

Managing Heterogeneous and Unanchored Expectations: A Monetary Policy Analysis*

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January 6, 2016

Abstract

We study monetary policy in a New Keynesian model with heterogeneity in expectations. Agents may choose from a continuum of forecasting rules and adjust their expectations based on relative past performance. The extent to which expectations are anchored to the fundamentals of the economy turns out to be crucial in determining whether the central bank (CB) can stabilize the economy. When expectations are strongly anchored, little is required of the CB for local stability. Only when expectations are unanchored, the Taylor principle becomes a necessary condition. More aggressive policy may however be required to prevent coordination on almost self-fulfilling optimism or pessimism. When the zero lower bound on the nominal interest rate (ZLB) is accounted for, the inflation target must furthermore be high enough, in order to prevent coordination on self-fulfilling liquidity traps and deflationary spirals.

JEL Classification: E52, E32, C62

Keywords: Interest Rate Rules, Liquidity Traps, Heterogeneous Expectations, Bounded Rationality

*The authors gratefully acknowledge the financial support of NWO (Dutch Science Foundation) Project No. 40614011 "Monetary and Fiscal Policy under Bounded Rationality and Heterogeneous Expectations", and the financial support from the EU FP7 project "Integrated Macro-Financial Modeling for Robust Policy Design" (MACFINROBODS, grant 612796).

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1 Introduction

Traditionally, monetary policy is modeled under the assumption of homogeneous rational expectation (e.g. Woodford (2003)). All agents are then assumed to have perfect knowledge and use perfect model consistent expectations to forecast future variables, such as inflation and output. Surveys of consumers and professional forecasters (e.g. Mankiw et al. (2003); Carroll (2003)) as well as laboratory experiments with human subjects (e.g. Pfajfar and Zakelj (2011); Assenza et al. (2014)) show however, that there is considerable heterogeneity in the forecasts of these macroeconomic variables. This raises the question whether policy implications that follow from models with a representative agent having rational expectations are accurate, or whether this assumption is so restrictive that rational expectations models do not reflect reality and may lead to other, perhaps sometimes misleading policy recommendations. In this paper we investigate monetary policy in macroeconomic models under the alternative paradigm of bounded rationality and heterogeneous expectations (Brock and Hommes, 1997). In particular, we study how monetary policy can manage a continuum of heterogeneous expectation rules by applying the Large Type Limit (LTL) concept of Brock et al. (2005) to the New Keynesian framework. The LTL concept also allows us to give precise conceptualization of the idea of "strongly anchored" expectations in an analytically tractable way.

The importance of bounded rationality and a behavioral approach to macroeconomics has recently been stressed in the books Akerlof and Shiller (2010) and De Grauwe (2012). Assenza et al. (2016) model animal spirits through heterogeneous expectations and show the emergence and amplification of boom and bust cycles in a macroeconomic model where lenders have heterogeneous expectations about the default probability of firms. The theory of bounded rationality and heterogeneous expectations has recently also been applied to the New Keynesian Dynamic Stochastic General Equilibrium (DSGE) setting in Branch and McGough (2010), De Grauwe (2011), Massaro (2013) and in Agliari et al. (2014). These models nicely match with empirical stylized facts of output and inflation

(De Grauwe, 2012).

We study policy implications for an inflation targeting central bank (CB) under this new paradigm and these empirically relevant heterogeneous expectations in an otherwise standard New Keynesian framework. In addition, we study the effect of the zero lower bound on the nominal interest rate when expectations are heterogeneous. Expectations are assumed to be anchored around fundamental values of the model, i.e., the rational expectation equilibrium values that would arise if *all* agents were rational. However, not all agents expect exactly these fundamental values. Instead, some agents have slightly higher expectations, while other agent expect values that are somewhat lower. This heterogeneity can be interpreted in two different ways. First of all, it could be that agents make small mistakes in their otherwise adequate predictions. Alternatively, agents base their expectations on publicly available information, but also take their own personal views about the economy and about animal spirits into account.

Our agents furthermore realize that their expectations may not be perfect, and that other agents (e.g. professional forecasters) might be better at predicting the future. For this reason agents will adjust their expectations upwards if agents with higher expectations turned out to be right in the past. They do this to correct for their apparent mistakes, and to benefit from other agents that might have better information or prediction skills. The heterogeneity in expectations will however always be present. Our benchmark model specification assumes a continuum of prediction values, normally distributed around the fundamental values. We study inflation-output dynamics under heterogeneous expectations, using the large type limit concept, initially introduced by Brock et al. (2005) and used in other macroeconomic settings by Anufriev et al. (2013) and Agliari et al. (2014). We also consider the implications of discrete expectation values, by studying a stylized example with 3 different expectation values: optimists, pessimists, and fundamentalists; and a richer model where agents only form expectations with a precision of 0.5%.

Under these heterogeneous expectations we study monetary policy under three

different interest rate rules. In the first rule, the central bank responds to expected future deviations of inflation and output gap from their targets; in the second rule, the CB can respond to contemporaneous deviations from target; and in the third rule, the CB responds to the deviations from targets that occurred in the previous period. These three Taylor type interest rate rules are also studied by Bullard and Mitra (2002), who compare rational expectation results with results obtained under adaptive learning. These authors, however, assume homogeneous agents. In contrast, our focus is on monetary policy under *heterogeneous expectations*.

We find that whether the economy can be stabilized depends both on monetary policy and on the anchoring of aggregate expectations. Only when expectations are unanchored, the Taylor principle is a necessary condition for stability. When aggregate expectations are somewhat anchored to the fundamental values of the economy, stability can be achieved with weaker monetary policy. The forward-looking and contemporaneous Taylor rules then work very well. If however the CB cannot observe contemporaneous values of inflation and output gap and must instead rely on lagged values, monetary policy can easily destabilize the economy by being too aggressive when expectations are strongly anchored.

In our benchmark model with a continuum of prediction values the fundamental target steady state is typically unique, and local stability implies global stability. However, if expectations are somewhat unanchored and monetary policy is relatively weak (i.e. the Taylor principle is just satisfied), the system has a near unit root, and optimistic and pessimistic expectations can be *almost self-fulfilling*. Convergence to the fundamental steady state will then occur only very slowly. In that case, a single shock can lead to persistently high or low expectations, and it may take a long time for the economy to recover and mean revert back to the fundamental equilibrium. Furthermore, when expectations are discrete, almost self-fulfilling optimistic and pessimistic expectations imply the existence of additional steady states. This multiplicity of equilibria disappears as monetary policy becomes more aggressive. Time series simulations of the benchmark model and the model where only multiples of 0.5% are allowed look almost indistinguishable,

and both models show high persistence when monetary policy is not very strong.

In the final part of the paper, we turn to the implications of the zero lower bound of the nominal interest rate. We find that due to this lower bound, low initial inflation and output gap can initiate a fall in both expectations and realizations, that either ends when the lowest possible expectations are reached, or goes on for ever if no lower limit on expectations exists. In the latter case, the system has entered a deflationary spiral. This change in dynamics occurs due to the appearance of an additional steady state. Benhabib et al. (2001*a,b*) first highlighted the appearance of an extra equilibrium when the ZLB is introduced in a rational expectations New Keynesian framework. Evans et al. (2008), Benhabib et al. (2014) and Hommes and Lustenhouwer (2015) furthermore find that the additional steady state leads to deflationary spirals for low initial conditions in their models with boundedly rational agents.

The recent financial crises has highlighted the importance of both a lower bound on the interest rate, and its relation with low, self-fulfilling expectations. In order to fully understand liquidity traps and to come up with policy recommendations it is of crucial importance to realistically model expectations. Mertens and Ravn (2014) discuss the distinction between a liquidity trap that is driven by low economic fundamentals, and a liquidity trap that is driven by expectations. In Evans et al. (2008) and Benhabib et al. (2014) liquidity traps are driven by expectations, but these expectations are formed by homogeneous agents. Hommes and Lustenhouwer (2015) study liquidity traps under heterogeneous expectations in a model with endogenous credibility of the central bank. In the current paper we construct a different heterogeneous agent model that allows us to directly study how expectation driven liquidity traps are affected by a combination of policy parameters and the magnitude of anchoring and heterogeneity in expectations.

We find that the central bank can prevent prolonged liquidity traps with a high enough inflation target. This lowers the values of inflation and output gap from which deflationary spirals occur, and may exclude the possibility of self-fulfilling liquidity trap steady states. Alternatively, liquidity traps can be prevented if

expectations are strongly enough anchored to fundamental values of the economy.

The paper is organized as follows. In Section 2 the model, interest rate rules, and expectation formation mechanisms are presented. In Section 3 the local stability of the fundamental steady state is analyzed. Section 4 considers uniqueness and global stability and the implications of discrete expectations. Section 5 focuses on the zero lower bound and liquidity traps, and Section 6 concludes.

2 Model Specification

2.1 Economic model and interest rate rules

We use a log linearized New Keynesian model in line with Woodford (2003). Micro-foundations for this model when expectations are heterogeneous can be found in Hommes and Lustenhouwer (2015). The full model is described by a New Keynesian Phillips curve describing inflation, an IS curve describing output gap, and a rule for the nominal interest rate. Output gap (x_t) and inflation (π_t) are given by

$$x_t = E_t x_{t+1} + \frac{1}{\sigma}(E_t \pi_{t+1} - i_t) + u_t, \quad (1)$$

and

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t, \quad (2)$$

where κ , σ and β are model parameters. e_t and u_t are shocks to the economy, which will be white noise in our simulations. In the analytical analysis below we abstract from these shocks, and study the deterministic skeleton of the model.

Finally, i_t is the nominal interest rate. We consider three different interest rate rules, where the central bank responds respectively to the expectations, the contemporaneous values and the lagged values of inflation and output gap.

The first interest rate rule we consider is a forward-looking Taylor type rule given by

$$i_t = \pi^T + \phi_1(E_t \pi_{t+1} - \pi^T) + \phi_2 E_t x_{t+1}. \quad (3)$$

We assume here that the central bank can observe private sector expectations. Expectations $E_t\pi_{t+1}$ and E_tx_{t+1} are based on period $t - 1$ information, and are formed at the end of period $t - 1$. It is therefore not unreasonable to assume that the CB can base its period t decision for the interest rate, i_t , on period t private sector expectations.

Hommes and Lustenhouwer (2015) show that a forward-looking rule of the form (3), can be used to minimize the following loss function under discretion

$$E_t \sum_{i=0}^{\infty} \beta^i [(\pi_{t+i})^2 + \mu(x_{t+i})^2], \quad (4)$$

where $\mu \geq 0$ is the relative weight that the central bank assigns to the minimization of the squared output gap compared to the squared inflation. Rotemberg and Woodford (1999) show that this loss function can be derived from a second order approximation of the utility function of a representative agents. The coefficients that minimize (4) are given by

$$\pi^T = 0, \quad \phi_1^{opt} = 1 + \frac{\sigma\kappa\beta}{\mu + \kappa^2}, \quad \phi_2^{opt} = \sigma \quad (5)$$

We will consider these coefficients as a benchmark case.

Abstracting from shocks and plugging (3) into (1), gives the following model

$$x_t = (1 - \frac{\phi_2}{\sigma})E_tx_{t+1} - \frac{\phi_1 - 1}{\sigma}(E_t\pi_{t+1} - \pi^T), \quad (6)$$

$$\pi_t = \beta E_t\pi_{t+1} + \kappa x_t. \quad (7)$$

Secondly, we look at a more traditional contemporaneous Taylor rule:

$$i_t = \pi^T + \phi_1(\pi_t - \pi^T) + \phi_2 x_t. \quad (8)$$

With this interest rate rule the model is given by (7) and

$$(1 + \frac{\phi_2}{\sigma})x_t = E_tx_{t+1} + \frac{1}{\sigma}(E_t\pi_{t+1} - \pi^T - \phi_1(\pi_t - \pi^T)). \quad (9)$$

Substituting for inflation, the output gap equation can be written as

$$x_t = \frac{\sigma}{\sigma + \phi_2 + \kappa\phi_1} E_t x_{t+1} - \frac{\beta\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1} E_t \pi_{t+1} + \frac{\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1} \pi^T. \quad (10)$$

Since it is perhaps not very realistic that the CB can respond to contemporaneous values of inflation and output gap, the final interest rate rule we consider is a Taylor rule with lagged values of inflation and output gap

$$i_t = \pi^T + \phi_1(\pi_{t-1} - \pi^T) + \phi_2 x_{t-1}. \quad (11)$$

The model is then given by (7) and

$$x_t = E_t x_{t+1} - \frac{\phi_2}{\sigma} x_{t-1} + \frac{1}{\sigma} (E_t \pi_{t+1} - \pi^T) - \frac{\phi_1}{\sigma} (\pi_{t-1} - \pi^T). \quad (12)$$

2.2 Heuristic switching model

We deviate from the rational expectations hypothesis, and do not assume all agents always exactly expect the same outcome of future variables. Instead, we assume that some heterogeneity is present, with some agents expecting values that are a bit higher and some agents expecting values that are a bit lower. This heterogeneity could be caused by agents making small mistakes. Alternatively, the heterogeneity can arise because some agents think they have reasons to be more optimistic or pessimistic in their predictions than is warranted by the publicly available information. More specifically, expectations are distributed around the rational expectations equilibrium values \bar{x} and $\bar{\pi}$, given in Appendix A.¹

We furthermore assume that when agents that were more optimistic or pessimistic in their prediction turn out to be right, then other agents will learn from this and adjust their expectations in the direction of the better performing agents. Agents may, for example, think that the correct agents had additional information available to them, or just had more skills in analyzing the economic environment.

¹When either $\pi^T = 0$, or $\beta = 1$ the rational expectations equilibrium values coincide with the targets of the central bank. In that case we can alternatively interpret our expectations as being formed by some agents that trust the central bank and expect inflation and output gap to be equal to their targets, while other agents expect that the central bank will not be able to exactly achieve its goals, but that inflation and output gap will be slightly higher or lower.

We implement this way of expectation formation with a heuristic switching model as in Brock and Hommes (1997), where agents switch between simple prediction rules, or heuristics. The heuristics in our model consists of deviations from the fundamental values of the economy. The fraction of agents using the heuristic with deviation, or bias, b_h in period t is updated according to the discrete choice model with multinomial logit probabilities (see Manski et al. (1981)) given by

$$n_t^h = \frac{e^{\omega U_{t-1}^h}}{\sum_{h=1}^H e^{\omega U_{t-1}^h}}. \quad (13)$$

Here H is the total number of prediction values, and U_t^h is the fitness measure of predictor h in period t , which we will assume to be minus the last observed squared prediction error.² ω is the intensity of choice. The higher the intensity of choice, the more sensitive agents become with respect to relative performance of prediction values, and the more agents will coordinate their forecasts.

Output gap expectations are now given by a weighted average of the predictions of all types

$$E_t \pi_{t+1} = \bar{\pi} + \sum_{h=1}^H b_h n_t^h = \bar{\pi} + \sum_{h=1}^H b_h \frac{e^{-\omega(\pi_{t-1} - b_h - \bar{\pi})^2}}{\sum_{h=1}^H e^{-\omega(\pi_{t-1} - b_h - \bar{\pi})^2}}. \quad (14)$$

This equation can be written as

$$E_t \pi_{t+1} = \bar{\pi} + \frac{\frac{1}{H} \sum_{h=1}^H b_h e^{-\omega(\pi_{t-1} - b_h - \bar{\pi})^2}}{\frac{1}{H} \sum_{h=1}^H e^{-\omega(\pi_{t-1} - b_h - \bar{\pi})^2}}. \quad (15)$$

Below we will first consider the limit of H going to infinity. Brock et al. (2005) show that in this case the dynamics can be closely approximated with the so called large type limit (LTL), where there is a continuum of prediction biases. We assume that the prediction biases b_h are normally distributed around zero (so that expectations are distributed around $\bar{\pi}$), with variance s^2 .

We can now approximate (15) by the LTL obtained by replacing sample means

² $U_{t-1}^h = -(\pi_{t-1} - E_{t-2}^h \pi_{t-1})^2$ for inflation, and $U_{t-1}^h = -(x_{t-1} - E_{t-2}^h x_{t-1})^2$ for output gap.

by population means:

$$E_t \pi_{t+1} = \bar{\pi} + \frac{\int b e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} e^{-\frac{b^2}{2s^2}} db}{\int e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} e^{-\frac{b^2}{2s^2}} db}. \quad (16)$$

The large type limit can also be interpreted in terms of Bayesian updating. Agents then try to learn in each period about the correct value of b , with $N(0, s^2)$ as their prior. The likelihood function (the distribution of π_{t-1} given the true value of b) is normal, with a variance inversely related to the intensity of choice (ω).³ This means that the intensity of choice (ω) is inversely related to the perceived noise with which agents can observe the correct value of b . This interpretation is in line with the random utility model underlying the multinomial logit probabilities given in (13) (see Anderson et al. (1992)).

In the online supplementary material it is shown that (16) reduces to

$$E_t \pi_{t+1} = \frac{\bar{\pi}}{2\omega s^2 + 1} + \frac{2\omega s^2}{2\omega s^2 + 1} \pi_{t-1}. \quad (17)$$

Similarly, for output gap we can write

$$E_t x_{t+1} = \frac{\bar{x}}{2\omega s^2 + 1} + \frac{2\omega s^2}{2\omega s^2 + 1} x_{t-1}, \quad (18)$$

In the LTL model, expectations thus are a linear combination of the fundamental values of the economy and of past realizations. The weights on these values determine to what extent *aggregate expectations* are *anchored* to the fundamentals of the economy. If the weight on the fundamental values is high (near 1), aggregate expectations are always close to the fundamentals of the economy. We refer to this situation as "strongly anchored expectations". If the weight on the funda-

³After the realization of π_{t-1} , the distribution over the biases b is updated according to (13) to

$$\phi(b) = \frac{e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} \frac{1}{s\sqrt{2\pi}} e^{-\frac{b^2}{2s^2}}}{\int_{-\infty}^{\infty} e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} \frac{1}{s\sqrt{2\pi}} e^{-\frac{b^2}{2s^2}} db} = \frac{\frac{\sqrt{2\omega}}{\sqrt{2\pi}} e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} \frac{1}{s\sqrt{2\pi}} e^{-\frac{b^2}{2s^2}}}{\int_{-\infty}^{\infty} \frac{\sqrt{2\omega}}{\sqrt{2\pi}} e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2} \frac{1}{s\sqrt{2\pi}} e^{-\frac{b^2}{2s^2}} db}.$$

The right hand side of this equation can be interpreted as a posterior distribution that agents attach to the correct value of b , after observing π_{t-1} . The prior distribution of b then equals the assumed normal distribution of prediction values ($\frac{1}{s\sqrt{2\pi}} e^{-\frac{b^2}{2s^2}}$), and the likelihood function equals $\frac{\sqrt{2\omega}}{\sqrt{2\pi}} e^{-\omega(\pi_{t-1}-b-\bar{\pi})^2}$. That is, π_{t-1} was generated with a normal distribution with variance $\frac{1}{2\omega}$.

mental values is low (near 0), aggregate expectations jump around considerably in response to shocks. We refer to this situation as "unanchored expectations".

In Equation (17) and (18) it can be seen that the weight on the fundamental values is strictly decreasing in the intensity of choice (ω) and the variance of the distribution of types (s^2). The intuition for this is that a higher intensity of choice allows more and faster changes of expectations and a higher variance of the distribution of types makes it more likely that these changes move expectations towards values far away from the fundamentals of the economy. Therefore, a higher ω and s^2 imply that aggregate expectations move more in response to shocks and thus become less strongly anchored.

3 Stability

When aggregate expectations equal $E_t\pi_{t+1} = \bar{\pi}$ and $E_tx_{t+1} = \bar{x}$, inflation and output gap are equal to their rational equilibrium values. These values comprise a steady state in the two dimensional dynamical system defined by (18), (17), (7) and either (6), (10) or (12), depending on the interest rate rule.

We find that in our economy the core task of monetary policy should be providing a feedback mechanism that prevents optimistic or pessimistic expectations from becoming self-fulfilling. Here, we define self-fulfilling optimistic expectations as high *aggregate* expectations that lead to realizations that are as high as (or perhaps even higher than) these expectations. If monetary policy fails to provide mean reversion of expectations, explosive drifts of inflation and output gap, away from their fundamental values $\bar{\pi}$ and \bar{x} may arise. The policy feedback mechanism must however not be so strong that it overreacts to so small fluctuations in the economy and thereby causes larger fluctuations in the opposite direction.

More technically speaking, monetary policy must aim at making the above mentioned steady state locally (and preferably also globally) stable. The specific restrictions on monetary policy that result in stability depend on both the monetary policy rule that is used, and on how strongly expectations are anchored to the fundamental values of the economy. This is discussed in detail below.

3.1 Forward-looking Taylor rule

The first feedback mechanism we consider is responding directly to observed expectations with the forward-looking Taylor rule given by (3). By responding strongly enough to expectations, the CB can make sure that high (low) expectations do not lead to too high (low) realizations of inflation and output gap, and therefore cannot become self-fulfilling.

However, if the CB responds too strongly, high inflation expectations lead to very low inflation realizations, which (depending on how strongly expectations are anchored), can lead to very low expectations in the subsequent period. These low expectations then again induce a strong policy response that leads to very high realizations of inflation, and high expectations in the subsequent period. This process of explosive overshooting then continues with both expectations and realizations reaching ever higher and lower values.

Proposition 1 formally states the conditions for stability of the fundamental steady state under the forward-looking Taylor rule. As expected, there is both a lower and an upper bound on how aggressive monetary policy responses can be. The conditions are presented in terms of the inflation policy coefficient, ϕ_1 , and are a function of the output gap policy coefficient, ϕ_2 , and the parameters ω and s^2 (which together determine how strongly expectations are anchored). The first condition ensures that the first eigenvalue of the dynamical system, λ_1 , is smaller than $+1$; and the second condition ensures that the other eigenvalue, λ_2 , is larger than -1 . Proof of Proposition 1 is given in Appendix B.1.

Proposition 1. *(See Figure 1) When the CB adheres to the forward-looking Taylor rule given by (3), the fundamental steady state is locally stable if and only if*

$$\phi_1 > 1 - \frac{1 + (1 - \beta)2\omega s^2}{2\omega s^2 + 1} \left(\frac{\sigma}{2\omega s^2 \kappa} + \frac{\phi_2}{\kappa} \right), \quad (\lambda_1 < +1), \quad (19)$$

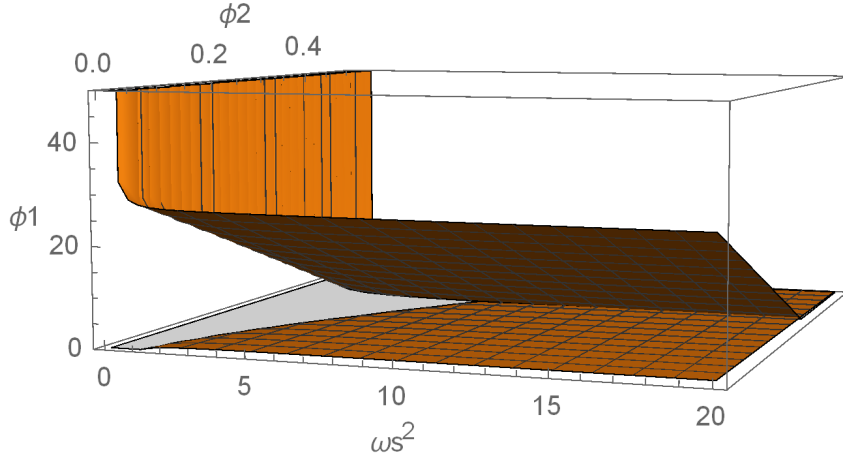


Figure 1: Stability region of fundamental steady state under the forward-looking Taylor rule. For policy parameters (ϕ_1, ϕ_2) and expectation parameters (ω, s^2) between the two surfaces, the fundamental steady state is locally stable.

and

$$\phi_1 < 1 + \frac{(1 + \beta)2\omega s^2 + 1}{2\omega s^2 + 1} \left(\frac{(4\omega s^2 + 1)\sigma}{2\omega s^2 \kappa} - \frac{\phi_2}{\kappa} \right), \quad (\lambda_2 > -1). \quad (20)$$

Figure 1 plots the two conditions for stability from Proposition 1 as a function of ωs^2 and ϕ_2 . The lower surface represents Condition (19), and the upper surface depicts Condition (20). In the figure, it can be seen that for low values of ωs^2 (where aggregate expectations are strongly anchored to the fundamental values), weak inflation policy ($0 < \phi_1 < 1$) does not lead to instability. For higher values of ωs^2 (unanchored expectations) however, the central bank must respond strongly enough to inflation in order to satisfy (19). If we let ωs^2 go to infinity, Condition (19) reduces to $\phi_1 > 1 - (1 - \beta)\frac{\phi_2}{\kappa}$, which is the well known Taylor principle that must be satisfied under rational expectations in order to obtain local determinacy.

Condition (20) also becomes more stringent as expectations become unanchored (higher ωs^2). For low values of ωs^2 the upper limit on ϕ_1 goes to infinity; and as ωs^2 goes to infinity it is required that $\phi_1 < 1 + (1 + \beta) \left(\frac{2\sigma}{\kappa} - \frac{\phi_2}{\kappa} \right)$. This condition coincides with the upper bound for local determinacy with a forward-looking Taylor rule under rational expectations (Bullard and Mitra, 2002). We can conclude that with a forward-looking Taylor rule, unanchored expectations (high values of ω and s^2) require the same restrictions on policy parameters as under

RE, while when expectations are anchored the region of stable policy parameters becomes larger.

It is further of interest whether the fundamental steady state is locally stable under the policy coefficients that minimize the loss function (4). Proposition 2 states that this always is the case. Its proof is given in Appendix B.2

Proposition 2. *When the CB adheres to the forward-looking Taylor rule given by (3), with ϕ_1 and ϕ_2 chosen as in (5) the fundamental steady state is always locally stable.*

3.2 Contemporaneous Taylor rule

Next, we consider the feedback mechanism of letting the interest rate respond to contemporaneous values of inflation and output (equation (8)). By responding strongly enough to these values any potential deviations from the fundamental values can be eliminated, including deviations caused by expectations. In that case, self-fulfilling drifts away from the fundamental steady state cannot arise.

Furthermore no matter how strongly the central bank responds, the overshooting mechanism discussed in the previous section can never occur with a contemporaneous Taylor rule. That is, a strong policy response to high inflation will never lead to low realized inflation. Indeed, we find that there is no upper bound on the policy coefficients for local stability of the fundamental steady state. This result is in line with rational expectations findings of Bullard and Mitra (2002).⁴

Proposition 3 states that when the CB uses (8), all that is required is that it responds strongly enough to inflation or output, no matter how expectations are formed. The proof is provided in Appendix B.3.

⁴Agliari et al. (2014) do find an upper limit on ϕ_1 with a contemporaneous Taylor in a model similar to ours. The reason for this is that in their model inflation expectations also depend on lagged output gap. Looking at Equations (2) and (10) it can be seen that for large ϕ_1 and ϕ_2 the effect of output gap expectations on dynamics goes to zero. Since in our model inflation expectations only depend on past inflation, the system then becomes one dimensional. The fundamental steady state will now always be locally stable since the coefficient on lagged inflation in this one dimensional system is smaller than 1 (since $\phi_1 > 1$). This reasoning would no longer hold if inflation expectations also depended on lagged output gap, as in Agliari et al. (2014).

Proposition 3. *(See Figure 2) When the CB adheres to the contemporaneous Taylor rule given by (11), the fundamental steady state is locally stable if and only if*

$$\phi_1 > \frac{2\omega s^2}{2\omega s^2 + 1} - \frac{1 + (1 - \beta)2\omega s^2}{2\omega s^2 + 1} \left(\frac{\sigma}{(2\omega s^2 + 1)\kappa} + \frac{\phi_2}{\kappa} \right), \quad (\lambda_1 < +1). \quad (21)$$

The bottom surface of Figure 2 plots Condition (21). This surface is (both qualitatively and quantitatively) very similar to condition (19), and again reduces to the Taylor principle when $\omega s^2 \rightarrow \infty$.

3.3 Lagged Taylor rule

Finally, we consider the case where the CB cannot observe contemporaneous values of inflation and output gap and instead responds to lagged values, by using Equation (11). The feedback mechanism to expectations then is an indirect one. If expectations are unanchored, there is a strong correlation between lagged values and expectations of inflation and output gap. In this case, responding to lagged values results in almost the same interest rate as would have been obtained by responding directly to observed expectations.

If, however, expectations are strongly anchored to the fundamental values, this correlation between expectations and lagged values disappears. On the one hand, this is not a problem since strongly anchored expectations also imply that expectations are already stable, so that there is no need for a feedback mechanism. However, when expectations are always fairly stable and uncorrelated with lagged values, there is a danger of destabilizing the economy by responding too strongly to these lagged values. This happens through a similar overshooting mechanism as described in Section 3.1: high inflation is, in the subsequent period, followed by a strong policy response, resulting in very low inflation. This again induces a strong policy response in the period after that, resulting in very high inflation, and so on.

Proposition 4 describes the conditions for local stability under the lagged Tay-

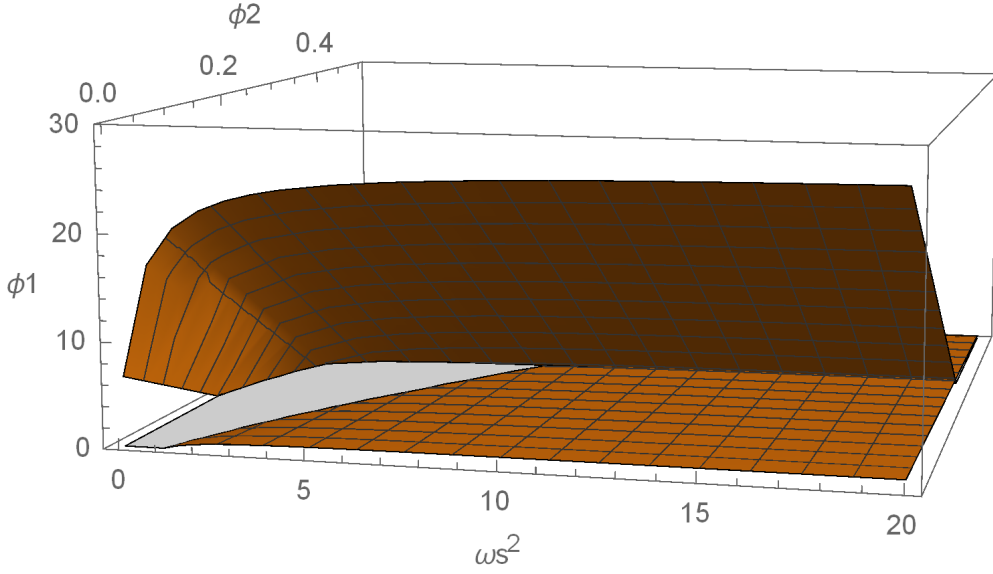


Figure 2: Stability region of fundamental steady state under the contemporaneous and lagged Taylor rule. For policy parameters (ϕ_1, ϕ_2) and expectation parameters (ω, s^2) between the two surfaces, the fundamental steady state is locally stable under the lagged Taylor rule. For the contemporaneous Taylor rule it is stable everywhere above the bottom surface.

lor rule. These conditions consist of a lower and an upper bound on the CB's policy response. The proof of the proposition is provided in B.4.

Proposition 4. (See Figure 2) *When the CB adheres to the lagged Taylor rule given by (8), the fundamental steady state is locally stable if and only if*

$$\phi_1 > \frac{2\omega s^2}{2\omega s^2 + 1} - \frac{1 + (1 - \beta)2\omega s^2}{2\omega s^2 + 1} \left(\frac{\sigma}{(2\omega s^2 + 1)\kappa} + \frac{\phi_2}{\kappa} \right), \quad (\lambda_1 < +1), \quad (22)$$

and

$$\phi_1 < \frac{2\omega s^2}{2\omega s^2 + 1} + \frac{(1 + \beta)2\omega s^2 + 1}{2\omega s^2 + 1} \left(\frac{(4\omega s^2 + 1)\sigma}{(2\omega s^2 + 1)\kappa} - \frac{\phi_2}{\kappa} \right), \quad (\lambda_2 > -1). \quad (23)$$

Note that Condition (22), is exactly the same as Condition (21), so that when it comes to reacting strongly enough to inflation, it does not matter whether a contemporaneous rule or a Taylor rule with lagged values is used.

Condition (22) and (23) are both plotted in Figure 2. The upper limit on ϕ_1 clearly differs from the one in Figure 1, by becoming more stringent instead of less stringent when expectations become more strongly anchored. The reason for this

is, as described above, that when expectations become more strongly anchored the correlation between expectations and lagged values decreases. Responding aggressively to lagged values then no longer serves the function of providing a feedback mechanism to deviations in expectations, but instead only destabilizes the economy by amplifying any fluctuations in the economy.

For the limiting case of no anchoring in expectations ($\omega s^2 \rightarrow \infty$) Condition (23) coincides with (20), and with the rational expectations conditions found by Bullard and Mitra (2002).

4 Multiple steady states and self-fulfilling expectations

In the previous Section we analyzed *local stability* of the fundamental steady state. Here we found that for unanchored expectations, local stability can be achieved under all three interest rate rules by satisfying the conditions for rational expectations local determinacy. That is, the Taylor principle and the same upper bounds that are presented by Bullard and Mitra (2002). When expectations are anchored to the fundamental values of the economy, the Taylor principle is no longer a necessary condition for local stability of the fundamental steady state. This does however not necessarily mean that convergence to this steady state will occur within a reasonable number of periods, or that convergence to the fundamental steady state occurs from all initial conditions.

In this section we investigate whether almost self-fulfilling expectations (i.e., expectations that lead to realizations that are close to the expected values) can hinder convergence to the fundamental steady state and how almost self-fulfilling expectations can be ruled out by adequately chosen monetary policy. In Section 4.1 we consider the benchmark LTL model with a continuum of prediction values. Here, almost self-fulfilling expectations take the form of a near unit root. In Sections 4.2 and 4.3 we look at almost self-fulfilling *steady states* that can arise when expectations are discrete.

4.1 Steady states in LTL model

Proposition 5 states that in the LTL model specified above, the fundamental steady state is typically unique. The proof of Proposition 5 is given in Appendix B.5.

Proposition 5. *The fundamental steady state in the LTL model is the unique steady state under the forward-looking Taylor rule unless ϕ_1 is exactly equal to the value of (19). For the lagged and contemporaneous Taylor rule the fundamental steady state is unique unless ϕ_1 exactly equals the value of (21). At these knife edge cases where one eigenvalue equals +1, there is a continuum of steady states. Steady state output gap corresponding to inflation level $\tilde{\pi}$ is then given by*

$$\tilde{x} = \frac{1}{\kappa} \left(\left(1 - \frac{2\omega s^2 \beta}{2\omega s^2 + 1}\right) \tilde{\pi} - \frac{\beta \tilde{\pi}}{2\omega s^2 + 1} \right). \quad (24)$$

From Proposition 5 it follows that if one eigenvalue equals +1, expectations are perfectly self-fulfilling for a continuum of inflation and output gap values (all comprising a steady state). When monetary policy is slightly more aggressive this eigenvalue is slightly smaller than +1 and the continuum of steady state disappears. Convergence to the fundamental steady state then occurs from all initial inflation and output levels. However, as long as the eigenvalue is close to +1 (near unit root), expectations are still *almost* self-fulfilling for a continuum of non-fundamental values, and convergence to the fundamental steady state will be very slow. This implies near random walks in inflation and output when shocks are added to the model.

Figure 3 plots the largest eigenvalue of the model with the forward-looking Taylor rule as a function of ϕ_1 , for different values of ϕ_2 . We here use the Woodford (1999) calibration with $\beta = 0.99$, $\sigma = 0.157$ and $\kappa = 0.024$. We furthermore set $\omega s^2 = 13.8$, to match the calibration of Section 4.3. Figure 3 illustrates that the largest eigenvalue can be close to 1 for a considerable range of values of ϕ_1 , especially when ϕ_2 is relatively high or relatively low. A near unit root therefore occurs quite generally in our model. For higher values of ωs^2 all curves are shifted upward, so that the largest eigenvalue becomes even higher, and for lower values of ωs^2 all curves of Figure 3 are shifted downward. The largest eigenvalues of the

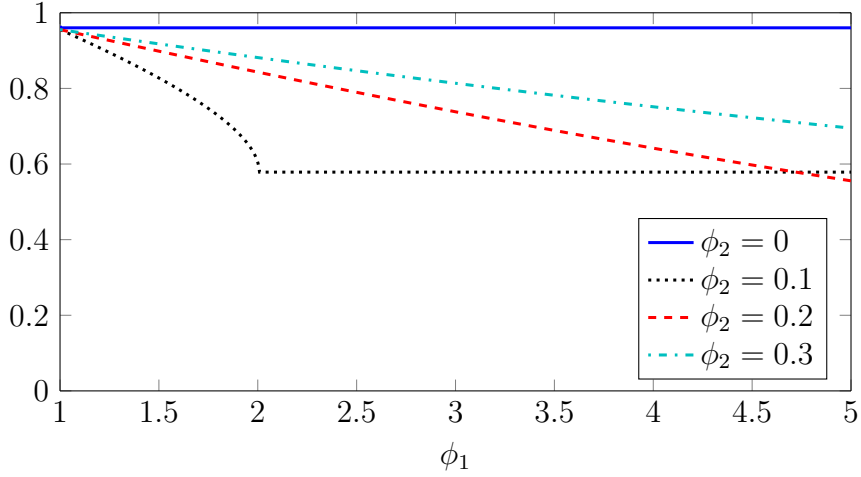


Figure 3: Largest eigenvalue of LTL model under forward-looking Taylor rule as function of ϕ_1 for different values of ϕ_2 , and for $\omega s^2 = 13.8$

models with the contemporaneous and lagged Taylor rule are similar and therefore not shown.

Furthermore, the fact that the non-fundamental steady states disappear when the eigenvalue no longer exactly equals +1 (and hence optimistic and pessimistic expectations can no longer be perfectly self-fulfilling) is heavily dependent on the assumption of a continuum of prediction values. When expectations are discrete, *almost* self-fulfilling expectations can still lead to the existence of multiple (stable) non-fundamental steady states. We illustrate this below, first with a stylized model with 3 prediction values, and then with a quantitatively more realistic model with 41 prediction values.

4.2 Steady states in the 3-type model

First consider a stylized example with three different prediction biases. Fundamentalists have a bias of zero and expect the rational expectations equilibrium values: $E_t^{fun} x_{t+1} = \bar{x}$ and $E_t^{fun} \pi_{t+1} = \bar{\pi}$. Then there are optimists and pessimists who have a bias of b and $-b$ respectively. Their expectations are given by $E_t^{opt} x_{t+1} = \bar{x} + b$, $E_t^{opt} \pi_{t+1} = \bar{\pi} + b$, $E_t^{pes} x_{t+1} = \bar{x} - b$, and $E_t^{pes} \pi_{t+1} = \bar{\pi} - b$. The above specification implies that the distribution of belief types is discrete uniform with mean zero and variance $\frac{2}{3}b^2$. Note that we allow an agent to be op-

timistic about one variable, while being pessimistic or fundamentalistic about the other, so that the fractions of agents that are optimistic and pessimistic (denoted respectively $n_t^{z,opt}$ and $n_t^{z,pes}$) may differ between the two variables ($z = x, \pi$). The fraction of fundamentalists of a variable equals $n_t^{z,fun} = 1 - n_t^{z,opt} - n_t^{z,pes}$. Aggregate expectations are given by

$$E_t x_{t+1} = \bar{x} + b(n_t^{x,opt} - n_t^{x,pes}), \quad (25)$$

$$E_t \pi_{t+1} = \bar{\pi} + b(n_t^{\pi,opt} - n_t^{\pi,pes}). \quad (26)$$

Finally, fractions are given by (13), with $h = opt, pes, fun$.

The fundamental steady state with $\pi_t = \bar{\pi}$ and $x_t = \bar{x}$ always exist in the 3-type model, no matter what interest rate rule is chosen. The fractions of optimists and pessimists in the fundamental steady state are for both variables equal to

$$\bar{n}^{opt} = \bar{n}^{pes} = \frac{1}{2 + e^{\omega b^2}}. \quad (27)$$

Because of the heterogeneity of our agents, these fractions will typically never be zero, and the highest fraction of agents that can have fundamentalistic expectations at any time is given by

$$1 - \bar{n}^{opt} - \bar{n}^{pes} = 1 - \frac{2}{2 + e^{\omega b^2}}. \quad (28)$$

This quantity crucially depends on the intensity of choice, ω , which can be seen as a measure of coordination of agents. When the intensity of choice equals zero there can never be any coordination of expectations. All expectations fractions are then always equal to $\frac{1}{3}$, and the model reduces to $x_t = \bar{x}$ and $\pi_t = \bar{\pi}$. That is, the system will always be in the fundamental steady state, and global stability is always achieved for any specification of monetary policy.

When ω goes to infinity agents coordinate perfectly, which implies that the fraction of fundamentalists in the fundamental steady state goes to 1. The expectations of all agents then equal those of a fully rational representative agent. However, infinite intensity of choice also facilitates the possibility of coordination

Table 1: Steady States of 3-type model with $\omega = +\infty$ together with conditions for existence.

Belief $\pi.x.$	Forward Looking	Contemporaneous/Lagged
Fun.Fun.	Always	Always
Opt.Opt.	$\phi_1 < 1 + \frac{\sigma}{2} - \phi_2$	$\phi_1 < \frac{2+\sigma-\phi_2}{2\beta+\kappa}$
Pes.Pes.	$\phi_1 < 1 + \frac{\sigma}{2} - \phi_2$	$\phi_1 < \frac{2+\sigma-\phi_2}{2\beta+\kappa}$
Pes.Opt.	$1 - \frac{\sigma}{2} + \phi_2 < \phi_1 < 1 + \frac{\sigma}{\kappa}(\beta - \frac{1}{2}) - \sigma + \phi_2$	$\frac{2-\sigma+\phi_2}{2\beta-\kappa} < \phi_1 < 2 - 2\sigma + \frac{\sigma+\phi_2}{\kappa}(2\beta - 1)$
Opt.Pes.	$1 - \frac{\sigma}{2} + \phi_2 < \phi_1 < 1 + \frac{\sigma}{\kappa}(\beta - \frac{1}{2}) - \sigma + \phi_2$	$\frac{2-\sigma+\phi_2}{2\beta-\kappa} < \phi_1 < 2 - 2\sigma + \frac{\sigma+\phi_2}{\kappa}(2\beta - 1)$
Opt.Fun.	$1 - \frac{\sigma}{2} < \phi_1 < 1 + \frac{\sigma}{2}$	$\frac{2-\sigma-\phi_2}{2\beta+\kappa} < \phi_1 < \frac{2+\sigma+\phi_2}{2\beta-\kappa}$
Pes.Fun.	$1 - \frac{\sigma}{2} < \phi_1 < 1 + \frac{\sigma}{2}$	$\frac{2-\sigma-\phi_2}{2\beta+\kappa} < \phi_1 < \frac{2+\sigma+\phi_2}{2\beta-\kappa}$
Fun.Opt.	$\sigma(1 - \frac{1}{2\kappa}) < \phi_2 < \frac{\sigma}{2}$	$(2\kappa - 1)\sigma - \kappa\phi_1 < \phi_2 < \sigma - \kappa\phi_1$
Fun.Pes.	$\sigma(1 - \frac{1}{2\kappa}) < \phi_2 < \frac{\sigma}{2}$	$(2\kappa - 1)\sigma - \kappa\phi_1 < \phi_2 < \sigma - \kappa\phi_1$

on non-fundamental steady states.

Since our dynamical system is linear in expectation fractions, the system for an arbitrary positive but finite value of the intensity of choice, is a convex combination of the system with zero intensity of choice, and the system with infinite intensity of choice. For this reason we first analyze this second limiting case below.

4.2.1 Steady states for infinite intensity of choice

In what follows it will be convenient to make the following assumptions about the model parameters: $0 < \kappa < 2\beta - 1$. This is not unreasonable since β is usually calibrated around 0.99, and most calibrations of κ are much lower than 1.⁵

In Proposition 6 it is stated that, for $\omega = +\infty$, nine different stable steady states can coexist. Proof of Proposition 6 is given in Appendix C.1.

Proposition 6. *(See Table 1) When $\omega = +\infty$ there are nine different locally stable steady states that each exist for some range of values of the policy parameters ϕ_1 and ϕ_2 . The fundamental steady state is the only steady state that exists for all parameter settings.*

The intuition behind the multiplicity of steady states is that there can be (almost) self-fulfilling coordination on optimism (Opt.), on fundamentalism (Fun.),

⁵See e.g. Schorfheide (2008)

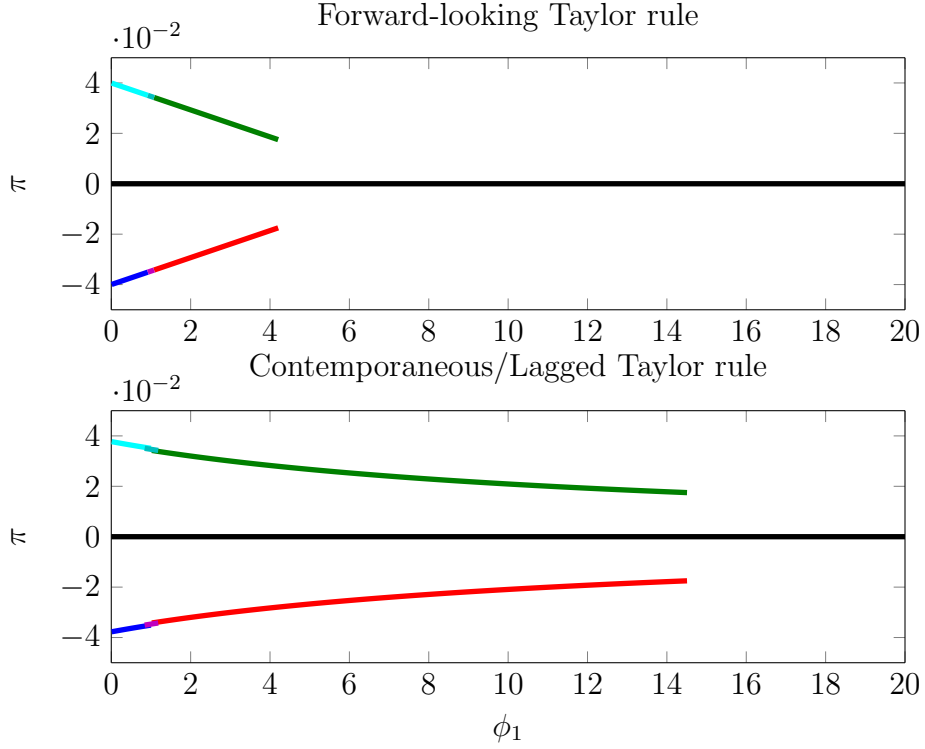


Figure 4: Steady states of 3-type model from Table 1 as a function of ϕ_1 for $\phi_2 = \sigma$. The black lines indicate the fundamental steady state and colored lines represent the non fundamental states with optimism or pessimism in inflation.

or on pessimism (Pes.). Since this can happen both for inflation and for output gap there are nine combinations of heuristics on which coordination can occur. This gives nine candidate steady states. Whether or not these steady states actually exist depends on whether expectations are sufficiently self-fulfilling to ensure that the heuristics where agents coordinate upon are indeed the best performing heuristics. This depends on the interest rate rule and policy parameters chosen by the central bank.

The first column of Table 1 summarizes the nine steady states that can exist. The first term indicates which heuristic is best performing with respect to inflation, and the second with respect to output gap. The second column of Table 1 states the conditions on the monetary policy parameters ϕ_1 and ϕ_2 , for which the corresponding steady state exist under the forward-looking Taylor rule. The final column gives the conditions for existence under the contemporaneous and lagged Taylor rule. In Appendix C.1 it is illustrated how these conditions are derived.

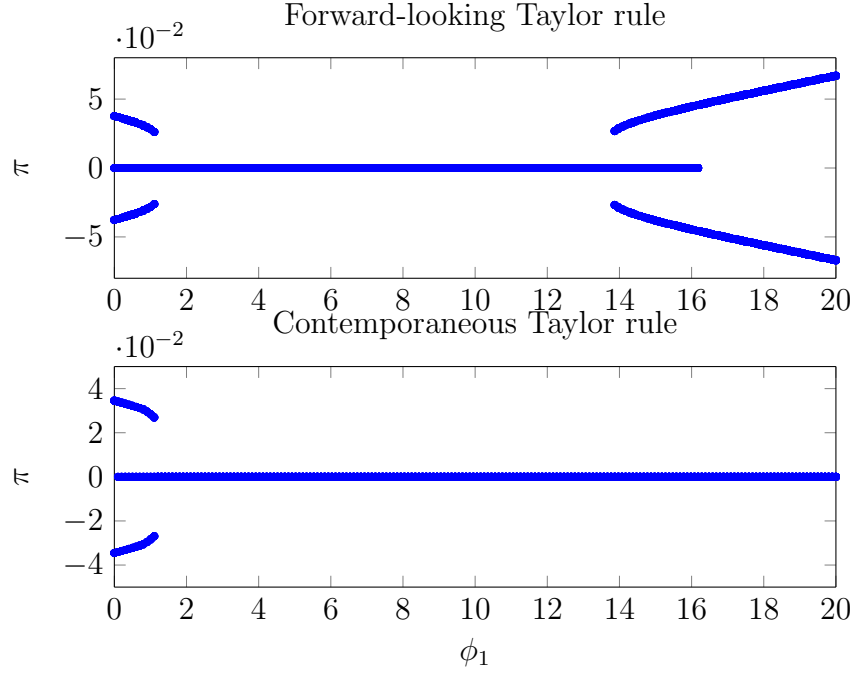


Figure 5: Bifurcation diagram in ϕ_1 for $\omega = 31709$. The upper right and lower right curves in the top panel represent a (stable) 2-cycle. All other curves depict (stable) steady states.

Figure 4 plots the inflation value of the first 7 steady states for different values of the parameter ϕ_1 in case $\phi_2 = \sigma$. The final two steady states of Table 1 do not exist in this case. We again use the Woodford (1999) calibration, and we set $\pi^T = 0$ and $b = 0.035/4$. For this value of the bias, optimists expect annualized inflation and output gap to be 3.5% above their fundamental values. The first panel of Figure 4 corresponds to the model with the forward-looking Taylor rule and the second panel to the models with the contemporaneous and lagged Taylor rules. Even though the fundamental steady state is always locally stable, it is not the only attractor. It can be seen that if the central banks wants to achieve uniqueness under the forward-looking rule it needs to respond with $\phi_1 > 4.5$ under this calibration while under the contemporaneous and lagged Taylor rule even more aggressive monetary policy is required.

4.2.2 Steady states for finite intensity of choice

As mentioned above, for strictly positive but finite intensity of choice the system is a convex combination of the systems with zero and infinite intensity of choice. This implies that the steady states presented in Table 1 still can exist for finite intensity of choice, but that they will exist for a smaller region of policy parameters. It is in this case therefore only a sufficient and no longer a necessary condition for uniqueness that the inequalities of Table 1 do not hold.

In Figure 5, a bifurcation diagram is plotted, with ϕ_1 as bifurcation parameter. The intensity of choice is calibrated such that $\bar{n}^{opt} = \bar{n}^{pes} = 0.075$, so that in the fundamental steady state only 85% of the agents are fundamentalists.⁶ It can be seen that the steady states of the previous section still exist under this lower calibration of the intensity of choice, but that policy implications are now less extreme than in the limiting case of infinite intensity of choice. It can further be seen in the figure that, as in the LTL model, the CB can also respond too strongly under the forward-looking Taylor rule. This leads to the existence of a two cycle, and causes the fundamental steady state to become unstable for high values of ϕ_1 . These two issues are discussed in the online supplementary material.

4.3 41 types

The above stylized 3-type model is not very realistic in the sense that it does not allow for different gradations in optimism and pessimism. Agents are forced to expect either exactly fundamental inflation or a fairly high or fairly low number. Some discreteness in expectations is however an empirically relevant phenomenon, since people do not form expectations with infinitely many decimals. Instead, humans prefer round numbers when reporting their expectations (a phenomenon labeled *digit preference*).⁷ It may therefore be desirable to construct a model where e.g. only multiples of 0.5% are allowed as expectations. In such a model, parameter settings that lead to almost self-fulfilling expectations will result in

⁶This requires $\omega = 31709$

⁷Curtin (2010)

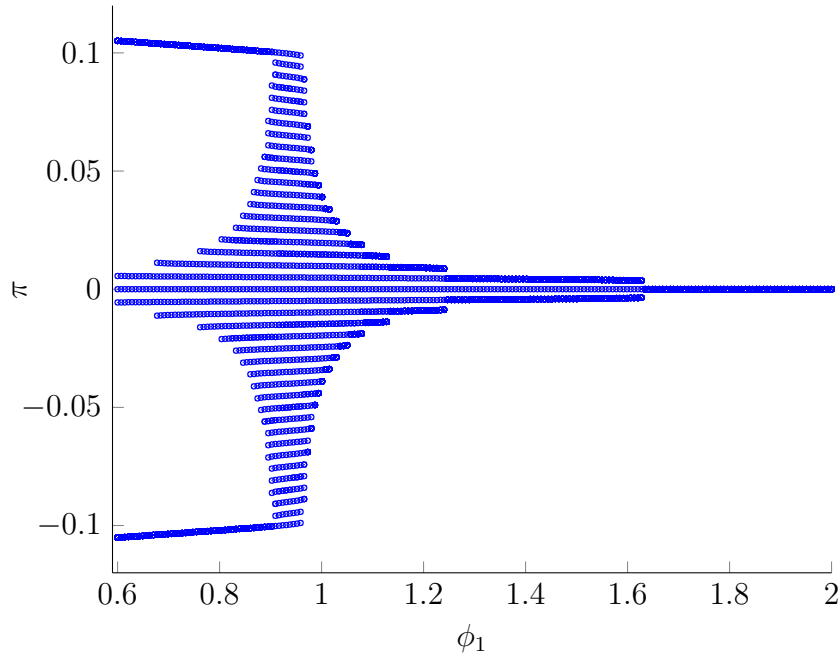


Figure 6: Bifurcation diagram of 41-type model in ϕ_1 for large intensity of choice (2 million). The blue lines represent steady states at the 41 different levels of inflation expectations.

multiple steady states in the same manner as in the 3-type model. While in the 3-type model conditions on self-fulfillingness are relatively weak, steady states must be very close to self-fulfilling in a model with a large number of different biases. This model will therefore contain characteristics of both the 3-type model and the LTL model specification.

Figure 6 plots a bifurcation diagram of a model with 41 types uniformly distributed around the fundamental values. These types are located at all multiples of 0.5% between 10% below and 10% above the fundamental value.⁸ The intensity of choice is chosen relatively high (2 million) to facilitate the existence of almost self fulfilling non-fundamental steady states. It can be seen that for each of the 41 inflation prediction values there is a range of ϕ_1 -values where this prediction comprises an almost self-fulfilling steady state. For lower intensity of choice this range becomes smaller, just as in the 3-type model, and at some point non-fundamental steady states only exist at a single value of ϕ_1 , just as in the LTL model.

Figure 7 shows a simulated time series of inflation and output gap of the

⁸Measured in annualized values.

LTL model, the 3-type model, and the model with 41 types. We again use the calibration of Woodford (1999) and set $s = 0.059/4$, so that the variance of the distribution of types in the LTL model, matches that of 41-type model. We calibrate the intensity of choice at $\omega = 63500$ to let the 41-type model match expectations from survey data.⁹ At this calibration, the interquartile range of the expectations distribution in the fundamental steady state is 1% (in annualized terms). Outside of this steady state the interquartile range then typically is 1.5%, and less for realized values close to the highest or lowest possible prediction value. This is in line with the findings of Mankiw et al. (2003), who show that the interquartile range of the Livingston Survey and the Survey of Professional Forecasters is around 1%.

Shocks to inflation and output gap are white noise, and have an annualized standard deviation of 1.5%. The interest rate rule that is used is the forward-looking Taylor rule, with $\phi_1 = 1.5$ and $\phi_2 = \sigma = 0.157$. This specification minimizes (4) when the weight on output gap is $\mu = 0.007$.

In the top panels of Figure 7 it can be seen that there are large drifts in inflation and output gap in the LTL model, even though there is no autocorrelation in shocks. This is due to the fact that, even though monetary policy is considerably more aggressive than required by the Taylor principle, the largest eigenvalue (0.882) is still relatively close to +1.

Turning to the the 3-type model in the middle panels of Figure 7, it can be seen that this model nicely captures the general sentiments in terms of optimism and pessimism that appear in the LTL specification, but that the 3-type model is not rich enough to correspond closely to the LTL model quantitatively. It can furthermore be seen that periods of optimism about inflation arise together with periods of pessimism about output gap. This is consistent with Table 1 from which it follows that for the current policy parameter setting there are three different steady states in the 3-type model: the Fun.Fun., the Opt.Pes., and the

⁹Note that the magnitude of the intensity of choice depends on the units of measurement of the data. 63500 should therefore not necessarily be seen as a high number. If we interpret the LTL model in terms of Bayesian updating, then $\omega = 63500$ implies that the perceived noise agents encounter in observing true values has a standard deviation of 1.12% of annualized inflation (see footnote 3).

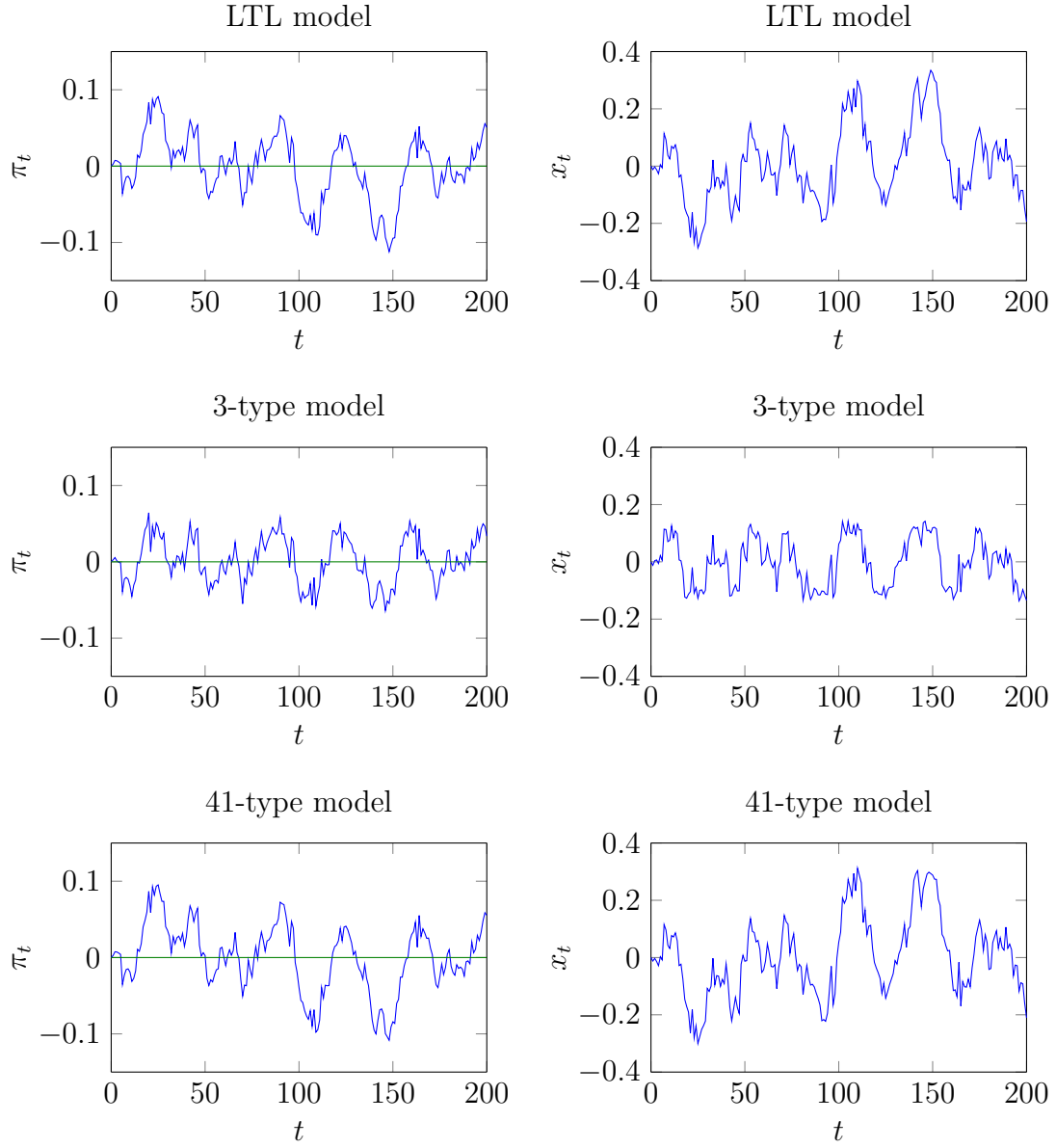


Figure 7: Simulated time series under three different model specifications with the forward-looking Taylor rule and $\pi^T = 0$, $\phi_1 = 1.5$ and $\phi_2 = \sigma = 0.157$.

Pes.Opt. steady state. This tells us that combinations of high output and low inflation expectations or vice versa are *almost* self-fulfilling. This implies that, in the presence of shock, in the 3-type model there is switching between coordination on these two levels and on the fundamental steady state, but also that in the LTL model temporary coordination on the same three levels arises.

Finally, the dynamics of the 41-type model are almost indistinguishable from the LTL dynamics. So even though this model consists of discrete expectations, its simulated time series are almost the same as those of the continuous expectations LTL model. We conclude that both the LTL model and the 3-type model show features that are also present in the 41-type model.

5 Zero lower bound on the interest rate

In this section the effect of a zero lower bound (ZLB) on the nominal interest rate is investigated. This lower bound turns out to have important consequences for the dynamics of our models. The global stability results of the previous section no longer hold when the ZLB is accounted for. Instead, prolonged liquidity traps can arise in the form of pessimistic steady states or even deflationary spirals with ever decreasing inflation and output gap.

Since the interest rate in our model, i_t is measured in percentage deviations from steady state, this variable is equal to $i_t = -i^* = \log(\beta)$ when the actual interest rate is at its zero lower bound. In normal times the interest rate is still given by one of the interest rate rules of Section 2.1. However, when this rule implies that the $i_t < -i^*$, then i_t is instead set equal to $-i^*$. When the zero lower bound on the interest rate is a binding constraint we say that the system is in the "ZLB region". Otherwise the system is in the "positive interest rate region".

In the ZLB region the model is described by

$$x_t = E_t x_{t+1} + \frac{1}{\sigma} E_t \pi_{t+1} + \frac{i^*}{\sigma}, \quad (29)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t. \quad (30)$$

5.1 LTL under the Zero Lower Bound

First, consider the benchmark LTL model with a continuum of prediction values. When the zero lower bound is binding, the policy coefficients ϕ_1 and ϕ_2 can no longer provide a feedback mechanism that prevents optimistic or pessimistic expectations from becoming self-fulfilling. Low inflation and output gap can then induce a self-fulfilling deflationary spiral with ever decreasing inflation and output gap. Whether this will occur or not for a given level of inflation and output gap depends on how strongly expectations are anchored to the fundamentals of the economy, and on the inflation target. Technically speaking, it depends on the existence and position of a non-fundamental steady state in the ZLB region. This is discussed in Proposition 7. Proof of the proposition is given in Appendix D.1.

Proposition 7. *The unique ZLB steady state of the LTL model is given by*

$$\pi^z = \frac{(\sigma(1 + (1 - \beta)2\omega s^2) + \kappa(2\omega s^2 + 1)) \bar{\pi} + \kappa(2\omega s^2 + 1)^2 i^*}{\sigma(1 + (1 - \beta)2\omega s^2) - \kappa(2\omega s^2 + 1)2\omega s^2}, \quad (31)$$

and

$$x^z = \frac{((1 - \beta)\frac{\sigma}{\kappa}(1 + (1 - \beta)2\omega s^2) + (2\omega s^2 + 1)) \bar{\pi} + (1 + (1 - \beta)2\omega s^2)(2\omega s^2 + 1)i^*}{\sigma(1 + (1 - \beta)2\omega s^2) - \kappa(2\omega s^2 + 1)2\omega s^2}. \quad (32)$$

For

$$\omega s^2 < \frac{1}{4} \left(\sqrt{(1 - (1 - \beta)\frac{\sigma}{\kappa})^2 + 4\frac{\sigma}{\kappa}} - (1 - (1 - \beta)\frac{\sigma}{\kappa}) \right), \quad (33)$$

the zero lower bound is not binding when inflation and output gap are equal to (31) and (32), so that the steady state does not exist. Recovery to the positive interest region occurs for all initial conditions in the ZLB region in this case.

If (33) does not hold, the steady state lies in the ZLB region and is an unstable saddle-point. All initial conditions above the stable eigenvector through (π^z, x^z) then imply recovery, while all initial conditions below it result in a deflationary spiral. This eigenvector has slope

$$-\frac{1 - (1 - \beta)\frac{\sigma}{\kappa} + \sqrt{(1 - (1 - \beta)\frac{\sigma}{\kappa})^2 + 4\frac{\sigma}{\kappa}}}{2\sigma}. \quad (34)$$

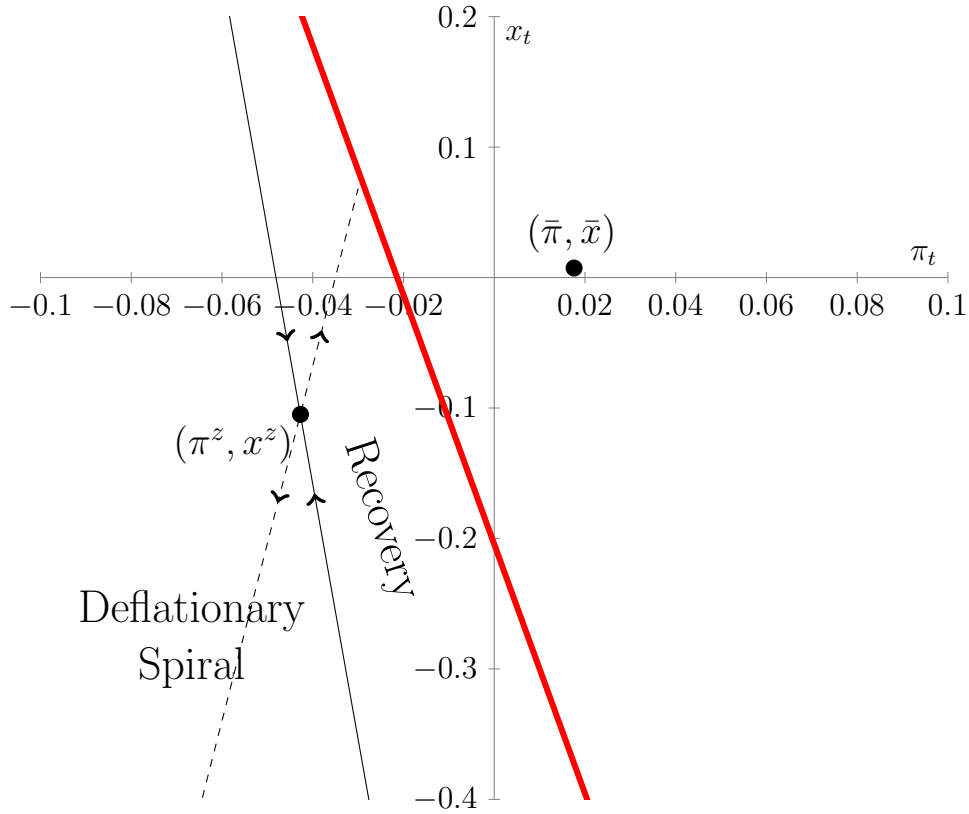


Figure 8: Regions of recovery and a deflationary spiral in the annualized (π, x) -plane for $\omega s^2 = 13.8$. The thick red line indicates the ZLB. The black dot just below 2% inflation indicates the target steady state, while the black dot at the bottom of the figure depicts the unstable ZLB saddle steady state from Proposition 7. The unstable eigenvector through this saddle is depicted by the dashed line, while the stable eigenvector is depicted by the solid black line. For initial conditions to the left of this black line a deflationary spiral occurs, and for initial conditions to the right of this line inflation and output gap will recover and increase to the positive interest rate region

Figure 8 illustrates the "recovery region" and the "deflationary spiral region" in the (π, x) -plane. Here we set $\pi^T = 2\%$, and otherwise use the calibration of the previous section, so that $\omega s^2 \approx 13.8$. The red line in Figure 8 depicts the zero lower bound. For values of inflation and output gap above this line the nominal interest rate is positive and convergence to the fundamental steady state $(\bar{\pi}, \bar{x})$ can occur. For combinations of inflation and output gap below this line the zero lower bound is a binding constraint. The black line indicates the stable eigenvector through the unstable saddle steady state of Proposition 7. Combinations of inflation and output gap above (i.e. to the right of) this line are not too low, so that recovery to

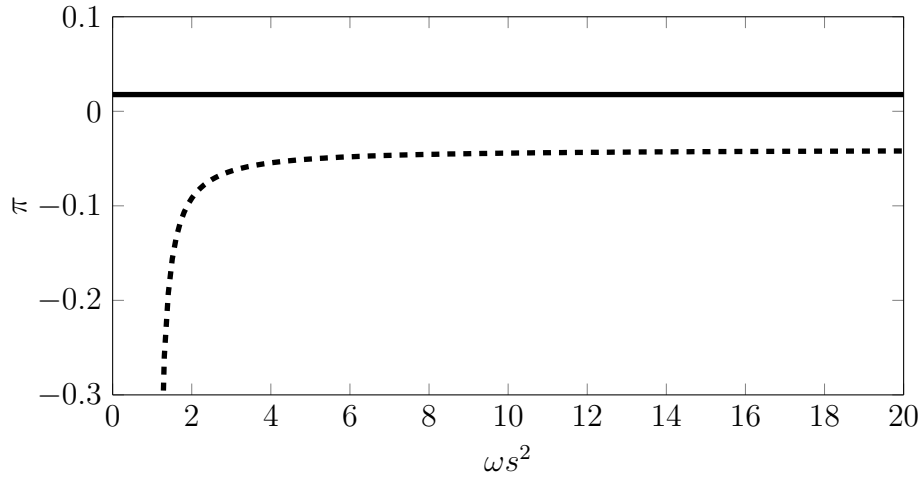


Figure 9: Stable target steady state (solid) and ZLB saddle steady state (dashed) for different levels of anchoring of expectations.

the positive interest region occurs. Inflation and output gap combinations below the black line lead to declining inflation and output gap and hence a deflationary spiral.

The position of the black line depends on the position of the ZLB steady state, which in turn depends on the anchoring of expectations and the inflation target. As can be seen in Equation (31), the inflation level of the ZLB steady state depends linearly on $\bar{\pi}$, and hence linearly on the inflation target. This implies that increasing the inflation target linearly moves the black line in Figure 8 down (to the left) and thereby linearly increases the recovery region and decreases the deflationary spiral region. So even when the ZLB is binding and the CB loses its control over the interest rate, it can still affect the economy with its inflation target.

The relation between the anchoring of expectations and the size of the deflationary spiral region is slightly more complex. Figure 9 shows how exactly the position of the ZLB steady state depends on the anchoring of expectations, with ωs^2 on the horizontal axis and inflation on the vertical axis. The dashed curve represents the ZLB saddle steady state, while the solid line depicts the stable target steady state.

It can be seen in Figure 9 that for low anchoring of expectations (high ωs^2)

the ZLB saddle steady state lies relatively close to the fundamental steady state. As ωs^2 increase towards infinity (unanchored expectations) the ZLB steady state approaches its limiting value of $(\pi^z, x^z) = (-i^*, -\frac{1-\beta}{\kappa}i^*)$, and a deflationary spiral becomes more likely. Decreasing ωs^2 initially only slowly changes the position of the steady state. However, as expectation become more strongly anchored (low ωs^2) the inflation level of the steady state is rapidly decreased. This implies a large movement of the black line in Figure 8, and a considerable decrease in the deflationary spiral region. Now, very low levels of inflation and output gap are needed for a deflationary spiral to occur. When expectations become even more strongly anchored, the ZLB saddle steady state disappears altogether, and the fundamental steady state becomes globally stable.

5.2 ZLB with finitely many expectation values

Above, we found that in the benchmark LTL model deflationary spirals can arise when the zero lower bound on the interest rate is introduced, and that these deflationary spirals can be prevented by strongly anchored expectations, or by a high inflation target. But how dependent are these results to the assumption of a continuum of prediction values?

Consider the 3-type model with fundamentalists, optimists and pessimists, and the model with 41-expectation values, discussed in Sections 4.2 and 4.3. In these models the possible values that can be taken by expectations is limited, and expectations cannot become unboundedly negative. In contrast with the LTL model, a deflationary spiral can therefore not occur. The Coordination on pessimistic expectations can however occur, in the form of almost self-fulfilling pessimistic steady states. We refer to such a steady state where the ZLB is binding as a "liquidity trap steady state".

The most interesting case of a pessimistic steady state arises when pessimistic expectations about at least one variable are more than self-fulfilling. That is, a steady state can exist with pessimistic expectations, where the realized values of inflation (and/or output gap) are even lower than the expectations. If the model

then would allow agents to decrease their expectations (for example by allowing for a larger number of different biases to choose from), this steady state would no longer be a steady state. Instead agents would choose lower expectations for the next period. This would decrease both inflation and output gap, and lead agents to choose even lower expectations for both variables in the period after that, again reducing inflation and output gap. This process would go on until the lowest possible expectations are chosen about at least one variable. In this sense reaching the steady state with the most pessimistic expectations in a model with bounded expectation values represents a liquidity trap similar to a deflationary spiral in the LTL model.

In Section 5.2.1 we consider such a steady state in the 3-type model and analyze its properties. We find that the anchoring of expectations and the level of the inflation target affect the possibility of a liquidity trap in a similar manner as in the LTL model. These qualitative features also carry over to liquidity trap steady states in a richer model with more types, like the one discussed in Section 4.3. In Section 5.2.2 we investigate, in this quantitatively more realistic model, how shocks in the economy can trigger self-fulfilling pessimistic coordination and how policy can prevent this.

5.2.1 Pessimistic steady states in 3-type model

When the ZLB is introduced to the model 3-type model of Section 4.2 the interest rate will be constraint by its lower bound when pessimistic expectations dominate.¹⁰ Conditions on existence of pessimistic steady states then change compared to Section 4.2. The most interesting liquidity trap steady state is the steady state where most agents have pessimistic expectations about both inflation and output gap (Pes.Pes.). Proposition 8 states the conditions for existence of this pessimistic steady state under the zero lower bound. Its proof is given in Appendix E.1

Proposition 8. *When the zero lower bound is binding the liquidity trap steady*

¹⁰Here it is assumed that the calibration is such that it is possible that the ZLB can become binding in the 3-type model. In the uninteresting case that this cannot happen, the model always behaves exactly as in Section 4.2.

state where pessimism is the best performing heuristic for both variables exists if and only if

$$\bar{\pi} + i^* < b \left(\sigma m_t^x + \min(m_t^\pi - \frac{\sigma}{2}, (1 + \beta \frac{\sigma}{\kappa}) m_t^\pi - \frac{\sigma}{2\kappa}) \right). \quad (35)$$

Here $m_t^\pi = n_t^{\pi, pes} - n_t^{\pi, opt}$ and $m_t^x = n_t^{x, pes} - n_t^{x, opt}$ lie between 0 and 1 and are increasing in the intensity of choice.

The condition in Proposition 8 is least stringent when the intensity of choice is infinite, and all agents are pessimistic. In this case $m_t^\pi = m_t^x = 1$, and the condition reduces to $\bar{\pi} + i^* < b(1 + \frac{\sigma}{2})$. As the intensity of choice is decreased, condition (35) becomes more stringent. In the other limiting case of $\omega = 0$ the condition reduces to $m_t^\pi = m_t^x = 0$, which can never be satisfied.

It can also be seen in Equation (35) that the condition on existence becomes more stringent as b , and thereby the variance of the distribution of types, decreases. Finally, it follows from Proposition 8 that the pessimistic steady state can also be made to disappear by increasing the inflation target, and thereby $\bar{\pi}$. We can therefore conclude that, as in the benchmark LTL model, a liquidity trap can be prevented by expectations that are strongly anchored around the fundamental values, or by a high inflation target.¹¹

5.2.2 Preventing liquidity traps in the 41-type model

We now turn to the question of whether relatively small shocks to the economy can trigger self-fulfilling pessimistic expectations, and how this can be prevented with appropriate policy measures. To study these questions we turn to the quantitatively more realistic model from section 4.3, with 41 multiples of 0.5% as possible biases.

In this model, liquidity trap steady states can also exist and simulations show that the conditions on existence of these steady states follow the same qualitative features as those in the 3-type model. That is, the possibility of a liquidity trap

¹¹Other liquidity trap steady states show these same qualitative features for conditions on existence. Results are available on request.

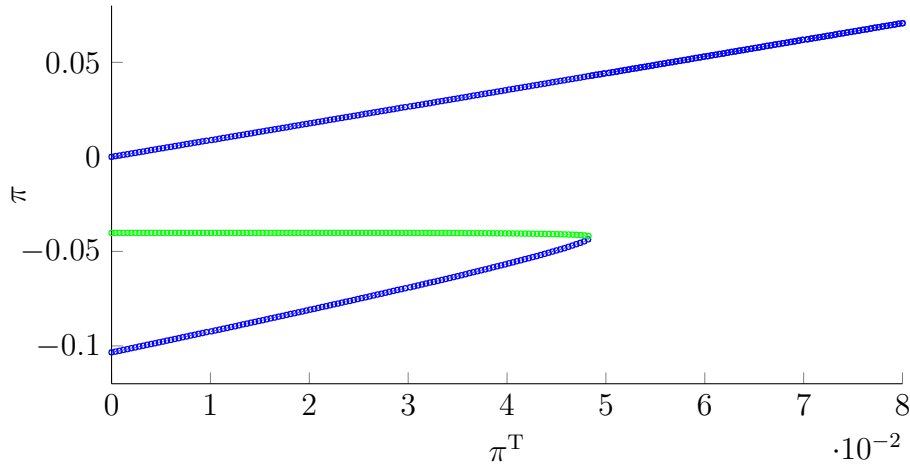


Figure 10: Bifurcation diagram of 41-type model in π^T for $\omega = 63500$. The upper blue curve represents the fundamental steady state, and the lower blue curve the liquidity trap steady state. The green curve depicts the unstable steady state that separates their basins of attraction.

disappears if expectations are strongly anchored or if the inflation target is high enough.

Figure 10 and 11 presents bifurcation diagrams of the 41-type model with the zero lower bound, with respectively π^T and ω as bifurcation parameters. The same calibration as in Section 4.3 is used. It can be seen that for low values of π^T and for high values of ω (weak anchoring of expectations), there exists two stable steady states (blue): the fundamental steady state at $\pi = \bar{\pi}$, and a liquidity trap steady state with low inflation, where pessimistic expectations dominate. The basin of attraction of these two steady states are separated by an unstable steady state (green). As π^T is increased or ω is decreased, the liquidity trap steady state comes closer to the basin of attraction of the fundamental steady state, and eventually ceases to exist. This implies that the fundamental steady state can be made globally stable with a high enough inflation target or with strongly anchored expectations.

Figure 12 illustrates how the 41-type model is affected by the zero lower bound, and how a raised inflation target can be used to prevent self-fulfilling coordination on pessimism. A similar figure could be made for a decreased intensity of choice. In the simulated time series the same random seed is used as in Figure 7.

The first column of Figure 12 shows the time series of inflation, output gap

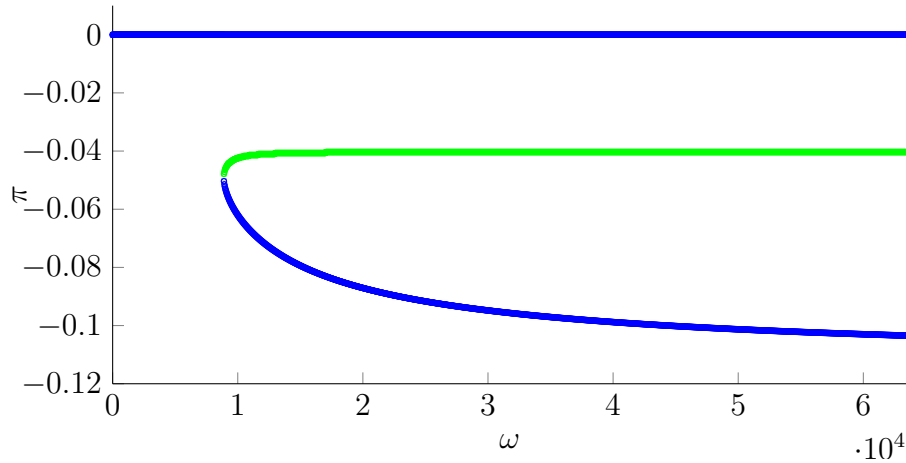


Figure 11: Bifurcation diagram of 41-type model in ω for $\pi^T = 0$. The upper blue curve represents the fundamental steady state, and the lower blue curve the liquidity trap steady state. The green curve depicts the unstable steady state that separates their basins of attraction.

and the nominal interest rate for the case of $\pi^T = 0$. The first part of the dynamics (where the nominal interest rate is positive) are exactly as in the top panels of Figure 7. However, in the bottom left panel of Figure 12 it can be seen that the wave of pessimism around period 100 results in a desired interest rate (blue) that is below its lower bound, so that the actual interest rate (green) is set at $-i^*$. The combination of low inflation and a nominal interest rate bounded by its lower bound implies a high real interest rate. This reinforces the wave of pessimism, and facilitates a self-fulfilling decline in inflation and output gap expectations that comes to a halt only when the lowest possible expectations about both variables (and thereby the liquidity trap steady state from Figure 10) are reached. This pessimistic steady state lies quite far from the basin of attraction of the fundamental steady state, so that even when series of positive shocks occur (around periods 120 and 160), the economy keeps moving back to the liquidity trap steady state.

In the middle panels of Figure 12 the annualized inflation target is increased to 2%. From Figure 10 we know that the pessimistic steady state then still exists, and we indeed observe that this steady state is reached around period 100. However, the steady state now lies closer to the basin of attraction of the fundamental steady state, so that recovery to the fundamental steady state occurs after a sequence

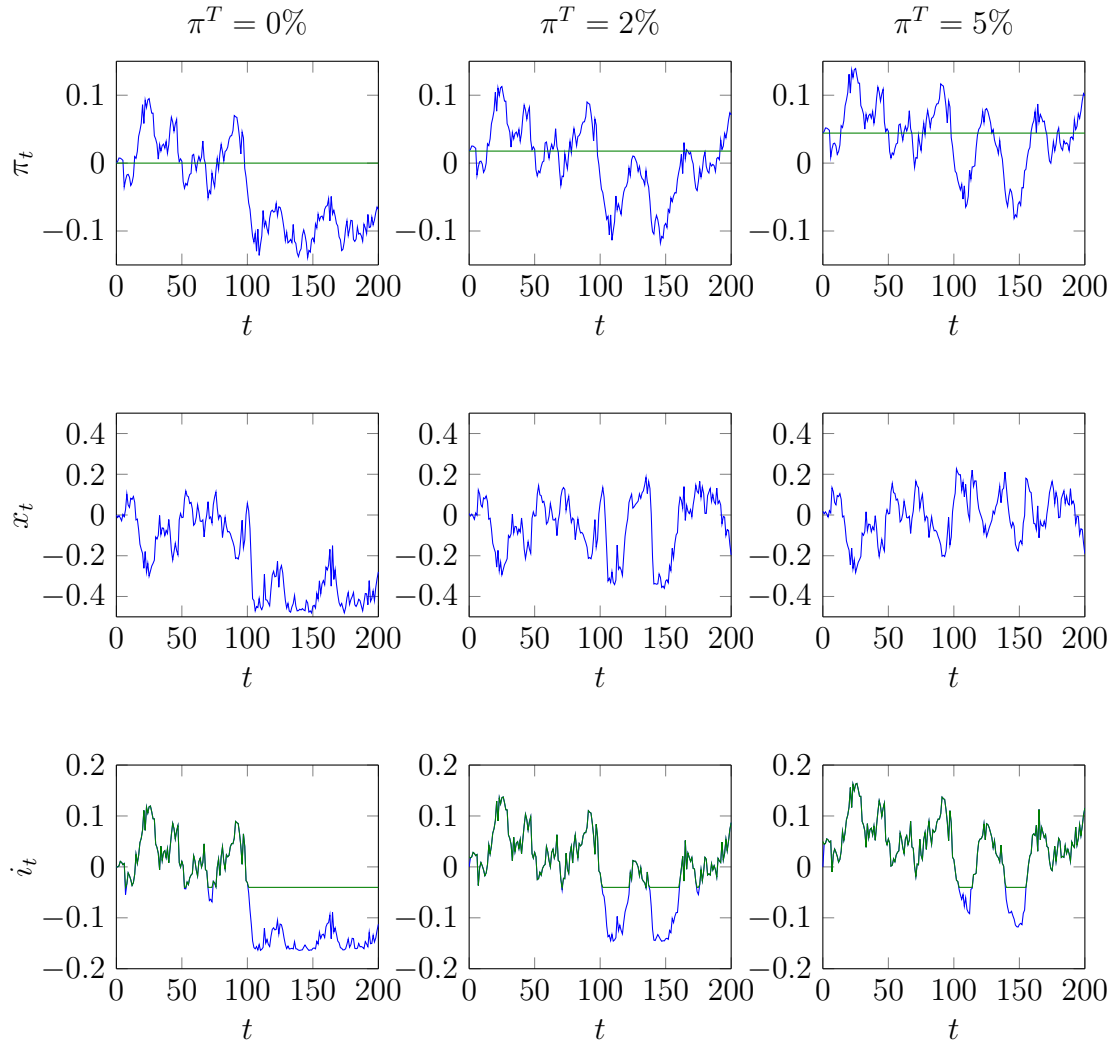


Figure 12: Simulated time series of 41-type model with the ZLB, for different values of the inflation target (π^T).

of positive shocks. A new wave of pessimism then leads to a new liquidity trap around period 150, but after some time recovery again occurs.

Finally, the right column of Figure 12 depicts the case where the inflation target is increased to 5%. As can be seen in Figure 10, the deflationary spiral steady state does not exist for this value of the policy parameter. In the bottom right panel of Figure 12 it can be seen that waves of pessimism still lead the zero lower bound to become binding. However, in the absence of shocks inflation and output gap now would always start to increase towards the fundamental steady state. Consequently, periods of low inflation and low output gap are less severe, and never last very long.

6 Conclusion

We study a New Keynesian macroeconomic model with heterogeneity in expectations. In this setup we compare three different interest rate rules and obtain a number of policy recommendations. To achieve local stability of the fundamental steady state, the central bank must prevent self-fulfilling coordination on optimistic or pessimistic expectations by responding strongly enough to (lagged) inflation and output gap, or their expectations. The Taylor principle is however only a necessary condition for local stability when expectations are unanchored. When aggregate expectations are strongly anchored to the fundamentals of the economy (because there is not much heterogeneity in expectations, and agents only slowly change their predictions) the CB is able to stabilize the economy with fairly weak monetary policy.

However, even when the fundamental steady state is locally stable, convergence to it may be quite slow due to almost self-fulfilling expectations and corresponding near unit root behavior. When expectations are discrete, these almost self-fulfilling expectations may furthermore lead to the existence of non-fundamental steady states where agents coordinate on optimism or pessimism. The Central bank can mitigate these problems with more aggressive policy than otherwise required. If the CB responds to lagged values of inflation and output gap (e.g. because it cannot observe contemporaneous values) it must however take care not to destabilize the economy with policy that is too aggressive.

When the zero lower bound on the nominal interest rate is taken into account, convergence to the targets of the CB cannot be guaranteed just by the reaction coefficients of the monetary policy rule. Negative shocks can now drive the economy to a liquidity trap with a zero interest rate and low inflation and output gap (expectations). If there is no lower limit on expectation values that agents may consider, a liquidity trap can take the form of a self-fulfilling deflationary spiral with ever decreasing inflation and output gap.

We find that prolonged liquidity traps can be prevented by increasing the inflation target, or by increasing the anchoring of expectations. While the latter

cannot be directly controlled by the CB, this does not mean that in the real world the anchoring of expectation is not affected by the actions of the central bank. Expectations might for example become more strongly anchored around the fundamental values after a decade of stable inflation and output gap. After such a time of stability agents would not be inclined to expect very high or very low inflation, even after a shock. The variance of the expectation values considered by agents, as well as the amount of switching between expectation values (intensity of choice) would then have been reduced by the performance of the central bank. On the other hand, if, during some years of economic turmoil, inflation and output are very volatile and stray far from their fundamental values, expectations will become more unanchored, which makes it more likely that the economy locks into a liquidity trap. It may then take a long time before the economy can recover from such an (almost) self-fulfilling equilibrium.

A Rational expectations equilibrium

Since there are no autocorrelated shocks in the model the rational expectations equilibrium path coincides (at least under determinacy) with the perfect foresight steady state of the model. We solve for the perfect foresight steady state values by filling in $E_t x_{t+1} = x_t = \bar{x}$ and $E_t \pi_{t+1} = \pi_t = \bar{\pi}$ in equations (7), (6), (10) and (12). Under all three interest rate specifications, this gives

$$\bar{\pi} = \frac{\kappa(\phi_1 - 1)}{(1 - \beta)\phi_2 + \kappa(\phi_1 - 1)} \pi^T \quad (\text{A.1})$$

$$\bar{x} = \frac{(1 - \beta)(\phi_1 - 1)}{(1 - \beta)\phi_2 + \kappa(\phi_1 - 1)} \pi^T \quad (\text{A.2})$$

B LTL without the ZLB

B.1 Proof Proposition 1

Under the forward-looking Taylor rule, the Jacobian in the fundamental steady state equals

$$B \begin{pmatrix} 1 - \frac{\phi_2}{\sigma} & -\frac{\phi_1 - 1}{\sigma} \\ \kappa(1 - \frac{\phi_2}{\sigma}) & \beta - \kappa \frac{\phi_1 - 1}{\sigma} \end{pmatrix},$$

with

$$B = \frac{2\omega s^2}{2\omega s^2 + 1}. \quad (\text{B.1})$$

The eigenvalues therefore are given by

$$\lambda_{1,2} = \frac{B}{2} \left(\left(1 + \beta - \frac{\phi_2}{\sigma} - \kappa \frac{\phi_1 - 1}{\sigma} \right) \pm \sqrt{\left(1 + \beta - \frac{\phi_2}{\sigma} - \kappa \frac{\phi_1 - 1}{\sigma} \right)^2 - 4\beta \left(1 - \frac{\phi_2}{\sigma} \right)} \right).$$

Local stability requires $\lambda_1 < 1$ and $\lambda_2 > -1$. By keeping only the square root on one side of the equation and taking squares, $\lambda_1 < 1$ can be written as

$$\phi_1 > 1 + \left(-\beta B - \frac{1}{B} + 1 + \beta \right) \frac{\sigma}{\kappa} + (\beta B - 1) \frac{\phi_2}{\kappa}, \quad (\text{B.2})$$

Filling in B from (B.1) results in Condition (19).

Similarly, $\lambda_2 > -1$ can be written as

$$\phi_1 < 1 + (1 + \beta + \beta B + \frac{1}{B})\frac{\sigma}{\kappa} - (1 + \beta B)\frac{\phi_2}{\kappa}, \quad (\text{B.3})$$

from which Condition (20) can be obtained.

B.2 Proof Proposition 2

When ϕ_1 and ϕ_2 equal the values given in (5), the eigenvalues reduce to

$$\lambda_{1,2} = \frac{B}{2} \left(\left(\beta - \frac{\kappa^2}{\mu + \kappa^2} \right) \pm \sqrt{\left(\beta - \frac{\kappa^2}{\mu + \kappa^2} \right)^2} \right), \quad (\text{B.4})$$

Since μ is the relative weight on output gap in the loss function and therefore is nonnegative, λ_2 reduces to zero and λ_1 becomes

$$\lambda_1 = B \left(\frac{\beta\mu - (1 - \beta)\kappa^2}{\mu + \kappa^2} \right). \quad (\text{B.5})$$

Both eigenvalues are therefore inside the unit circle as long as $\lambda_1 < 1$, which is satisfied since $0 < \beta < 1$, and $B < 1$.

B.3 Proof Proposition 3

Under the contemporaneous Taylor rule, the Jacobian in the fundamental steady state equals

$$B \begin{pmatrix} \frac{\sigma}{\sigma + \phi_2 + \kappa\phi_1} & -\frac{\beta\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1} \\ \kappa \frac{\sigma}{\sigma + \phi_2 + \kappa\phi_1} & \beta - \kappa \frac{\beta\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1} \end{pmatrix}$$

The eigenvalues therefore are given by

$$\lambda_{1,2} = \frac{B}{2} \left(\left(\frac{(1 + \beta)\sigma + \beta\phi_2 + \kappa}{\sigma + \phi_2 + \kappa\phi_1} \right) \pm \sqrt{\left(\frac{(1 + \beta)\sigma + \beta\phi_2 + \kappa}{\sigma + \phi_2 + \kappa\phi_1} \right)^2 - 4\beta \left(\frac{\sigma}{\sigma + \phi_2 + \kappa\phi_1} \right)} \right).$$

$\lambda_1 > \lambda_2$ always holds, so local stability requires $\lambda_1 < 1$ and $\lambda_2 > -1$. The first condition can be written as

$$\phi_1 > B + (-\beta B^2 + (1 + \beta)B - 1)\frac{\sigma}{\kappa} + (\beta B - 1)\frac{\phi_2}{\kappa}. \quad (\text{B.6})$$

Filling in B from (B.1) gives Condition (21).

$\lambda_2 > -1$ be written as

$$-4\beta\left(\frac{\sigma}{\sigma + \phi_2 + \kappa\phi_1}\right) < \frac{4}{B^2} + \frac{4}{B} \frac{(1 + \beta)\sigma + \beta\phi_2 + \kappa}{\sigma + \phi_2 + \kappa\phi_1}, \quad (\text{B.7})$$

which is always satisfied for $\phi_1, \phi_2 > 0$.

B.4 Proof Proposition 4

Under the lagged Taylor rule, the Jacobian in the fundamental steady state equals

$$\begin{pmatrix} B - \frac{\phi_2}{\sigma} & \frac{B}{\sigma} - \frac{\phi_1}{\sigma} \\ \kappa(B - \frac{\phi_2}{\sigma}) & B(\beta + \frac{\kappa}{\sigma}) - \frac{\kappa\phi_1}{\sigma} \end{pmatrix}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2} \left(B(1 + \beta + \frac{\kappa}{\sigma}) - \frac{\phi_2 + \kappa\phi_1}{\sigma} \pm \sqrt{(B(1 + \beta + \frac{\kappa}{\sigma}) - \frac{\phi_2 + \kappa\phi_1}{\sigma})^2 - 4\beta B(B - \frac{\phi_2}{\sigma})} \right).$$

Local stability again requires $\lambda_1 < 1$ and $\lambda_2 > -1$. The first condition can be written as

$$\phi_1 > B + (-\beta B^2 + (1 + \beta)B - 1)\frac{\sigma}{\kappa} + (\beta B - 1)\frac{\phi_2}{\kappa}. \quad (\text{B.8})$$

$\lambda_2 > -1$ can be written as

$$\phi_1 < B + (\beta B^2 + (1 + \beta)B + 1)\frac{\sigma}{\kappa} - (\beta B + 1)\frac{\phi_2}{\kappa}. \quad (\text{B.9})$$

Filling in B from (B.1) in (B.8) and (B.9) results in Conditions (22) and (23) respectively.

B.5 Proof Proposition 5

It follows from Equation (6) that in a steady state the model under the forward-looking Taylor rule satisfies

$$x = \frac{\frac{\bar{x}}{2\omega s^2 + 1}(1 - \frac{\phi_2}{\sigma}) - \frac{\phi_1 - 1}{\sigma}(\frac{2\omega s^2}{2\omega s^2 + 1}\pi + \frac{\bar{\pi}}{2\omega s^2 + 1} - \pi^T)}{(1 - (1 - \frac{\phi_2}{\sigma})\frac{2\omega s^2}{2\omega s^2 + 1})}, \quad (\text{B.10})$$

Plugging in in (7), using the definitions of \bar{x} and $\bar{\pi}$ and rearranging results in

$$\pi \left(1 - \beta \frac{2\omega s^2}{2\omega s^2 + 1} + \kappa \frac{(\phi_1 - 1)2\omega s^2}{\sigma + 2\omega s^2 \phi_2} \right) = \bar{\pi} \left(1 - \beta \frac{2\omega s^2}{2\omega s^2 + 1} + \kappa \frac{(\phi_1 - 1)2\omega s^2}{\sigma + 2\omega s^2 \phi_2} \right) \quad (\text{B.11})$$

This has as a solution $\pi = \bar{\pi}$, so that the fundamental steady state where average expectations equal the rational expectations equilibrium values always exists.

Alternatively, if the part in brackets in (B.11) is 0, any inflation level is a steady state. This is the case if and only if

$$\phi_1 = 1 - \frac{1 + (1 - \beta)2\omega s^2}{2\omega s^2 + 1} \left(\frac{\sigma}{2\omega s^2 \kappa} + \frac{\phi_2}{\kappa} \right), \quad (\text{B.12})$$

which is exactly the value where the fundamental steady state loses stability and one eigenvalue equals +1 (see Proposition 1).

For any inflation level $\tilde{\pi}$, corresponding steady state output gap \tilde{x} then follows from (7) and is given by (24).

Under the contemporaneous and lagged Taylor rules we can derive in the same way from (10) (or (12)) and (7) that the fundamental steady state ($\pi = \bar{\pi}$) always exists and that any inflation level can comprise a steady state if and only if

$$\phi_1 = \frac{2\omega s^2}{2\omega s^2 + 1} - \frac{1 + (1 - \beta)2\omega s^2}{2\omega s^2 + 1} \left(\frac{\sigma}{(2\omega s^2 + 1)\kappa} + \frac{\phi_2}{\kappa} \right). \quad (\text{B.13})$$

which is again exactly the bifurcation value where the fundamental steady states loses stability and one eigenvalue equals +1 (see Proposition 1). Steady state output gap corresponding to inflation $\tilde{\pi}$ is again equal to (24).

C 3-type model without the ZLB

C.1 Proof Proposition 6

Each of the nine combinations of optimism, pessimism and fundamentalism about inflation and output gap comprises a steady state if and only if under that particular combination of expectations, realized inflation and output gap are such that these expectations have the highest fitness measure. In the online supplementary

material it is shown that all nine steady states are locally stable when $\omega \rightarrow \infty$.

Consider the case where all agents are fundamentalistic with respect to output gap and optimistic with respect to inflation. Under the forward-looking Taylor rule the model then reduces to

$$x_t = -\frac{\phi_1 - 1}{\sigma}b, \quad (\text{C.1})$$

$$\pi_t = \pi^T + b - \kappa \frac{\phi_1 - 1}{\sigma}b. \quad (\text{C.2})$$

Under the contemporaneous and lagged rule the model reduces to

$$x_t = -\frac{\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1}b, \quad (\text{C.3})$$

$$\pi_t = \pi^T + b - \kappa \frac{\phi_1 - 1}{\sigma + \phi_2 + \kappa\phi_1}b. \quad (\text{C.4})$$

The steady state exists if and only if

$$-\frac{b}{2} < x_t < \frac{b}{2}, \quad (\text{C.5})$$

and

$$\pi_t > \pi^T + \frac{b}{2}. \quad (\text{C.6})$$

The conditions on output gap reduce to the conditions given in Table 1 and the conditions on inflation are then satisfied as well (since $\kappa < 1$).

The conditions on existence of all other steady states presented in Table 1 can be derived in the same way.

D Zero lower bound LTL

D.1 Proof Proposition 7

When the zero lower bound is binding the LTL model becomes

$$x_t = \frac{\bar{x}}{2\omega s^2 + 1} + \frac{2\omega s^2}{2\omega s^2 + 1}x_{t-1} + \frac{1}{\sigma} \frac{\bar{\pi}}{2\omega s^2 + 1} + \frac{1}{\sigma} \frac{2\omega s^2}{2\omega s^2 + 1}\pi_{t-1} + \frac{i^*}{\sigma}, \quad (\text{D.1})$$

$$\pi_t = \beta \frac{\bar{\pi}}{2\omega s^2 + 1} + \beta \frac{2\omega s^2}{2\omega s^2 + 1} \pi_{t-1} + \kappa x_t. \quad (\text{D.2})$$

Solving for the steady state of this model results in (31) and (32). Steady state output gap and inflation both are negative if and only if

$$\sigma(1 + (1 - \beta)2\omega s^2) - \kappa(2\omega s^2 + 1)2\omega s^2 < 0, \quad (\text{D.3})$$

which can be rewritten as (33).

The eigenvalues of the system defined by (D.1) and (D.2) are given by

$$\lambda_{1,2} = \frac{\omega s^2}{2\omega s^2 + 1} \left(\left(1 + \beta + \frac{\kappa}{\sigma}\right) \pm \sqrt{\left(1 + \beta + \frac{\kappa}{\sigma}\right)^2 - 4\beta} \right). \quad (\text{D.4})$$

This implies that the steady state is an unstable saddle if and only if

$$\omega s^2 > \frac{1}{\beta - 1 + \frac{\kappa}{\sigma} + \sqrt{\left(1 + \beta + \frac{\kappa}{\sigma}\right)^2 - 4\beta}}, \quad (\text{D.5})$$

which, after some algebraic manipulation, reduces to (33). Therefore, when (33) does not hold the system has a unique attractor that lies outside the ZLB region. This implies that from all initial conditions inflation and output gap will go towards this attractor and cross the zero lower bound. Recovery then always occurs.

When (33) holds, initial conditions below the stable eigenvector through the steady state given by (31) and (32) lead to ever decreasing inflation and output gap, while initial conditions above it lead to increasing inflation and output gap, and thereby to recovery. The slope of this eigenvector is given by (34).

E Zero lower bound 3-type model

E.1 Proof Proposition 8

In the ZLB region inflation and output gap are given by

$$x_t = n_t^{x,opt} b - n_t^{x,pes} b + \frac{\bar{\pi}}{\sigma} + \frac{n_t^{\pi,opt} b}{\sigma} - \frac{n_t^{\pi,pes} b}{\sigma} + \frac{i^*}{\sigma}, \quad (\text{E.1})$$

$$\pi_t = \beta(\bar{\pi} + n_t^{\pi, opt}b - n_t^{\pi, pes}b) + \kappa x_t, \quad (\text{E.2})$$

It therefore follows from (E.1) and (E.2) that pessimism remains the best performing heuristic if and only if

$$\bar{x} + (n_t^{x, opt} - n_t^{x, pes})b + \frac{\bar{\pi} + n_t^{\pi, opt}b - n_t^{\pi, pes}b + i^*}{\sigma} < \bar{x} - \frac{b}{2}, \quad (\text{E.3})$$

and

$$(\bar{\pi} + n_t^{\pi, opt}b - n_t^{\pi, pes}b)(\beta + \frac{\kappa}{\sigma}) + \kappa(\bar{x} + b(n_t^{x, opt} - n_t^{x, pes})) + \frac{\kappa}{\sigma}i^* < \bar{\pi} - \frac{b}{2}. \quad (\text{E.4})$$

These conditions together reduce to Equation (35).

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