market equilibrium stability, emerging endogenous fluctuations and the welfare effects of innovation policy.

Our approach allows us to identify two potential sources of market instability. The first one is operational in nature and is driven through adaptive quantity-setting by firms endowed with imperfect expectations of what their competitors will do. Given its source, we will call this phenomenon 'production instability' throughout the paper. The second type of instability in our model stems from adaptive choices of firms over two alternative production technologies. This choice, we show, can generate 'technological instability'. Our analysis reveals that these two types of decisions can together generate bounded endogenous market fluctuations. We find that market instability associated to bounded rationality in the production decision-making dimension is unaffected by boundedly rational technological choices. This is not the case for technological instability: boundedly rational production decisions will amplify it compared to the case of perfectly rational production decisions. For a policy maker, our model tells a cautionary tale: when the market is unstable, innovation policy that would optimize social welfare at equilibrium can generate large welfare losses compared to no interference.

Our first line of inquiry, relating to production instability, follows from the seminal paper of Theocharis (1960). He showed the Cournot-Nash equilibrium in a quantity-setting oligopoly to be dynamically unstable when more than three firms compete by supplying, in each period, naive best-response quantities to the output produced by competitors one period before. The importance of this result was reflected by the quick emergence of work extending, qualifying and containing it, Fisher (1961); McManus and Quandt (1961); Hahn (1962). With the development of chaos and bifurcation theory and the wide acknowledgment of their relevance for economic theory due to Grandmont (1985) and Brock and Hommes

(1997), Theocharis' inquiries were revisited in work celebrating endogenous cycle formation and chaotic dynamics in oligopoly markets, see Agliari et al. (2000); Droste et al. (2002); Bischi and Lamantia (2004); Hommes et al. (2011); Bischi and Lamantia (2012); Kopel et al. (2014); De Giovanni and Lamantia (2015). A frequent feature in these models is the introduction of behavioral heterogeneity - be it in terms of expectations, Droste et al. (2002); Hommes et al. (2011) or of the objective functions, Kopel et al. (2014); De Giovanni and Lamantia (2015). Our model also focuses on firm heterogeneity, but here it is of a technological nature.

As such, our analysis of technological instability also relates our paper to the work of Nelson and Winter (1982). From their perspective, bounded rationality and firms' technological heterogeneity - both central to our analysis - are crucial for understanding industrial dynamics and economic growth, particularly when aiming to explicitly integrate innovation and technological progress. Their approach inspired a rich literature of Agent-Based simulation models. Some of its highlights are surveyed in Dawid (2006).

Also inspired by Nelson and Winter (1982), Hommes and Zeppini (2014) and Diks et al. (2013) analyzed models where firms choose between alternative R&D strategies: innovation and imitation. Much like their analysis the work presented here employs a simpler formulation of the technological dimension, one that allows for a more general analysis of the model combining the classical tools of game theory and dynamic system analysis with bifurcation theory and numerical simulation. Despite differences in the naming of the two competing technologies¹, our market-setup closely resembles the one used in Hommes and Zeppini (2014) and Diks et al. (2013). The model analyzed here is different in two respects. Firstly, we consider a setting where firms strategically compete in an oligopoly market whereas Hommes and Zeppini (2014) assume an infinite firm population with no market power. Secondly, the work presented here relaxes the assumption of optimal quantity setting and investigates the evolutionary competition between a new and an old technology acknowledging that firms may also be boundedly rational in their production decisions, not only in their technology choices. Our work is also related to Ding et al. (2014) and Ding et al. (2015), where firms learn how to invest based on realizations of profit margins while, as in Hommes and Zeppini (2014), the market clears for Nash quantities every period. In contrast, our model draws a clear line of separation between technological choices and production decisions. This allows us to disentangle two different sources of instability as well as to study their interaction.

The paper is organized as follows. In Section 2 we begin by deriving some preliminary results for a market setup where firms choose production technology and are engaged in static

¹Where they speak of innovation versus imitation we contrast between an innovative strategy and a standard strategy.

Cournot competition. We characterize the Nash equilibrium of the model and show that, depending on model parameters - in particular, the cost of using the innovative technology - we can either obtain an equilibrium with all firms employing the same technology - innovative or standard - or a mixed equilibrium where fractions of the firm population use different technologies. This is summed up in Proposition 1. Casting our static model in a suite of dynamic settings we then move on to investigate the stability properties of the market equilibrium when firms make boundedly rational decisions, illustrating our results with numerical simulations. We consider three separate cases: (i) technology distribution is exogenous and production is determined by Cournot adjustment (i.e. best-response dynamics) - Section 3; (ii) output is always at the Cournot-Nash equilibrium but firms switch technologies based on average profitability - Section 4; (iii) output follows Cournot dynamics and technology is chosen based on profitability - Section 5. For all three scenarios we establish necessary and sufficient conditions for stability in the steady state, in Propositions 2, 3 and 4. Finally, in Proposition 5 of Section 6, we characterize an optimal innovation policy tailored on the Nash equilibrium of the model. We numerically illustrate its effects on the equilibrium outcome, contrasting them to its economic impact when the equilibrium it aims to adjust is dynamically unstable.

2 An infinite population Cournot oligopoly game with technological choice

Consider a firm population of unit size producing a homogeneous good. Firms play a two stage sequential game. In the first stage, all firms simultaneously choose one of two available production technologies. In the second stage, which takes place after technology decisions become public knowledge as the fractions of the population using each technology, firms simultaneously decide on an output level, q . Because we consider only two technological alternatives this means that, in the second stage, a share of the firm population, z , will produce with the *standard* technology, s , while the remaining $1 - z$ firms use the *innovative* technology, i . Cost functions associated with these two technologies are given by

$$c_s(q) = \frac{1}{2}d_s q^2,$$

for standard firms and

$$c_i(q) = K + \frac{1}{2}d_i q^2$$

for innovative firms. We assume $d_s > d_i \geq 0$, with $d_s - d_i$ representing the marginal cost advantage of innovative firms and $K > 0$ the fixed investment required for using the innovative production technology. The amount K is paid before production and will be treated throughout our analysis as a sunk cost.

Market competition is set-up to translate the classical Cournot oligopoly game to an infinite and heterogeneous population. Each firm is randomly matched to $N - 1$ other firms in the population and their joint second-stage output clears a linear (inverse) demand:

$$P(Q_N) = 1 - Q_N,$$

where Q_N is the sum of quantities produced by the N firms that are matched together.

The game is solved by backward induction. We begin by determining the competitively optimal output decisions in stage two for any given population shares. Then, by comparing the realized profits of the two technological strategies, we can establish what the Nash equilibrium regarding both choices is.

Depending on the production technology used, the expected average profit of a firm who knows the population shares of standard and innovating firms, z and $1 - z$, is computed as the probability weighted sum, over all possible market compositions, of the profits realized in each particular scenario, with k standard competitors and $N - 1 - k$ innovating competitors:

$$\begin{aligned} \pi_s(z) &= \sum_{k=0}^{N-1} \binom{N-1}{k} z^k (1-z)^{N-1-k} \left[1 - q_s - [kq_s + (N-1-k)q_i] - \frac{1}{2}d_s q_s \right] q_s \\ \pi_i(z) &= \sum_{k=0}^{N-1} \binom{N-1}{k} z^k (1-z)^{N-1-k} \left[1 - q_i - [kq_s + (N-1-k)q_i] - \frac{1}{2}d_i q_i \right] q_i - K. \end{aligned}$$

The only term in the above expression that depends on the summation index, k , accounts for the possible output of competitors in each possible market composition, $[kq_s + (N-1-k)q_i]$. Simplifying accordingly we obtain:

$$\begin{aligned} \pi_s(z) &= \left[1 - q_s - \bar{Q}_{N-1}(z) - \frac{1}{2}d_s q_s \right] q_s \\ \pi_i(z) &= \left[1 - q_i - \bar{Q}_{N-1}(z) - \frac{1}{2}d_i q_i \right] q_i - K, \end{aligned} \tag{1}$$

where $\bar{Q}_{N-1}(z) = \sum_{k=0}^{N-1} \binom{N-1}{k} z^k (1-z)^{N-1-k} [kq_s + (N-1-k)q_i]$ is the average competing output that a firm expects to face from the firms with which it will be matched, given population shares z and $1-z$. As we are assuming an infinite population, this expectation will be the same for both a standard or innovating firm. Restricting our attention to the quasi-symmetric equilibrium, where all firms of one type produce the same output, we can compute:

$$\begin{aligned} \bar{Q}_{N-1}(z) &= \sum_{k=0}^{N-1} \binom{N-1}{k} z^k (1-z)^{N-1-k} [kq_s + (N-1-k)q_i] \\ &= (N-1)[zq_s + (1-z)q_i] \end{aligned} \quad (2)$$

Plugging (2) into (1) and maximizing with respect to own quantities we obtain the reaction functions:

$$q_j(z) = R_j(\bar{Q}_{N-1}(z)) = \max \left\{ \frac{1 - (N-1)[zq_s + (1-z)q_i]}{2 + d_j}, 0 \right\}; \quad j \in \{i, s\}. \quad (3)$$

Solving for quantities and denoting, $Z = \frac{z}{2+d_s} + \frac{1-z}{2+d_i}$, we find the quasi-symmetric Cournot-Nash quantities as a function of z :

$$q_s^N(z) = \frac{1}{2 + d_s} \frac{1}{Z(N-1) + 1} \quad \text{and} \quad q_i^N(z) = \frac{1}{2 + d_i} \frac{1}{Z(N-1) + 1}. \quad (4)$$

Notice that $q_i^N(z) = \frac{2+d_s}{2+d_i} q_s^N(z) > q_s^N(z)$: innovators will always produce more than standard firms.

Plugging Nash quantities back into (1) we obtain the expected average Cournot-Nash profits:

$$\begin{aligned} \pi_s^N(z) &= \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \frac{1}{2+d_s} \\ \pi_i^N(z) &= \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \frac{1}{2+d_i} - K \end{aligned} \quad (5)$$

The choice of production technology in the first stage of the game will depend on the relation between the profit functions given in equation (5). Proposition 1, proven in Appendix A, is based on a comparison of the two profit functions in (5) depending on z and on model parameters. It allows identifying the quasi-symmetric Nash equilibria of the game in terms of output q_s and population-wide technology distribution, z . Because there are only two

available technologies and, according to equation (4), the quantities produced by the two types of firms are linearly dependent, $q_i^N(z) = \frac{2+d_s}{2+d_i} q_s^N(z)$, the Nash equilibrium of the two-stage game is fully characterized by a pair, (q_s^*, z^*) , of standard firm output and population share. While the model has four parameters, it is convenient to focus on the cost of innovation parameter, K . The results outlined below will serve as a benchmark for the dynamic analysis of Sections 4 and 5.

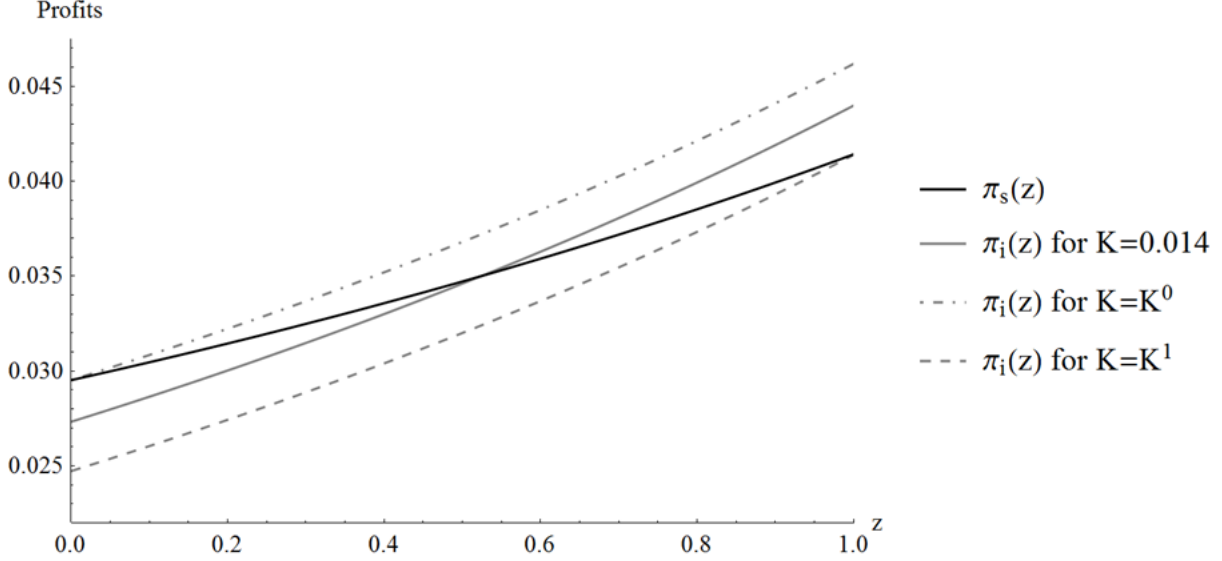
Proposition 1. *For all $N \geq 2$, $d_s > d_i > 0$, and $K^0 \equiv \frac{(2+d_i)(d_s-d_i)}{2(2+d_s)(N+1+d_i)^2} > 0$, $K^1 \equiv \frac{(2+d_s)(d_s-d_i)}{2(2+d_i)(N+1+d_s)^2} > K^0$ one the following three cases applies:*

- (a) $(q_s^*, z^*) = \left(\frac{2+d_i}{(2+d_i)(1+d_i+N)}, 0 \right)$ iff. $K < K^0$;
- (b) $(q_s^*, z^*) = \left(\sqrt{\frac{2(2+d_i)K}{(d_s-d_i)(2+d_s)}}, \frac{2+d_s}{d_s-d_i} - \frac{(2+d_i)}{(N-1)(d_s-d_i)} \left[\sqrt{\frac{(d_s-d_i)(2+d_s)}{2(2+d_i)K}} - (2+d_s) \right] \right)$ iff. $K^0 \leq K \leq K^1$;
- (c) $(q_s^*, z^*) = \left(\frac{1}{(1+d_s+N)}, 1 \right)$ iff. $K > K^1$.

The results of Proposition 1 can be intuitively grasped by inspecting Figure 1. The profits of innovative firms always slope steeper in z than they do for standard firms.² Changing K results only in a vertical shift of the innovative firms' profit curve. Therefore, if the two profit curves intersect, they will do so for a unique value, z^* . For $K < K^0$ innovative firms make strictly higher profits than standard firms, for any z , therefore all firms will innovate in the Nash equilibrium. For $K > K^1$ standard firms always make higher profits and the Nash equilibrium has all firms using the standard strategy. Otherwise, for a value of K in between the two bounds, there will be a unique interior point of intersection, z^* , where the two strategies generate equal profits. Any population shares with a smaller fraction of standard firms than z^* cannot be a Nash equilibrium since innovative firms will have an incentive to switch to the standard strategy (leading to an increase in z). Likewise, we cannot have a Nash equilibrium with more than z^* standard firms since in such a population a standard firm would want to switch to the innovative strategy (resulting in a decrease in z). From the expressions of z^* and q_s^* in Proposition 1 we can also immediately derive the comparative statics of the equilibrium with respect to innovation costs: equilibrium quantities (for both types of firms) are increasing in K and so is the equilibrium share of standard firms, z^* . Also, the interior equilibrium fraction of standard firms, z^* , is a strictly concave function of K .

²Because $Z = -z \frac{d_s-d_i}{(2+d_s)(2+d_i)} - \frac{1}{2+d_i}$ appears only in the denominator of 5 and is decreasing in z , both profits will be increasing in z . Furthermore, the slope of $\pi_i^N(z)$ is always $\frac{2+d_s}{2+d_i}$ times larger than that of $\pi_s^N(z)$.

Figure 1: Second-stage Nash profits as a function of z for $d_s = 1.5$, $d_i = 0.5$, $N = 4$ and different levels of K . The (first-stage) equilibrium share of standard firms is found at the intersection between $\pi_s(z)$ and $\pi_i(z)$.



At first glance, it may seem surprising that equilibrium output for both technological strategies increases in K , particularly given that quantities are strategic substitutes in Cournot oligopoly games. Inspecting the best-response functions for quantities, equation (3), we notice that they are identical save for the denominator which tells us that standard firms will react less aggressively to whatever the expected output by competitors is. While own output is strategically decreasing in the expected output of the competitors, it is not clear how competitors' output would react to changes in innovation costs, K . Intuitively, raising K would lead to a drop in the share of the more generously producing innovators. This, in turn, would leave a gap in the supply. This gap leaves room for extra production by both the (now fewer) innovators and standard firms. Intuitively, this is how equilibrium output increases for both types when fixed innovation costs, K , increase.

The effect of innovation costs on equilibrium firm output is best explained when we inspect the effect of K on total output at equilibrium, \bar{Q}_N^* which, regarding K , behaves exactly like the output by a firm's competitors at equilibrium, since $\bar{Q}_N^* = \frac{N}{N-1} \bar{Q}_{N-1}^*$. As both z^* and q_s^* are increasing in K , the effect of fixed innovation costs on the average industry quantity produced in equilibrium is a priori ambiguous: higher K means higher production by the standard firms but also less innovative firms who, at any given parameter combination, produce more than the standard ones. According to Corollary 1 it is the latter effect that is stronger overall. Therefore even if both types of firms would produce more when there are

less innovators - as a consequence of the innovative strategy being more expensive - overall the population mix effect is stronger, leading to lower average total industry output when innovation costs are higher.

Corollary 1. *In a mixed population equilibrium, average total industry output is given by:*

$$\bar{Q}_N^* = \frac{N}{N-1} [1 - (2 + d_s) q_s^*]$$

and decreases in K . Average total industry profits are:

$$TIP = N \frac{2 + d_i}{d_s - d_i} K.$$

Quite remarkably, once first-stage decisions on technology are internalized, the equilibrium quantity produced by an individual firm - no matter its technology - will not depend on the number of oligopoly firms, N , nor will its average equilibrium profit. The equilibrium share of standard firms, z^* , is however increasing in N .³ This means that an increase in the intensity of competition will lead, in our model, to less innovation, a result which has a Schumpeter mark II flavor⁴, with more intensive competition having a stifling effect on innovation. Indeed, given z , second stage profits are decreasing in N , see equation (5). As N increases, the part of the profit function that does not depend on fixed innovation costs decreases for both types of firms, but this decrease will be $\frac{2+d_s}{2+d_i}$ times stronger for innovative firms. In other words, moving from N to $N+1$ firms will lead to a flattening of the second stage profit curves given in (5), but the flattening will be more pronounced for $\pi_i^N(z)$ than for $\pi_s^N(z)$. This means that the profit functions will cross - if at all - for a higher z as shown in Figure 2. So, in the first stage more firms will choose the standard technology when the number of oligopoly firms increases.

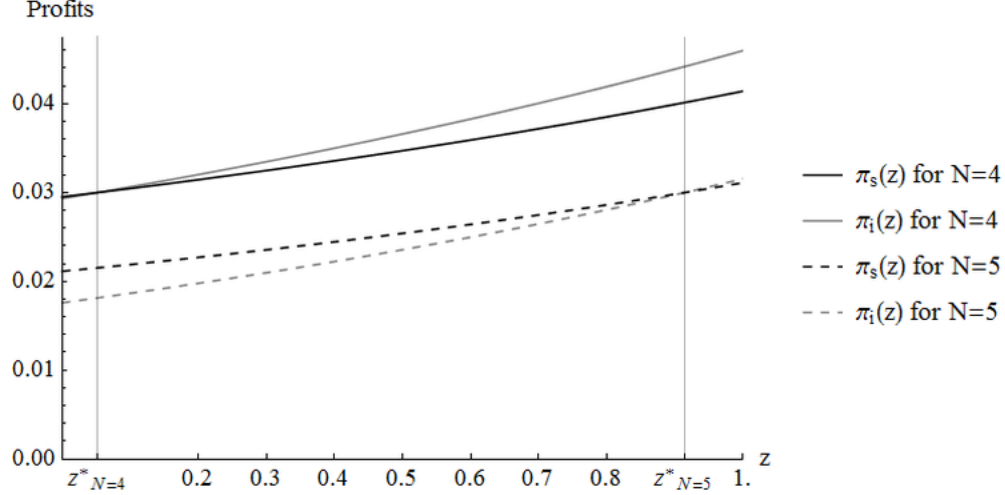
The comparative statics of the equilibrium share of standard firms with regard to marginal production costs are not always monotonic. We can show that the share of standard firms is

³It is clear from Proposition (1)(b) that z^* depends on N only through the denominator of the fraction multiplying the expression in square brackets, $E = (2 + d_s) - \frac{1}{q_s^*}$. Even though this expression does not depend itself on N , we cannot immediately determine its sign. However, we notice that, through q_s^* , it is monotonic in K . More precisely, as long as there is a $z^* \in [0, 1]$ we have $2 + d_s - \frac{1}{q_s^*(K^0)} < E < 2 + d_s - \frac{1}{q_s^*(K^1)}$ which simplifies to $\frac{2+d_s}{2+d_i} (1 - N) < E < 1 - N$, meaning that E is always negative. Therefore z^* is increasing in N . Notice also that $E = (2 + d_s) - \frac{1}{q_s^*} = -\frac{1-(2+d_s)q_s^*}{q_s^*} = -\frac{\bar{Q}_{N-1}^*}{q_s^*}$.

⁴Schumpeter's first conjecture, Schumpeter (1934), is that higher intensity of market competition between firms spurs innovation. This conjecture is often referred to as 'mark I'. He later stated, in Schumpeter (1942), in what is also called Schumpeter's mark II conjecture, that less acute competition gives firms the slack they need in order to divert resources to innovation.

always decreasing in d_s , as expected, but its relation to d_i depends on the relation between parameters d_s, d_i and N . Specifically z^* will be always increasing in d_i when the standard technology is not excessively more inefficient than the innovative technology, or, if the efficiency gap between the two technologies is high, when the number of oligopoly firms, N , is not too large; see also Appendix E for details.

Figure 2: The effects of increasing N on the equilibrium share of standard firms. $K = 0.012$, $d_s = 1.5$, $d_i = 0.5$.



To illustrate our results in such a way that allows comparisons between different model versions we will use one leading numerical example throughout the paper. Occasionally, we will examine the robustness of the insights offered by this numerical example by making variations of some key parameters.

Leading numerical example: $N = 4$, $d_s = \frac{3}{2}$ and $d_i = \frac{1}{2}$. From Proposition 1:

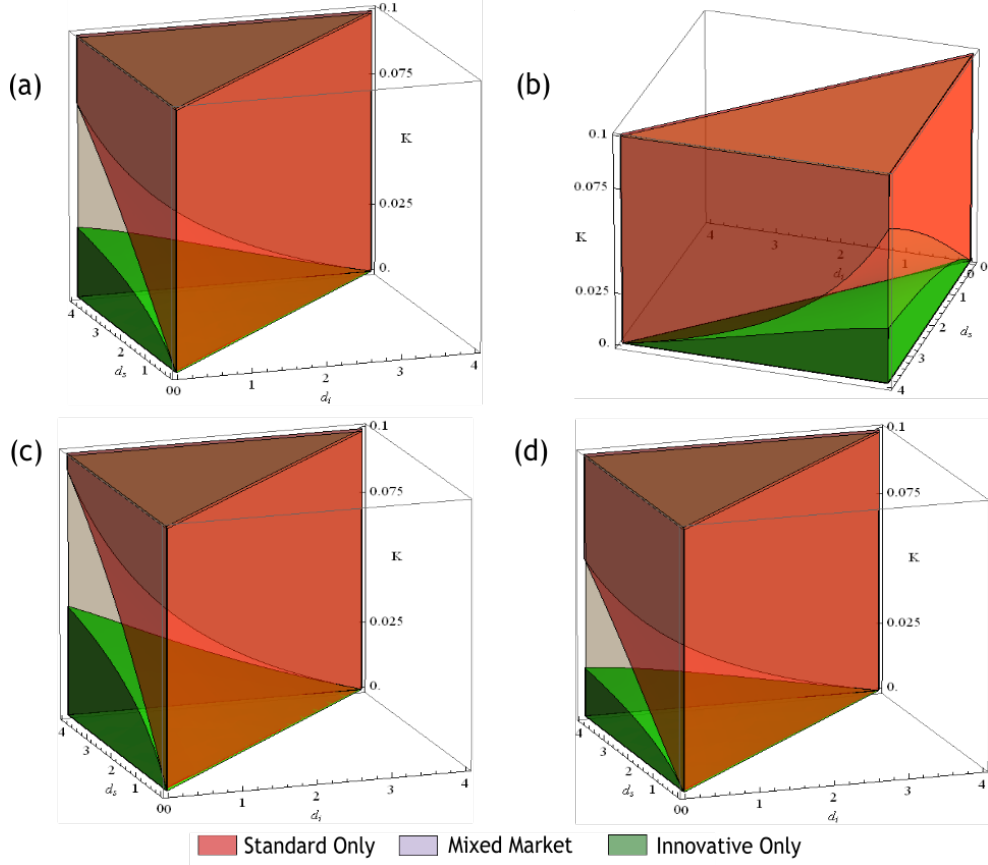
$$z^* = \frac{77}{12} - \frac{1}{3}\sqrt{\frac{35}{8K}},$$

with $K^0 = \frac{10}{847} = 0.0118$ and $K^1 = \frac{14}{845} \approx 0.0166$.

Proposition 1 gives a characterization of the full range of equilibria that our model can have in a static setting, with cases (a) and (c) corresponding to Nash equilibria with only one type of firm, standard or innovative, respectively. Case (b) corresponds to parameter combinations where both types of firms are present in equilibrium. Figure 3 provides a graphical illustration of these results for a particular choice for the number of firms $N = 4$. In accordance with the expressions for K^0 and K^1 , provided by Proposition 1, when N

increases we observe only an upward shift of the subset of the (d_s, d_i, K) parameter space where the mixed equilibrium exists, while its shape remains qualitatively the same.

Figure 3: Type of Nash equilibria in (d_i, d_s, K) space for $N = 4$ in (a) and (b), $N = 3$ in (c), $N = 5$ in (d)



Having characterized the Nash equilibrium of the static game, we turn our attention to the dynamic features of the model, analyzing what happens when firms repeatedly take decisions on which technology to use and/or on what quantity to produce. These decisions can be either fully rational or adaptive. In the next section, we first consider a version of the model where technology is exogenous (i.e. z is fixed) thus quantities produced are the only source of economic dynamics - forming naive expectations of the quantity supplied by competitors, each firm maximizes profits with respect to the quantity they produce. Next, in Section 4, we endogenize technological choice, but constrain the production behavior of the firms to supplying the Nash equilibrium quantities corresponding to the current shares of standard and innovative firms. Finally, we investigate the dynamic system where quantity follows the best-response dynamics and firms can also switch between production technologies.

3 Non-evolutionary best-response quantity dynamics

In this section we assume that the population shares are fixed and known to the firms before they make their production decisions. Quantity decisions follow a Cournot adjustment process where, before producing for period t , firms make a naive forecast, $\bar{Q}_{N-1,t}$, of how much their competitors will produce. The forecast will be based on known population shares, z and $1 - z$, and past production behavior $q_{s,t-1}$ and $q_{i,t-1}$. Given these expectations, each firm will produce a profit maximizing quantity, $q_{j,t}$, $j \in \{s, i\}$ that is given by the same best response functions in 3 adapted here for a dynamic setting:

$$q_{j,t} = R_j(\bar{Q}_{N-1,t}(z)) = \max \left\{ \frac{1 - (N-1)[zq_{s,t-1} + [1-z]q_{i,t-1}]}{2 + d_j}, 0 \right\}; j \in \{i, s\}$$

Notice that again $q_{i,t} = q_{s,t} \frac{2+d_s}{2+d_i}$ - the relation we found in our static analysis between Nash quantities holds in this dynamic set-up as well. Thus the dynamic model can be entirely characterized by a one-dimensional map. Specifically, the quantity dynamics expressed in q_s become:

$$\begin{aligned} q_{s,t} &= R_s(\bar{Q}_{N-1,t}(z)) \\ &= \max \left\{ \frac{1 - (N-1) \left[z + (1-z) \frac{2+d_s}{2+d_i} \right] q_{s,t-1}}{2 + d_s}, 0 \right\} \\ &= \max \left\{ \frac{1}{2 + d_s} - (N-1) \left[\frac{z}{2 + d_s} + \frac{1-z}{2 + d_i} \right] q_{s,t-1}, 0 \right\} \end{aligned} \quad (6)$$

So, for any fixed level of $z \in [0, 1]$, solving for the steady state quantity q_s^* gives:

$$\bar{q}_s^* = \frac{1}{2 + d_s} \frac{1}{\left(\frac{z}{2+d_s} + \frac{1-z}{2+d_i} \right) (N-1) + 1} \text{ and } \bar{q}_i^* = \frac{1}{2 + d_i} \frac{1}{\left(\frac{z}{2+d_s} + \frac{1-z}{2+d_i} \right) (N-1) + 1}, \quad (7)$$

the same values as the Cournot-Nash quantities derived in (4). Proposition 2, which goes without proof, describes the stability properties of this equilibrium.

Proposition 2. *The steady state of the dynamic system in (6) is the Cournot-Nash equilibrium of the static game and it is globally stable iff.:*

$$N < \widehat{N}(z) = 1 + \frac{(2 + d_i)(2 + d_s)}{2 + (1 - z)d_s + zd_i} \quad (8)$$

otherwise the dynamics converge to a two cycle .

Given that the map in (6) is linear, the price and quantity time-series generated by this model will either converge to the Nash Equilibrium or diverge and hit the non-negativity constraint on production settling into a two cycle oscillating between 0 and $\frac{1}{2+d_s}$. In a subspace of the parameter space defined by $(N - 1) \left[\frac{z}{2+d_s} + \frac{1-z}{2+d_i} \right] = 1$ there will be a bounded two-cycle with quantity oscillating between any initial value q_0 and $\frac{1}{2+d_s} - q_0$.

Proposition 2 also provides insight into the effects of model parameters on the stability of the system. It implies that increases in marginal costs, d_s and d_i will have a stabilizing effect on market dynamics whereas an increase in the share of firms that use the innovative technology will (weakly) reduce the stability of the system. The proposition also generalizes the Theocharis result: for $d_s = d_i = 0$ we will only have stability for $N - 1 < 2$ or $N = 2$.

Clearly, when d_s and d_i are sufficiently high the equilibrium is stable regardless of z . Likewise, for sufficiently high N and low enough d_s the equilibrium will always be unstable for all z . In between these two extremes, there exists an interesting section of the parameter space where the equilibrium is stable for large z (few innovators) and unstable for low z (many innovators) .

An interesting parameter region is obtained when d_s and d_i are such that the equilibrium is stable for $z = 1$ and unstable for $z = 0$. This requires $\widehat{N}(1) = 3 + d_s > N > 3 + d_i = \widehat{N}(0)$ which is the case in our leading numerical example. Obviously, if $\widehat{N}(0) < N < \widehat{N}(1)$ there exists a $\widehat{z} \in]0, 1[$ such that the fixed point \bar{q}_s^* is unstable for $z \in [0, \widehat{z}[$ and stable for $z \in]\widehat{z}, 1]$, where:

$$\widehat{z} = \frac{(2 + d_s)(N - 3 - d_i)}{(d_s - d_i)(N - 1)} \in]0, 1[\quad (9)$$

4 Evolutionary dynamics with Nash players

In this section, we assume that agents always play the Nash equilibrium strategy in terms of the quantity they supply, but switch production technologies based on their past performance. The adjustment of the share of firms employing a given technology will be governed by a replicator equation and provide the only source of dynamic behavior for the model analyzed in this section. We first describe the model set-up in general terms and then analyze

two alternative specifications for the replicator equation: a sluggish replicator where only a fraction of firms can change strategy in each period and a noisy replicator where all firms switch at once except for a share of firms that randomly and equiprobably mutate to one of the two technological strategies.

4.1 General set-up

To be more precise on the structure of the problem faced by firms and the timing of decision-making, in time period t , the following steps happen in order:

1. Firms pay any fixed costs associated to their newly chosen production technology (i.e. innovators pay K).
2. Firms find out the current population shares, they know z_t .
3. Firms produce the Nash equilibrium quantities corresponding to the current population shares, $q_s^N(z_t)$ and $q_i^N(z_t)$. Firms are randomly matched in groups of N . The market clears and average profits $\pi_s^N(z_t)$ and $\pi_i^N(z_t)$ are realized.
4. Update technological strategy: each firm randomly samples another firm from the firm population, if the selected firm obtained a higher profit, the updating firm imitates its R&D strategy with probability proportional to the profit difference. This determines the population shares in the next period, z_{t+1} .

A strategy revision protocol like the one described in the fourth step of the setup above can be approximated by an aggregate replicator equation describing how, for every current population state, z_t , average realized profits $\pi_s^N(z_t)$ and $\pi_i^N(z_t)$ will determine the population state in next period, z_{t+1} , see Schlag (1998); Lahkar and Sandholm (2008); Hofbauer and Sandholm (2009). Because we will explore two variants of the replicator equation, at this phase, we generically refer to it by $G(\pi_s^N(z_t), \pi_i^N(z_t), z_t)$.

Formally, the dynamic system defined above is a one-dimensional map and is fully described by the following equations:

$$\begin{aligned}\pi_s^N(z_t) &= \frac{1}{2} \left(\frac{1}{Z_t(N-1)+1} \right)^2 \frac{1}{2+d_s} \\ \pi_i^N(z_t) &= \frac{1}{2} \left(\frac{1}{Z_t(N-1)+1} \right)^2 \frac{1}{2+d_i} - K \\ z_{t+1} &= G(\pi_s^N(z_t), \pi_i^N(z_t), z_t),\end{aligned}\tag{10}$$

where $Z_t = \frac{z_t}{2+d_s} + \frac{1-z_t}{2+d_i}$.

Each firm knows the current fraction of standard firms, z_t , and sets the Nash equilibrium quantities corresponding to the current population shares and its own technology, expecting all competitors to do likewise. In this sense, firms have rational expectations because their expectation of competitors' quantity is based on the actual population shares and the realized Nash quantities. An alternative interpretation would be that the quantity and technology adjustment processes take place on different time scales with technology being updated less often while quantities have sufficient time to converge to the Nash equilibrium between subsequent technology decisions.

In the standard replicator specification the updated strategy shares depend linearly upon payoffs normalized by average population fitness. An alternative specification is the Adjusted Logit Dynamics derived in Dindo and Tuinstra (2011). While maintaining the same desirable behavioral properties, it has the advantage of generating smoother dynamics and working equally well when payoffs are negative - as may well be the case in our model. Applied to our model with only two alternative strategies, the Adjusted Logit replicator is given by:

$$G[\pi_s^N(z_t), \pi_i^N(z_t), z_t] = \frac{z_t \exp(\theta \pi_s^N(z_t))}{z_t \exp(\theta \pi_s^N(z_t)) + (1 - z_t) \exp(\theta \pi_i^N(z_t))}. \quad (11)$$

In the following subsections we explore the dynamic properties of the one-dimensional model described at the beginning of this section when the evolution of population shares is driven by one of two possible extensions of the above replicator equation.

4.2 Dynamics with sluggish replicator

We now consider a particular form of the dynamic system in (10) with a sluggish replicator equation, meaning that only a share $1 - \delta$ of the firm population updates their production technology:

$$\begin{aligned} z_{t+1} &= G_\delta(z_t) \\ &= \delta z_t + (1 - \delta) \frac{z_t \exp(\theta \pi_s^N(z_t))}{z_t \exp(\theta \pi_s^N(z_t)) + (1 - z_t) \exp(\theta \pi_i^N(z_t))}. \end{aligned} \quad (12)$$

This specification boils down to the Adjusted Logit Replicator for $\delta = 0$.

In the analysis that follows it will be handy to denote the difference between the average

profits realized by the two types of firms setting Nash quantities by:

$$\psi(z) = \pi_s^N(z) - \pi_i^N(z) = K - \frac{(2 + d_i)(d_s - d_i)(2 + d_s)}{2[(2 + d_s)(1 + d_i + N) - (d_s - d_i)(N - 1)z]^2}. \quad (13)$$

Proposition 3 sums up the dynamic properties of the evolutionary model where firms set Nash equilibrium quantities and update their production technology based on past performance. In conjunction with Proposition 1, it allows us to identify the steady state and make a global analysis of its stability.

Proposition 3. *Depending on the cost of using the innovative technology K , the dynamic system specified by equations (10) and (12) can have the following steady states and corresponding stability properties governed by the intensity of choice parameter, θ , and K :*

- (a) $z_0 = 0$, $\forall K > 0$, which is globally stable for $K < K^0$ and unstable for $K > K^0$, $\forall \theta$;
- (b) $z_1 = 1$, $\forall K > 0$, which is unstable for $K < K^1$ and globally stable for $K > K^1$, $\forall \theta$;
- (c) $z^* \in]0, 1[$, for $K^0 < K < K^1$, which is globally stable for $\theta < \hat{\theta}$ and unstable for $\theta > \hat{\theta}$, where:

$$\hat{\theta} = -\frac{2}{(1 - z^*)z^*\psi'(z^*)(1 - \delta)}.$$

Notice that the instability threshold, $\hat{\theta}$, for the interior equilibrium, z^* , as defined in Proposition 3, depends on z^* and therefore also depends on the value of K whenever $K^0 < K < K^1$. While $z_0 = 0$ and $z_1 = 1$ are always steady states of the system, they are (globally) stable if and only if they are also Nash equilibria of the static game, see Proposition 1.

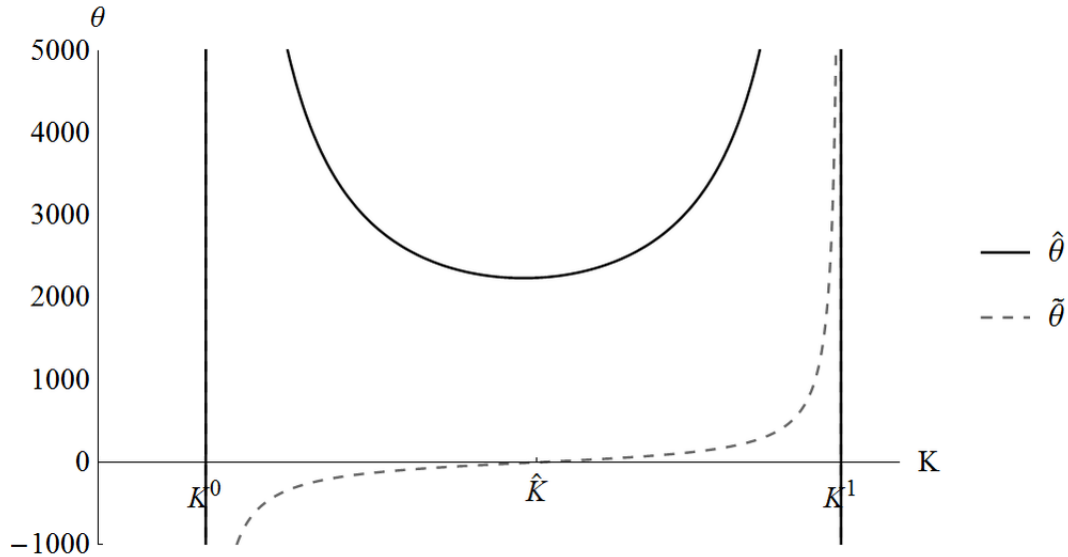
Interestingly, the stability properties of the Nash equilibrium change smoothly as we increase K , while its nature qualitatively shifts from a homogeneous to a mixed and then again to a homogenous population equilibrium. Figure 4 illustrates a nonlinear relation between on the one hand, the instability thresholds $\hat{\theta}$ and $\tilde{\theta}$ - which is the equivalent of $\hat{\theta}$ for the model with technological switching and best-response quantity dynamics, see Proposition 4 on page 32 - and the fixed cost of innovation on the other. When $K \in]K^0, K^1[$ approaches either of the interval boundaries, $\hat{\theta}$ tends to infinity. This feature is a direct result of the updating mechanism described by the replicator in (11).⁵ When only few firms of either one type are

⁵If, for instance, we would have used the standard logit updating equation which is based on best-response behavior rather than imitation of better performing strategies and, as a consequence, the exponentiated payoffs are not weighted by population shares, then we would not have had the same result of increased stability of the mixed equilibrium close to the boundaries.

present in the equilibrium population, the probability that, following a perturbation, a firm of the predominant type will encounter a more profitable firm of the scarce type to imitate its strategy is smaller compared to a situation when strategies are evenly distributed at equilibrium. This means that around an equilibrium that is closer to the boundary there will be comparatively less switching of strategies and therefore smoother readjustment towards the equilibrium following a shock. The bifurcation diagrams in Figure 6 confirm that the interior steady state, z^* , tends to be more stable when K is close to K^0 or K^1 and unstable for values of K located towards the center of the interval.

Notice also that the instability threshold, $\hat{\theta}$, is increasing in δ . Quite intuitively, the smaller the share of firms, $1-\delta$, that update their strategy the less likely will be the emergence of overshooting behavior. Therefore, we find the standard Adjusted Logit replicator to be more unstable than its sluggish extension.

Figure 4: Instability thresholds $\hat{\theta}$ and $\tilde{\theta}$ as a function of K for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$

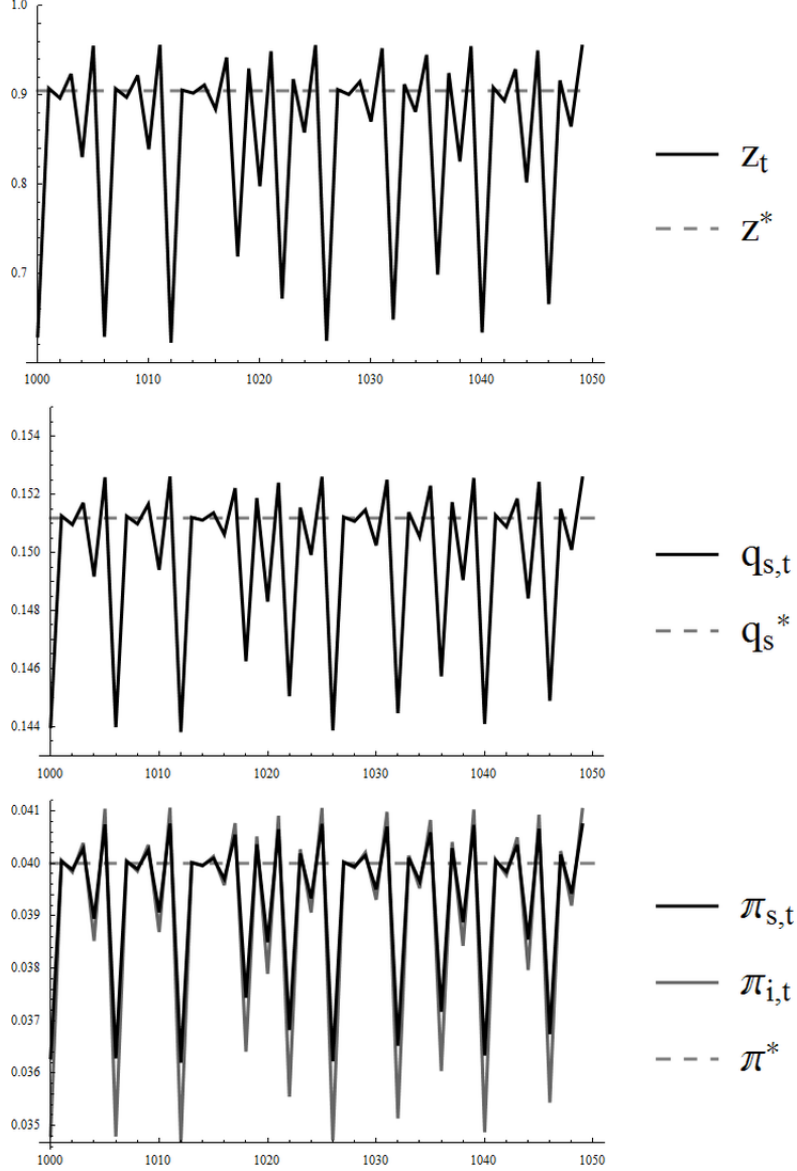


Numerical analysis⁶ confirms the analytical results above showing the market can be destabilized by overshooting in strategy adjustment, for high θ . Although all market variables, quantity, profits and strategy shares, alternate above and below the steady state, they never manage to converge to it. Inspecting the time-series plotted in Figure 5 we see that

⁶With the exception of the numerical analysis in Section 6 and the bifurcation diagrams with randomized initial conditions in Figures 17, 23 and 24, all numerical simulation results presented in this paper were powered by the unmatched speed of execution of the *E&F Chaos*, the user-friendly software for nonlinear dynamic analysis developed at CeNDEF, University of Amsterdam. For a presentation of its capabilities see Diks et al. (2008).

the economic mechanics of this result is fairly simple. When z_t is below its steady state value, as is the case in period 1000, both types of firms will underproduce compared to the mixed equilibrium output. This happens because quantities are strategic substitutes in Cournot games and expected average competing output is decreasing in z . Therefore, firms will expect competitors to produce more than \bar{Q}_{N-1}^* when $z_t < z^*$ and react accordingly by producing too little. This means that second-stage Nash profits in (10) will both be below the steady state value, but more so for innovators who will not be able to compensate their fixed innovation costs by selling a large enough output: as shown in Section 2, we will have $\pi_s^N(z_t) > \pi_i^N(z_t)$. This determines a switch towards the standard strategy, but because the intensity of choice is too high, the steady state z^* is overshoot and now the opposite is true, with $\pi_s^N(z_t) < \pi_i^N(z_t)$ determining another overzealous switch towards innovation.

Figure 5: Evolutionary strategy switching by sluggish Adjusted Logit replicator and Nash quantity setting: chaotic time series after a relaxation time of 1000 periods for the share of standard firms, z_t , standard firm output, $q_{s,t}$, and profits of standard and innovating firms, $\pi_{s,t}$ and $\pi_{i,t}$. Parameters are based on the leading numerical example, with $K = 0.016$ with $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 10,000$, and $\delta = 0.25$.



Interestingly, when starting close to the steady state, successive fluctuations increase in amplitude, this is to be expected, since the further away we are from the steady state, the higher the profit difference, see also Figure 1. However, once z_t moves further away from the

steady state⁷, fluctuations will undergo a strong dampening, and although still overshooting z^* , z_t will come very close to it before again starting to oscillate away.⁸

Comparing results for different choices of the sluggishness parameter, δ , outlines a remarkable difference between the model dynamics with the standard Adjusted Logit compared to the sluggish replicator. Figures 6 and 7 show how only a small amount of sluggishness is enough to tame the system into a less complicated pattern. According to the results depicted in Figure 8 the small pocket of unruly dynamics generated by the Adjusted Logit replicator disappear for $\delta > 0.05$. As we further increase δ , we can have, for sufficiently high θ , qualitatively different types of dynamics. This is illustrated by Figures 6, 7 and especially 8. For low levels of δ , an over-shooting two-cycle exists for a considerable range of values of $K \in]K^0, K^1[$ provided the intensity of choice, θ , is high-enough. As δ increases, cycles of higher order as well as chaotic dynamics - as displayed in Figure 5 - become possible while at the same time the range of the fluctuations becomes smaller.⁹ As δ approaches 1, stability is eventually restored. This parallels the result of Diks et al. (2013) where introducing memory in the fitness function used for strategy updating in an evolutionary model of innovation and imitation quantitatively reduced system instability (lower amplitude of fluctuations) while also increasing it in a qualitative sense (creating a bifurcation route to chaos). Although our model differs in the exact specification of fitness and of the evolutionary dynamics¹⁰, increasing δ in our model has very similar effects on the shape of the dynamic map as increasing the weight of past profits has on the one-dimensional map analyzed by Diks et al. (2013, p. 812): it simply shifts weight towards the linear increasing component of a map that, as in their paper, is “a convex combination of a linear increasing and a non-linear decreasing map”, see also Figure 12 for a graphical representation of the map for various parameter combinations.

⁷When the steady state is below $\frac{1}{2}$ a mirroring scenario ensues with larger overshooting *above* the steady state followed by a return to small but increasingly ample oscillations around the steady state.

⁸It is hard to give a convincing economic interpretation to this observation, but inspection of the dynamic map, see representation in the lower right corner of Figure 12, hints that what we observe is generated by its asymmetric tent-like shape with a flat-sloping, increasing linear component given by δz_t and the locally decreasing nonlinear component $(1 - \delta) \frac{z_t \exp(\theta \pi_s^N(z_t))}{z_t \exp(\theta \pi_s^N(z_t)) + (1 - z_t) \exp(\theta \pi_i^N(z_t))}$.

⁹Inspection of Figure 8 shows a fairly large parameter range where the system converges to a stable three-cycle, for sufficiently high θ . According to Li and Yorke (1975) this is sufficient grounds to infer that our one-dimensional system exhibits chaotic fluctuations.

¹⁰Here, a fraction δ of the population does not update its strategy whereas in their model the fitness function which evaluates the performance of a strategy is a weighted average of past profits. Translated to our notation, the evolutionary dynamics used by Diks et al. (2013) would read:

$$z_{t+1} = \frac{\exp(\theta U_{s,t})}{\exp(\theta U_s(z_t)) + \exp(\theta U_i(z_t))}$$

with $U_{j,t} = \omega U_{j,t-1} + (1 - \omega) \pi_j^N(z_t)$.

Figure 6: Evolutionary strategy switching by sluggish Adjusted Logit replicator with Nash quantity setting: bifurcation diagrams for the share of standard firms, z_t , over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 10,000$, for various levels of sluggishness (a) $\delta = 0$ (b) $\delta = 0.05$; (c) $\delta = 0.25$; (d) $\delta = 0.5$; (e) $\delta = 0.75$.

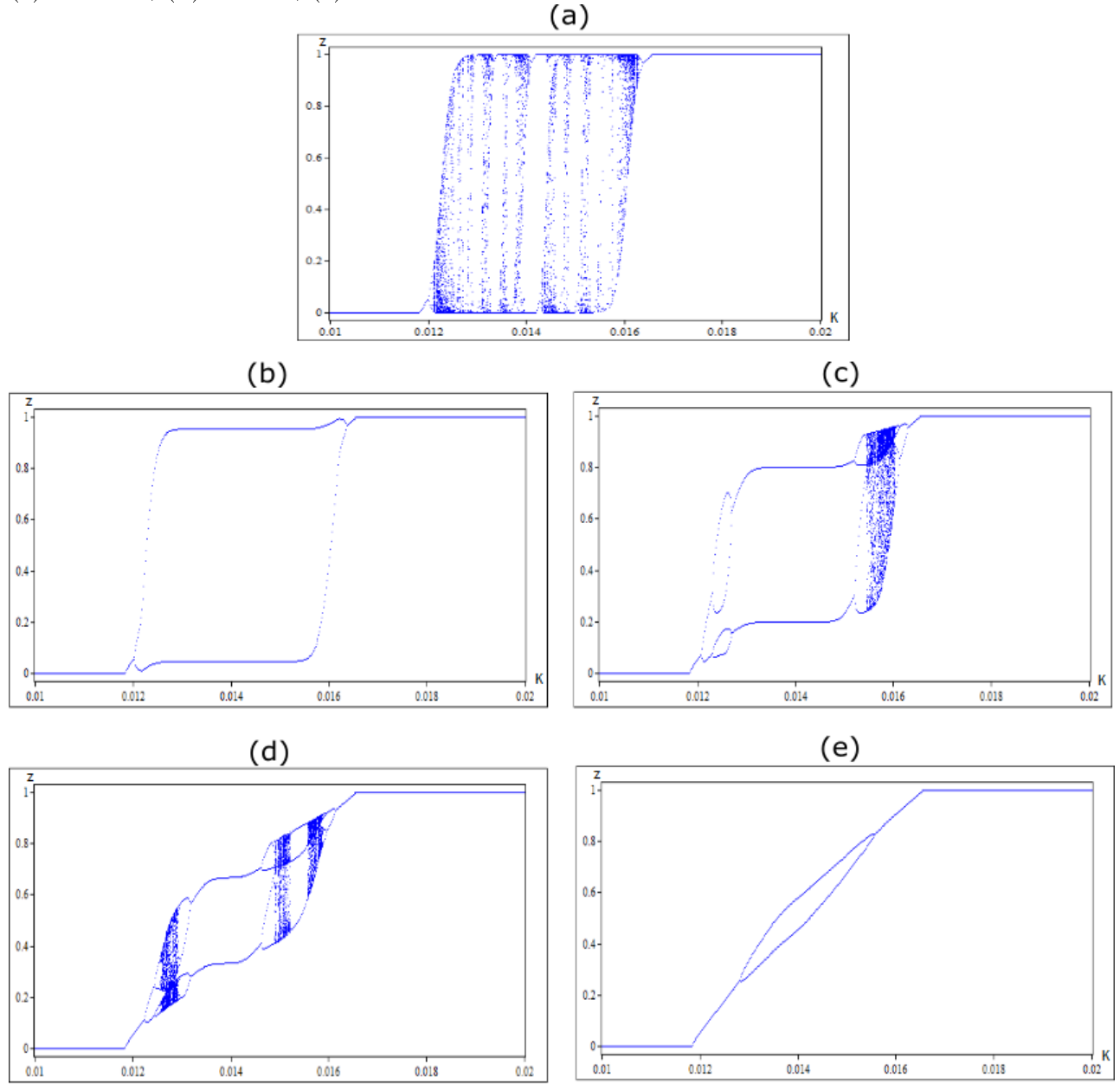


Figure 7: Evolutionary strategy switching by sluggish Adjusted Logit replicator and Nash quantity setting: bifurcation diagram in (θ, K) space for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $z_0 = 0.5$ and (a) $\delta = 0$, (b) $\delta = 0.01$, (c) $\delta = 0.10$, (d) $\delta = 0.5$.

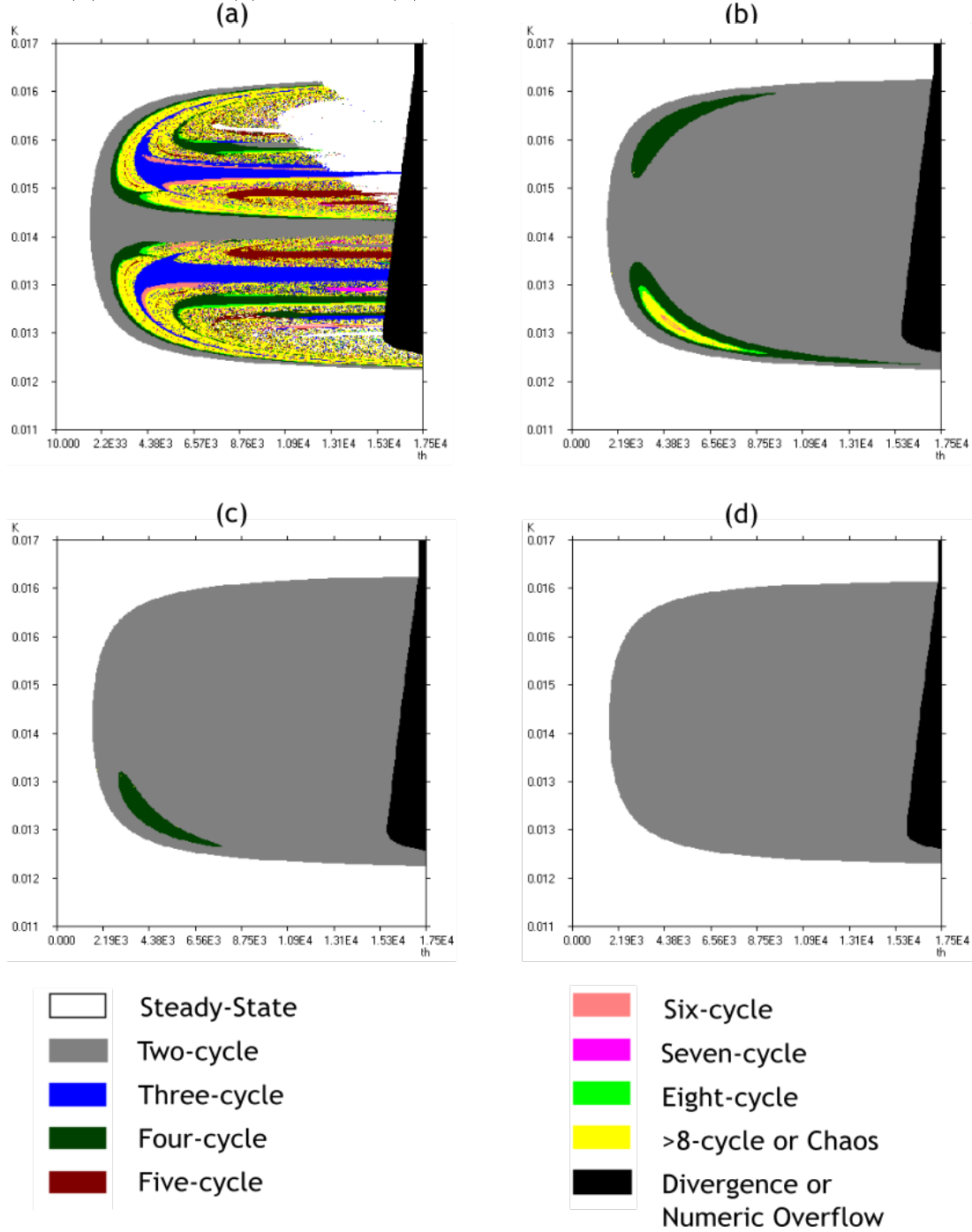
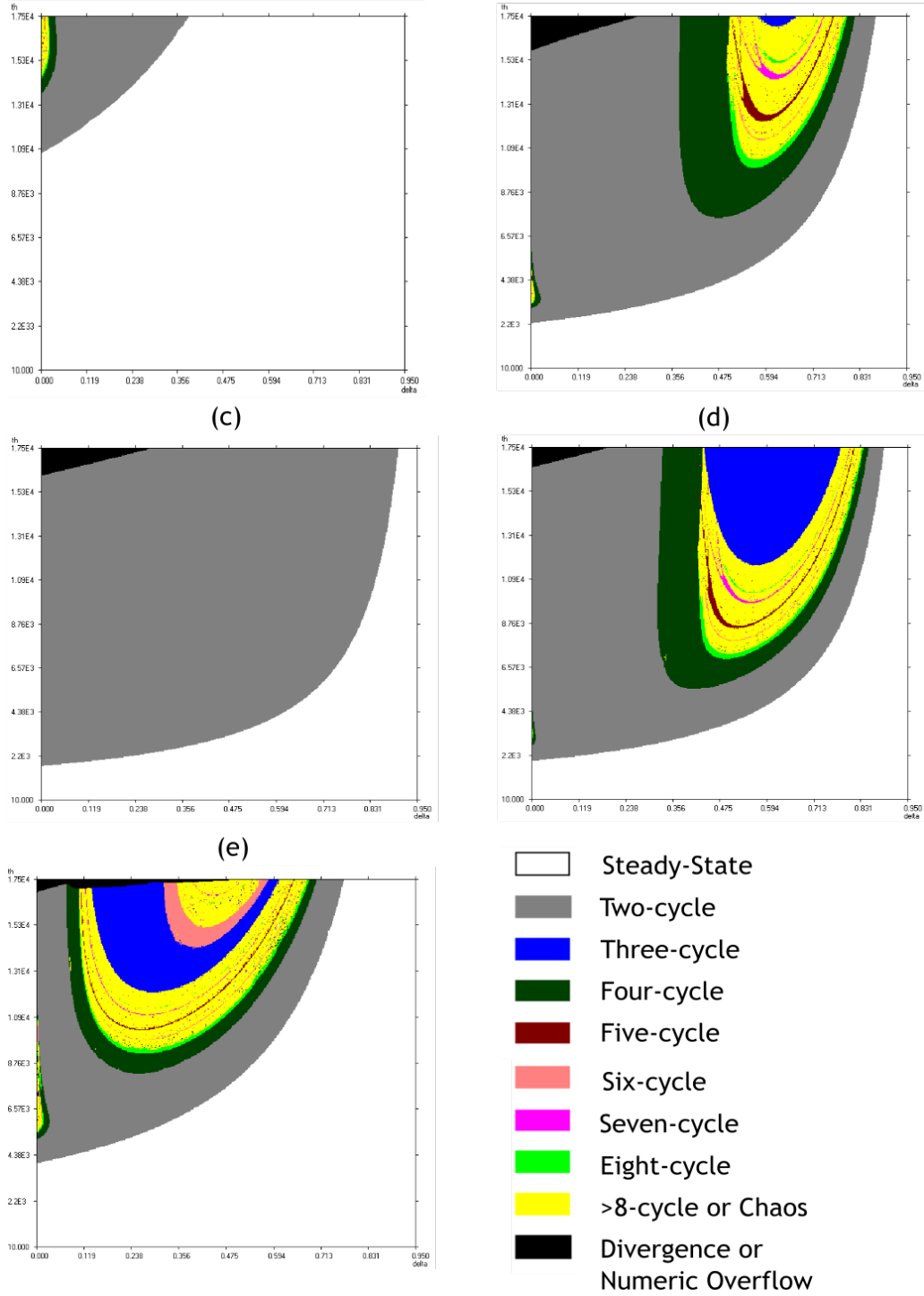


Figure 8: Evolutionary strategy switching by sluggish Adjusted Logit replicator and Nash quantity setting: bifurcation diagram in (δ, θ) space for, $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $z_0 = 0.5$ and (a) $K = 0.012$, (b) $K = 0.013$, (c) $K = 0.014$, (d) $K = 0.015$, (e) $K = 0.016$,



In examining Figure 7(a) one may be surprised to notice the upper right portion of the parameter space that seems to converge to a stable steady state according to our numerical results. This may seem even more surprising if we take into account that this apparent convergence tends to happen for higher intensity of choice, ($\theta > \sim 10000$). However, on closer inspection, we find that the system does not actually converge to the mixed Nash equilibrium steady state there, but instead it becomes “stuck” in the steady state with standard firms only, $z = 1$. As argued above, this state cannot be a stable steady state of the system as long as it is not the Nash equilibrium. What we actually observe in Figure 7(a) is a numerical issue that the standard Adjusted Logit Replicator is prone to fall prey to. When, as far as machine precision can distinguish, z_t becomes equal to 1 the replicator equation in (11) becomes a trivial equality and the system can no longer change its state. It is useful to note that while the sluggish extension to the replicator equation in (14) also becomes a trivial equality for $z_t = 1$, the fact that a portion of the firms, δ , does not update their strategy, is enough to keep the system far enough from the border so that it does not become stuck in the same way.

4.3 Dynamics with noisy replicator

An alternative to the sluggish replicator dynamic analyzed in the previous subsection is the following specification:

$$G_\rho(\pi_s^N(z_t), \pi_i^N(z_t), z_t) = \rho + (1 - 2\rho) \frac{z_t \exp(\theta \pi_s^*(z_t))}{z_t \exp(\theta \pi_s^*(z_t)) + (1 - z_t) \exp(\theta \pi_i^*(z_t))} \quad (14)$$

This specification also reduces to the Adjusted Logit replicator when we set $\rho = 0$.

Here the whole population can change strategy in every period, however only a share $1 - 2\rho$ does so according to the profitability of each strategy. The remaining, 2ρ firms take up a strategy at random, which results in the fact that there will be at least ρ firms in every period employing each strategy. This means that $z_i = \rho$ and $z_s = 1 - \rho$ are the two extreme values that z_t can take, but they can only be asymptotically approached for very high intensity of choice when one of the pure strategies is dominant, that is when $\theta \rightarrow \infty$. For $\pi_s^*(z) > \pi_i^*(z)$, $\forall z$ (i.e. $K < K^0$) we have $z \rightarrow \rho$ and conversely for $K > K^1$ we will have $z \rightarrow 1 - \rho$.

Computing the fixed points of the map can no longer be done analytically, since it would involve solving for z in:

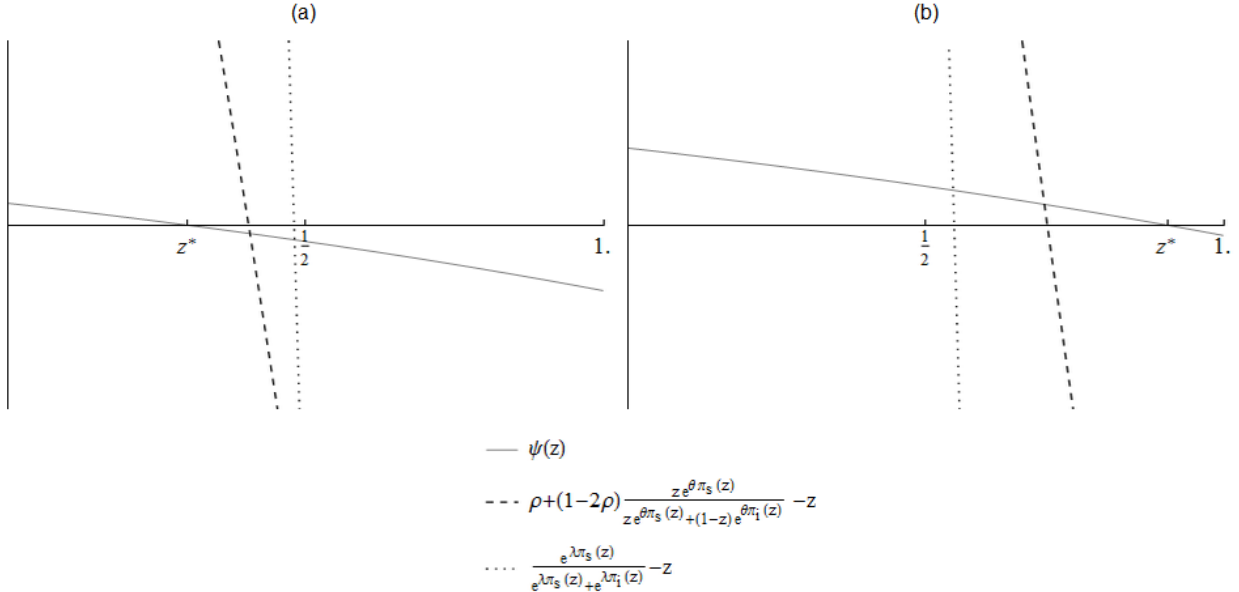
$$z = \rho + (1 - 2\rho) \frac{z \exp(\theta \pi_s^*(z))}{z \exp(\theta \pi_s^*(z)) + (1 - z) \exp(\theta \pi_i^*(z))}.$$

Notice also, that the interior equilibrium point where both types of firm make the same profits for $K^0 < K < K^1$, z^* , as defined in Proposition 1 is no longer a steady state of the dynamic system. This is obvious when either $z^* < \rho$ or $z^* > 1 - \rho$. For $\rho < z^* < 1 - \rho$, whenever $z_t = z^*$ the map in (14) boils down to:

$$z_{t+1} = z^* + \rho(1 - 2z^*)$$

meaning that only when $z^* = \frac{1}{2}$ will the dynamic map also be at a steady state. Otherwise, the steady state will be above z^* when $z^* < \frac{1}{2}$ and below z^* when $z^* > \frac{1}{2}$. Exactly how far z^* is from the steady state of the noisy replicator will also depend on the size of the noise parameter ρ .

Figure 9: Evolutionary strategy switching and Nash quantity setting: comparison of the fixed point of the noisy Adjusted Logit replicator (intersection of dashed line with the horizontal axis) against the Quantal Response Equilibrium (intersection of the dotted line with the horizontal axis). Both intersections are found between the Nash equilibrium with mixed population, z^* and $\frac{1}{2}$. Parameters: $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $K = 0.013$ in panel (a) and $K = 0.016$ in panel (b); $\rho = 0.05$, $\theta = 100$, $\lambda = 100$.

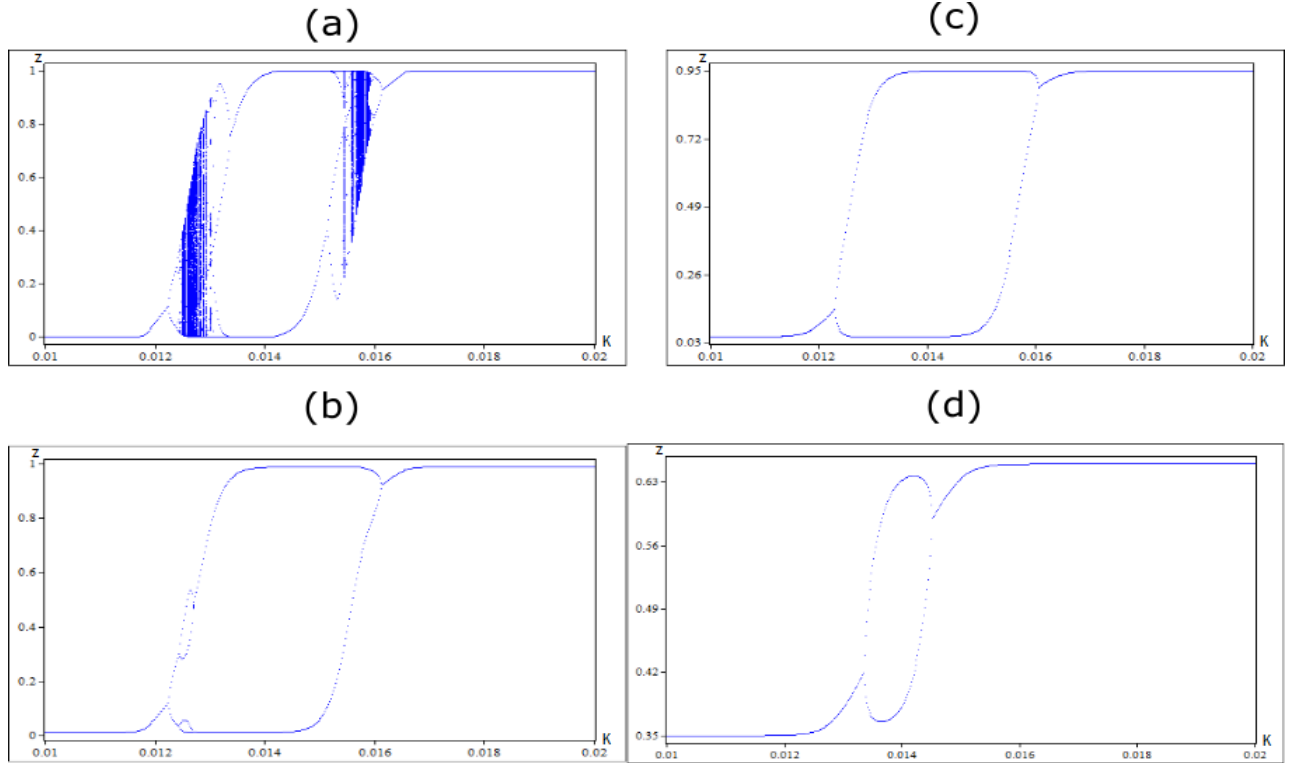


In this sense, the noisy Adjusted Logit replicator relates to the Quantal Response Equilibrium of McKelvey and Palfrey (1993). In both cases, the fixed point is shifted from the mixed

Nash equilibrium towards $\frac{1}{2}$, see Figure 9. Moreover, the extreme case where the noise parameter, $\rho = \frac{1}{2}$, leads the same outcome of random choice between the two strategies as setting the rationality parameter of their specification, λ , to zero.

In Figure 10 and 11 we can see bifurcation diagrams of the system with noisy replicator. A two-cycle typically occurs for large enough θ and when noise is very small we can also observe higher order cycles and chaotic fluctuations¹¹, however increasing ρ greatly reduces the complexity of the dynamics generated by the system in general, as well as it reduces the range of K for which the interior fixed point is unstable.

Figure 10: Evolutionary strategy switching by noisy Adjusted Logit replicator and Nash quantity setting: bifurcation diagrams over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 5,000$, for various levels of noise (a) $\rho = 0.001$; (b) $\rho = 0.01$; (c) $\rho = 0.05$; (d) $\rho = 0.35$. The variable represented is z_t after $T = 1000$ iterations.



¹¹This is most likely a legacy of the standard Adjusted Logit replicator, see also Figure 6(a).

Figure 11: Evolutionary strategy switching by noisy Adjusted Logit replicator and Nash quantity setting: bifurcation diagram in (θ, K) space for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $z_0 = 0.5$ and (a) $\rho = 0$, (b) $\rho = 0.005$, (c) $\rho = 0.01$, (d) $\delta = 0.35$.

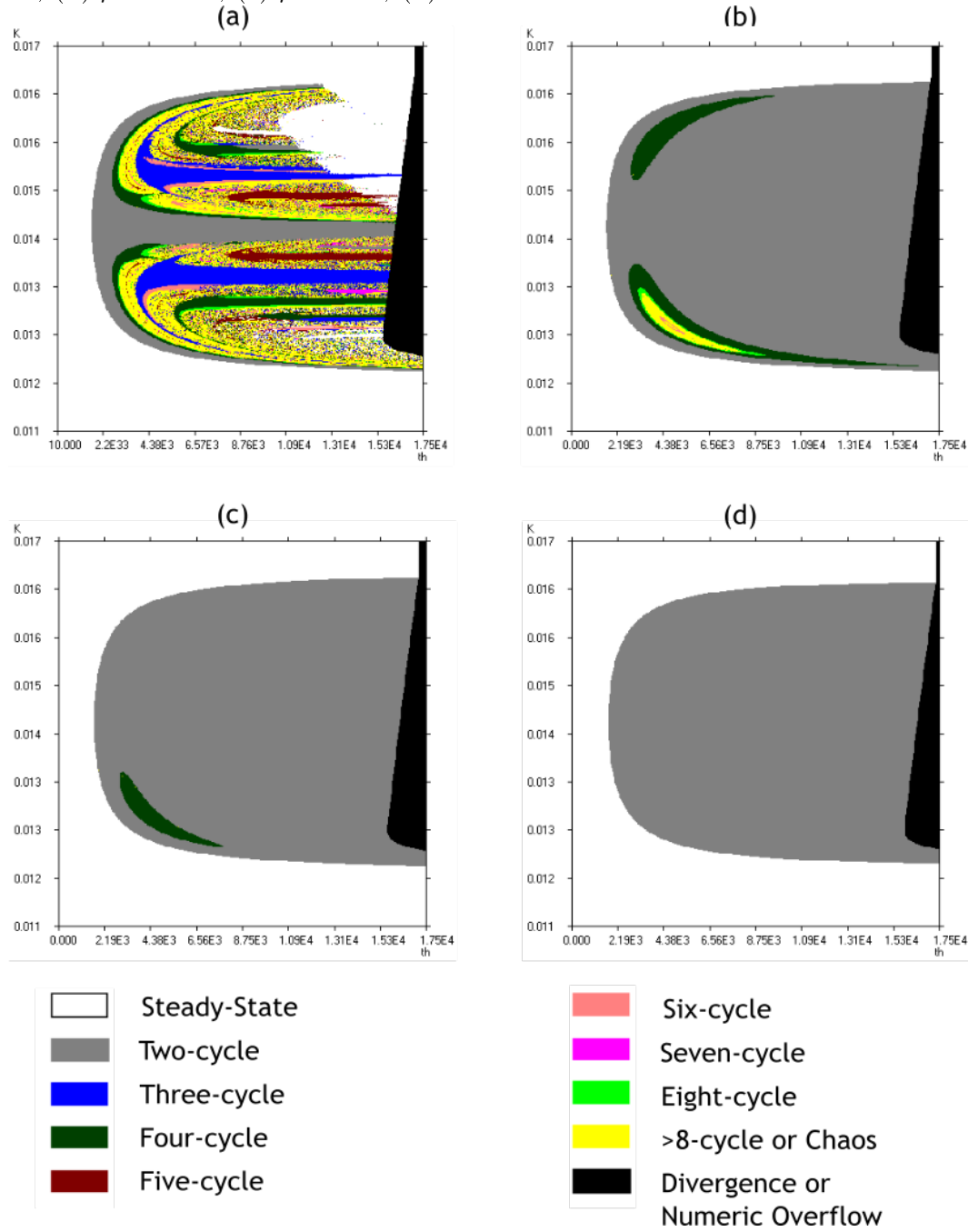
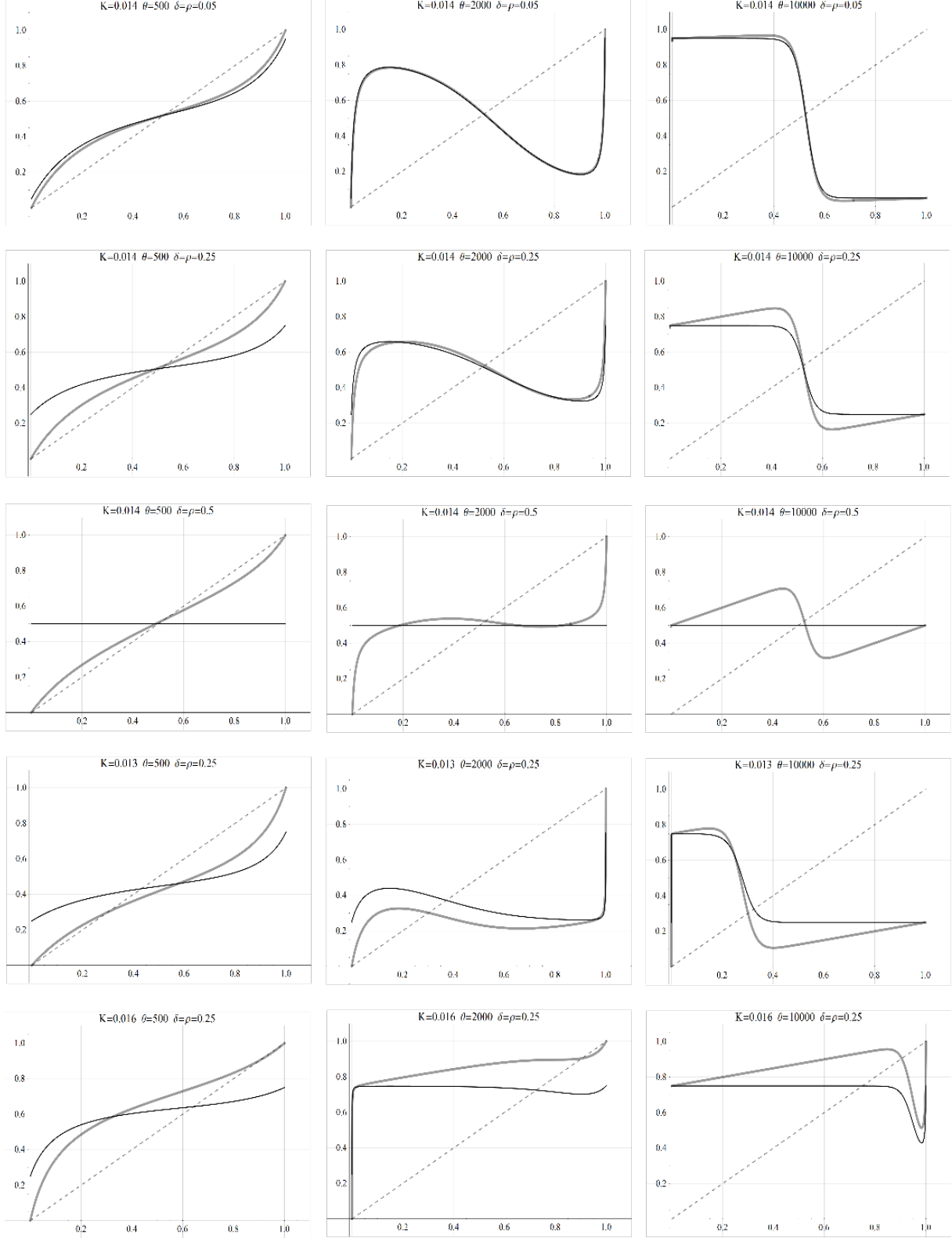


Figure 12: Evolutionary strategy switching with Nash quantities: comparing the replicator maps for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$ and various combinations of parameters K , θ , δ and ρ . Noisy replicator in black, sluggish replicator in gray.



Even though the maps of the two evolutionary processes, the noisy replicator and the sluggish replicator, bear a striking analytical and graphical resemblance, see Figure 12, the dynamics generated by them are visibly different. Most importantly, while decreasing the amplitude of fluctuations, raising the parameter ρ does not increase the complexity of the dynamics as an increase in δ does for the sluggish replicator. This is because increasing δ also affects the slope of the exterior segments of the map, located before the first inflection point and after the third inflection point. Contrarily, increasing ρ leaves these segments parallel to the horizontal axis and increases their size, making it easier for the system to converge to a stable two-cycle. In both cases, increasing δ and ρ seem to reduce the absolute value of the slope of the map at the intersection with the first diagonal - making the mixed steady state more stable - but this effect is stronger for the noisy replicator map than for the sluggish replicator one.

5 Evolutionary dynamics with best reply quantities

In this subsection, we assume that agents play the best response to the quantity they expect their competitors will supply *and* switch production technologies based on their past performance. Firm behavior, in time period t , can be broken down into four steps that take place in the following order:

1. Firms pay any fixed costs associated with their newly chosen production technology (i.e. innovators pay K).
2. Firms find out the average output in the previous period:

$$\bar{Q}_{N,t-1} = N(z_{t-1}q_{s,t-1} + (1 - z_{t-1})q_{i,t-1})$$

and, based on it, they construct their myopic expectations of how much their competitors will produce in t , $\bar{Q}_{N-1,t}^e = \frac{N-1}{N}\bar{Q}_{N,t-1}$.

3. Firms produce the best response quantity $q_{s,t} = R_s(\bar{Q}_{N-1,t}^e)$ and $q_{i,t} = R_i(\bar{Q}_{N-1,t}^e)$. They are randomly matched in groups of N . Market clearing for each group leads to the realization of firm profits in time period t .
4. Firms update their technological strategy: each firm randomly samples another from the firm population, if the selected firm obtained a higher profit, the updating firm imitates its R&D strategy with probability proportional to the profit difference.

Notice that, compared to the event sequencing in Section 4 step 2 differs in what information is known. Whereas in the model with Nash quantities firms know current period, t , population shares and produce the corresponding “second-stage” Nash quantities, here firms only know the output in the past period, $t - 1$, which incorporates information on past population shares, z_{t-1} instead of current population shares z_t . In this sense the firms of this section are not only lacking the Nash “formula” for competitively optimum production, but they also lack some of the information on which the utilization of one such formula rests.

The best reply quantity dynamics are given by:

$$q_{j,t} = R_j \left(\bar{Q}_{N-1,t}^e(z_{t-1}, q_{s,t-1}) \right) = \max \left\{ \frac{1 - (N-1) [z_{t-1} q_{s,t-1} + (1 - z_{t-1}) q_{i,t-1}]}{2 + d_j}, 0 \right\}; j \in \{i, s\}$$

As in the previous sections, the above equation implies that $q_{i,t} = q_{s,t} \frac{2+d_s}{2+d_i}$, and again, we have $q_{i,t} > q_{s,t}$; innovators always produce more than standard firms. The dynamics can therefore be modeled by a two-dimensional map specifying best-response quantity dynamics and evolutionary dynamics for the population shares.

In explicit term, the dynamical system under investigation is given by:

$$\begin{cases} q_{s,t} = R_s \left(\bar{Q}_{N-1,t}^e(z_{t-1}, q_{s,t-1}) \right) = \max \left\{ \frac{1}{2 + d_s} - (N-1) \left[\frac{z_{t-1}}{2 + d_s} + \frac{1 - z_{t-1}}{2 + d_i} \right] q_{s,t-1}, 0 \right\} \\ z_t = G(\pi_{s,t-1}, \pi_{s,t-1}; z_{t-1}) \end{cases} \quad (15)$$

where

$$\pi_{s,t-1}(z_{t-1}, q_{s,t-1}) = \left[1 - q_{s,t-1} - (N-1) \left[z_{t-1} + (1 - z_{t-1}) \frac{2 + d_s}{2 + d_i} \right] q_{s,t-1} - \frac{1}{2} d_s q_{s,t-1} \right] q_{s,t-1} \quad (16)$$

and

$$\pi_{i,t-1}(z_{t-1}, q_{s,t-1}) = \left[1 - q_{i,t-1} - (N-1) \left[z_{t-1} + (1 - z_{t-1}) \frac{2 + d_s}{2 + d_i} \right] q_{s,t-1} - \frac{1}{2} d_i q_{i,t-1} \right] q_{i,t-1} - K \quad (17)$$

are firm profits realized in $t - 1$ and $G(\pi_{s,t-1}, \pi_{s,t-1}; z_{t-1})$ generically specifies how current population shares depend on past profits.

As we did in the previous section, it is convenient to denote the profit differential of the two strategies by:

$$\Psi(z, q) = \pi_s(z, q) - \pi_i(z, q). \quad (18)$$

Notice that, in contrast to $\psi(z)$ from (13), $\Psi(z, q)$ is now a function of two variables that drive the dynamics of our model.

5.1 Dynamics with sluggish Replicator

We first investigate the steady states of (15) and their stability when:

$$G_\delta(\pi_{s,t}, \pi_{i,t}, z_t) = \delta z_t + (1 - \delta) \frac{z_t \exp(\theta \pi_{s,t})}{z_t \exp(\theta \pi_{s,t}) + (1 - z_t) \exp(\theta \pi_{i,t})}. \quad (19)$$

As in the previous section, see Proposition 3, the population structures with homogeneous technology, z_0 and z_1 - with corresponding values for q_s are steady states of the two-dimensional system as well. Therefore for all $K^0 < K < K^1$ we have three steady states:

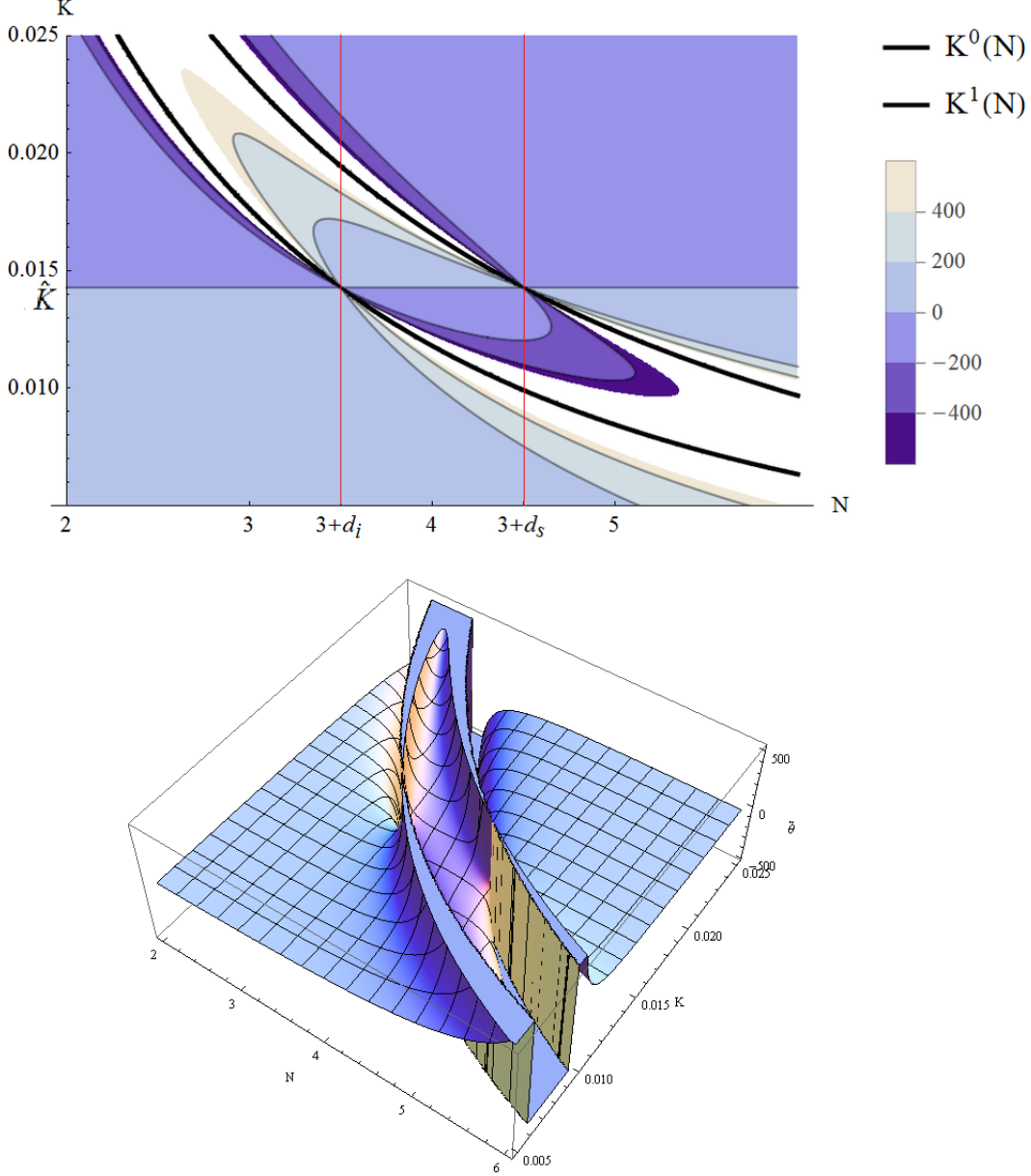
$$(z_{SS}, q_{SS}) \in \left\{ \left(0, \frac{2 + d_i}{(2 + d_s)(1 + d_i + N)} \right), \left(1, \frac{1}{(1 + d_s + N)} \right), \{z^*, q^*\} \right\},$$

where z^* is defined by Proposition 1. When there is no mixed population Nash equilibrium, $\{z^*, q^*\}$, the dynamic system will have only two steady states.

As long as they do not coincide with the Nash equilibrium, the border steady states with homogenous population cannot be stable. The homogeneous populations states are fixed points of the map only due to the population weighted specification of the replicator, while profits of the two types of firms are unequal with $\pi_s \left(0, \frac{2 + d_i}{(2 + d_s)(1 + d_i + N)} \right) > \pi_i \left(0, \frac{2 + d_i}{(2 + d_s)(1 + d_i + N)} \right)$ and $\pi_s \left(1, \frac{1}{(1 + d_s + N)} \right) < \pi_i \left(1, \frac{1}{(1 + d_s + N)} \right)$. Therefore, for any small perturbation in population shares the replicator equation will drive the system away from the border.

As for the stability of the steady state with a mixed population, we can establish necessary and sufficient conditions for stability. The interior equilibrium where both types of firms are present loses stability either for a sufficiently high intensity of choice - technological instability, or may never be stable when the quantity best-reply function is unstable due to high N - production instability. Proposition 4 establishes necessary and sufficient conditions for technological instability as long as the system is not production unstable.

Figure 13: Contour Plot of $\tilde{\theta}$ in (N, K) space with 3D representation below. Parameters: $d_s = 1.5$, $d_i = 0.5$, $\delta = 0.25$.



Proposition 4. *If a Nash equilibrium with mixed firm population, (z^*, q^*) , exists, it is a stable steady state of the system in (15) with replicator specification (19) iff. $\theta < \tilde{\theta} = \frac{2(N-1)(2+d_i)(2q_s^*(2+d_s)-1)}{(1-\delta)(2+d_s)q_s^*[(2+d_s)(1+d_i+N)q_s^*-2-d_i][1-(1+d_s+N)q_s^*]}$*

As θ is an intensity of choice parameter, it should be constrained to $\theta > 0$, otherwise firms will switch towards the less profitable strategy. Notice that from the expression of $\tilde{\theta}$, we

cannot be certain it is always positive. It is arranged to display only positive factors with the exception of the expression $2q_s^*(2 + d_s) - 1$, which is of ambiguous sign. On closer inspection we obtain the following corollary which is essentially a study of whether the threshold $\tilde{\theta}$ is positive or negative.

Corollary 2. *Depending on the relation between parameters, N , K , d_s and d_i the stability of the equilibrium with mixed firm population fall in one of the following three cases:*

- (a) *When $N > 3 + d_s$, the mixed equilibrium is never stable ($\tilde{\theta} < 0$);*
- (b) *When $N < 3 + d_i$, the mixed equilibrium can be stable for a sufficiently low, but positive, intensity of choice, $0 < \theta < \tilde{\theta}$;*
- (c) *When $3 + d_i < N < 3 + d_s$, there is a threshold, $\hat{K}(\theta)$ such that the mixed equilibrium will be stable for a sufficiently low intensity of choice, $0 \leq \theta < \tilde{\theta}$ if $K > \hat{K}$ and never stable for $K < \hat{K}$. For the limit case $\theta = 0$ we have $\hat{K}(\theta) = \frac{d_s - d_i}{8(2 + d_s)(2 + d_i)}$*

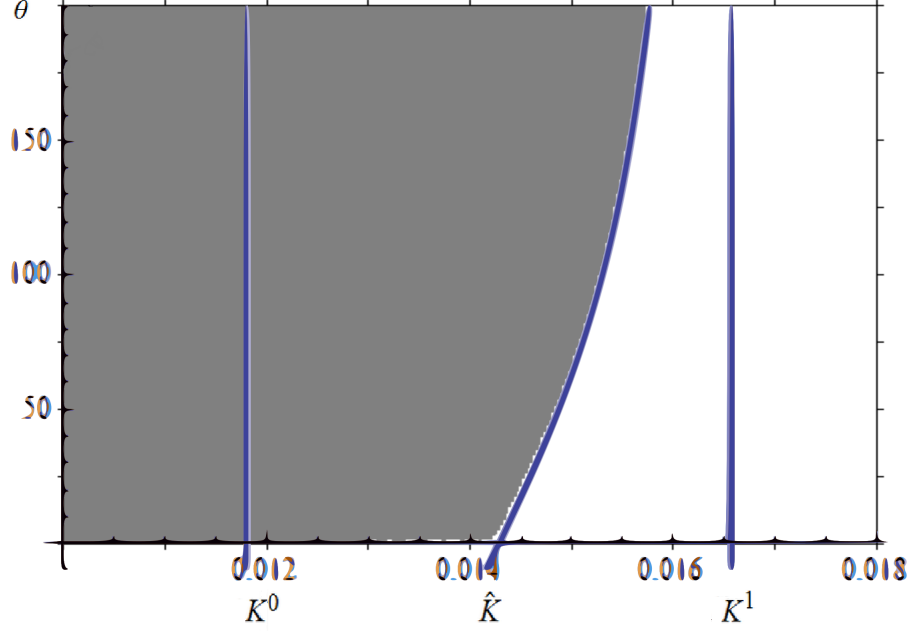
Figures 13, 14 and 15 illustrate Proposition 4 and Corollary 2 focusing on our leading numerical example. In Figures 15 and 14 we use both numerical simulation and the analytical results derived above for verification. Notice that our leading numerical example falls under case (c) of Corollary 2. In inspecting Figure 13 one should keep in mind that only the region between the curves $K^0(N) = \frac{(2 + d_i)(d_s - d_i)}{2(2 + d_s)(N + 1 + d_i)^2}$ and $K^1(N) = \frac{(2 + d_s)(d_s - d_i)}{2(2 + d_i)(N + 1 + d_s)^2}$ characterizes the stability of the mixed equilibrium. Below $K^0(N)$ the unique Nash equilibrium consists of innovative firms only while above $K^1(N)$ we only have standard firms. The boundaries N^0 and N^1 in Figure 15 refer to the positive inverses with respect to N of $K^0(N)$ and $K^1(N)$ respectively.

The results summarized by Corollary 2 above correspond to Proposition 2 discussed in Section 3. It is interesting to compare the threshold \hat{K} with the threshold \hat{z} that we identified in Section 3, where the only source of dynamics was production best-reply, by comparing to \hat{z} defined in equation (9). It turns out that we have precisely:

$$z^*(\hat{K}) = \hat{z}. \quad (20)$$

Comparing Corollary 2 against Proposition 2 we can conclude that technology choices do not alter the stability properties of the system on the dimension capturing production instability via parameter N . The boundary case of no technological instability, $\theta = 0$, corresponds exactly to the model with exogenously given population shares. This is not surprising, since

Figure 14: Bifurcation diagram in (K, θ) space with $\tilde{\theta}$ superimposed. Parameters: $d_s = 1.5$, $d_i = 0.5$, $N = 4$, $\delta = 0.25$. The bifurcation diagram separates regions of the parameter space converging to a stable steady state (white) from regions converging to a two cycle (gray) after 10000 iterations from system initialization very close to the steady state: $(q_{s,t=0}, z_{t=0}) = (q_s^* \pm 10^{-5}, z^* \pm 10^{-5})$.



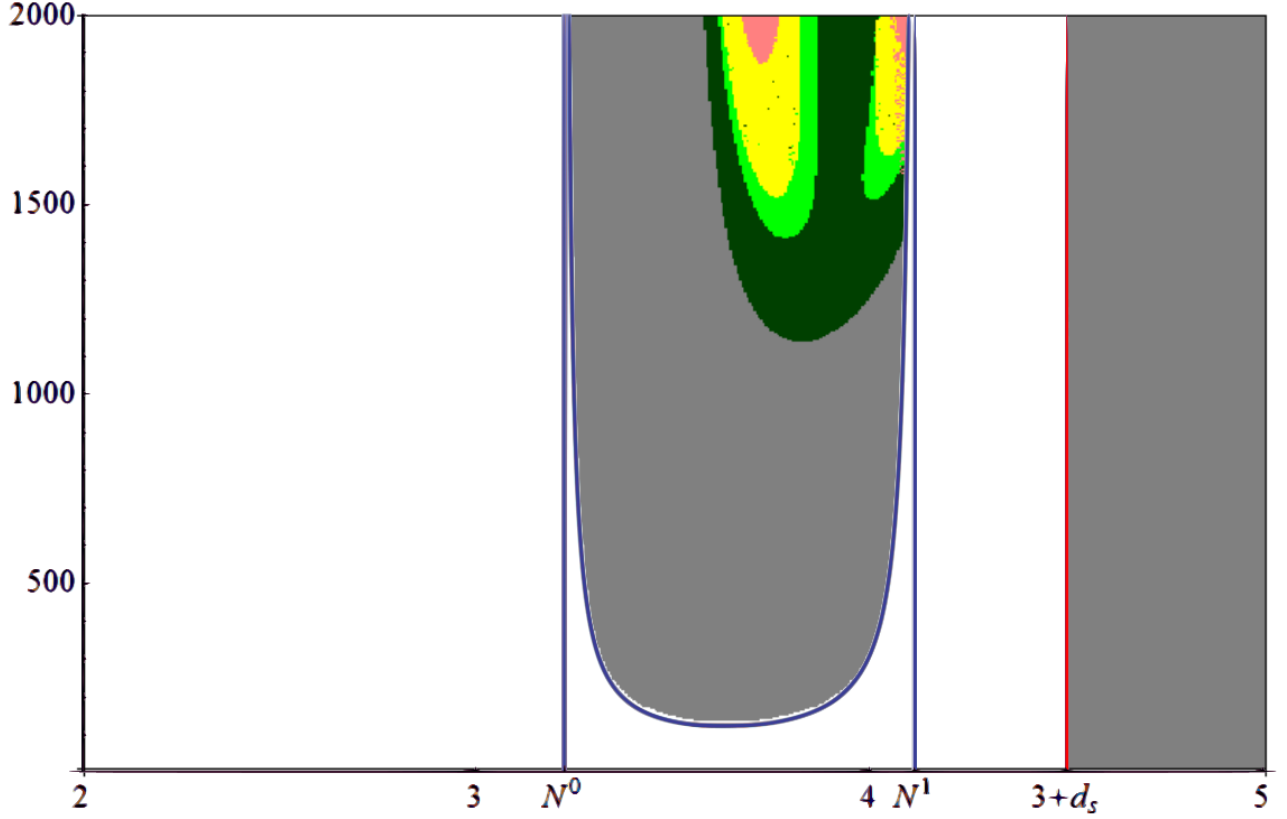
(15) collapses to only one equation for quantity dynamics with constant $z_t = z_{t=0}$, when the sluggish Adjusted Logit replicator with $\theta = 0$ is plugged in for $G(\pi_{s,t-1}, \pi_{s,t-1}; z_{t-1})$.

By comparing $\tilde{\theta}$ defined in the above Proposition and $\hat{\theta}$, derived in Proposition 3 we obtain the following corollary result:

Corollary 3. *It always holds that $\hat{\theta} > \tilde{\theta}$.*

This means that technological instability of the mixed equilibrium is always amplified when firms set best response quantities compared to when they set Nash quantities. This is illustrated in Figure 4 for our leading numerical example.

Figure 15: Bifurcation diagram in (N, θ) space with $\tilde{\theta}$ superimposed . Parameters: $d_s = 1.5$, $d_i = 0.5$, $K = 0.016$, $\delta = 0.25$. The bifurcation diagram displays system convergence after 10000 iterations from system initialization very close to the steady state: $(q_{s,t=0}, z_{t=0}) = (q_s^* \pm 10^{-5}, z^* \pm 10^{-5})$.



5.2 Noisy Replicator

The two dimensional system under consideration here is:

$$\begin{cases} q_{s,t+1} = R_s(Q_{t+1}^e) = \max \left\{ \frac{1}{2+d_s} - (N-1) \left[\frac{z_t}{2+d_s} + \frac{1-z_t}{2+d_i} \right] q_{s,t}, 0 \right\} \\ z_{t+1} = \rho + (1-2\rho)z(t) \frac{\exp(\theta\pi_{s,t})}{z(t)\exp(\theta\pi_{s,t}) + (1-z_t)\exp(\theta\pi_{i,t})} \end{cases} \quad (21)$$

Much like in Section 4.3 we cannot solve analytically for the steady states of the system when the evolution of population shares is governed by the noisy replicator equation, nor can we make any formal statements about their stability. However, numerical simulations of the above equations can provide a robustness check, at least in qualitative terms, of the results obtained for the sluggish replicator.

5.3 Numerical Analysis

5.3.1 Sluggish Replicator

Figure 17 shows bifurcation diagrams of the sluggish replicator model over parameter K for different values of δ . We notice that this time around, the amount of sluggishness in strategy adjustment has an ambiguous effect on the complexity of the dynamics. An increase from $\delta = 0.05$ to $\delta = 0.25$ eliminates all cycles of higher period than 4, but a further increase to $\delta = 0.5$ supports even chaotic dynamics. Also, δ seems to no longer have an impact on the range of fluctuations as it did in the model with Nash quantities.

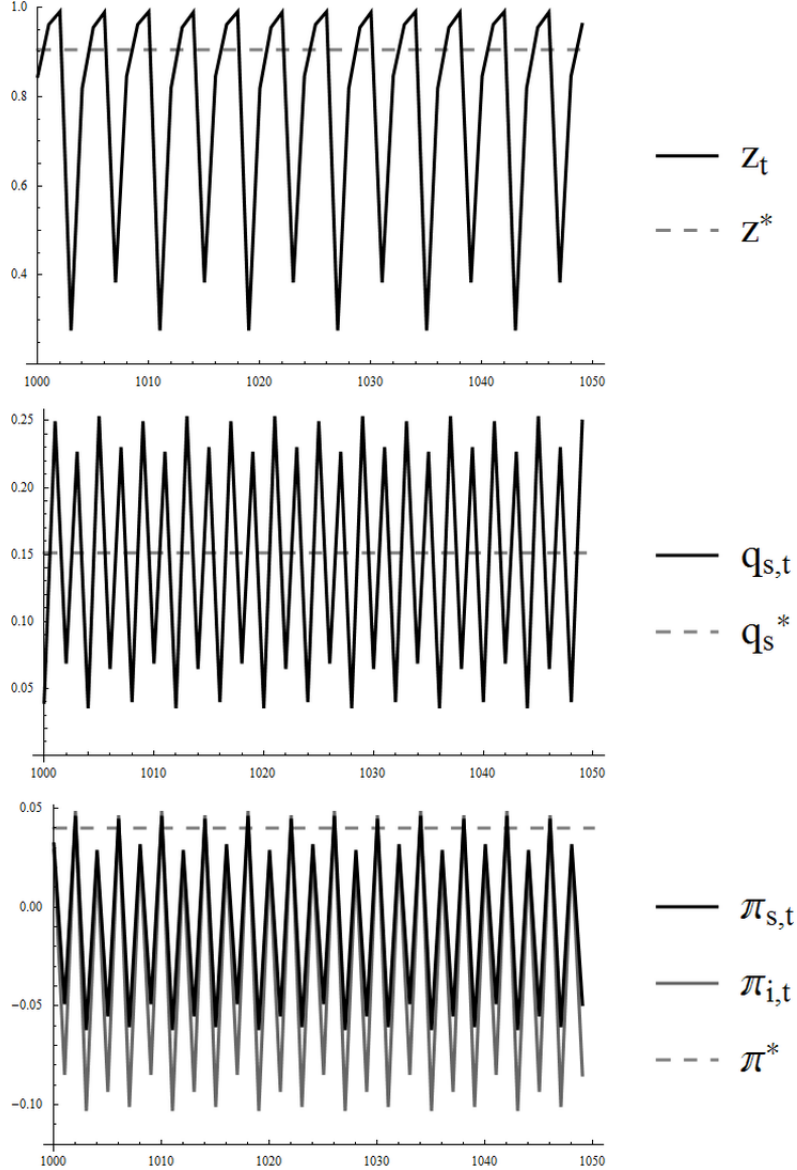
Inspecting the time series generated by the model in Figure 16 we first notice that market variables no longer gravitate around the steady state, as was the case with Nash output in Section 4. Comparing the times-series of profits with those of quantities and share of standard firms respectively we notice that quantity variation seems to account for much more of the profit variation than does the dynamics of the share of standard firms. Still, the role of population shares is not negligible: it is through the dynamics of z_t that quantity dynamics, in contrast to the results of pure best-reply dynamics of Section 3, are bounded away from the two-cycle generated by the non-negativity constraint we imposed on the best-reply function, yet without converging to the steady state. According to Corollary 2 we are in a parameter region where, in the absence of technological switching, the market would converge to the steady state mixed equilibrium. We actually do observe a dampening of the fluctuations of $q_{s,t}$ for the first four periods plotted, but, because the intensity of choice is so high, as soon the profit of being an innovator becomes even slightly higher than the profits of the standard strategy, as is the case in $t = 1002$, the share of standard firms subsequently plummets below the threshold¹² \hat{z} in $t = 1003$. This then leads to the wider fluctuation in $q_{s,t}$ in period $t = 1004$. This pattern then repeats over the next four periods with only slight variation in the exact values of market variables.

Comparing, Figure 18 with Figure 11 we first notice that the upper and lower bounds on fixed innovation costs, K^1 and K^0 no longer play the same clear cut role in distinguishing between unstable regions and parameter combinations converging to a homogenous population equilibrium. We are not surprised to see this for $K < K^0$ since our analytical results already clarified that the steady state with only innovating firms cannot be stable, even when it coincides with the Nash equilibrium because of production instability. Even for $K > K^1$ the diagrams often show non-convergence to the steady state in spite of the fact that we know the steady state with standard firms is locally stable. What we find instead, as we

¹²See equation(20).

show in more detail below, is that initial conditions are important in determining whether or not the system will converge to this steady state.

Figure 16: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: eight-cycle time series after a relaxation time of 1000 periods for the share of standard firms, z_t , standard firm output, $q_{s,t}$, and profits of standard and innovating firms, $\pi_{s,t}$ and $\pi_{i,t}$. Parameters are based on the leading numerical example, with $K = 0.016$ with $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 3200$, and $\delta = 0.25$.



When population dynamics are driven by the standard Adjusted Logit Replicator, $\delta = 0$, the model has the same problem we saw in the previous section: it becomes stuck in the

homogeneous population state with standard firms because z_t moves too close to its upper bound for machine precision to be able to distinguish it from 1. The difference here is that this can happen for much smaller θ . This is because quantity fluctuations can create much larger gaps between the profits of the two strategies, especially in favor of the standard firms. Another remarkable difference is the presence of a limit cycle for small δ and θ , see the smaller yellow region that slashes into the gray two-cycle area in panels (b) and (c) of Figure 18. This cycle is similar to the limit cycle that separates the two-cycle from the region that converges to the steady state with standard firms only in panel (a) of Figure 18.

Figure 19 displays phase diagrams of the system in the (q_s, z) space, for different values of K . When the innovative technology is cheap (e.g. $K = 0.005$), q_s runs a 4-cycle that hits the lower boundary¹³ of 0 output every 4 periods. As we increase K , a period-doubling bifurcation occurs, then a strange attractor with fractal structure appears, see Figure 20, on and off, until at $K = 0.01248$ it breaks off into a 12-cycle and then a 6-cycle at $K = 0.01412$.

At $K = 0.0162$ the same complicated attractor emerges again and for $K = 0.01748$ the attractor settles back into an 8-cycle which goes through consecutive period halving bifurcations at $K = 0.0175$ and $K = 0.01795$. At $K = 0.0194$ the two-cycle loses stability and the only attractor left is the stable steady state that had emerged at $K = K^1$.

In Figure 17, the initial values for z and q_s are random draws, so multiple attractors can be present in each diagram. This is indeed the case for both $\delta = 0.05$ and $\delta = 0.25$ when $K > K^1$. While $(z^*, q^*) = (1, 0.1538)$ becomes a stable steady state¹⁴ it is only locally stable and most initial conditions actually converge to the complex attractor, see Figure 21(a). Moreover, for $\delta = 0.05$ a second complex attractor appears for $0.0181 < K < 0.0183$ as can be better seen from the zoom-in in panel (b) of Figure 17 and in Figure 21(b). Interestingly, when the two complex attractors coexist their basins of attraction are fractal. Two complex attractors with fractal basins also coexist at $K = 0.01624$ one is a 6-cycle the other is a higher order cycle, see Figures 19 and 22.

¹³Up to $K = 0.0112$, the attractors constantly hit the 0 output bound we impose on the best-reply function in (15). For higher K this no longer happens and the quantity dynamics generated by system 15 are intrinsically bounded. Encouragingly, whether or not the extra restriction $q_s \geq 0$ is binding the attractor or limit cycle looks very similar.

¹⁴The best-response dynamics is locally stable for a population consisting of only standard firms and standard firms make higher profits than innovative firms for $K > K^1$.

Figure 17: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: bifurcation diagrams for q_s over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\delta = 0.05$, (a) and (b) panels $\delta = 0.25$ in panel (c) and $\delta = 0.5$ in panel (d) and. Initial values $(z_{t=0}, q_{s,t=0})$ are drawn from a bivariate uniform distribution with support on $[0, 1] \times [0, 0.3]$.

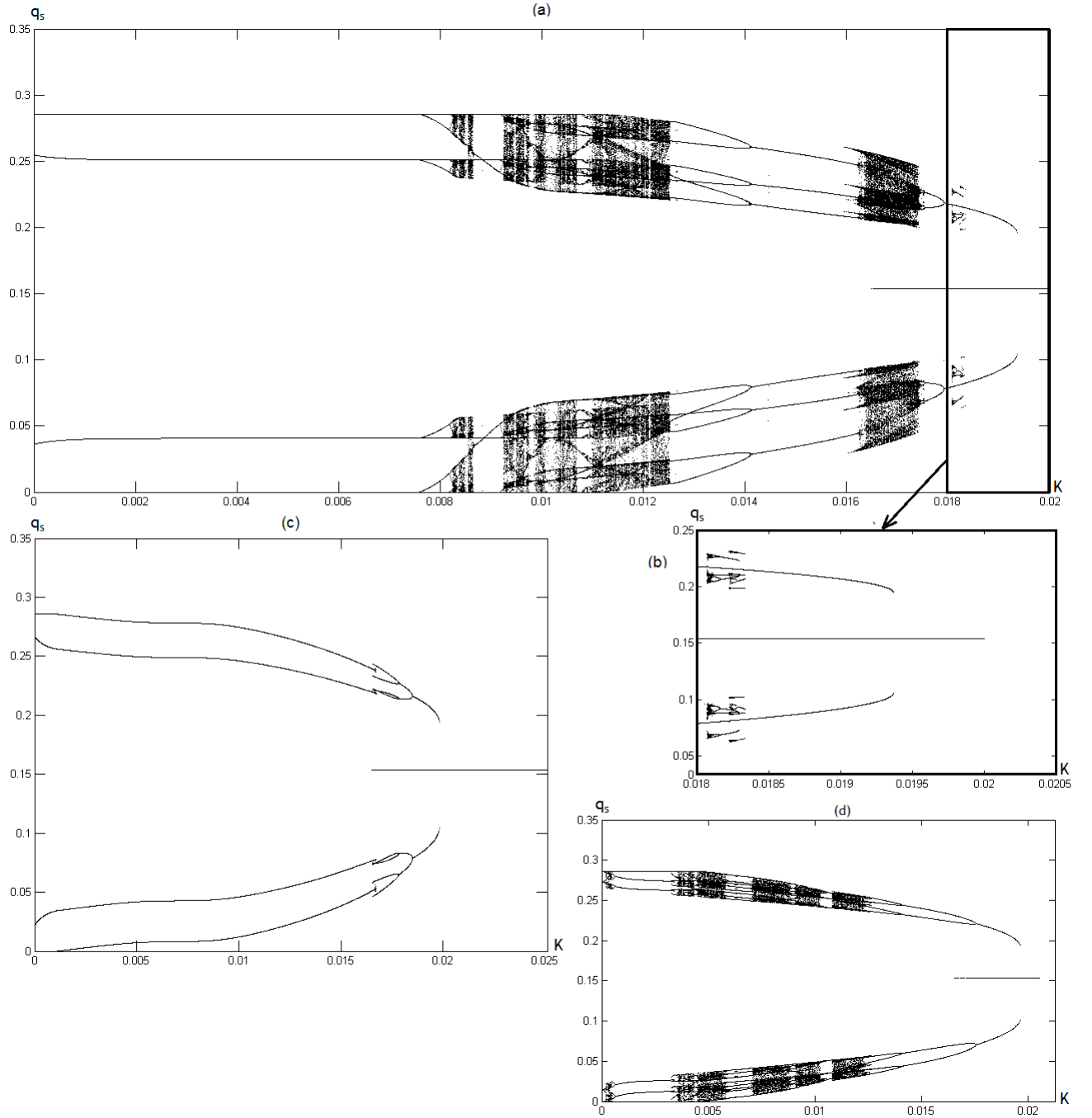


Figure 18: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: bifurcation diagram in (θ, K) space for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $(z_{t=0}, q_{s,t=0}) = (0.5, 0.15)$ and (a) $\delta = 0$, (b) $\delta = 0.01$, (c) $\delta = 0.1$, (d) $\delta = 0.25$, (e) $\delta = 0.5$.

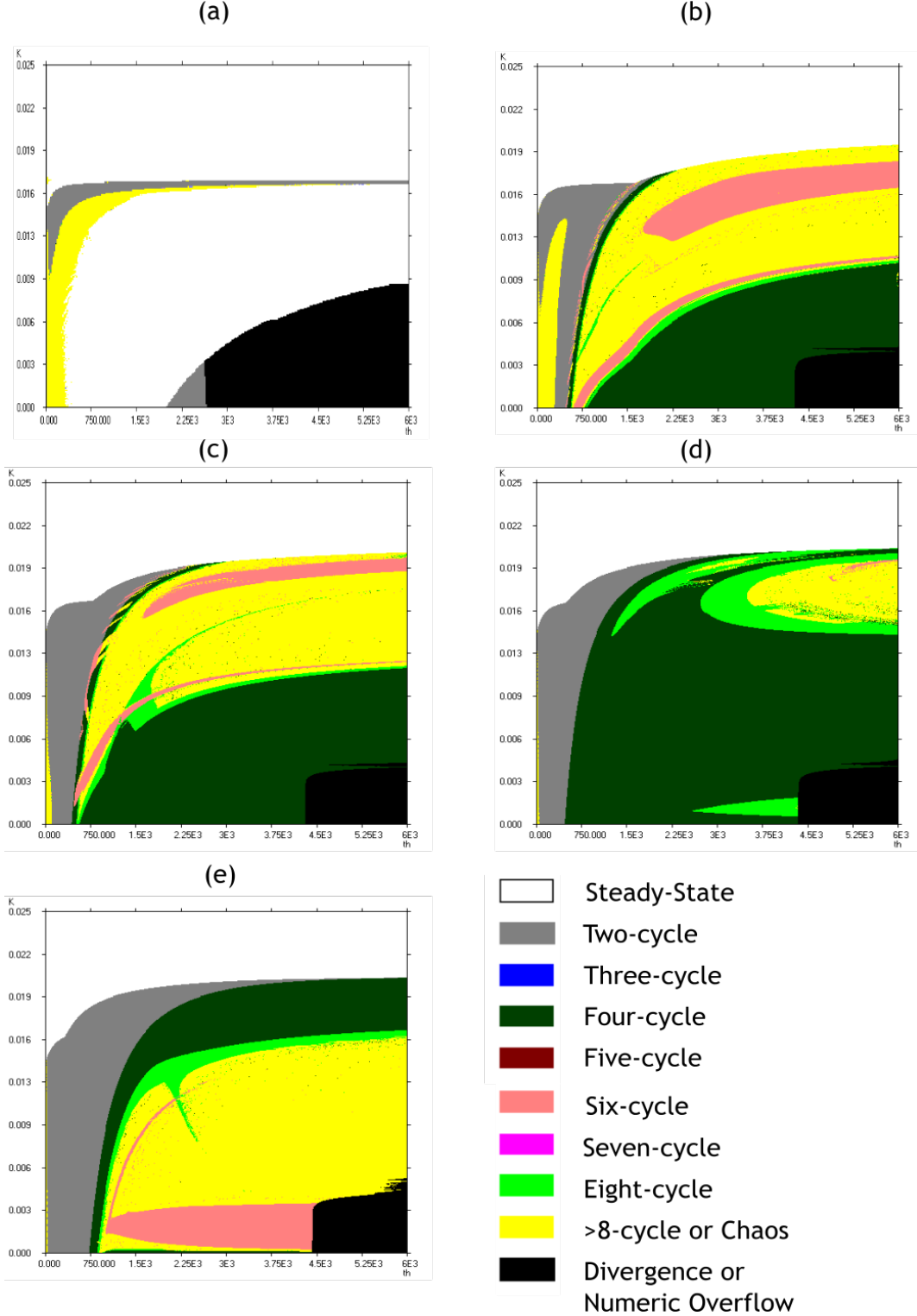


Figure 19: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: phase diagrams for $(q_s(t), z(t))$ with quantity on the horizontal axis and share of standard firms on the vertical axis. $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\delta = 0.05$

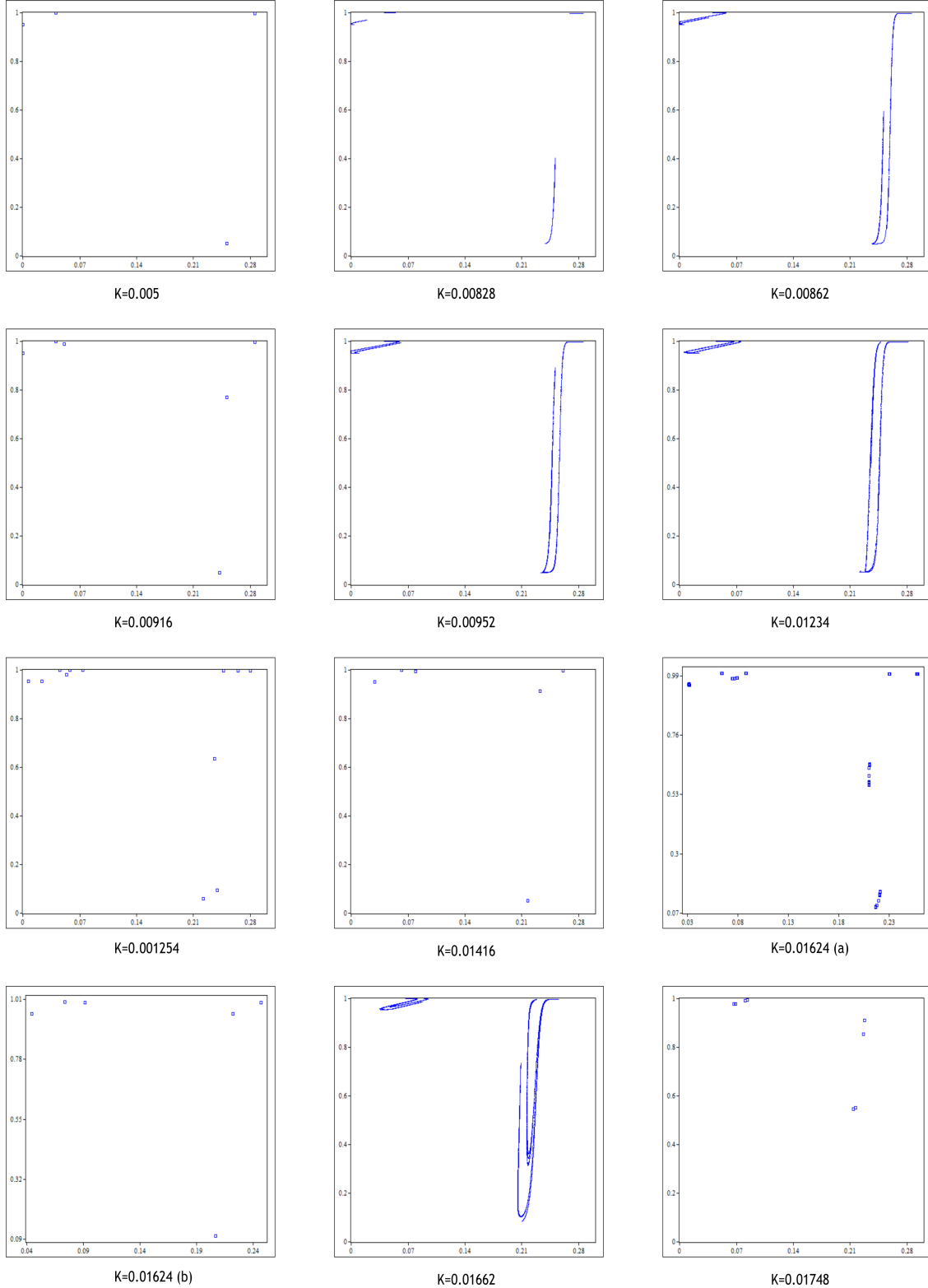
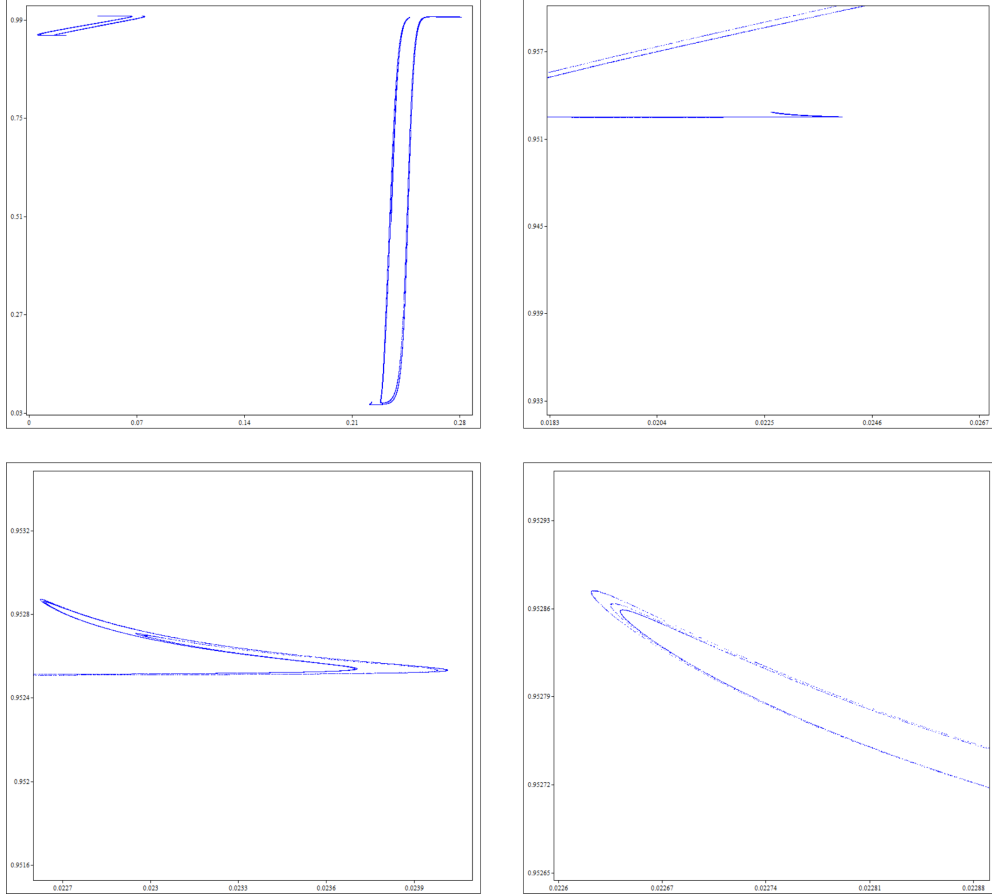


Figure 20: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: zoom-in reveals fractal structure of strange attractor. Phase diagram for $(q_s(t), z(t))$ with quantity on the horizontal axis and share of standard firms on the vertical axis. $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\delta = 0.05$, $K = 0.01234$



5.3.2 Noisy Replicator

Figures 23 and 24 show bifurcation diagrams for the two dimensional system with mutation in (21). Compared to the system with sluggish adjustment, the dynamics seem more tame. In particular there is no coexistence of complex attractors, but we do still observe, as shown by Figure 25, the coexistence of a nontrivial attractor and the locally stable steady state for $K > K^1$ with basins of attraction very similar to those plotted in Figure 21. Instead of the strange attractor observed with sluggish adjustment, the most complicated dynamics generated with the noisy replicator is a limit cycle, see Figure 26.

Figure 21: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: co-existing attractors. Basins of attraction for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\delta = 0.05$, (a) $K = 0.018$ and (b) $K = 0.01815$

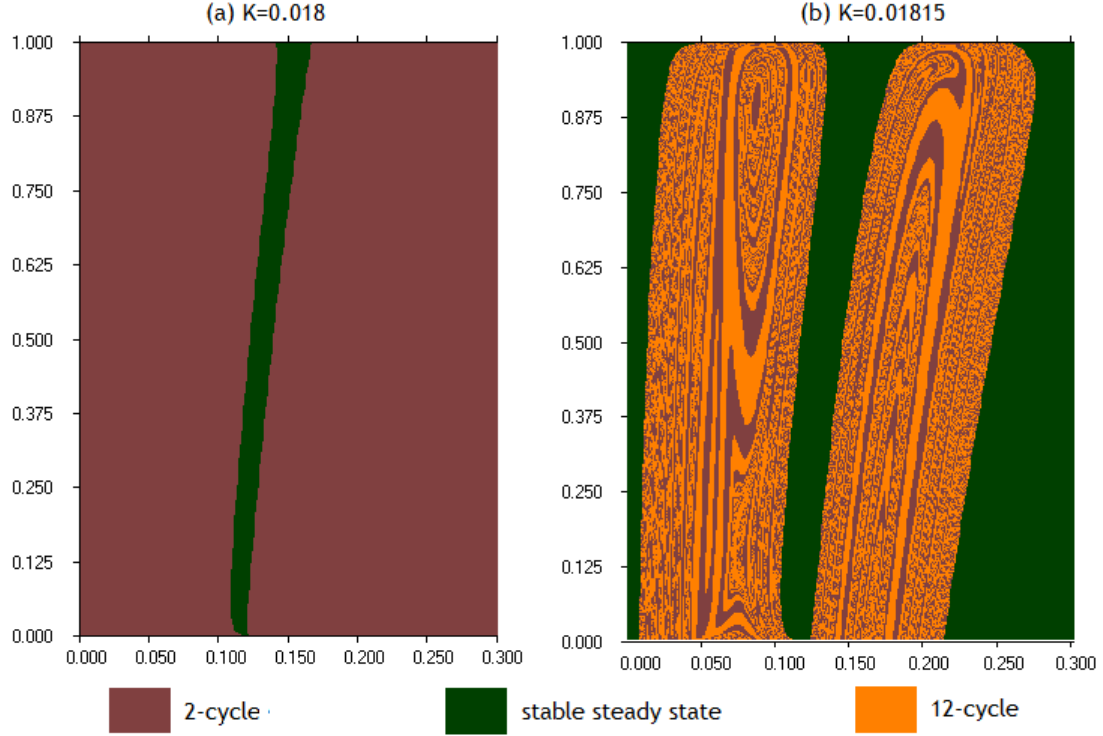


Figure 22: Best-response dynamics with evolutionary switching given by sluggish Adjusted Logit replicator: co-existing attractors. Basin of attraction for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\delta = 0.05$, $K = 0.01624$

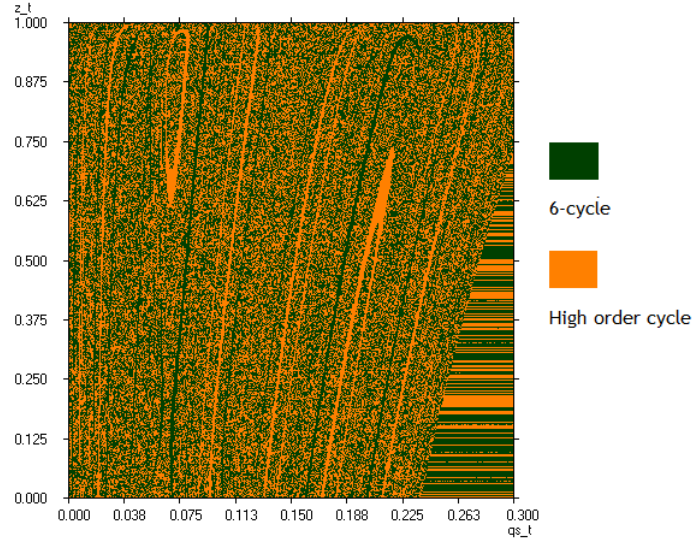


Figure 23: Best-response dynamics with evolutionary switching given by noisy Adjusted Logit replicator: bifurcation diagrams for q_s over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\rho = 0.05$ in panel (a), $\rho = 0.25$ in panel (b). Initial values $(z_{t=0}, q_{s,t=0})$ are drawn from a bivariate uniform distribution with support on $[0, 1] \times [0, 0.3]$.

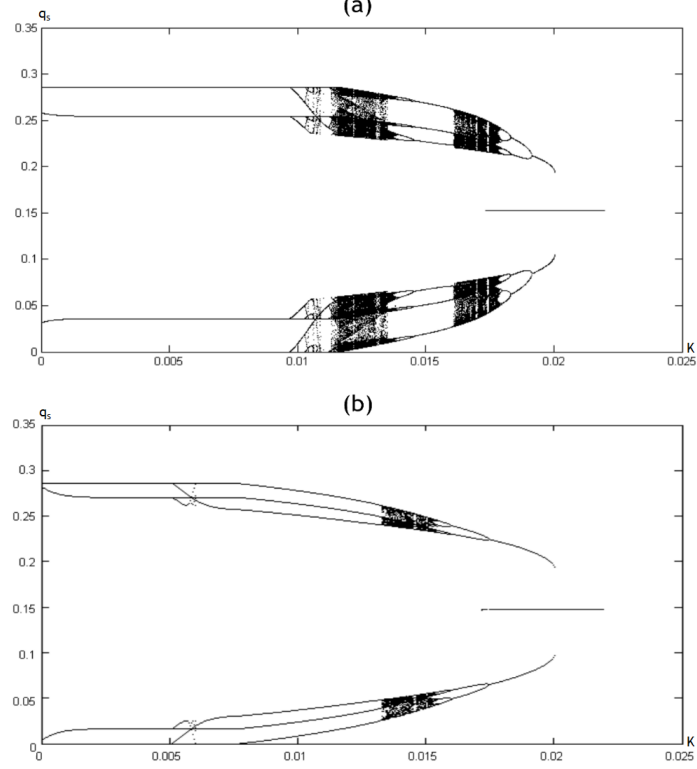


Figure 24: Best-response dynamics with evolutionary switching given by noisy Adjusted Logit replicator: bifurcation diagrams for q_s over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 1,000$. $\rho = 0.05$ in panel (a), $\rho = 0.25$ in panel (b). Initial values $(z_{t=0}, q_{s,t=0})$ are drawn from a bivariate uniform distribution with support on $[0, 1] \times [0, 0.3]$.

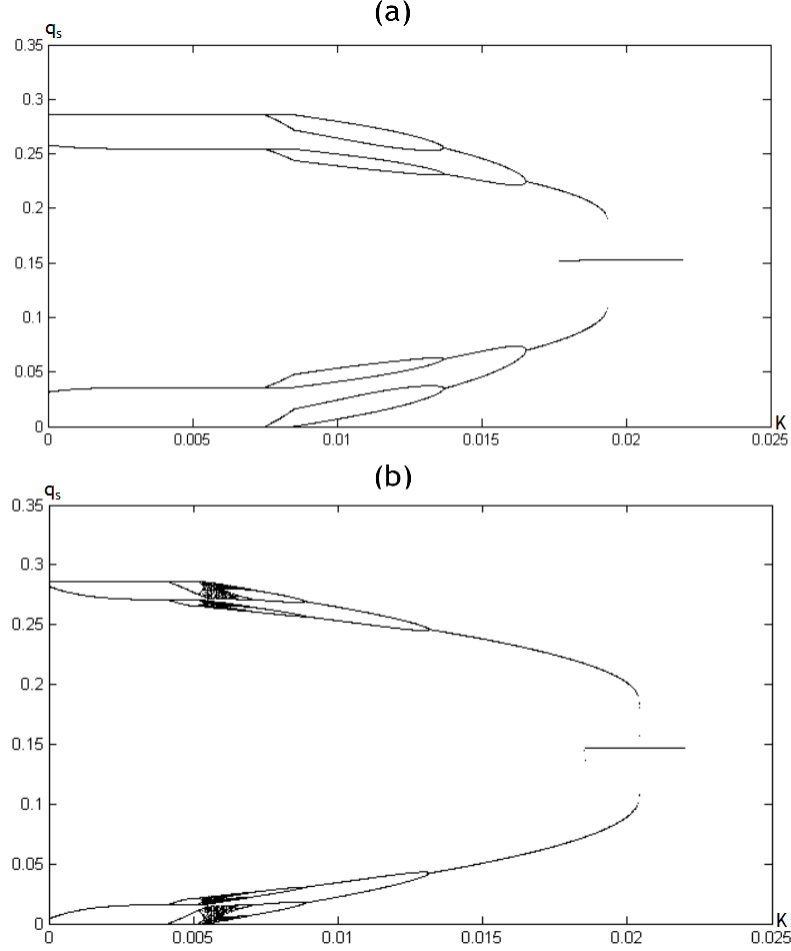


Figure 25: Best-response dynamics with evolutionary switching given by noisy Adjusted Logit replicator: co-existing attractors. Basins of attraction for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\rho = 0.05$, $K = 0.018$

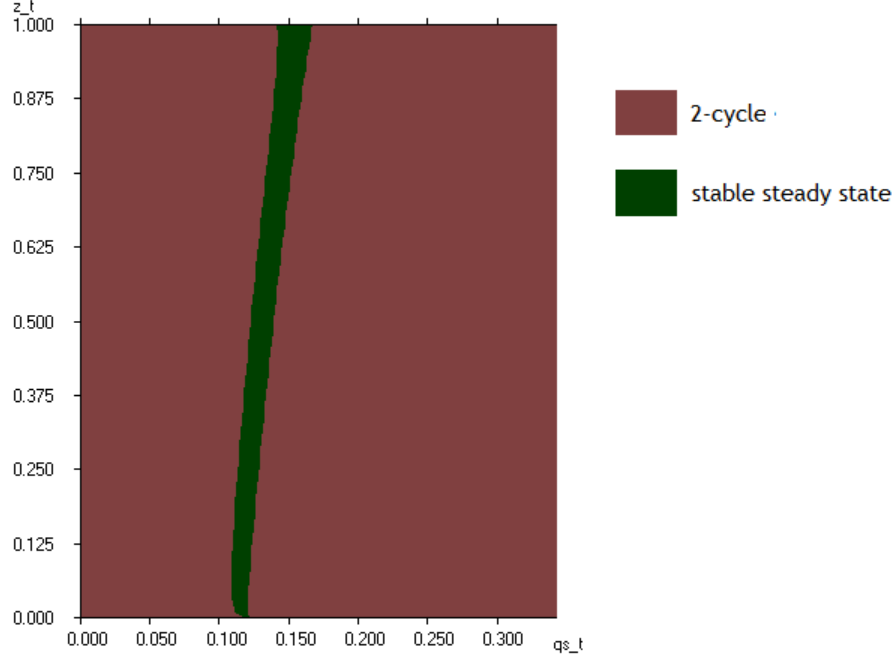
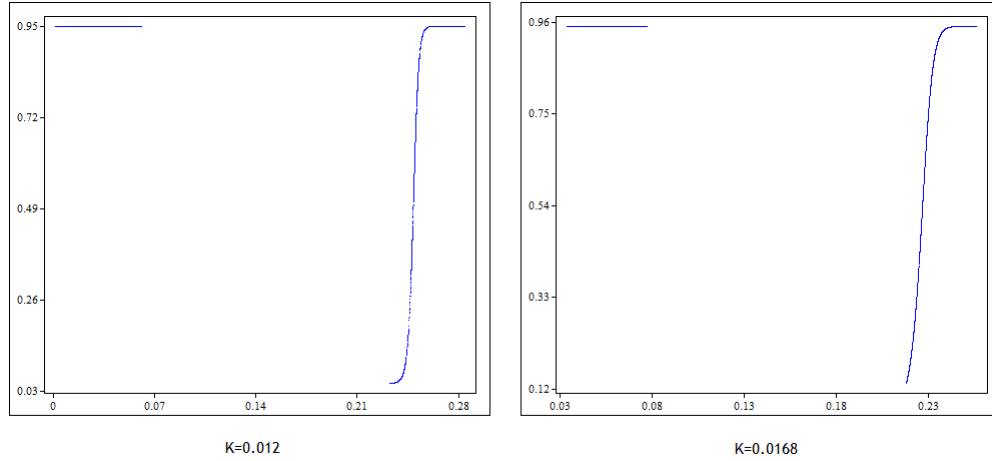


Figure 26: Best-response dynamics with evolutionary switching given by noisy Adjusted Logit replicator: limit cycle. Phase diagrams over parameter K , for $N = 4$, $d_s = \frac{3}{2}$, $d_i = \frac{1}{2}$, $\theta = 2,000$. $\rho = 0.05$ for different values of K



6 Welfare and innovation policy

We consider here a variation of the core model that includes innovation policy under the form of a tax and/or a subsidy. In this context we compare welfare outcomes as a function of policy at both the equilibrium and, when the equilibrium is not stable, along the trajectories generated by the model.

With taxes (or subsidies) to standard firms τ_s and to innovators, τ_i , the profit functions for the two strategies become:

$$\begin{aligned}\tilde{\pi}_s &= [1 - q_s - \bar{Q}_{N-1}(z) - \frac{1}{2}d_s q_s] q_s + \tau_s \\ \tilde{\pi}_i &= [1 - q_i - \bar{Q}_{N-1}(z) - \frac{1}{2}d_i q_i] q_i - K + \tau_i\end{aligned}\tag{22}$$

Notice that firms' best replies remain unchanged by the introduction of taxation, it is only the average profits for the two distinct technology strategies that change. This essentially means our previous analysis around parameter K , established by Proposition 1, can be applied to the profit functions defined above in (22) to obtain a similar result by simply substituting $\tilde{K} = K - \tau_i + \tau_s$ for K . Likewise, we will extend the notation with a tilde for all quantities pinned down by Proposition 1: by substituting \tilde{K} for K in the expression of the interior equilibrium fraction of firms, z^* , we obtain the equilibrium share of standard firms in the presence of innovation policy: $\tilde{z}^* = \frac{2+d_s}{d_s-d_i} + \frac{(2+d_i)}{(N-1)(d_s-d_i)} \left[2 + d_s - \frac{1}{\tilde{q}_s^*} \right]$, where $\tilde{q}_s^* = \sqrt{\frac{2(2+d_i)\tilde{K}}{(d_s-d_i)(2+d_s)}}$. As was the case without policy, the latter expression for \tilde{z}^* only characterizes the Nash equilibrium share of firms as long as $K^0 \leq \tilde{K} \leq K^1$, where K^0 and K^1 are the lower and upper bounds for innovation costs defined in Proposition 1. When $K^0 > \tilde{K}$, $\tilde{z}^* = 0$ and when $K^1 < \tilde{K}$, $\tilde{z}^* = 1$. Another implication of Proposition 1 is that, by setting taxes and/or subsidies, the policy maker can change the nature of the market equilibrium between the three possible cases: standard firms only, mixed firm population and innovative firms only.

By evaluating Nash equilibrium profits in (5) at \tilde{z}^* and adding τ_s for standard firms and τ_i for innovators we obtain the average realized equilibrium profits with policy for the two types of firms:

$$\begin{aligned}\tilde{\pi}_s &= \pi_s^N(\tilde{z}^*) + \tau_s \\ \tilde{\pi}_i &= \pi_i^N(\tilde{z}^*) + \tau_i.\end{aligned}\tag{23}$$

Without the tax/subsidy component, the average profits, $\pi_s^N(\tilde{z}^*)$ and $\pi_i^N(\tilde{z}^*)$ can be

used¹⁵ for computing total industry profits at an equilibrium given by $\tilde{z}^* \in [0, 1]$, :

$$\begin{aligned} TIP(\tilde{z}^*) &= N [\tilde{z}^* \pi_s^N(\tilde{z}^*) + (1 - \tilde{z}^*) \pi_i^N(\tilde{z}^*)] \\ &= \frac{N}{2} \frac{(2 + d_i)(2 + d_s)(2 + d_s(1 - \tilde{z}^*) + d_i \tilde{z}^*)}{[(2 + d_s)(N + 1 + d_i) - \tilde{z}^*(d_s - d_i)(N - 1)]^2} - KN(1 - \tilde{z}^*). \end{aligned} \quad (24)$$

Average consumer surplus can be computed as the average utility obtained by consumers at equilibrium:

$$CS = \frac{1}{2} \sum_{k=0}^N \binom{N}{k} (\tilde{z}^*)^k (1 - \tilde{z}^*)^{N-k} [k \tilde{q}_s^* + (N - k) \tilde{q}_i^*]^2.$$

Using Proposition 1 we can express \tilde{q}_s^* as a function of \tilde{z}^* :

$$\tilde{q}_s^* = \frac{2 + d_i}{(2 + d_s)(N + 1 + d_i) - \tilde{z}^*(d_s - d_i)(N - 1)}$$

and compute consumer surplus as a function of the equilibrium share of standard firms:

$$CS(\tilde{z}^*) = \frac{N}{2} \frac{[(d_s - d_i)^2 (1 - \tilde{z}^*) \tilde{z}^* + N(2 + d_s(1 - \tilde{z}^*) + d_i \tilde{z}^*)^2]}{[(2 + d_s)(N + 1 + d_i) - \tilde{z}^*(d_s - d_i)(N - 1)]^2}. \quad (25)$$

We denote by $W(\tilde{z}^*)$ total welfare at a regulated equilibrium where:

$$W(\tilde{z}^*) = TIP(\tilde{z}^*) + CS(\tilde{z}^*).$$

Unfortunately, the expressions for total industry profits and consumer surplus defined above are so complex that it is impossible to obtain a tractable analytical solution to welfare maximization - a \tilde{K} which maximizes $W(\tilde{z}^*)$. Proposition 5 offers instead a generic characterization of optimal policy in terms of the model parameters.

Proposition 5. *Depending on innovation cost parameter K , a benevolent social planner will optimally induce a Nash equilibrium by setting taxes and/or subsidies, τ_s and τ_i such that:*

(a) *Standard firms are driven out of the market, when $K < \tilde{K}^0$.*

(b) *No innovators are operating in the market, when $K > \tilde{K}^1$.*

¹⁵Alternatively we could compute welfare based on the expressions for equilibrium profits and output as functions of the model parameters given in Corollary 1. Using these leads to identifying \tilde{K}^* from the solution to a third order equation in $\sqrt{\tilde{K}}$. Unfortunately the expression is very complex and we prefer expressing welfare as a function of \tilde{z} , which gives the advantage of easy inspection of the border scenarios in Proposition 5.

(c) Firm population shares are driven to a unique $\tilde{z}_w = \arg \max W(\tilde{z}^*)$, $\tilde{z}_w \in]0, 1[$ when $\tilde{K}^0 < K < \tilde{K}^1$.

The thresholds $\tilde{K}^0 < \tilde{K}^1$ are given by:

$$\tilde{K}^0 = \frac{(d_s - d_i)}{2(2 + d_s)^2(N + 1 + d_i)^3} [(d_s + d_i + d_i d_s)(N + 1 + d_i) + 2(2 + d_i)(3 + d_i + d_s + N)]$$

$$\tilde{K}^1 = \frac{(d_s - d_i)}{2(2 + d_i)^2(N + 1 + d_s)^3} [(d_s + d_i + d_i d_s)(N + 1 + d_s) + 2(2 + d_s)(3 + d_i + d_s + N)]$$

and it always holds that $\tilde{K}^1 > K^1$.

The results collected in the above proposition, which is proven in Appendix D, are based on the strict concavity of $W(\tilde{z}^*)$ and on the monotonic relation between \tilde{K} and \tilde{z}^* . Because \tilde{z}^* is a monotonically increasing function of $\tilde{K} = K + \tau_s - \tau_i$, optimal policy will be represented by any combination of τ_s and τ_i that achieves the \tilde{K} which corresponds to the \tilde{z}^* that maximizes $W(\tilde{z}^*)$. When case (a) of Proposition 5 applies, the policy maker will want to set τ_s and τ_i such that $\tilde{K} \leq K^0$, effectively driving standard firms out of the market. If innovation costs are in the intermediate range, $\tilde{K}^0 < K < \tilde{K}^1$ - as described by case (c) - what the policy maker should do, depends on whether \tilde{K}_w - the \tilde{K} that corresponds to the optimal \tilde{z}_w - is greater or smaller than K . When $\tilde{K}_w < K$, the policy maker favors the innovative technology, $\tau_s < \tau_i$ and conversely, he favors the standard technology by setting $\tau_s > \tau_i$ to achieve \tilde{K}_w which corresponds to the equilibrium population shares, \tilde{z}_w , where welfare is maximized.

There are three possible rankings for the boundaries involved in establishing the results of Proposition 5:

1. $K^0 < \tilde{K}^0 < K^1 < \tilde{K}^1$. In this situation, case (c) of Proposition 5 requires that the regulator, whenever she intervenes, will do so in order to stimulate innovation, see also Figure 27.
2. $\tilde{K}^0 < K^0 < K^1 < \tilde{K}^1$. In this situation, \tilde{K} , as a function of K , will cross the first diagonal (i.e. $\tilde{K} = K$) for some $K \in [\tilde{K}^0, \tilde{K}^1]$, meaning that in case (c) of Proposition 5 the regulator intervenes to suppress innovation for small K , and stimulate innovation for larger K , see also Figure 28.

3. $K^0 < K^1 < \tilde{K}^0 < \tilde{K}^1$. In this situation, \tilde{K} will never be equal to K for any $K \in [\tilde{K}^0, \tilde{K}^1]$. Whenever she intervenes, the regulator will stimulate innovation, setting $\tilde{K} < K$.

Figure 27 illustrates how these results apply to our leading numerical example. We assume that the social planner takes minimal action: when the socially optimal population shares are also achieved by the market with no intervention, she does nothing: $\tau_s = \tau_i = 0$ and when the social optimum is attained with homogeneous population $\tilde{z}_w = 0$ or $\tilde{z}_w = 1$, she sets τ_s and τ_i such that $\tilde{K} = K^0$ or $\tilde{K} = K^1$ respectively. For the parameter configuration in our leading numerical example we have: $K^0 < \tilde{K}^0 < K^1 < \tilde{K}^1$. For this configuration the optimal policy is always in support of the innovative firms, $\tau_s < \tau_i$, for any K . For our leading numerical example this is true because we have $K^0 < \tilde{K}^0$. However, as Proposition 5 suggests, this is not always the case. For instance, when $N = 14$, $d_s = 6.5$ and $d_i = 0.5$, we have $\tilde{K}^0 < K^0 < K^1 < \tilde{K}^1$ and the social planner will set $\tau_s > \tau_i$ for smaller K and $\tau_s < \tau_i$ for larger K , see Figure 28.

Proposition 5 describes optimal innovation policy in a static environment or in a dynamic environment where the Nash equilibrium is globally attractive. However, the actual effects of policy on the economic trajectories observed in a market where agents are boundedly rational may in fact differ substantially from what is expected to obtain in equilibrium, specifically when the equilibrium is not stable. Without the assumption of a stable equilibrium the welfare along a dynamic trajectory of variables z_t and $q_{s,t}$ can be computed by averaging over the realized industry profits and consumer surplus. For each period the expressions in (24) and (25) become:

$$TIP_t = N [z_t \pi_{s,t} + (1 - z_t) \pi_{i,t}]$$

and

$$CS_t = \frac{N q_{s,t}^2 [(d_s - d_i)^2 (1 - z_t) z_t + N(2 + d_s + d_i z_t - d_s z_t)^2]}{2(2 + d_i)^2}.$$

Figure 27: (a) $\tilde{K} = K + \tau_s - \tau_i$ with optimal welfare policy; (b) Equilibrium share of standard firms; (c) Welfare in Equilibrium; as a function of fixed costs of innovation, K , for $N = 4$, $d_s = 1.5$, and $d_i = 0.5$.

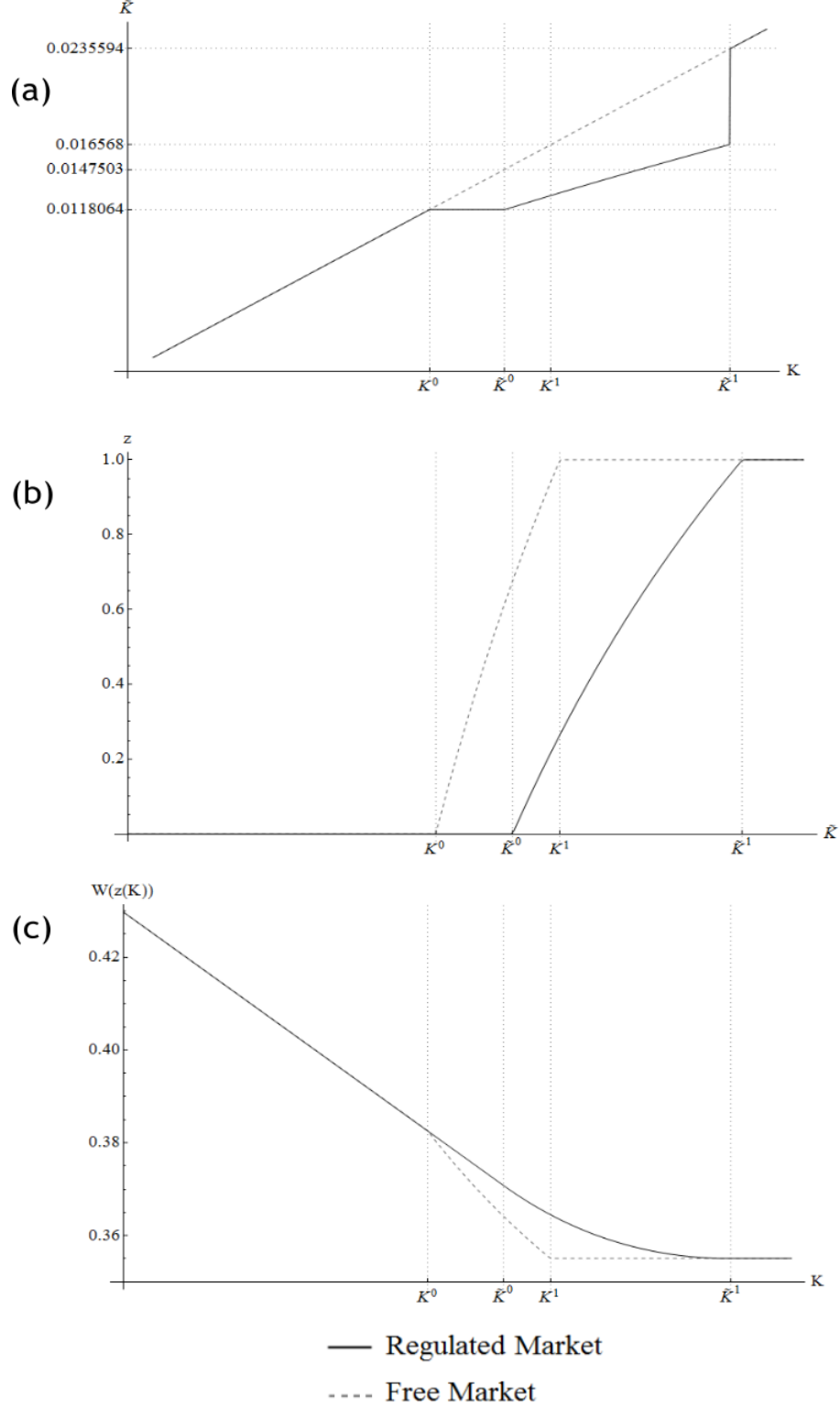


Figure 28: (a) $\tilde{K} = K + \tau_s - \tau_i$ with optimal welfare policy; (b) Equilibrium share of standard firms; (c) Welfare in Equilibrium; as a function of fixed costs of innovation, K , for $N = 14$, $d_s = 6.5$, and $d_i = 0.5$.

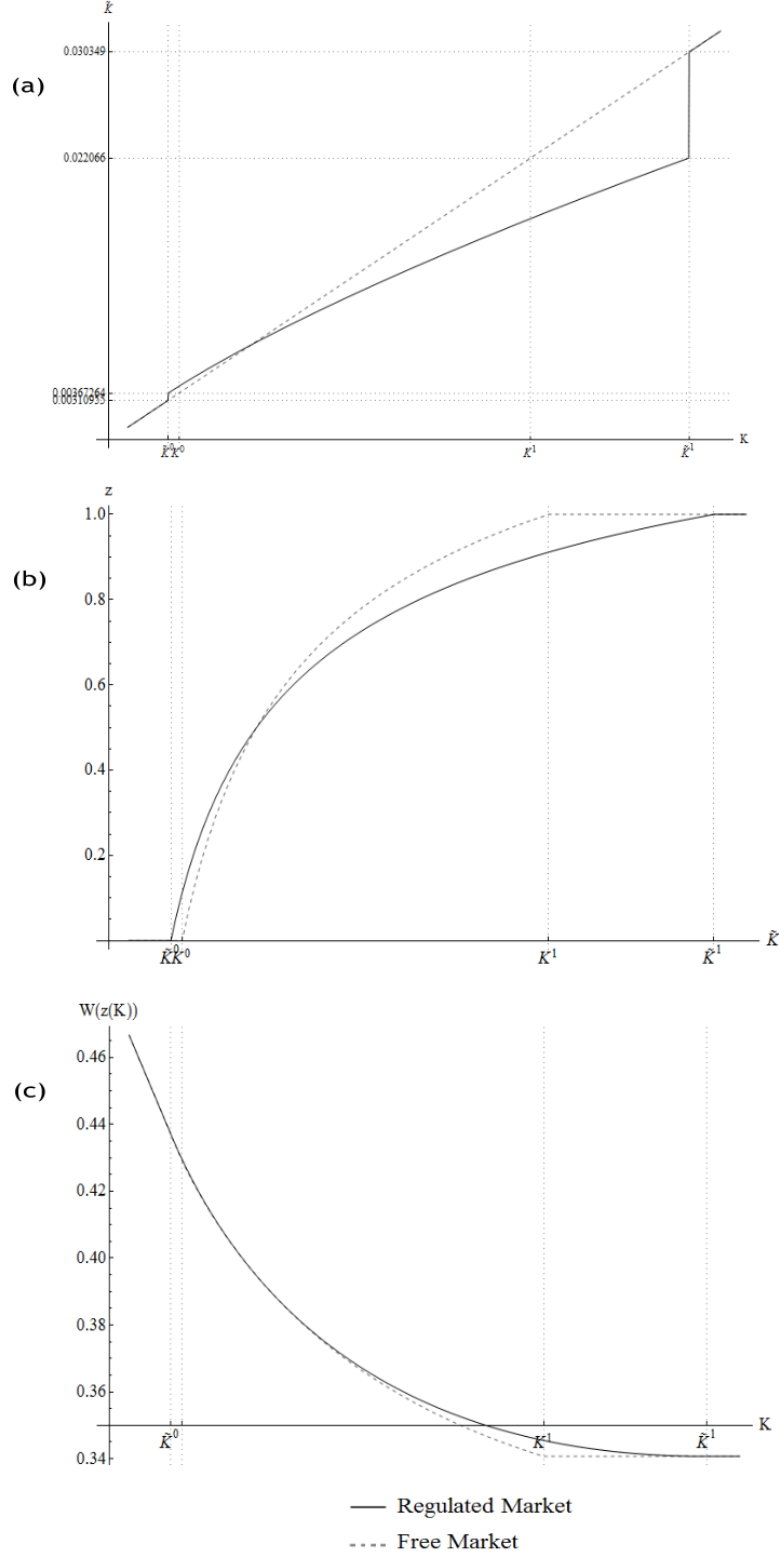


Figure 29 shows that innovation policy tuned for the Nash equilibrium of the model is not adequate when the model equilibrium is unstable. In fact we see that for most values of K the economy suffers a welfare loss. This happens because the Nash equilibrium tends to be unstable for lower values of K and stable for higher values. By setting $\tilde{K} < K$ the policy maker drives the market towards instability and/or wider fluctuations which generate lower welfare outcomes than in equilibrium. The Nash equilibrium first becomes stable without policy and later on the policy too sets \tilde{K} high enough for stability to occur. This is why we observe a small region where policy still results in an improvement of welfare when $\theta = 60$. This however occurs when $\theta = 2000$ since the Nash equilibrium no longer stable at any $z < 1$. Moreover, with higher technological instability the welfare loss due to an ill-calibrated policy seems to be larger.

Comparing policy effects between the model version where both technology and quantity are boundedly rationally set with the model where quantities are set at the Nash equilibrium allows us to gauge the share of policy failure that is due to technological instability. In Figure 30 we see that equilibrium calibrated innovation policy can still be ill-suited but not by far as harmful as it was under production instability. In fact, welfare in the regulated market comes quite close to the optimal welfare the regulator thinks she is aiming for. Comparing time series in Figures 5 and 16 as well as bifurcation diagrams in Figures 6 and 17 gives us a sense that the regulatory failure is mostly due to production instability. Whereas fluctuations in the model with Nash quantities are around the equilibrium this is no longer the case in the two-dimensional model and we see that the range of fluctuations is strictly decreasing in K . Therefore, when $\tilde{K} < K$ as is the case in our leading numerical example, the policy maker, whenever she intervenes, drives the industry towards more instability.

Figure 29: Welfare effects of innovation policy in the dynamic two-dimensional model as a function of fixed costs of innovation, K , for $N = 4$, $d_s = 1.5$, and $d_i = 0.5$, (a) $\theta = 60$; (b) $\theta = 2000$. Welfare is averaged over 3000 time-periods after a relaxation of a 1000 periods for the realized values of $q_{s,t}$ and z_t along their trajectories.

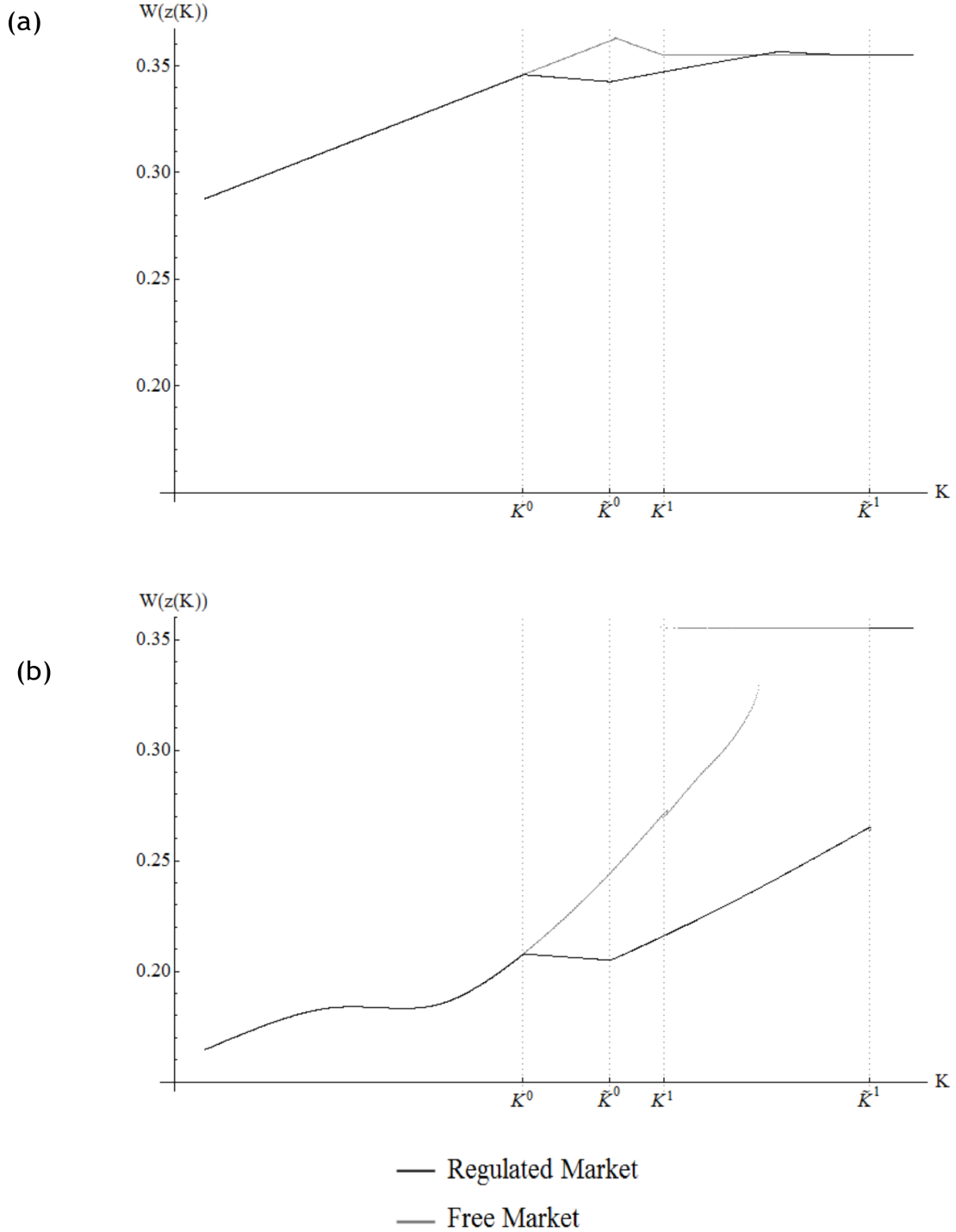
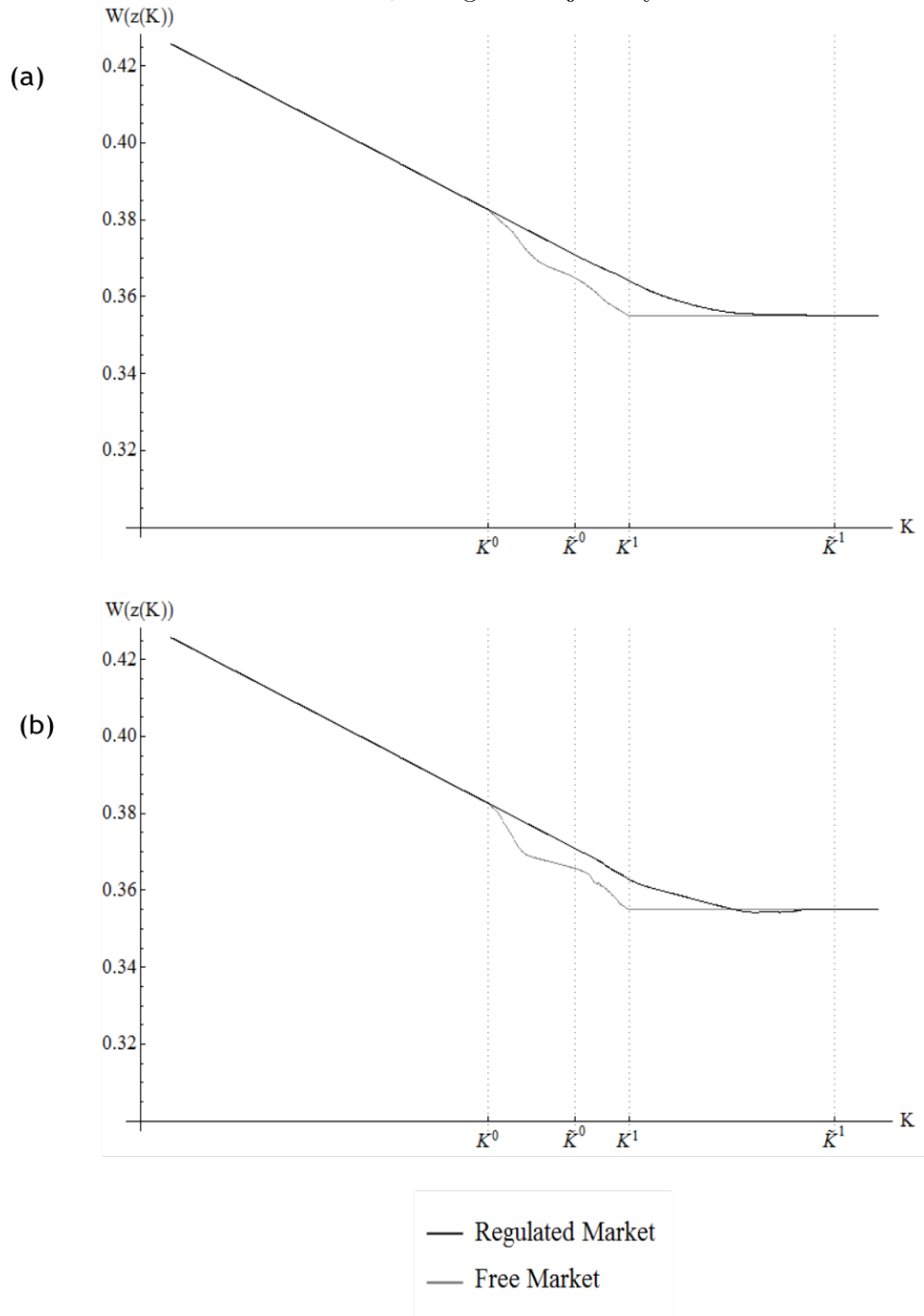


Figure 30: Welfare effects of innovation policy in the dynamic two-dimensional model as a function of fixed costs of innovation, K , for $N = 4$, $d_s = 1.5$, and $d_i = 0.5$, $\delta = 0.25$ and (a) $\theta = 4000$; (b) $\theta = 10000$. Welfare is averaged over 3000 time-periods after a relaxation of a 1000 periods for the realized values of z_t along its trajectory.



7 Discussion and conclusion

Our model analyzed the role of both production decisions and R&D decisions in the generating endogenous market fluctuations. For specific parameter combinations, namely when intensity of choice is high and there is a relatively large number of firms competing in oligopoly, the decision-making process may generate exotic dynamic phenomena such as chaotic fluctuations and coexisting nontrivial attractors with fractal basins. Endogenous fluctuations have important repercussions on the policy implications of our analysis: policy-making under the assumption of a stable market equilibrium may have adverse effects if the stability assumption does not hold or no longer holds. This result parallels the one obtained by Tuinstra et al. (2014) showing that trade barriers that reduce welfare in equilibrium may have stabilizing effects on endogenous market fluctuations and therefore improve welfare when the equilibrium is unstable. Moreover, when the instability of the market equilibrium is acknowledged by the policy-maker, welfare improving policy may still remain a challenge. If multiple basins of attraction co-exist and their frontiers are fractal, foreseeing the effects of policy remains a daunting task because, in practice, one cannot expect to estimate the current state of the system with sufficient accuracy to be able to tell in which basin it is located.

While Hommes et al. (2011) show that explosive best response dynamics can become bounded when nested in an evolutionary heuristic switching model, we show that the same can be obtained by a model with evolutionary switching between production technologies. To the extent that technological heterogeneity is a more widely accepted feature of economic reality than behavioral heuristic heterogeneity, our analysis has the potential to broadcast to a broader if not only different audience the argument that Theocharis' result on oligopoly instability is still relevant.

On the technical side we offer some insight into how different versions of the Adjusted Logit replicator compare by using both its sluggish extension with only a share of the population updating strategies each period and its noisy version where each period a share of the population switches strategy at random. We find that that the sluggish version has the advantage better tractability, by maintaining the property of having fixed-points that coincide with the Nash-equilibria. While similar in terms of stability, the mutation replicator seems to generate less complicated dynamics than its sluggish cousin. Another most important lesson, however, is that in numerical simulation, the standard Adjusted Logit replicator can get stuck at steady states that are not economically meaningful, while using its sluggish or noisy extensions avoid such inconvenience.

A number of extensions to the present work can be envisaged. Innovation enters our

model in a particular way that is mostly consistent with licensing of an external technology: firms have to pay a fixed cost each period in order to use the innovative technology. Kamien and Tauman (1986) examined a static model where a patent holder licenses a cost-reducing innovation to a an oligopolistic market. They find that the a fixed-fee licensing scheme is proffered by the licensor compared to a royalty scheme. On the other hand, when auctioning of the innovation is also possible, Kamien et al. (1992), find that an auction may be preferred by the licensor, in particular, when the innovation is more substantive. Examining empirical evidence, Vishwasrao (2007) finds that in the presence of sales fluctuations fixed-fee licensing is the preferred method. Therefore, our way of modeling innovation costs is adequate when the cost advantage of the innovative technology is not extremely high and when quantity dynamics are unstable, which, as we have shown, is consistent with our results. However, the model used here cannot address long-term technological change. To have such power, we would have to consider a scenario where the conditions by which innovation is brought to the market change over time as a result of market outcomes. Such a scenario is investigated by Hommes and Zeppini (2014) where repeated use of the innovative technology over time can drive the associated production cost down. They find that depending on the elasticity of demand and model parameters either, market breakdown (exclusion of the innovative technology) or technological progress (exclusion of the standard technology) or balanced technological change (change in the equilibrium share of innovators) can occur. It would be perhaps interesting to test the robustness of their results in our setting with oligopoly competing firms.

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Appendix A - Proof of Proposition 1 and Corollary 1

Proof. Profits are equal when $\pi_s^N(z) = \pi_i^N(z)$. Solving for K we obtain

$$\begin{aligned} K^* &= \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \frac{1}{2+d_i} - \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \frac{1}{2+d_s} \\ &= \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \left[\frac{1}{2+d_i} - \frac{1}{2+d_s} \right] \\ &= \frac{1}{2} \left(\frac{1}{Z(N-1)+1} \right)^2 \left[\frac{d_s - d_i}{(2+d_i)(2+d_s)} \right], \end{aligned}$$

which is positive $\forall d_s > d_i$.

Expressing K^* as a function of Z gives:

$$Z^* = \frac{1}{N-1} \left(\sqrt{\frac{d_s - d_i}{2(2+d_i)(2+d_s)K}} - 1 \right)$$

We have thus:

$$\frac{z^*}{2+d_s} + \frac{1-z^*}{2+d_i} = \frac{1}{N-1} \left(\sqrt{\frac{d_s - d_i}{2(2+d_i)(2+d_s)K}} - 1 \right).$$

.Multiplying both sides by $(2+d_s)(2+d_i)$ we obtain

$$z^* = \frac{2+d_s}{d_s-d_i} - \frac{1}{N-1} \left(\sqrt{\frac{(2+d_i)(2+d_s)}{2(d_s-d_i)K}} - \frac{(2+d_i)(2+d_s)}{d_s-d_i} \right) \quad (26)$$

By using (26), we can compute the equilibrium quantity for a standard firm when technologies are equally profitable as function of the model parameters (7):

$$q_s^* = \sqrt{\frac{2(2+d_i)K}{(d_s-d_i)(2+d_s)}} \quad (27)$$

We can also rewrite z^* as a function of q_s^* :

$$z^* = \frac{2+d_s}{d_s-d_i} + \frac{(2+d_i)}{(N-1)(d_s-d_i)} [2+d_s-(q_s^*)^{-1}]$$

Note that z^* is increasing in K , as expected; the more expensive it is to use the innovative technology, the smaller will be the equilibrium share of firms using it.

Rewriting (5) in function of z , we can compute Nash equilibrium profits for any given population shares, z .

$$\pi_s^N(z) = \frac{(2+d_i)^2(2+d_s)}{2((2+d_s)(1+d_i+N) - (d_s-d_i)(N-1)z)^2} \quad (28)$$

$$\pi_i^N(z) = \frac{(2+d_i)(2+d_s)^2}{2[(2+d_s)(1+d_i+N) - (d_s-d_i)(N-1)z]^2} - K \quad (29)$$

Notice that average profits, for both types of firms will be strictly increasing and convex in z . The share of standard firms appears only in the denominator in the expression, $(2+d_s)(1+d_i+N) - (d_s-d_i)(N-1)z = d_s + (1+d_i+N) + ((1-z)d_s + zd_i)(N-1)$, which is strictly positive $\forall 0 \leq z \leq 1$, $N \geq 2$ and $d_s < d_i$ and also decreasing in z as it linearly depends on the population weighted average of marginal costs

Notice also that $\frac{\partial \pi_s^N(z)}{\partial z} = \frac{2+d_i}{2+d_s} \frac{\partial \pi_i^N(z)}{\partial z}$ which means that the profit function of the innovative firms slopes steeper in z than the profit function of standard firms. This implies that if the profit functions cross for some $z > 0$ they will only do so once, assuring the uniqueness of z^* .

The interior point defined in part (b) of the proposition exists when the two profit function cross for some $z \in]0, 1[$. This is equivalent to imposing $\pi_s^N(1) - \pi_i^N(1) < 0 < \pi_s^N(0) - \pi_i^N(0)$, meaning that condition $z^* \in]0, 1[$ is equivalent to:

$$K^0 = \frac{(2+d_i)(d_s-d_i)}{2(2+d_s)(N+1+d_i)^2} < K < \frac{(2+d_s)(d_s-d_i)}{2(2+d_i)(N+1+d_s)^2} = K^1 \quad (30)$$

which correspond to the two threshold levels for the fixed cost of investing in the new technology that are defined by the proposition.

Thus, when $K^0 < K < K^1$, for $z \in (0, z^*)$, we have $\pi_s^N(z) > \pi_i^N(z)$ and for $z \in (z^*, 1)$, it is $\pi_s^N(z) < \pi_i^N(z)$.

Moreover condition $\pi_s^N(1) - \pi_i^N(1) > 0$ is equivalent to $K > K^1$, standard firms are always better off than innovators because of the high cost to innovate.

The opposite holds if $\pi_s^N(0) - \pi_i^N(0) < 0$, i.e. for $0 < K < K^0$, where innovating dominates using the standard technology.

(Corollary 1)

By expressing equilibrium profits as a function of q_s^* as:

$$\pi_s^N(q_s^*) = \frac{1}{2} (2 + d_s) (q_s^*)^2 \quad (31)$$

$$\pi_i^N(z) = \frac{1}{2} \frac{(2 + d_s)}{(2 + d_i)} (2 + d_s) (q_s^*)^2 - K \quad (32)$$

and substituting for $q_s^* = \sqrt{\frac{2(2+d_i)K}{(d_s-d_i)(2+d_s)}}$ we find equilibrium profits are linearly increasing in K for both types of firms:

$$\pi_s^{N*} = \frac{(2 + d_i)}{(d_s - d_i)} K \quad (33)$$

$$\pi_i^{N*} = \frac{(2 + d_s)}{(d_s - d_i)} K - K = \pi_i^N(q_s^*) \quad (34)$$

This means that average total industry profits are given by:

$$TIP = N (z^* \pi_s^{N*} + (1 - z^*) \pi_i^{N*}) = N \frac{(2 + d_i)}{(d_s - d_i)} K$$

Average industry output is given by

□

$$\begin{aligned} \bar{Q}_N^* &= N [z^* q_s^* + (1 - z^*) q_i^*] \\ &= N q_s^* \left[\frac{2 + d_s}{2 + d_i} - z^* \frac{d_s - d_i}{2 + d_i} \right] \\ &= \frac{N}{N - 1} [1 - (2 + d_s) q_s^*], \end{aligned}$$

Appendix B - Proof of Proposition 3

Proof. Obviously the points $z_0 = 0$ and $z_1 = 1$ are equilibria for the map (12). Moreover, when a pure strategy (innovating or using the standard technology) dominates the other for all z , then the evolutionary dynamics make the system converge to that dominating strategy. For instance consider the system for $0 < K < K^0$ so that innovation always dominates the standard strategy. From $\psi(z_0) < 0$ and computing:

$$g'(z_0) = \delta + (1 - \delta)e^{\theta\psi(0)} \in (0, 1).$$

Also from $\psi(z_1) < 0$ and

$$g'(z_1) = \delta + (1 - \delta)e^{-\theta\psi(1)} \in (1, +\infty)$$

we have that $z_0 = 0$ is locally asymptotically stable and $z_1 = 1$ is unstable (the reasoning when innovating is dominated by going standard is analogous).

For any $K^0 < K < K^1$, according to Proposition 1, $\psi(z_0) > 0 \rightarrow g'(z_0) > 1$ and $\psi(z_1) < 0 \rightarrow g'(z_1) > 1$.

Now let us consider the inner equilibrium z^* . First of all, there is only one such interior equilibrium, as $\pi_i^*(z) = \pi_s^*(z)$ has a unique solution for $z > 0$, if any at all - this is assured by the monotonicity of $\psi(z)$. The eigenvalue of the dynamic system at the interior steady state is

$$g'(z^*) = 1 + (1 - z^*)z^*\theta(1 - \delta)\psi'(z^*)$$

and so z^* always loses stability through a flip bifurcation when the intensity of choice θ is sufficiently high, as the loss of stability occurs at

$$\hat{\theta} = -\frac{2}{(1 - z^*)z^*\psi'(z^*)(1 - \delta)} > 0 \quad (35)$$

due to overshooting around the equilibrium.

We have so far discussed the local stability of the equilibria, but we can also argue for global stability. For z_0 and z_1 this is trivial since in cases (a) and (c) all trajectories will monotonically converge to z_0 and z_1 respectively, because the profit of one type of firm is always larger regardless of z . In case (b) things will be only slightly more complicated, with all trajectories starting below (above) z^* being driven upwards (downwards) as case (b) of Proposition 1 implies. As long as there is only mild overshooting around z^* , i.e. $\theta < \hat{\theta}$ the

oscillations will be dampened and the population share of standard firms will converge to z^* . \square

Appendix C - Proof of Proposition 4 and Corollaries 3 and 2

Proof. Consider the Jacobian matrix at the interior steady state, which, by taking into account that profits for the two strategies are equal at equilibrium, can be written as follows:

$$J = \begin{pmatrix} -(N-1) \left(\frac{1-z^*}{2+d_i} + \frac{z^*}{2+d_s} \right) & (N-1) \left(\frac{1}{2+d_i} - \frac{1}{2+d_s} \right) q_s^* \\ (1-z^*)z^*(1-\delta)\theta \frac{\partial \Psi(q_s^*, z^*)}{\partial q_s} & 1 + (1-z^*)z^*(1-\delta)\theta \frac{\partial \Psi(q_s^*, z^*)}{\partial z} \end{pmatrix} \quad (36)$$

We can write the Jacobian as a function of the basic model parameters and q_s^* . This entails substituting in for z^* as a function of equilibrium quantity, $z^* = \frac{2+d_s}{d_s-d_i} + \frac{(2+d_i)}{(N-1)(d_s-d_i)} \left(2 + d_s - \frac{1}{q_s^*} \right)$ and working out the derivatives of the profit differential $\Psi(q_{s,t}, z_t)$, as defined in (18), with respect to $q_{s,t}$ and z_t evaluated at q_s^* and z^* into:

$$\frac{\partial \Psi(q_s^*, z^*)}{\partial q_s} = \frac{(d_s - d_i) (1 - (2 + d_s) q_s^*)}{2 + d_i}$$

and

$$\frac{\partial \Psi(q_s^*, z^*)}{\partial z} = -\frac{2 (d_s - d_i) (N - 1) K}{(2 + d_i) (2 + d_s)}.$$

Making the above substitutions into (36) and simplifying yields:

$$\begin{aligned} J_{11} &= -\frac{1 - (2 + d_s) q_s^*}{(2 + d_s) q_s^*} \\ J_{12} &= (N - 1) \left(\frac{1}{2 + d_i} - \frac{1}{2 + d_s} \right) q_s^* = \frac{2K (N - 1)}{(2 + d_s)^2 q_s^*} \\ J_{21} &= (1 - \delta) \theta \frac{[1 - (2 + d_s) q_s^*] [(2 + d_s) (1 + d_i + N) q_s^* - (2 + d_i)] [1 - (1 + d_s + N) q_s^*]}{(d_s - d_i) (N - 1)^2 (q_s^*)^2} \\ J_{22} &= 1 - (1 - \delta) \theta \frac{2K [(2 + d_s) (1 + d_i + N) q_s^* - (2 + d_i)] [1 - (1 + d_s + N) q_s^*]}{(2 + d_s) (d_s - d_i) (N - 1) (q_s^*)^2}. \end{aligned} \quad (37)$$

Where J_{ij} is the element in the i -th row and j -th column of the Jacobian matrix. Notice that although J_{12} was already in (36) an expression in model parameters and q_s^* only in , we

used that $q_s^* = \frac{(q_s^*)^2}{q_s^*} = \frac{2(2+d_i)K}{(2+d_s)(d_s-d_i)} \frac{1}{q_s^*}$ to bring it to a more compact form.

Denoting by $E_1 = 1 - (2 + d_s) q_s^*$, $E_2 = [(2 + d_s) (1 + d_i + N) q_s^* - (2 + d_i)]$ and $E_3 = [1 - (1 + d_s + N) q_s^*]$ the reoccurring patterns we can further simplify the terms of the Jacobian Matrix to:

$$\begin{aligned} J_{11} &= -\frac{E_1}{(2 + d_s) q_s^*} \\ J_{12} &= \frac{2K (N - 1)}{(2 + d_s)^2 q_s^*} \\ J_{21} &= (1 - \delta) \theta \frac{E_1 E_2 E_3}{(d_s - d_i) (N - 1)^2 (q_s^*)^2} \\ J_{22} &= 1 - (1 - \delta) \theta \frac{2K E_2 E_3}{(2 + d_s) (d_s - d_i) (N - 1) (q_s^*)^2}. \end{aligned}$$

The expressions E_1 , E_2 and E_3 are all positive as long as we are talking about an interior equilibrium as can be verified by evaluating them at the boundary values K^0 and K^1 :

$$\begin{aligned} E_1 &\geq E_1|_{K=K^1} = 1 - \frac{2 + d_s}{1 + d_s + N} > 0 \\ E_2 &\geq E_2|_{K=K^0} = 0 \\ E_3 &\geq E_3|_{K=K^1} = 0. \end{aligned}$$

This is convenient because it allows us to determine the sign of three of the Jacobian terms: $J_{11} < 0$, $J_{12} > 0$, $J_{21} > 0$.

By further denoting $E_4 = \frac{E_2 E_3}{(d_s - d_i) (N - 1) (q_s^*)^2}$ we can conveniently rewrite the Jacobian matrix in a compact and analytically tractable form:

$$J = \begin{pmatrix} -\frac{E_1}{(2 + d_s) q_s^*} & \frac{2K(N-1)}{(2 + d_s)^2 q_s^*} \\ (1 - \delta) \theta \frac{E_1 E_4}{(N-1)} & 1 - (1 - \delta) \theta \frac{2K E_2 E_3}{(2 + d_s)} \end{pmatrix}. \quad (38)$$

The determinant of the Jacobian matrix turns out to be equal to the first term on the diagonal:

$$Det J = -\frac{E_1}{(2 + d_s) q_s^*} = 1 - \frac{1}{(2 + d_s) q_s^*} < 0$$

and its trace is:

$$Tr J = 1 - \frac{E_1}{(2 + d_s) q_s^*} - (1 - \delta) \theta \frac{2K E_2 E_3}{(2 + d_s)}.$$

The discriminant of the eigen vector polinomial, $p(\lambda) = \lambda^2 - \text{tr}J + \text{Det}J$, is

$$\Delta = \frac{4E_1(2+d_s)q_s^* + [E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4)]^2}{(2+d_s)^2(q_s^*)^2} > 0.$$

Therefore the eigenvalues of the Jacobian must be real always.

Finally, we can write the eigenvalues of the Jacobian matrix as:

$$\begin{aligned}\lambda_1 &= -\frac{1}{2(2+d_s)q_s^*} \left(\sqrt{(E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4))^2 + 4E_1(2+d_s)q_s^*} \right. \\ &\quad \left. + [E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4)] \right) \\ \lambda_2 &= \frac{1}{2(2+d_s)q_s^*} \left(\sqrt{(E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4))^2 + 4E_1(2+d_s)q_s^*} \right. \\ &\quad \left. - [E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4)] \right)\end{aligned}$$

The complexity of the above expressions make it hard to establish stability of the system by comparing the eigenvalues to ± 1 but they do offer some valuable insight. To ease the exposition we denote by $E_5 = [E_1 - q_s^*(2+d_s - 2(1-\delta)\theta KE_4)]$ and $E_6 = 4E_1(2+d_s)q_s^*$. While the sign of E_5 is ambiguous, E_6 is always positive.

First of all, it is clear that $\lambda_1 = -\frac{1}{2(2+d_s)q_s^*} \left(\sqrt{E_5^2 + E_6} + E_5 \right)$ is negative. When E_5 is positive, the term in brackets is surely positive, when E_5 is negative the term under the square root will surely be large enough to offset E_5 , so the term in brackets is always positive.

The same reasoning can be applied to determine that $\lambda_2 = \frac{1}{2(2+d_s)q_s^*} \left(\sqrt{E_5^2 + E_6} - E_5 \right)$ is always positive.

Knowing that the eigenvalues are always real and of opposite sign is informative because, having ruled out all other possible configurations, we can establish that the system will be stable if and only if the characteristic polinomial is positive at both -1 and 1 :

$$\begin{cases} 1 + \text{tr}J + \text{det}J > 0 \\ 1 - \text{tr}J + \text{det}J > 0 \end{cases} \quad (39)$$

This is equivalent to

$$\begin{cases} \frac{2(2q_s^*(2+d_s) - q_s^*(1-\delta)\theta E_4 K - 1)}{(2+d_s)q_s^*} > 0 \\ \frac{2(1-\delta)E_4\theta K}{2+d_s} \geq 0 \end{cases}$$

This means that the positive eigenvalue is always stable whereas the negative eigenvalue

is stable if and only if:

$$2q_s^* (2 + d_s) - q_s^* (1 - \delta) \theta E_4 K - 1 > 0,$$

or, expanding E_4 and isolating θ to one side, when :

$$\theta < \tilde{\theta} = \frac{2(N-1)(2+d_i)(2q_s^*(2+d_s)-1)}{(1-\delta)(2+d_s)q_s^*[(2+d_s)(1+d_i+N)q_s^*-2-d_i][1-(1+d_s+N)q_s^*]}.$$

(Corollary 2)

Notice that $\tilde{\theta} = \frac{(N-1)(d_s-d_i)(2q_s^*(2+d_s)-1)q_s^*}{(1-\delta)KE_2E_3}$ is not necessarily positive, its sign will be the same as the sign of $E_7 = 2q_s^*(2+d_s) - 1$ which depends on the relation between model parameters. E_7 attains its maximum value when $K = K^1$ so in order for the interior equilibrium to be stable, at positive θ , for some K , we must have $E_{7|K=K^1} > 0$, or $3 + d_s > N$. Therefore the interior equilibrium is never stable for $N > 3 + d_s$. The minimum value of E_7 is attained at $K = K^0$ allowing us to conclude that the interior equilibrium can always be stable - for low enough θ - if $E_{7|K=K^0} > 0$, or $3 + d_i > N$.

When $3 + d_i < N < 3 + d_s$, provided that θ is sufficiently low for the equilibrium to be stable at admissible value of $K \in [K^0, K^1]$, there will be some \hat{K} such that for $K < \hat{K}$ the mixed equilibrium is unstable at any $\theta > 0$ while for $K > \hat{K}$ the mixed equilibrium can be stable for some $\theta < \tilde{\theta}$. We can compute \hat{K} by solving for $\tilde{\theta} = 0 \leftrightarrow 2q_s^*(2+d_s) - 1 = 0$:

$$\hat{K} = \frac{d_s - d_i}{8(2+d_s)(2+d_i)}$$

Note that \hat{K} does not depend on N .

(Corollary 3)

Expressing $\hat{\theta}$ as a function of q_s^* and using the same substitutions for E_2 and E_3 we obtain:

$$\hat{\theta} = \frac{2(2+d_i)(N-1)}{(1-\delta)(2+d_s)q_s^*E_2E_3}$$

Computing the ratio of the two thresholds we obtain:

$$\frac{\hat{\theta}}{\tilde{\theta}} = \frac{1}{2(2+d_s)q_s^*-1} > 1 \leftrightarrow 2E_1 > 0,$$

which always holds.

□

Appendix D - Proof of Proposition 5

Proof. Examining the first and second derivatives of $W(z)$:

$$\begin{aligned}\frac{\partial W(z)}{\partial z} &= NK - N(d_s - d_i) \left\{ \frac{(2 + d_i)(2 + d_s)(3 + d_i + d_s + N)}{[(2 + d_s)(N + 1 + d_i) - z(d_s - d_i)(N - 1)]^3} \right. \\ &\quad \left. + \frac{d_s + d_i + d_i d_s}{2[(2 + d_s)(N + 1 + d_i) - z(d_s - d_i)(N - 1)]^2} \right\} \\ \frac{\partial^2 W(z)}{\partial z^2} &= - (d_s - d_i)^2 (N - 1) N \left\{ \frac{3(2 + d_i)(2 + d_s)(3 + d_i + d_s + N)}{[(2 + d_s)(N + 1 + d_i) - z(d_s - d_i)(N - 1)]^4} \right. \\ &\quad \left. + \frac{(d_i + d_s + d_i d_s)[(2 + d_s)(N + 1 + d_i) - z(d_s - d_i)(N - 1)]}{[(2 + d_s)(N + 1 + d_i) - z(d_s - d_i)(N - 1)]^4} \right\},\end{aligned}$$

we notice that the only increasing component of the welfare function with respect to the equilibrium share of standard firms, \tilde{z}^* , is given by the element that accounts for the fixed costs of innovation paid by the innovative firms, $KN(1 - \tilde{z}^*)$. The remaining components of the welfare function, consumer surplus and profits (net of innovation costs) are strictly decreasing in z . Furthermore, $W(\tilde{z}^*)$ is strictly concave, therefore it will have a unique, local maximum for $\tilde{z}^* \in [0, 1]$, which will be either at one of the boundaries of the unit interval where:

$$\begin{aligned}W(0) &= N \left[\frac{2 + d_i + N^2}{2(1 + d_i + N)^2} - K \right] \\ W(1) &= N \frac{2 + d_s + N^2}{2(1 + d_s + N)^2}\end{aligned}$$

or at an interior point, $W(z_w)$.

Model parameters will determine which one of the above three possibilities actually characterizes the model. Notice that concavity of $W(z)$ implies that we always have $W'(0) > W'(1)$. We can further investigate the required relations for each case, focusing again on parameter K . However, the analysis is hardly tractable:

(i) When maximum welfare obtains at $z = 0$ we will have $W'(0) < 0$ which requires that:

$$K < \frac{(d_s - d_i)}{2(2 + d_s)^2(N + 1 + d_i)^3} [(d_s + d_i + d_i d_s)(N + 1 + d_i) + 2(2 + d_i)(3 + d_i + d_s + N)] = \hat{K}^0$$

(ii) When maximum welfare obtains at $z = 1$ we must have $W'(1) > 0$ which requires

that:

$$K > \frac{(d_s - d_i)}{2(2 + d_i)^2(N + 1 + d_s)^3} [(d_s + d_i + d_i d_s)(N + 1 + d_s) + 2(2 + d_s)(3 + d_i + d_s + N)] = \hat{K}^1$$

(iii) Finally, when $W'(0) > 0$ and $W'(1) < 0$ which is equivalent to $\hat{K}^0 < K < \hat{K}^1$ the welfare optimum will be at some interior $z_w \in]0, 1[$.

Given that $W(z)$ is concave, we can be sure that $\hat{K}^0 < \hat{K}^1$, but we may want to verify. Because the expression in square brackets for \hat{K}^1 is visibly greater than its homologue for \hat{K}^0 , it all boils down to showing that the factors before the expressions in square brackets satisfy:

$$\frac{(d_s - d_i)}{2(2 + d_s)^2(N + 1 + d_i)^3} < \frac{(d_s - d_i)}{2(2 + d_i)^2(N + 1 + d_s)^3} \Leftrightarrow \left(\frac{2 + d_i}{2 + d_s}\right)^2 < \left(\frac{N + 1 + d_i}{N + 1 + d_s}\right)^3$$

where the latter expression is always true for $d_i < d_s$ and $N \geq 2$.

Finally, we may be interested to compare \hat{K}^0 and \hat{K}^1 with K^0 and K^1 defined in Proposition 1. However, the only parameter-independent relation we can establish is:

$$\hat{K}^1 - K^1 = \frac{(d_s - d_i)[8 + d_i(3 + d_s - N) + d_s(5 + d_s + N)]}{2(2 + d_i)^2(N + 1 + d_s)^3} > 0 \Leftrightarrow \hat{K}^1 > K^1$$

We also have that $\tilde{K}^0 > K^0$ iff.

$$N < \frac{8 + 5d_i + d_i^2 + 3d_s + d_i d_s}{d_s - d_i}$$

and that $K^1 > \tilde{K}^0$ iff.:

$$(2 + d_s)^3(1 + d_i + N)^3 > (2 + d_i)(1 + d_s + N)^2(d_i^2(3 + d_s) + 4(3 + N) + d_s(N + 5) + d_i[11 + 3N + d_s(4 + N)])$$

□

Appendix E - Comparative statics

Here we derive comparative statics of z^* with respect to production costs, d_s and d_i - one would expect z^* to be decreasing in d_s and increasing in d_i . Straightforward differentiation leads to complex formulae that do not provide a basis for determining the sign of the derivatives, but notice that the derivatives of z^* with respect to parameters d_s and d_i are strictly decreasing and increasing in K respectively:

$$\frac{\partial^2 z^*}{\partial d_s \partial K} = - \frac{(2 + d_i)^2}{4\sqrt{2} (d_s - d_i)^2 \sqrt{\frac{(2+d_i)(2+d_s)}{2(d_s-d_i)K}} K^2 (N-1)} < 0$$

$$\frac{\partial^2 z^*}{\partial d_i \partial K} = \frac{(2 + d_s)^2}{4\sqrt{2} (d_s - d_i)^2 \sqrt{\frac{(2+d_i)(2+d_s)}{2(d_s-d_i)K}} K^2 (N-1)} > 0$$

Which means that evaluating the derivatives of z^* with respect to d_s and d_i at some boundary value of K will translate into boundary values for the derivatives of interest. Applying this logic yields that $\frac{\partial z^*}{\partial d_s} < \frac{\partial z^*}{\partial d_s} |_{K=K^0} = -\frac{(2+d_i)(1+d_i+N)}{2(d_s-d_i)^2(N-1)} < 0$, as expected. Unfortunately, $\frac{\partial^2 z^*}{\partial d_i} > \frac{\partial^2 z^*}{\partial d_i} |_{K=K^0} = \frac{(2+d_s)(3d_s+d_s d_i+N(2+2d_i-d_s)+2)}{2(2+d_i)(d_s-d_i)^2(N-1)} \geq 0$, is of ambiguous sign. We can only say that, z^* is increasing in d_i if $(2 + 2d_i - d_s) > 0$, or, otherwise, if $(2 + 2d_i - d_s) < 0$, when $N < \frac{3d_s+d_s d_i+2}{-(2+2d_i-d_s)}$. This means z^* is increasing in d_i when the standard technology is not excessively more inefficient than the innovative technology, or, if the latter is the case, when the number of firms competing in oligopoly is not too large.