When to spend the carbon budget? Three approaches to global climate mitigation

Daan in 't Veld[a]
[a] PBL Netherlands Environmental Assessment Agency, P.O. box 30314, 2500 GH, The Hague.
Daan.intVeld@pbl.nl.

Abstract
The concept of a carbon budget implies that when CO_2 emissions exceed the budget, a temperature threshold is triggered. This raises the question how and when to spend the remaining carbon budget wisely. Economists have proposed that the carbon budget should be depleted following Hotelling's rule. In a simple, general model with a dynamic resource stock, I theoretically investigate three solution structures: the Hotelling solution (derived for fossil reserves), the Faustmann solution (for forestry), and the greedy solution (consuming as much as possible). The question is under which conditions each of these solution structures is optimal. While the Hotelling solution is a useful approach to steer away from greediness, the Faustmann approach could become more appropriate when the world comes closer to a climate-neutral economy. This alternative would imply a yearly emission ceiling, depending on the natural carbon cycle.

Keywords: carbon budget; Hotelling-Faustmann model; synthesis; climate mitigation; dynamic resources.
JEL codes: C61, Q32, Q54

1. Introduction

The IPCC Sixth Assessment Report estimates a remaining carbon budget of 1350 Gt CO_2 for limiting global warming to 2.0°C (see Figure 1). This means that emitting more than 1350 Gt CO_2 will likely increase global temperatures to more than 2 degrees on average. Similarly, for a 50 percent likelihood to limit global warming to 1.5°C, the remaining carbon budget is 500 Gt CO_2. If we accept such a carbon budget, the main economic question is how we distribute the available 'resource' over time, space and people.

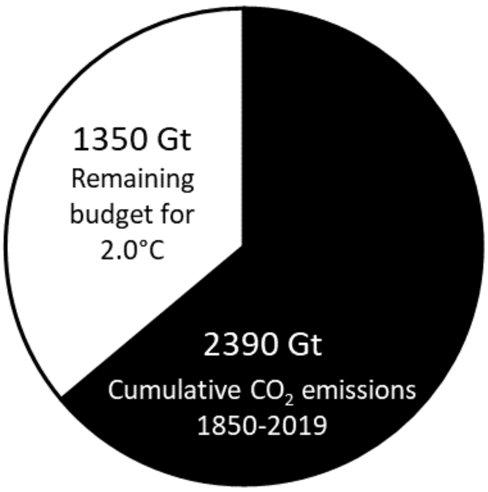

Figure 1. The remaining carbon budget for a 50% likelihood to stay below 2.0°C. Source of numbers: IPCC (2021: 29).

Economists elaborating on this topic have proposed that the carbon budget should be depleted following Hotelling's rule (Mattauch et al., 2019; Bednar et al., 2021). This rule posits that "the price of an exhaustible resource must grow at a rate equal to the rate of interest, both along an efficient extraction path and in a competitive resource industry equilibrium" (Devarajan and Fisher, 1981). Originally derived from fossil reserves – where Hart and Spiro (2011) have argued that it does not describe oil extraction well – the model of

Hotelling have been applied to many fields including climate economics. From a collective, global perspective, Hotelling's rule suggests that the carbon budget should be consumed at a constant rate every year.

The question posed in this paper is: *is Hotelling's rule optimal under a dynamic resource stock*? Or, under what circumstances is the Hotelling approach – implying a decreasing resource stock – justified? While calculated carbon budgets have been noted to depend on linearity (Schurer et al., 2017), alternatives for the Hotelling rule to consume the budget have not been discussed.

In a conceptual, analytical model, I show two alternative approaches: the Faustmann solution and the greedy solution. To start with the latter, the greedy solution is a simple benchmark that consumes the entire budget immediately. The Faustmann solution, transferred from forest economics (Macura, 2013) into the realm of climate change, points at emitting the amount of carbon is absorbed by Nature in a given year.

2. Background

This paper synthesises two analytical results in the field of natural resource economics. Focusing on the optimal depletion of exhaustible resources, the synthesis starts with Hotelling's rule of maximising economic rent (Hotelling, 1931). In Appendix A I provide the fundamental assumptions underlying Hotelling's rule, to investigate its general applicability. The resulting conceptual framework supports environmental policymaking by investigating the robustness of the proposed approach for climate mitigation.

The Hotelling solution applies to resources that are static (for example, an oil reserve), but many resources are dynamic (for example, a forest, or a carbon budget). Under certain conditions, the optimal economic usage of such dynamic resources is to keep the size of the resource constant over time, by using, in each time period, only the amount of resource that is regenerated by Nature. This solution is known as the Faustmann (1849) solution in a branch of the literature called `forest economics'.

In this paper I synthesise these two solution structures, together with a greedy solution, into a combined Hotelling-Faustmann model for one-person economies. (A related goal was set by Vukina et al. (2001), who applied a steady-state version of such a model to data on empirical tree prices.) I analyse the dynamic aspects of this model in theory, in order to investigate what type of solutions can arise in more complex environments.

3. Model

3.1 Hotelling-Faustmann model

Assume a single decision-maker – in economic terms: a social planner – disposing of a resource which may fluctuate over time. Also assume that the amount available of this resource (ignoring, for now, consumption) is an exogenous process described as

$$z_{t+1} = f_{\alpha\beta}(z_t) \equiv \alpha\, z_t(1 - \beta z_t). \tag{1}$$

with z_t the size of the resource (scaled to 1), and $\alpha \in (0,4)$ and $\beta \in [0,1]$ are the parameters governing the dynamics.

The 'logistic' difference equation (1), combining an exponential growth rate α and a death rate β, is universally applicable to many dynamic processes (van den Noort, 1992). The logistic growth model has a long history in biology (Verhulst, 1838) and is "arguably the simplest nonlinear difference equation" (May, 1976). In the context of climate economics, rate α captures that the carbon budget may grow, for example by natural carbon sinks; while β sets a limit to this growth when z_t exceeds $\frac{1}{2\beta}$.

The process (1) captures two special cases, namely for $\beta = 0$ and for $\beta = 1$.

Special case $\boldsymbol{\beta = 0}$: The system $z_{t+1} = f_{\alpha\beta}(z_t)$ with $\beta = 0$ is a simple (exponential) system, with the following types of behaviour.

1. For $\alpha = 1$, the resource z_t is static, so the social planner faces a "cake-eating" problem (see Dasgupta and Heal, 1974; Adda and Cooper, 2003).
2. For $\alpha > 1$, the resource z_t increases with rate α.We will rule out this case below (by Assumption 1).
3. For $0 < \alpha < 1$, the resource z_t is decreasing over t and converges to a trivial equilibrium, $z^* = 0$.

Special case $\boldsymbol{\beta = 1}$: The system $z_{t+1} = f_{\alpha\beta}(z_t)$ with $\beta = 1$ is a one-dimensional nonlinear map. For $\beta = 1$ and $1 < \alpha < 2.7$, growth is stable (van den Noort, 1992), while for $\alpha > 3$ the map exhibits complex behaviour in a mathematical sense (Hommes, 2013: p.43).

One more, technical assumption is made to assure the dynamics of the system are bounded.

Assumption 1. [Confinement of dynamics]
We assume

$$\alpha < \alpha^{max}(\beta) \equiv \begin{matrix} 1/(1-\beta) & if\ \beta \leq ½ \\ 4\beta & if\ \beta > ½ \end{matrix}$$

Proposition 1. *Under Assumption 1, the dynamics of the system* $z_{t+1} = f_{\alpha\beta}(z_t)$ *with* $z_0 \in (0,1)$ *are confined to* $z_t \in (0,1)$ *for all* t.

Assumption 1 and Proposition 1 are derived from checking that $f_{\alpha\beta}(z_t) < 1$ for $z = 1$ (the highest value of z) and for $z = \frac{1}{2\beta}$ (the highest value of $f_{\alpha\beta}(z)$).

The social planner's maximisation problem is

$$\begin{aligned} \max_{\boldsymbol{x}} U(\boldsymbol{x}) = \quad & \max_{\boldsymbol{x}} \textstyle\sum_{t=0}^{\infty} \delta^t u(x_t) \\ & \text{s.t. } x_t \leq z_t \\ & \quad z_{t+1} = f_{\alpha\beta}(z_t - x_t) \end{aligned} \tag{2}$$

in which $\boldsymbol{x} = \{x_t\}_{t=0}^{\infty}$ is the consumption path, and δ is the discount factor. We denote with capital letter U the overall utility over all periods, and with small letter u the (contemporaneous) utility in a particular period.

The last line of equation (2) introduces the dynamics with the logistic map $f_{\alpha\beta}(z_t)$. Denote the auxiliary variable $\widetilde{z_t} \equiv z_t - x_t$, distinguishing between the resource level at the beginning of the period, z_t, and at the end, $\widetilde{z_t}$.

3.2 Underlying assumptions
To investigate its general applicability, I discuss the ideas underlying the model in Appendix A. The Hotelling-Faustmann model depends on two main 'presuppositions', complemented by two specifications and some (technical) assumptions to clarify the exposition.

4. Results
4.1 Theoretical results
Within the model of Section 3, I derive the Hotelling solution as follows. All the proofs can be found in Appendix B.

Proposition 2. [The general Hotelling solution]. The general Hotelling solution x_t^H for the social planner's problem is defined by

$$u'(x_t^H) = \delta\, f'_{\alpha\beta}(\tilde{z}_t)\, u'(x_{t+1}^H). \tag{3}$$

So consumption decreases with the discount factor and with the derivative of the map at the point $\widetilde{z_t}$, depending on the derivative of the utility function.

I will show that the Hotelling solution is not the optimal strategy in all cases. Specifically, I define two alternatives as follows.

Definition 1 [The Faustmann solution]. The Faustmann solution is defined as $x_t = \max(x^F)$ for all $t > 0$, such that $z^F = f_{\alpha\beta}(z^F - x^F)$ for all $t > 0$.

The Faustmann solution is a fixed consumption level, that allows the resource in every period to grow back to the same level. There exist a range of these paths with a fixed consumption level, and the Faustmann path is the one that the consumption is highest. Under equation (1), the definition implies a Faustmann solution for $\alpha > 1$ equal to $x_t = x^F \equiv \frac{(\alpha-1)^2}{4\alpha\beta} z_0$ for all $t > 0$, with $x_0^F = z_0 - (z^F - x^F)$.

Definition 2 [The greedy solution]. The greedy solution is defined as $x_0^g = z_0$, implying $x_t^g = 0$ for $t > 0$.

I consider the results for three standard (and simple) utility functions: linear, logarithmic and quadratic. For every utility function, I investigate which of the three solutions is optimal: Hotelling, Faustmann, or greedy.

Example 1. *If $u(x_t) = x_t$, then the optimal solution is the greedy solution for $\alpha < 1$ or for δ sufficiently small; for $\alpha > 1$ and δ sufficiently large, the Faustmann solution is optimal.*

The outcomes of Example 1 are illustrated in Figure 2 within the (α, β)-space. Figure 3 shows the results in more detail for $\beta = 1$. The linear utility function implies that consumption is additive over time.

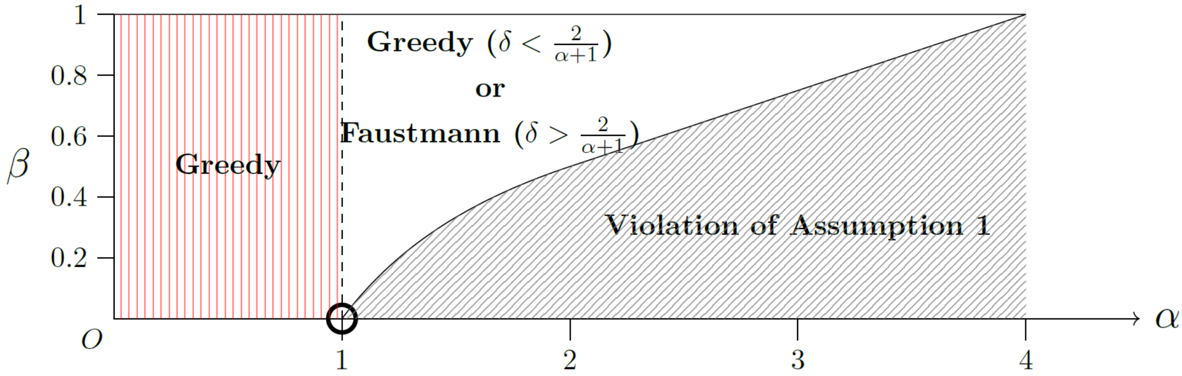

Figure 2. Optimal solutions for $u(x_t) = x_t$. White area = the optimal solution depends on δ; red area = greedy solution is optimal; grey area = dynamics are not bounded; circle = special static case ($\alpha = 1, \beta = 0$).

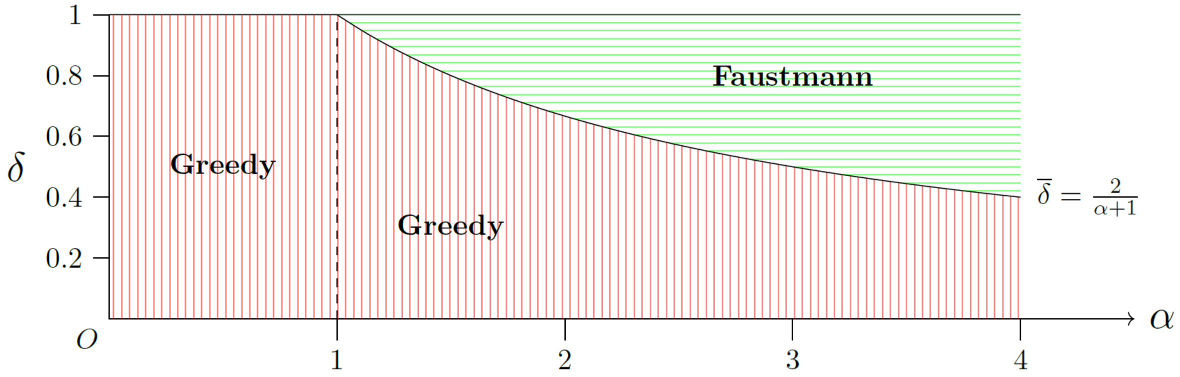

Figure 3. Optimal solutions for for $u(x_t) = x_t$ and $\beta = 1$. Red area = greedy solution is optimal; green area = Faustmann solution is optimal.

Example 2. *If $u(x_t) = log\ (x_t)$, then the optimal solution is the Hotelling solution either for $\alpha < 1$ or for δ sufficiently small; for $\alpha > 1$ and δ sufficiently large, the Faustmann solution is optimal.*

The outcomes of Example 2 are illustrated in Figure 4. The logarithmic utility function implies that consumption is multiplicative over time.

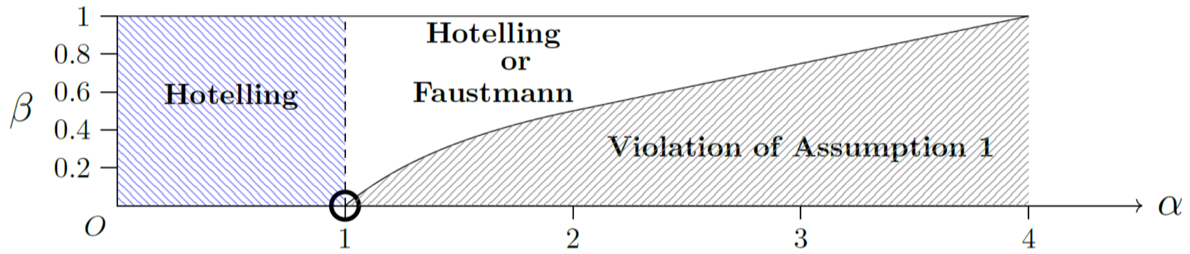

Figure 4. Optimal solutions for $u(x_t) = log\ (x_t)$. White area = the optimal solution depends on δ; blue area = Hotelling solution is optimal; grey area = dynamics are not bounded; circle = special static case ($\alpha = 1, \beta = 0$).

Example 3. *If $u(x_t) = 2x_t - x_t^2$, then the optimal solution is the Faustmann solution for $\alpha > 1$ and δ sufficiently large; for δ sufficiently small, the Hotelling solution is optimal; and in other cases, it is the greedy solution.*

The outcomes of Example 3 are illustrated in Figure 5. The quadratic utility function is intended as an intermediate case between multiplicative and additive consumption.

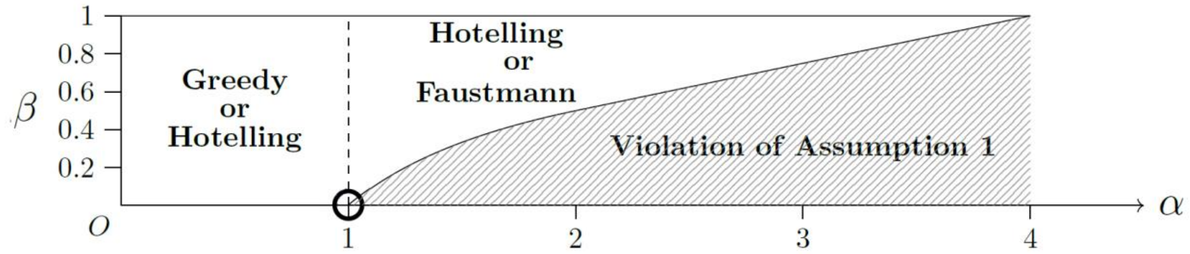

Figure 5. Optimal solutions for $u(x_t) = 2x_t - x_t^2$. White area = the optimal solution depends on δ; grey area = dynamics are not bounded; circle = special static case ($\alpha = 1, \beta = 0$).

Simulations with different utility functions confirm the results of Example 3 in a qualitative way. The parameter space is divided in three areas in which the Hotelling, Faustmann and greedy solutions are optimal. For $\alpha < 1$, the Faustmann solution does not exists, and either Hotelling or greedy strategies work best. For $\alpha > 1$, the choice is in most cases between Hotelling and Faustmann solutions.

4.2 Implications for the carbon budget

Since 1850, global carbon emissions have risen on average 2 percent per year, reaching a level above 40 Gt per year after 2012 (see Figure 6). In 2020, the year of the covid lockdowns, the decrease was 4.7 percent, before rebounding in 2021 to 2019 levels (UNEP, 2022). In order to stay within the remaining carbon budget of 1350 Gt for 2.0°C, emissions would have to be cut with around 3 percent per year. How can we understand these empirical facts within the (simplified) theoretical framework provided here?

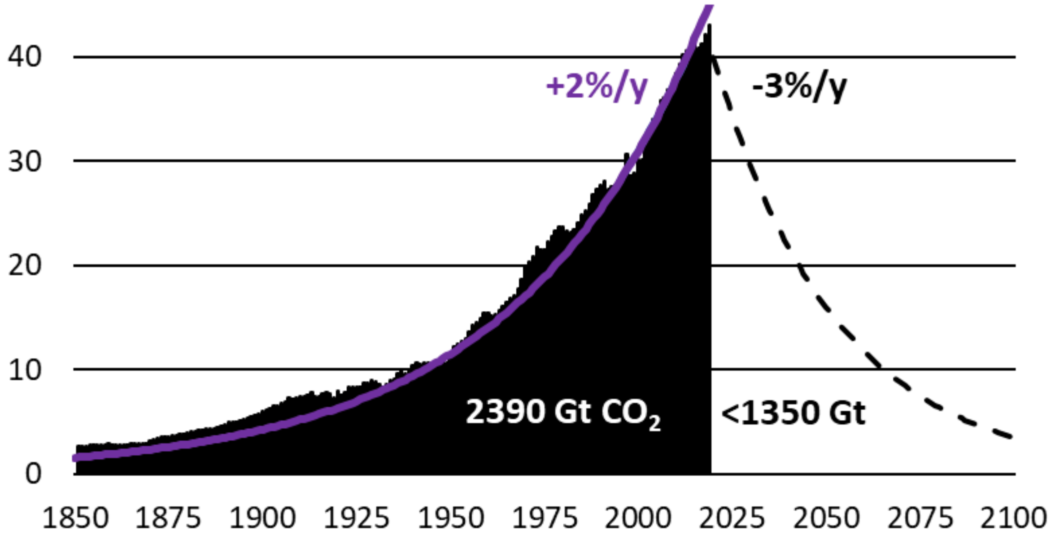

Figure 6. Historic CO_2 emissions have increased at 2 percent per year, reaching levels over 40 Gt per year after 2012. Staying within the remaining carbon budget for 2.0°C would require cuttings of almost 3 percent per year.

Clearly, the 2 percent annual increase of carbon emissions have been driven by growth of population and economic activity. To model this economic constraint, consider an additional equation

$$x_t \leq \gamma_t$$

in the maximisation problem (2). The economic potential to use the carbon budget increases by $\gamma = 1.02$ per year. When γ_t is low, clearly it is optimal to use all carbon available under the constraint. This is a greedy solution, which can be reformulated as using $x_t^g = \min\,(z_t, \gamma_t)$ at time t.

Under which conditions is it optimal to shift to a non-greedy strategy? Let us first consider the Faustmann alternative. If we assume a carbon cycle process following equation (1) and optimality of Faustmann's rule, then a level of 40 Gt and a carbon budget of (2390 + 1350 =) 3740 Gt suggests that $\frac{x^F}{z_0} = \frac{(\alpha-1)^2}{4\alpha\beta} \approx 0.011$. This implies $\alpha \in (1;\ 1.23)$ and $\beta = \frac{(\alpha-1)^2}{0.043\alpha}$. Also implied is a lower bound on the discount rate δ, to ensure that the Faustmann solution gives a higher pay-off than the other solutions. So, for example, parameter values of $\alpha = 1.05$, $\beta = 0.05$ and $\delta = 0.98$ could be consistent with a Faustmann solution.

Instead, if we assume optimality of Hotelling's rule and logarithmic utility, then a 3 percent annual decrease implies by $\frac{x_{t+1}^H}{x_t^H} = \delta\, f'_{\alpha\beta}(\tilde{z}_t)$ that $\delta\, f'_{\alpha\beta}(\tilde{z}_t) \approx 0.97$. Under equation (1), this suggests $\alpha < 1.02$. (Note that β would have to be close to 0 to satisfy Assumption 1.) So to facilitate a shift from greed to the Hotelling solution, we require parameter values α and β implying a close to static system, which seems realistic and thus possible.

Figure 7 illustrates how the shift from a greedy to a Hotelling solution could take place. In the initial periods, when the economic potential to emit is low, the greedy solution is always optimal. The black line represents the choice x_t^*, which increases with increases with $\gamma = 1.02$ per year. At some point, the Hotelling (blue line) and Faustmann (green line) solutions become feasible. In Figure 7, the choice shifts to Hotelling, indicating a relative low discount rate δ.

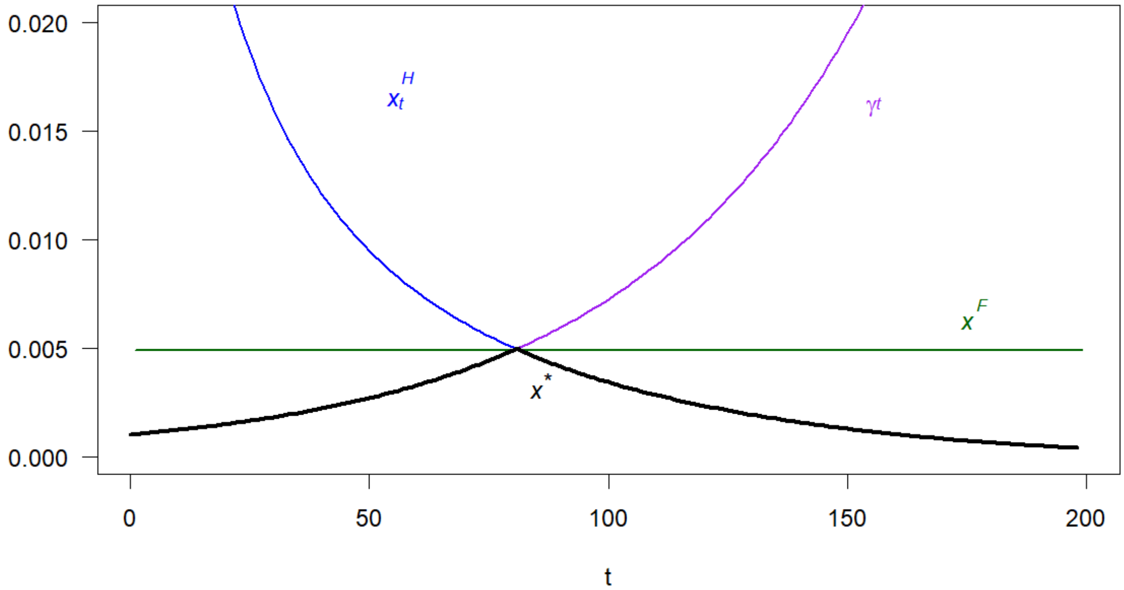

Figure 7. Illustration of Hotelling (blue), Faustmann (green) and greedy (purple) solutions over time. Hotelling: numerical optimisation of $U(\boldsymbol{x}^H)$ over different starting values x_0^H. Faustmann: $x_t = x^F$ for $t > 0$ (x_0^F is not depicted). Greedy: maximal value γ_t. Parameter values: $\alpha = 1.02$; $\beta = 0.02$; $\delta = 0.97$; $z_0 = 1$; $\gamma = 1.02$; $\gamma_0 = 0.001$. The thick black line indicates a possible shift from a greedy to a Hotelling solution.

5. Concluding remarks

Is Hotelling's rule optimal under a dynamic carbon budget ? In a simple, but general model with a dynamic resource stock, the answer is shown to be yes. The static, fixed resource stock is retained as a special case in the model. When the utility function is convex, the Hotelling solution is optimal when the discount rate is sufficiently small, and for many parameter values specifying the resource dynamics.

Yet the model does point to an alternative, Faustmann solution. Under the Faustmann rule, every year no more carbon is emitted than can be absorbed by natural carbon sinks. The Hotelling, Faustmann and greedy approaches have been brought explicitly together in the model.

The social planner discounts the future in a way that is standard for economists, but is not undisputed. When there is uncertainty about climate dynamics, standard maximisation may not be appropriate. Also, future generations receive increasingly small weights compared to current generations. For these reasons, alternative formulations of the optimisation problem, such as maximin (Rawls, 1971), might be more appropriate.

One could argue the world is currently following a greedy strategy concerning CO_2. We emit as much as the economy allows, to rapidly approach a global warming of 1.5 or even 2 degrees Celsius. While climate change is commonly perceived as "the greatest market failure the world has ever seen" (Stern, 2007), the model shows that the greedy approach could also be 'rational' in a one-person world. This occurs in particular if the social planner is very impatient.

The Hotelling approach to consume the carbon budget is useful to steer away from greediness. This Hotelling approach makes clear that every tonne emitted CO_2 adds up towards higher temperature. As the world comes closer to a climate-neutral economy, however, a Faustmann-type of approach could become more appropriate. This would mean annual carbon ceilings (as advocated by, for example, KBT, 2021), depending on the natural carbon cycle.

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Appendix A: Presuppositions

This Appendix aims to provide insight into the underlying assumptions of the model. The Hotelling-Faustmann model presented in Section 2 ultiimately depends on two main ‘presuppositions’, complemented by two specifications.

Preceding any economic model is the idea of a human society that pursues certain goals: for example, provision of food for all members of society (Reuten, 2019). For this purpose, society requires the creating and transformation of goods (or physical capital).

The two main presuppositions deal with (1) the character of economic decision-making and (2) and its timing, as explained below. The specifications are related to natural resource (1a) use in one-person economies (1b), the focus of this paper.

Presupposition 1 (economics)
“Economics deals with decisions for an optimal creation and transformation of goods, by combining elements of nature, human activity, cultivated nature, and previously created instruments."

Specification 1a (natural resources)
“Natural resource economics deals with the optimal use of nature, taking human activity, cultivated nature, and previously created instruments as given."

Specification 1b (one-person economics)
“One-person economics deals with a (hypothetical) `society' of a single person and its optimal activities."

Presupposition 2 (time)
“Decisions about activities of creation and transformation of goods deal with the timing of these activities. Choosing an activity means acting on one particular form of goods, and not on another; and it means acting on that form now, and not in the future."

With these presuppositions and specifications in mind, the stylised model of optimal use of a limited resource unfolds itself. All assumptions in the model either follow from the presuppositions or from the specifications; or are technical assumptions clarifying the exposition (equation (1), Assumption 1 and the utility functions).

Appendix B: Proofs

Proof of Proposition 2

The maximisation problem (2) expressed as the Bellman equation reads

$$V(z_t) = \max_{x_t \in [0,z_t]} \left[u(x_t) + \delta\, V\left(f_{\alpha\beta}(z_t - x_t) \right) \right]$$
$$= \max_{\tilde{z}_t \in [0,z_t]} \left[u(z_t - \tilde{z}_t) \quad + \quad \delta\, V\left(f_{\alpha\beta} \tilde{z}_t \right) \right].$$

The first-order condition of the second Bellman equation (taking derivatives of $\tilde{z}_t$) leads to the Euler equation

$$u'(x_t^H) = \delta\, f'_{\alpha\beta}(\tilde{z}_t)\, u'(x_{t+1}^H). \qquad (3')$$

We call a solution satisfying the Euler equation (3') for all $t > 0$ the *Hotelling solution*, denoted by $\{x_t^H\}_{t=0}^{\infty}$.

Proof of Example 1

With linear utility and $\delta < 1$, smoothing consumption is never optimal, so the Hotelling solution is not relevant in this case. For $\alpha < 1$, the Faustmann solution does not exist, so then the greedy solution is optimal.

For $\alpha > 1$, the greedy solution delivers $U_{lin}^g = u(z_0) = z_0$. And the Faustmann solution delivers $U_{lin}^F = u(x_0^F) + \sum_{t=1}^{\infty} \delta^t\, u(x^F) = z_0 \left(\frac{2\alpha\beta - \alpha + 1}{2\alpha\beta} + \frac{\delta}{1-\delta} \frac{(\alpha-1)^2}{4\alpha\beta} \right)$. This implies that the Faustmann solution delivers more when $\frac{\delta}{1-\delta} \frac{(\alpha-1)^2}{4\alpha\beta} > 1 - \frac{2\alpha\beta - \alpha + 1}{2\alpha\beta}$ or $\delta > \bar{\delta}_{lin} \equiv \frac{2}{\alpha+1}$.

So the optimal solution is the greedy solution either for $\alpha < 1$ or for δ sufficiently small; for $\alpha > 1$ and δ sufficiently large, the Faustmann solution is optimal.

Proof of Example 2
As $u(0)$ goes to minus infinity, the greedy solution is never optimal under logarithmic utility. For $\alpha < 1$, the Faustmann solution does not exist, so then the Hotelling solution is optimal.

For $\alpha > 1$, the Faustmann solution delivers $U_{log}^{F} \;= u(x_0^F) + \sum_{t=1}^{\infty} \delta^t\, u(x^F) = \frac{\log z_0}{1-\delta} + \log \frac{2\alpha\beta-\alpha+1}{2\alpha\beta} + \frac{\delta}{1-\delta} \log \frac{(\alpha-1)^2}{4\alpha\beta}$. The Hotelling solution delivers $U_{log}^{H} = \frac{\log z_0}{1-\delta} + \sum_{t=0}^{\infty} \delta^t \log\left(\frac{x^H}{z_0}\right)$. Equation (3) implies $\frac{x_{t+1}^H}{x_t^H} = \delta\, f_{\alpha\beta}'(\tilde{z}_t)$. The Faustmann and Hotelling solutions can not be compared algebraically. For δ larger than a certain $\bar{\delta}_{log}$, it is clear that keeping a fixed consumption level will be more worthwhile than optimal depletion.

So the optimal solution is the Hotelling solution either for $\alpha < 1$ or for δ sufficiently small; for $\alpha > 1$ and δ sufficiently large, the Faustmann solution is optimal.

Proof of Example 3
For $\alpha < 1$, the Faustmann solution does not exist, so then either greedy or the Hotelling solution is optimal.

For $\alpha > 1$, the greedy solution delivers $U_{qua}^{g} = u(z_0) = 2z_0 - z_0^2$; the Faustmann solution $U_{qua}^{F} \;= u(x_0^F) + \sum_{t=1}^{\infty} \delta^t\, u(x^F) = \; 2z_0 \left(\frac{2\alpha\beta-\alpha+1}{2\alpha\beta} + \frac{\delta}{1-\delta}\frac{(\alpha-1)^2}{4\alpha\beta}\right) - z_0^2 \left(\left(\frac{2\alpha\beta-\alpha+1}{2\alpha\beta}\right)^2 + \frac{\delta}{1-\delta}\frac{(\alpha-1)^4}{16\alpha^2\beta^2}\right)$; and the Hotelling solution $U_{qua}^{H} \;= \sum_{t=0}^{\infty} \delta^t (2x^H - (x^H)^2)$. Equation (3) implies $x_{t+1}^H = 1 - \frac{1-x_t^H}{\delta\, f_{\alpha\beta}'(\tilde{z}_t)}$. For $\delta > \bar{\delta}_{qua}$, the Faustmann solution is clearly optimal, as it retains a fixed consumption level in the future. With $\delta < \bar{\delta}_{qua}$ and $\alpha > 1$, the Hotelling solution is just an improved greedy solution, that takes into account the convexity of the utility function as well as the growth factor α. So the greedy solution is not optimal is $\alpha > 1$.

So the optimal solution is the Faustmann solution for $\alpha > 1$ and δ sufficiently large; for δ sufficiently small, the Hotelling solution is optimal; and in other cases, it is the greedy solution.