

# A note on uniqueness of clearing prices in financial systems

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## Abstract

The Eisenberg and Noe (2001) model of the financial system is generalized to the case where default is solved by means of a *bankruptcy rule*. For regular financial networks a unique vector of clearing prices exists if only the bankruptcy rule is strongly monotonic. This shows uniqueness of the clearing prices on regular financial networks for the class of equal sacrifice rules by Young (1988), and many variations of the proportional rule as in Csóka and Herings (2018). This paper disentangles the role of network topology from the way defaults are solved.

**Keywords:** Financial networks, Systemic risk, Contagion, Clearing algorithm, Rationing, Proportional Rule, Constrained Equal Award Rule  
**JEL Classification:** C79, D31, D81, M41.

## 1 Introduction

In the aftermath of the financial crisis in 2008 following the subprime mortgage crises in the US, the delicate ways the players in the financial industries are intertwined has been discussed as the main source of the world wide spread of the financial shock. And still the intricate way these players are connected is a main concern amongst economists and policymakers. Governments and central banks took extraordinary measures to bend the impact of the crisis through monetary stimulation programmes and quantitative easing, leaving society with costs exceeding 10 trillion dollars. Now these economic accommodations are at the verge of being revoked, the induced outflow (or lack of inflow) may result in liquidity

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disruptions which could eventually lead to similar detrimental effects to the financial institutions as surfaced in the years after 2008. And it is believed that the impact of those disruptions be amplified by the fact that worldwide debts levels hit an all-time high.

The major lesson for the architecture of the financial network is that it cannot only be seen as a means by which institutions and firms may diversify their risk exposures, but that instead it may also be the main cause for the amplification of risk. The dependencies within the network may cause shocks to spread by contagion, and lead to a cascade of defaults – if not (again) prevented by public institutions. See for instance the overview of Glasserman and Young (2016) or Caccioli et al. (2018) which try to disentangle the problem by discussing various ways correlations between nodes in the financial system play a role. In this paper we will further investigate the rather simple yet seminal model of a financial system as proposed by Eisenberg and Noe (2001). Here a financial system is characterized by the liability structure (who is liable to whom, and to what extent) and a description of the aggregate external cash inflow per node, say firm or financial institution. The authors aim at clearing this market at a specific moment in time, by determining a scheme of simultaneous *clearing prices* that define the payments of each of the nodes to others. In this way a net value for each node is defined. More specifically, given such payment scheme, there are two types of nodes – the ones with a positive net value who will be able to pay all their liabilities and those with a negative net value that cannot. A node is said to *default* in the latter case, if the total inflow of cash, i.e., the external cashflow plus the payments to the node by others, minus the total sum of liabilities of the node is negative. These *Eisenberg and Noe clearing prices* are constructed such that (i) no node pays more than it has available, and (ii) a defaulting node will make a maximal payment equal to its total cash inflow. Eisenberg and Noe (2001) also propose an iterative procedure by which the clearing prices may be calculated, and in this process defaults may occur at different stages mimicking the indirect way financial institutions may be affected by earlier defaults. The model allows to interpret the phase in which a financial institution defaults as a measure of its resilience to default; the earlier a node defaults – if at all – the more financial instability it can be credited. Other measures of financial instability and assessment of systemic risk are found in Elsinger et al. (2006), Acemoglu et al. (2015), Battiston et al. (2012).

Crucial assumption in Eisenberg and Noe (2001) is the principle of proportionality; in case of a defaulting node, the corresponding clearing price is shared proportional to the liabilities of the node to the others. Groote-Schaarsberg et al. (2018) and Csóka and Herings (2018) show in a continuous and discrete setting, respectively, that the assumption of proportionality in solving defaulting situa-

tions is not crucial at all, as the idea of clearing prices is still meaningful for other bankruptcy rules. In accordance with Eisenberg and Noe (2001) both aforementioned works stress the fact that clearing prices may not be unique – but the resulting allocation is. This means that the net equity for an agent is the same for each of those vectors of clearing prices. Besides that, the set of vectors of clearing prices is well-structured as it is a completely ordered lattice with a smallest and a largest element.

Groote-Schaarsberg et al. (2018) show that within the continuous formulation of the model uniqueness of clearing prices is guaranteed for *hierarchical* structures, i.e., problems that relate to an upper triangular matrix of liabilities. As an example they discuss supply chains which may have this hierarchical structure. In particular this means that uniqueness of clearing prices is related to a network specific characteristic. In this paper, I show that the clearing prices related to *strictly monotonic* bankruptcy rules are unique for the *regular* financial networks discussed by Eisenberg and Noe (2001). The set of rules that are strictly monotonic in the estate component is rich and includes for example the *equal sacrifice rules* introduced by Young (1988) – which were proposed in a taxation model. Regularity of the network requires for each specific node that the aggregate operating cash flow corresponding to the set of nodes it can reach through the liability network is positive. Importantly, regularity is a pure network characteristic, independent from the bankruptcy rule that is used. So the contribution of this paper is also that in studying for vulnerabilities of the financial system, network driven effects are disentangled from the way defaults are settled. Next, I will show that for bankruptcy rules that are even strongly monotonic the iterative procedure suggested by Eisenberg and Noe (2001) is converging in finitely many steps so that it may be used to calculate the vector of clearing prices. A strongly monotonic bankruptcy rule sees to it that an agent with a positive claim on a specific agent is always credited with a minimal but positive fraction of additional available payment under default. Basically this monotonicity property makes the iterated mapping contracting, so that on the domain of prices there will be the one fixed point we are looking for. The monotonicity property is a sufficient condition for the results, though not necessary. As an example I discuss the financial systems corresponding to the constrained equal award rule, which is not strictly monotonic, and show that clearing prices may still be unique.

The uniqueness result also has some say in papers that explore other generalizations of Eisenberg and Noe’s model. Consider for example the model including defaulting costs by Rogers and Veraart (2013), or the model where financial institutes reinsure themselves through *credit default swaps* as in Schuldenzucker et al. (2016) (see also Elliott et al. (2014)). Also it allows to generalize the characterization of Nash equilibria in the 2 stage game proposed by Allouch and Jalloul

(2018), where the players have the choice in the first period to save or invest an amount of capital. This game is easily generalized to general bankruptcy rules. Uniqueness of the clearing prices assures that the players do not need to overcome a possible coordination problem. Then the equilibria may be characterized as discussed by Allouch and Jalloul (2018) for the proportional rule, in terms of the choice in period 1 to default or not. The analysis for other monotonic bankruptcy rules is similar as the induced games also show strategic complementarities.

The question of uniqueness of clearing prices is also addressed by Csósak and Herings (2018), who present a discrete model that allows for decentralized clearing of the financial system. This model accommodates practical situations where it is hard to retrieve all necessary information or where defaults are not filed simultaneously due to timing elements. The authors concentrate on methods used in practice, which are often a mixture of priority and proportional rules. The authors also conclude that uniqueness of clearing prices is not guaranteed for the discrete and decentralized model - and not for the limiting continuous framework that results from letting the smallest unit of account go to zero. A procedure is discussed which calculates the smallest vector of clearing prices in finitely many steps for the discrete model - which may not converge in the limiting continuous model. The result in this paper may be used to study for decentralized pricing schemes in a continuous setup.

## 2 The general framework and results

### 2.1 Mathematical prerequisites

Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean vector space. Special vector is the zero vector  $\mathbf{0}$  with all zero coordinates. Denote the set of all non-negative vectors by  $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$ . Below we will use  $N = \{1, 2, \dots, n\}$  for some integer  $n > 1$  as notation for a set of agents. With slight abuse of notation we will sometimes choose to denote  $\mathbb{R}^N$  by  $\mathbb{R}^n$ . For any two vectors  $x, y \in \mathbb{R}^n$  we define vectors  $x \wedge y, x \vee y \in \mathbb{R}^n$  such that for all  $i$

$$\begin{aligned}(x \wedge y)_i &:= \min\{x_i, y_i\} \\ (x \vee y)_i &:= \max\{x_i, y_i\}\end{aligned}$$

In addition we define  $x^+ := x \vee \mathbf{0}$  where  $\mathbf{0}$  is the zero vector in  $\mathbb{R}^n$  such that  $\mathbf{0}_i = 0$  for all  $i$ . We will write  $x \leq y$  iff  $x_i \leq y_i$  for all  $i$ , and  $x < y$  if  $x_i < y_i$  for all  $i$ . Then using this, we define  $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$  as the set of all non-negative vectors, whereas  $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n : x > \mathbf{0}\}$ .

Denote by  $\|\cdot\|$  the  $\ell^1$  norm on  $\mathbb{R}^n$  so that for all  $x \in \mathbb{R}^n$  we have

$$\|x\| := \sum_{i=1}^n |x_i|.$$

The  $j$ -th row of a matrix  $M$  is denoted  $M_j$ .

## 2.2 Bankruptcy rules

Formally, a bankruptcy problem<sup>1</sup> for a set of agents  $N = \{1, 2, \dots, n\}$  is an ordered tuple  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  where  $c$  stands for the vector of *justified claims* of the agents and  $E$  is the available *estate*, such that  $\sum_{i \in N} c_i \geq E$ . Denote by  $\mathcal{C}$  the set of all bankruptcy problems for  $N$ . A *bankruptcy rule* is a mapping  $r : \mathcal{C} \rightarrow \mathbb{R}^N$  such that for all  $(c, E) \in \mathcal{C}$

- (a)  $\mathbf{0} \leq r(c, E) \leq c$ ,
- (b)  $\sum_{i \in N} r_i(c, E) = E$ .

Before discussing some of the most popular rules in the literature of bankruptcy problems I will discuss some monotonicity properties for rules that are at the core of the results presented in this paper:

**Monotonicity**  $E \rightarrow r(c, E)$  is non-decreasing, i.e., for  $E < E'$  it holds  $r(c, E) \leq r(c, E')$  for all  $c$ .

So monotonicity is a rather weak property that ascertains that no agents suffers when there is more to divide. A stronger version is the following:

**Strict monotonicity**  $r$  is monotonic and  $E \rightarrow r_i(c, E)$  is strictly increasing if  $c_i > 0$ .

The reason that we restrict our attention to  $i$  with  $c_i > 0$  is that it leaves room to allocate 0 to zero claimants, regardless the size of the estate.

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<sup>1</sup>Here I chose to use the term bankruptcy problem, but in fact the rationing problems as in Moulin (2002) or taxation problems in Young (1988) are of the same mathematical structure. Solution concepts within these fields of the literature on distributive justice can usually easily be transferred and interpreted. For overviews, see Thomson (2015) and Moulin (2002).

**Example 1** The *proportional rule*  $r^p$  is the strictly monotonic bankruptcy rule defined for non-trivial problems by

$$r_i^p(c, E) = \begin{cases} \frac{c_i}{\sum_{j \in N} c_j} E & \text{if } \sum_{j \in N} c_j > 0, \\ 0 & \text{else .} \end{cases}$$

Young (1988) discusses the rule within the taxation setting as a flat tax. It is used by Eisenberg and Noe (2001) in order to define settlements after a firm defaults.  $\triangleleft$

**Example 2** The *constrained egalitarian rule*  $r^{\text{CEA}}$  is the parametric rule defined by  $r_i^{\text{CEA}}(c, E) = \min\{c_i, \lambda\}$  such that  $\lambda$  solves  $\sum_{j \in N} \min\{c_j, \lambda\} = E$ . This rule is not strictly monotonic as increases in estate may not be strictly beneficial for the smaller claimants.  $\triangleleft$

Well-known in the literature on taxation problems (see Young (1988), Lambert and Naughton (2009)) is the class of strictly monotonic rules which are referred to as *equal sacrifice rules*. Such rules make use of a notion of utility for income, modelled by a strictly increasing and continuous function  $U : (0, \infty) \rightarrow \mathbb{R}$  where  $U(x)$  is interpreted as the utility of an agent at income  $x \in (0, \infty)$ . A bankruptcy rule  $r$  is said to *equalize absolute sacrifice relative to  $U$*  if for all rationing problems  $(c, E) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}$  we have

$$t = r(c, E) \Leftrightarrow \text{there is } \lambda \geq 0 \text{ such that for all } i \in N, c_i > 0 \implies U(c_i) - U(t_i) = \lambda.$$

Notice that for such bankruptcy rules we have, using existence of  $U^{-1}$ ,

$$c_i > 0 \implies r_i(c, E) = t_i = U^{-1}(U(c_i) - \lambda)$$

where  $\lambda$  is such that  $\sum_{i \in N} t_i = E$ . Again, it is important to realise that  $r_i(c, E) = 0$  if only  $c_i = 0$  (which is an implication of the definition of a bankruptcy rule part (a)).

**Example 3** The proportional rule is an equal sacrifice method corresponding to utility function  $U(x) = \log(x)$ . To see this, for each bankruptcy problem  $(c, E)$  and  $t = r^p(c, E)$  we have

$$\ln(c_i) - \ln(t_i) = \lambda \iff \frac{c_i}{t_i} = e^\lambda \iff t_i = \frac{c_i}{\sum_{j \in N} c_j} E = r_i^p(c, E) \text{ with } \lambda = \ln \frac{c(N)}{E}.$$

**Example 4** Consider the utility function  $U(x) = -x^\alpha$  where  $\alpha < 0$  is fixed. Each  $\alpha$  defines implicitly an equal sacrifice bankruptcy rule  $t = r^\alpha(c, E)$ :<sup>2</sup>

$$U(c_i) - U(t_i) = \lambda \Leftrightarrow r^\alpha(c, E) := t_i = U^{-1}(U(c_i) - \lambda) = (c_i^\alpha + \lambda)^{1/\alpha}. \quad (1)$$

For instance, for  $\alpha = -1$  the equation (1) simplifies to the parametric rule

$$r_i^\alpha(c, E) = \frac{c_i}{\lambda c_i + 1}.$$

The class of rules  $\{r^\alpha\}_\alpha$  is naturally related to  $r^p$  and the (not strictly monotonic)  $r^{\text{CEA}}$  as it can be shown that

$$r^{\text{CEA}} = \lim_{\alpha \rightarrow -\infty} r^\alpha \text{ and } r^p = \lim_{\alpha \rightarrow 0} r^\alpha. \quad \triangleleft$$

Note that for (strictly) monotonic bankruptcy rules  $r$  the right derivative  $\frac{\partial^+}{\partial E} r_i(c, E)$  exists for all bankruptcy problems and is non-negative. Below we will focus on strictly monotonic rules with the following property:

**Strong monotonicity** Bankruptcy rule  $r$  is *strongly monotonic* if it is monotonic and for each  $i \in N$  with  $c_i > 0$  and each interval  $I = [0, E^*]$  there exists  $\alpha_i > 0$  such that

$$\frac{\partial^+}{\partial E} r_i(c, E) \geq \alpha_i \text{ for } E \in I. \quad (2)$$

So strong monotonicity states that each increase of the available estate results in a minimal fair share of the increment for each agent with non-zero claim. Basically, strong monotonicity rules out a kind of exotic rules which are strictly monotonic and do allow for zero derivatives. Where most of the generalizations of Eisenberg and Noe (2001) only make use of strict monotonicity, the following implication of strong monotonicity is most useful for computational issues that we will discuss later on. By (2) we have

$$\sum_{i \in N: c_i > 0} \alpha_i \leq \sum_{i \in N: c_i > 0} \frac{\partial^+}{\partial E} r_i(c, E) = 1. \quad (3)$$

## 2.3 The economic model

Consider a group of  $n$  economic agents, each defined by having a certain level of financial liabilities towards other agents in this group. In this way we constitute a financial network with dependent actors, where the connections or relations of agents within the network are shaped through the nominal liabilities an agent

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<sup>2</sup>See Young (1988) Theorem 2.

has to other agents in the system. In particular, these liabilities represent the binding financial promises of agents to others in the system. This structure can be represented by an  $n \times n$  matrix  $L$ , where  $L_{ij}$  stands for the nominal liability of agent  $i$  to agent  $j$ . We will assume that these liabilities are all non-negative and no agent has a liability to herself, so that the main diagonal of  $L$  consists of zeroes. So, for all  $i, j \in N$  we assume that  $L_{ii} = 0$  and  $L_{ij} \geq 0$ . Besides, we will assume that each of the agents also has some cash inflow from sources outside the liabilities network: let  $e_i \geq 0$  be the *operating cash flow* received by agent  $i$ . Let  $p_i$  represent the total of payments by agent  $i$  to the other agents in the system, and let  $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}_+^n$  be the summarizing vector of total payments made by the agents. Let  $\tau \in \mathbb{R}_+^n$  be the vector that summarizes the total nominal obligations of the agents in the system, i.e., for  $i \in N$  let

$$\tau_i := \sum_{j=1}^n L_{ij}. \quad (4)$$

This *total obligation* vector  $\tau$  summarizes agent-wise the payment levels required to satisfy all the contractual liabilities in the network. We will assume that all liabilities have the same maturity date at which they become due and should be paid for. Suppose the financial system is cleared using a vector of payments  $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}_+^n$  where  $p_i$  stands for the payment of agent  $i$  to clear his obligations. We will assume that no agent pays more than the total of his obligations, or  $p \leq \tau$ . On the other hand, each agent  $i$  has some justified claim  $L_{ji}$  on  $p_j$ . In particular this means that for each  $j \in N$  the ordered pair  $(L_j, p_j)$  is a classic *bankruptcy problem* or *claims problem* as in Thomson (2015). Now suppose that we use a strictly monotonic bankruptcy rule  $r$  to determine per agent how his payment  $p_j$  is split amongst the claimants and their claims as given by  $L_j$ . This means that from payment  $p_j$  by agent  $j$ , agent  $i$  obtains  $r_i(L_j, p_j)$ . Then the *total cash flow* to agent  $i$  equals the sum of the payments received from other agents plus the operating cash flow, which then is given by

$$\sum_{j=1}^n r_i(L_j, p_j) + e_i.$$

A *financial system* is characterized by an ordered triple  $(L, e, r)$ , where  $L$  is an  $n \times n$  matrix of liabilities (nonnegative and diagonal is zero),  $e \geq \mathbf{0}$  is the vector of external cash flows, and  $r$  is a bankruptcy rule.

## 2.4 Clearing payment vectors for a financial system

Crucial in the clearing of a financial system is the determination of the individual payments. Below we will focus on the question whether payment vectors exist,



that see to a clearing of the financial system such that two minimal requirements are satisfied. First we will require from a payment vector that it expresses the idea of limited liability: no agent should pay more than the total of his cash inflow. Second, in principal we strive for the situation where each agent is first held responsible for all its liabilities, and in case he cannot make the necessary payments all his cash inflow is used to cover his obligations to the creditors. The following definition is the Eisenberg and Noe (2001) version, only now for general bankruptcy rules:

**Definition 1** *A clearing payment vector for the financial system  $(L, e, r)$  is a vector  $p^* \in [0, \tau]$  that satisfies*

$$(a) \text{ Limited Liability: } p_i^* \leq \sum_{j=1}^n r_i(L_j, p_j^*) + e_i$$

(b) *Absolute Priority: Either  $p_i^* = \tau_i$  or*

$$p_i^* = \sum_{j=1}^n r_i(L_j, p_j^*) + e_i.$$

Then any clearing payment vector  $p^*$  satisfies the following condition, that will be central in this paper:

$$p_i^* = \left( \sum_{j=1}^n r_i(L_j, p_j^*) + e_i \right) \wedge \tau_i. \quad (5)$$

So – no different from Eisenberg and Noe (2001) – we conclude that the clearing vector  $p^*$  is a fixed point of the map,  $\Phi(\cdot, L, e, r) : [0, \tau] \rightarrow [0, \tau]$  defined by the coordinate mappings

$$\Phi_i(p, L, e, r) = \left( \sum_{j=1}^n r_i(L_j, p_j) + e_i \right) \wedge \tau_i. \quad (6)$$

**Theorem 1 (Tarski (1955))** *Let  $(A, \leq)$  be any complete lattice<sup>3</sup> and suppose  $f : A \rightarrow A$  is monotonically increasing, i.e., for all  $x, y \in A, x \leq y$  implies  $f(x) \leq f(y)$ . Then the set of all fixed points of  $f$  is a complete lattice with respect to the order  $\leq$ .*

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<sup>3</sup>A partially ordered set  $(A, \leq)$  is a *complete lattice* if every subset  $U$  of  $A$  has both an infimum and a supremum in  $(A, \leq)$ . This holds for the partially ordered set  $[0, \tau]$ .

Theorem 1 in Eisenberg and Noe (2001) generalizes to our case, as long as we constrain ourselves to increasing division rules. The existence of clearing prices for general bankruptcy rules is already done in Groote-Schaarsberg et al. (2018). For monotonic bankruptcy rules we have a stronger statement.

**Theorem 2** *Consider a financial system  $(L, e, r)$  with a monotonic bankruptcy rule  $r$ . Then:*

- (a) *There is a greatest and a least clearing payment vector,  $p^+$  and  $p^-$ .*
- (b) *Under all clearing vectors, the value of equity for each agent is the same, that is, if  $p'$  and  $p^*$  are any two clearing vectors, then for all  $i \in N$*

$$\left( \sum_{j=1}^n r_i(L_j, p'_j) + e_i - \tau_i \right) \vee 0 = \left( \sum_{j=1}^n r_i(L_j, p^*_j) + e_i - \tau_i \right) \vee 0.$$

*Proof:* The mapping  $\Phi$  defined by (6) is increasing on the partially ordered set  $[0, \tau]$ . Also  $([0, \tau], \leq)$  is a complete lattice, so that the implication of Tarski's fixed point theorem (see Tarski (1955)) is that the set of fixed points of  $\Phi$  is a complete lattice with respect to  $\leq$ . In particular  $\Phi$  has a greatest fixed point  $p^+$ , and a smallest fixed point  $p^-$ . So this proves (a).

For part (b) let  $p'$  be any clearing vector. It is necessary and sufficient to show that the value of equity is the same under  $p'$  and  $p^+$ . First note that by monotonicity of  $r$ ,  $p^+ \geq p'$  implies that for all  $i$

$$\left( \sum_{j=1}^n r_i(L_j, p^+_j) + e_i - \tau_i \right) \vee 0 \geq \left( \sum_{j=1}^n r_i(L_j, p'_j) + e_i - \tau_i \right) \vee 0. \quad (7)$$

Because  $p^+$  and  $p'$  are both clearing vectors, it also must be the case that for all  $i$

$$\begin{aligned} \left( \sum_{j=1}^n r_i(L_j, p^+_j) + e_i - \tau_i \right) \vee 0 &= \sum_{j=1}^n r_i(L_j, p^+_j) + e_i - p^+_i, \\ \left( \sum_{j=1}^n r_i(L_j, p'_j) + e_i - \tau_i \right) \vee 0 &= \sum_{j=1}^n r_i(L_j, p'_j) + e_i - p'_i. \end{aligned}$$

Now summing up the right-hand side over  $i = 1, \dots, n$  yields

$$\sum_{i=1}^n \left( \sum_{j=1}^n r_i(L_j, p^+_j) + e_i - p^+_i \right) = \sum_{j=1}^n p^+_j + \sum_{i=1}^n e_i - \sum_{i=1}^n p^+_i$$

$$\begin{aligned}
&= \sum_{i=1}^n e_i = \sum_{j=1}^n p'_j + \sum_{i=1}^n e_i - \sum_{i=1}^n p'_i \\
&= \sum_{i=1}^n \left( \sum_{j=1}^n r_i(L_j, p'_j) + e_i - p'_i \right).
\end{aligned}$$

But then this means that in (7) we should have equality as well.  $\square$

### 3 Characterizing the clearing prices

Eisenberg and Noe (2001) characterize vectors of clearing prices using the notion of a surplus set, i.e., a set of agents  $S$  with no external obligations and a positive aggregate operation cash flow:

**Definition 2** *A set  $S \subset N$  is a surplus set if for all  $(i, j) \in S \times S^c$  we have  $L_{ij} = 0$  and  $\sum_{i \in S} e_i > 0$ .*

**Lemma 1** *If  $p$  is a clearing vector for the financial system  $(L, e, r)$ , then it is not possible for all agents in a surplus set to have zero equity value.*

*Proof:* Suppose  $S$  is a surplus set for  $(L, e, r)$ . Denote by  $P_i^+$  the sum of all of the external payments from  $S^c$  to node  $i \in S$ . Since  $S$  is a surplus set, its members do not make payments to agents in  $S^c$ . Then zero equity for all nodes in  $S$  at clearing vector  $p$  implies that

$$p_i = \sum_{j \in S} r_i(L_j, p_j) + e_i + P_i^+, \text{ for all } i \in S. \quad (8)$$

Then summing up the equations (8) over  $S$  yields

$$\sum_{i \in S} p_i = \sum_{j \in S} \sum_{i \in S} r_i(L_j, p_j) + \sum_{i \in S} (e_i + P_i^+) = \sum_{j \in S} p_j + \sum_{i \in S} (e_i + P_i^+). \quad (9)$$

Here, the second equality is due to the fact that  $S$  is a surplus set, so that the payments made by members in  $S$  are redistributed amongst  $S$ . Now (9) implies  $0 = \sum_{i \in S} (e_i + P_i^+)$ , contradicting the assumption that  $\sum_{i \in S} e_i > 0$ .  $\square$

**Definition 3** *A liability matrix  $L$  is associated the directed liability graph where there is a directed edge from  $i$  to  $j$ ,  $i \rightarrow j$ , if  $L_{ij} > 0$ . For each  $i \in N$ , define the risk orbit  $\mathcal{O}(i)$  as the set of nodes  $j \in N$  such that there exists a directed path in the liability graph from  $i$  to  $j$ .*

**Lemma 2** Suppose  $p$  is a clearing vector for  $(L, e, r)$ . Let  $\mathcal{O}(i)$  be a risk orbit that satisfies  $\sum_{k \in \mathcal{O}(i)} e_k > 0$ . Then there must be an agent  $j \in \mathcal{O}(i)$  that has positive equity, or

$$\tau_j < \sum_{k \in N} r_j(L_k, p_k) + e_j.$$

*Proof:* It is easily seen that  $\mathcal{O}(i)$  is a surplus set. Then Lemma 1 shows  $\mathcal{O}(i)$  should contain an agent with positive equity.  $\square$

**Definition 4** The ordered pair  $(L, e)$  is a regular financial structure if each of its risk orbits  $\mathcal{O}(i)$  is a surplus set.

**Theorem 3** If a financial system  $(L, e, r)$  is such that  $(L, e)$  is regular and  $r$  is strictly monotonic, then the greatest and least clearing vectors are the same, i.e.,  $p^+ = p^-$ .

*Proof:* Assume  $p^+ \geq p^-$  and  $p^+ \neq p^-$ . Define  $E_j^+$  and  $E_j^-$ , respectively, as the equity of agent  $j$  under  $p^+$  and  $p^-$ , respectively. From Theorem 1 we get that  $E_j^+ = E_j^-$  for all  $j$ . Absolute priority implies that for agents  $j$  with positive equity it must hold  $p_j^+ = p_j^- = \tau_j$ . These are the agents that first should pay back all their obligations. So, for agents  $j$  with  $p_j^+ > p_j^-$  must be zero equity agents. And in particular there must be such an agent, say agent  $i$ , such that  $p_i^+ > p_i^-$ . Due to regularity of the financial structure and Lemma 2, the risk orbit  $\mathcal{O}(i)$  contains a node with positive equity. So there is  $\ell \in \{1, 2, \dots, n\}$  and a path

$$i = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{\ell-1} \rightarrow i_\ell = m$$

as part of  $\mathcal{O}(i)$  so that all agents on the path have zero equity, and  $m$  is the agent with positive equity value. Claim:  $p_{i_k}^+ - p_{i_k}^- > 0$  for  $k = 0, 1, \dots, \ell - 1$ . We will use a proof by induction. First, for  $k = 0$  the claim holds. Now take  $k \leq \ell - 1$  and assume the claim is true for  $t < k$ . Since all agents  $i_t$  with  $t \leq k$  are zero equity agents, their payments equal their inflows. Especially this holds for agent  $i_k$ , so that

$$\begin{aligned} p_{i_k}^+ &= \sum_{j=1}^n r_{i_k}(L_j, p_j^+) + e_{i_k} \\ p_{i_k}^- &= \sum_{j=1}^n r_{i_k}(L_j, p_j^-) + e_{i_k} \end{aligned}$$

Then

$$p_{i_k}^+ - p_{i_k}^- = \sum_{j=1}^n (r_{i_k}(L_j, p_j^+) - r_{i_k}(L_j, p_j^-)).$$

By the induction hypothesis we have that  $p_{i_{k-1}}^+ - p_{i_{k-1}}^- > 0$ , and since  $i_{k-1} \rightarrow i_k$  we have  $L_{i_{k-1}i_k} > 0$  so that by strict monotonicity of  $r$  it must be that  $r_{i_k}(L_j, p_j^+ \geq r_{i_k}(L_j, p_j^-)$  for all  $j \in N$  and  $r_{i_k}(L_{i_{k-1}}, p_{i_{k-1}}^+) > r_{i_k}(L_{i_{k-1}}, p_{i_{k-1}}^-)$ . Then we may conclude that  $p_{i_k}^+ - p_{i_k}^- > 0$ . We claim that  $E_m^+ > E_m^-$ . This follows from the following consideration. By definition we have

$$E_m^+ - E_m^- = \sum_{j=1}^n (r_m(L_j, p_j^+) - r_m(L_j, p_j^-)) - (p_m^+ - p_m^-). \quad (10)$$

Because  $m$  is a node with positive equity value, absolute priority implies that  $p_m^+ = p_m^-$ . Because  $i_{\ell-1} \rightarrow m$  it holds that  $L_{i_{\ell-1}m} > 0$  and thus  $r_m(L_{i_{\ell-1}}, p_{i_{\ell-1}}^+) > r_m(L_{i_{\ell-1}}, p_{i_{\ell-1}}^-)$  by strict monotonicity – as we just have shown that  $p_{i_{\ell-1}}^+ > p_{i_{\ell-1}}^-$ . Then by the fact that  $p^+ \geq p^-$  and (10) it follows  $E_m^+ - E_m^- > 0$ , which establishes our claim. But this in turn conflicts with the result in Theorem 2 that the value of equity at all nodes is the same under all clearing prices, and in particular under  $p^-$  and  $p^+$ . This shows that the clearing price must be unique.  $\square$

Consider  $S^\Phi = \{p \in [0, \tau], \Phi(p) \leq p\}$ , the set of *supersolutions* of the operator  $\Phi$ . In addition define for  $p \in S^\Phi$  the *default set under  $p$* ,  $D(p) \subset N$  by  $i \in D(p) \Leftrightarrow \Phi_i(p) < \tau_i$ . Note that  $D(p) \neq N$ . For fixed  $q \in S^\Phi$ , define the map  $p \rightarrow F(p | q)$  by

$$F_i(p | q) = \begin{cases} \sum_{j \in D(q)} r_i(L_j, p_j) + \sum_{j \notin D(q)} r_i(L_j, \tau_j) + e_i & \text{if } i \in D(q), \\ \tau_i & \text{if } i \notin D(q). \end{cases} \quad (11)$$

So,  $F(p | q)$  describes the payments for all agents such that agents  $i$  non defaulting under  $q$  pay all their obligations  $\tau_i$  and the defaulting agents under  $q$  pay  $p_i$ .

**Theorem 4** *For each financial system  $(L, e, r)$  such that  $(L, e)$  is regular and  $r$  is a strongly monotonic, the map  $F(\cdot | q)$  has a unique fixed point for each supersolution  $q$ .*

*Proof:* Take  $q \in S^\Phi$ . We will show that  $F(\cdot | q)$  defines a contraction on  $[0, \tau]$ , so that by Banach's Theorem we have a unique fixed point. Consider  $p, p' \in [0, \tau]$ . First of all, for all  $j \in D(q)$  there is  $t(j) \in N \setminus D(q)$  such that  $L_{jt(j)} > 0$ . Then since  $r$  is strongly monotonic we may apply (2) to interval  $[0, \tau_j]$  and ascertain the existence of  $\alpha_{t(j)} \in (0, 1)$  such that  $|r_{t(j)}(L_j, p_j) - r_{t(j)}(L_j, p'_j)| \geq \alpha_{t(j)} |p_j - p'_j|$ . Now let  $\alpha = \min_{j \in D(q)} \alpha_{t(j)}$ . Then according to  $r$  the agents in  $D(q)$  take maximally the remaining fraction  $(1 - \alpha) \in (0, 1)$  of the change in  $p_j - p'_j$ . Using this in (\*) below, we get

$$\|F(p | q) - F(p' | q)\| = \sum_{i \in N} |F_i(p | q) - F_i(p' | q)|$$

$$\begin{aligned}
&= \sum_{i \in D(q)} |F_i(p \mid q) - F_i(p' \mid q)| + \sum_{i \notin D(q)} |F_i(p \mid q) - F_i(p' \mid q)| \\
&= \sum_{i \in D(q)} |F_i(p \mid q) - F_i(p' \mid q)| = \sum_{i \in D(q)} \left| \sum_{j \in D(q)} (r_i(L_j, p_j) - r_i(L_j, p'_j)) \right| \\
&\leq \sum_{i \in D(q)} \sum_{j \in D(q)} |r_i(L_j, p_j) - r_i(L_j, p'_j)| \stackrel{*}{\leq} \sum_{j \in D(q)} (1 - \alpha) |p_j - p'_j| \\
&\leq (1 - \alpha) \sum_{j=1}^n |p_j - p'_j| = (1 - \alpha) \|p - p'\|,
\end{aligned}$$

which shows that  $F(\cdot \mid p')$  is a contraction.  $\square$

The importance of Theorem 4 lays in the fact that we now may use the mapping  $F$  to actually calculate the unique clearing vector for financial systems  $(L, e, r)$ . For any supersolution  $q$  we may calculate the fixed point  $f(q)$  of the mapping  $p \mapsto F(p \mid q)$ . In particular, this shows that we may define inductively the sequence of payment vectors

$$p^0 = \tau, p^j = f(p^{j-1}). \quad (12)$$

This is the sequence that Eisenberg and Noe (2001) in the model with  $r = r^p$  refer to as the *fictitious default sequence* and the machinery producing the sequence is called the *fictitious default algorithm*. We may show that at each step in the algorithm a supersolution is calculated, and that the algorithm does what we actually want: calculating the clearing vector. This is summarized in the lemma below, the proof of which is almost identical to that of Eisenberg and Noe (2001)<sup>4</sup>:

**Lemma 3** *Consider the fictitious default algorithm applied to a financial system  $(L, e, r)$  with regular  $(L, e)$  and strongly monotonic division rule  $r$ . Then the sequence  $\{p^j\}$  defined by (12) is decreasing to the clearing vector in at most  $n$  iterations of the algorithm.*

*Proof:* The logic is almost the same as that in Eisenberg and Noe (2001). First we will show that for all  $j$  the vector  $p^j$  is a supersolution for  $\Phi$ , and that  $p^j$  is non-increasing in  $j$ . This is done using mathematical induction. Firstly, for  $j = 0$  it is clear that  $p^0 = \tau$  is a supersolution for  $\Phi$ . Secondly, suppose the assertion is true for  $p^k$ . Note that  $F(p^k \mid p^{k-1}) = p^k$ . Because  $p^k \in S^\Phi$ , we must have  $\sum_{j \in N} r_i(L_j, p_j^k) + e_i \leq p_i^k$  for agents  $i$  defaulting under  $p^k$ . This implies, together

<sup>4</sup>Again, instead of writing the mappings  $F$  and  $\Phi$  vector-wise, we focus at single coordinates instead. The rest is identical

with the definition of  $F$  that  $\Phi(p^k) = F(p^k | p^k)$ . Now we invoke our induction hypothesis that  $p^k$  is a supersolution to  $\Phi$ , and so for  $F(\cdot | p^k)$ . This implies that the fixed point of  $F(\cdot | p^k), p^{k+1}$ , is (weakly) smaller than  $p^k$ . This shows the first part of our claim. Now, since  $p^{k+1} \leq p^k$  the set of defaulting agents under  $p^k$  is not smaller than that under  $p^{k+1}$ .

- If these sets of nodes are the same, we conclude that  $\Phi(p^{k+1}) = F(p^k | p^k)$ , and since  $p^{k+1}$  is defined as fixed point of  $F(\cdot | p^k)$ , this shows that  $p^{k+1}$  is a fixed point of  $\Phi$ . In particular, it is a supersolution for  $\Phi$ .
- If the set of defaulting agents is larger under  $p^{k+1}$  than under  $p^k$ , then it must be that some agents pay their obligations in full under  $p^k$  and default under  $p^{k+1}$ , and the other agents either default or not both under  $p^{k+1}$  and  $p^k$ . Thus, for those nodes  $i$  with changing payments, i.e., for which default occurs under  $p^{k+1}$  but not under  $p^k$ , we have  $\Phi(p^{k+1})_i < p_i^{k+1}$  and for all other nodes  $j$  we have  $\Phi(p^{k+1})_j = p_j^{k+1}$ . This shows that  $p^{k+1}$  is a supersolution to  $\Phi$ .

And this concludes our proof by induction as also we have shown that  $\{p^j\}$  is a weakly decreasing sequence. Now we turn to convergence of the sequence. The reasoning above also shows that if the set of defaulting nodes is the same under both  $p^{j+1}$  and  $p^j$ , then

- $p^j$  is a fixed point of  $\Phi$ , and
- the sequence will remain constant after  $p^{j+1}$ .

If  $p^j$  fails to be a fixed point of the map  $\Phi$  then an agent that does not default under  $p^j$  will do so under  $p^{j+1}$ . This means that the number of defaulting agents will increase at the next iteration. Because there are only  $n$  agents, and at most  $n - 1$  of them can default in any supersolution, the payment vector resulting from the algorithm may change at most  $n$  times. Because the sequence is constant only at fixed points of  $f$ , the clearing vector is reached after at most  $n$  iterations.  $\square$

**Example 5** Consider the following financial system  $(L, e, r)$  with

$$L = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}, e = \mathbf{0}.$$

Then for  $r = r^{\text{CEA}}$  there is a continuum of  $p^t = (2, t, t)$  where  $t \in [4, 5]$ . Note that  $r^{\text{CEA}}$  is not strictly monotonic, and so not strongly monotonic. For the strongly monotonic proportional rule  $r = r^p$  we have a unique vector of clearing prices:  $p = (2, \frac{5}{2}, \frac{5}{2})$ .

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