Supplementary Material for "Localizing Strictly Proper Scoring Rules"

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Abstract

This supplementary material for "Localizing Strictly Proper Scoring Rules" complements the main paper with proofs, additional derivations, Monte Carlo analyses and empirical results. For context, notation and definitions, see the main paper. In the first section, we provide comprehensive proofs of the main theoretical results. In the second section, we provide additional proofs for other findings highlighted in the main text. Derivations related to various semi-local scoring rules are elaborated in the third section, focusing on their properties and localized variants. In the fourth section, the detailed Monte Carlo simulation analyses are presented, examining the size and power properties of a selection of conditional and censored scoring rules. Finally, the last section of this supplement contains extensive tables that provide the underlying results for the empirical performance summarized in the main paper.

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Contents

A Proofs . . . 3

A.1 Proof of Theorem 2 . . . 3

A.2 Proof of Theorem 3 . . . 6

B Additional Proofs . . . 11

B.1 Proof of the censored density in Equation (3) . . . 11

B.2 Proof of Corollary 1 . . . 12

B.3 Proof of Corollary 2 . . . 13

C Derivations for Table 1 . . . 14

C.1 LogS . . . 15

C.2 PsSphS_α . . . 16

C.3 PowS_α . . . 19

D Monte Carlo Simulation . . . 21

D.1 Size . . . 21

D.2 Power . . . 22

E Additional Tables . . . 28

E.1 Risk management . . . 28

E.2 Inflation . . . 31

E.3 Climate . . . 35

A Proofs

A.1 Proof of Theorem 2

For clarity of exposition, we first prove the main ingredients of the proof via two lemmas and a corollary.

Lemma A1. *Consider the generalized censored scoring rule defined in Definition 5. Then, $\forall w \in \mathcal{W}$ and $\mathrm{H} \in \mathcal{H}$, the following identity holds:*

$$\int_{\mathcal{Y}} S^\flat_{w,\mathrm{H}}(\mathrm{F}, y)\mathrm{P}(\mathrm{d}y) = \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w,\mathrm{H}}(\mathrm{d}y).$$

Proof. The result follows by rearranging the integral on the left-hand side. Specifically,

$$\begin{aligned}
\int_{\mathcal{Y}} S^\flat_{w,\mathrm{H}}(\mathrm{F}, y)\mathrm{P}(\mathrm{d}y) &= \int_{\mathcal{Y}} \Big(w(y)S(\mathrm{F}^\flat_{w,\mathrm{H}}, y) + \big(1 - w(y)\big)\int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, q)\mathrm{H}(\mathrm{d}q)\Big)\mathrm{P}(\mathrm{d}y) \\
&= \int_{\mathcal{Y}} w(y)S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}(\mathrm{d}y) + \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, q)\int_{\mathcal{Y}} \big(1 - w(y)\big)\mathrm{P}(\mathrm{d}y)\mathrm{H}(\mathrm{d}q) \\
&= \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}_w(\mathrm{d}y) + \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\bar{P}_w\mathrm{H}(\mathrm{d}y) \\
&= \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\big(\mathrm{P}_w(\mathrm{d}y) + \bar{P}_w\mathrm{H}(\mathrm{d}y)\big) \\
&= \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w,\mathrm{H}}(\mathrm{d}y).
\end{aligned}$$

□

Lemma A2. *Consider two distributions P and F on the same measurable space $(\mathcal{Y}, \mathcal{G})$. On the same space, let their censored counterparts $\mathrm{P}^\flat_{w,\mathrm{H}}$ and $\mathrm{F}^\flat_{w,\mathrm{H}}$ be given by Definition 5 and suppose that Assumption 1 holds. Then,*

$$\mathrm{F}^\flat_{w,\mathrm{H}}(E) = \mathrm{G}^\flat_{w,\mathrm{H}}(E),\ \forall E \in \mathcal{G} \iff \mathrm{F}(E \cap \{w > 0\}) = \mathrm{G}(E \cap \{w > 0\}),\ \forall E \in \mathcal{G}.$$

Proof. " $\implies$ " We start with the most challenging direction, for which Assumption 1 is of critical importance. Let $\tilde{E}$ be an element of $\mathcal{G}$ satisfying the conditions given in Assumption 1. First, note that

$$
\begin{aligned}
&\mathrm{F}^{\flat}_{w,\mathrm{H}}(E) = \mathrm{G}^{\flat}_{w,\mathrm{H}}(E), \qquad \forall E \in \mathcal{G} \\
&\implies \mathrm{F}^{\flat}_{w,\mathrm{H}}\left(E \cap \tilde{E}\right) = \mathrm{G}^{\flat}_{w,\mathrm{H}}\left(E \cap \tilde{E}\right), \qquad \forall E \in \mathcal{G} \\
&\implies \int_{\mathcal{Y}} (1-w)\,\mathrm{dF} \cdot \mathrm{H}\left(E \cap \tilde{E}\right) = \int_{\mathcal{Y}} (1-w)\,\mathrm{dG} \cdot \mathrm{H}\left(E \cap \tilde{E}\right), \qquad \forall E \in \mathcal{G} \\
&\implies \int_{\mathcal{Y}} (1-w)\,\mathrm{dF} \cdot \mathrm{H}\left(\tilde{E}\right) = \int_{\mathcal{Y}} (1-w)\,\mathrm{dG} \cdot \mathrm{H}\left(\tilde{E}\right), \\
&\implies \int_{\mathcal{Y}} (1-w)\,\mathrm{dF} = \int_{\mathcal{Y}} (1-w)\,\mathrm{dG},
\end{aligned}
$$

where we have used the closure of σ-algebras under countable intersections. Then, exploit this equality to conclude

$$
\begin{aligned}
&\mathrm{F}^{\flat}_{w,\mathrm{H}}(E) = \mathrm{G}^{\flat}_{w,\mathrm{H}}(E), \qquad \forall E \in \mathcal{G} \\
&\implies \int_{\mathcal{Y}} w(y)\mathbb{1}_{y\in E}\mathrm{F}(\mathrm{d}y) = \int_{\mathcal{Y}} w(y)\mathbb{1}_{y\in E}\mathrm{G}(\mathrm{d}y), \qquad \forall E \in \mathcal{G} \\
&\implies \mathrm{F}(E \cap \{w > 0\}) = \mathrm{G}(E \cap \{w > 0\}), \qquad \forall E \in \mathcal{G}.
\end{aligned}
$$

" $\impliedby$ " The other direction is somewhat trivial. Indeed,

$$
\begin{aligned}
&\mathrm{F}(E \cap \{w > 0\}) = \mathrm{G}(E \cap \{w > 0\}), \qquad \forall E \in \mathcal{G} \\
&\implies \int_{\mathcal{Y}} w(y)\mathbb{1}_{y\in E}\mathrm{F}(\mathrm{d}y) = \int_{\mathcal{Y}} w(y)\mathbb{1}_{y\in E}\mathrm{G}(\mathrm{d}y), \qquad \forall E \in \mathcal{G} \\
&\implies \int_{\mathcal{Y}} (1-w)\mathrm{dF} = \int_{\mathcal{Y}} (1-w)\mathrm{dG},
\end{aligned}
$$

and the two implied results jointly imply $\mathrm{F}^{\flat}_{w,\mathrm{H}}(E) = \mathrm{G}^{\flat}_{w,\mathrm{H}}(E),\ \forall E \in \mathcal{G}, \forall \mathrm{H} \in \mathcal{H}$. $\square$

Corollary A3. *Let Assumption 1 be satisfied. Then, the generalized censored scoring rule defined in Definition 5 is localizing* $\forall \mathrm{H} \in \mathcal{H}$.

Proof. Suppose that $\mathrm{F}(E \cap \{w > 0\}) = \mathrm{G}(E \cap \{w > 0\})$, $\forall E \in \mathcal{G}$. Then, by Lemma A2, $\mathrm{F}^{\flat}_{w,\mathrm{H}}(E) = \mathrm{G}^{\flat}_{w,\mathrm{H}}(E), \forall E \in \mathcal{G}$, whence it follows that $S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y) = S^{\flat}_{w,\mathrm{H}}(\mathrm{F}, y), \forall y \in \mathcal{Y}$. $\square$

We now turn to the main body of the proof. The definition of a strictly locally proper scoring rule (Definition 3) and the underlying concepts it relies on, namely, a localizing weighted scoring rule (Definition 2) and propriety (Definition 1), require us to establish a list of three conditions. Specifically, $\forall \mathrm{H} \in \mathcal{H}$: (i) $S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y)$ must be localizing relative to $\mathcal{W}$, (ii) $S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y)$ must be proper relative to $\mathcal{P}$, $\forall w \in \mathcal{W}$, and (iii) the 'if and only if' statement in Definition 3. We prove them one by one.

(i) $S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y)$ is localizing relative to $\mathcal{W}$, $\forall \mathrm{H} \in \mathcal{H}$, by Corollary A3.

(ii) Fix an arbitrary $w \in \mathcal{W}$ and $\mathrm{H} \in \mathcal{H}$. Since $\mathcal{P}^{\flat}_{w,\mathrm{H}} \subseteq \mathcal{P}^{\flat}$, S is strictly proper relative to $\mathcal{P}^{\flat}_{w,\mathrm{H}}$, i.e.,

$$\int_{\mathcal{Y}} S(\mathrm{P}^{\flat}_{w,\mathrm{H}}, y)\mathrm{P}^{\flat}_{w,\mathrm{H}}(\mathrm{d}y) \geq \int_{\mathcal{Y}} S(\mathrm{F}^{\flat}_{w,\mathrm{H}}, y)\mathrm{P}^{\flat}_{w,\mathrm{H}}(\mathrm{d}y), \quad \forall \mathrm{P}^{\flat}_{w,\mathrm{H}}, \mathrm{F}^{\flat}_{w,\mathrm{H}} \in \mathcal{P}^{\flat}_{w,\mathrm{H}}, \tag{A.1}$$

which is, by definition of the class $\mathcal{P}^{\flat}_{w,\mathrm{H}} \equiv \{[\mathrm{P}]^{\flat}_{w,\mathrm{H}}, \mathrm{P} \in \mathcal{P}\}$, equivalent to

$$\int_{\mathcal{Y}} S([\mathrm{P}]^{\flat}_{w,\mathrm{H}}, y)[\mathrm{P}]^{\flat}_{w,\mathrm{H}}(\mathrm{d}y) \geq \int_{\mathcal{Y}} S([\mathrm{F}]^{\flat}_{w,\mathrm{H}}, y)[\mathrm{P}]^{\flat}_{w,\mathrm{H}}(\mathrm{d}y), \quad \forall \mathrm{P}, \mathrm{F} \in \mathcal{P}, \tag{A.2}$$

and hence, by Lemma A1, also

$$\int_{\mathcal{Y}} S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y)\mathrm{P}(\mathrm{d}y) \geq \int_{\mathcal{Y}} S^{\flat}_{w,\mathrm{H}}(\mathrm{F}, y)\mathrm{P}(\mathrm{d}y), \quad \forall \mathrm{P}, \mathrm{F} \in \mathcal{P}. \tag{A.3}$$

Therefore, $S^{\flat}_{w,\mathrm{H}}(\mathrm{P}, y)$ is proper relative to $\mathcal{P}$ by Definition 1.

(iii) Since S is strictly proper relative to $\mathcal{P}^\flat$ and hence $\mathcal{P}^\flat_{w,\mathrm{H}}$, it also follows that, $\forall w \in \mathcal{W}$ and $\mathrm{H} \in \mathcal{H}$,

$$\int_{\mathcal{Y}} S(\mathrm{P}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w,\mathrm{H}}(\mathrm{d}y) = \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w,\mathrm{H}}(\mathrm{d}y) \iff \mathrm{P}^\flat_{w,\mathrm{H}} = \mathrm{F}^\flat_{w,\mathrm{H}},$$

and thus, by Lemma A2,

$$\int_{\mathcal{Y}} S(\mathrm{P}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w}(\mathrm{d}y) = \int_{\mathcal{Y}} S(\mathrm{F}^\flat_{w,\mathrm{H}}, y)\mathrm{P}^\flat_{w,\mathrm{H}}(\mathrm{d}y) \iff \mathrm{P}(E \cap \{w > 0\}) = \mathrm{F}(E \cap \{w > 0\}),$$

$\forall E \in \mathcal{G}$, and hence, by Lemma A1, also

$$\int_{\mathcal{Y}} S^\flat_{w,\mathrm{H}}(\mathrm{P}, y)\mathrm{P}(\mathrm{d}y) = \int_{\mathcal{Y}} S^\flat_{w,\mathrm{H}}(\mathrm{F}, y)\mathrm{P}(\mathrm{d}y) \iff \mathrm{P}(E \cap \{w > 0\}) = \mathrm{F}(E \cap \{w > 0\}),$$

which is the desired 'if and only if' statement of Definition 3.

But then, as we have verified each of the listed conditions (i) to (iii), we have shown that $S^\flat_{w,\mathrm{H}}(\mathrm{P}, y)$ is strictly locally proper relative to $(\mathcal{P}, \mathcal{W})$, $\forall \mathrm{H} \in \mathcal{H}$. $\square$

A.2 Proof of Theorem 3

We start by rephrasing the hypotheses. Since the densities f_{jt} must integrate to one on $A_t \cup A_t^c$, the null hypothesis implies that these densities integrate to $\mathrm{F}_{jt}(A_t^c)$ on A_t^c. Therefore, the implied specification on A_t^c can be summarized as

$$\frac{\mathrm{F}_{jt}(A^c)}{\mathrm{H}_{jt}(A^c)} h_{jt} \mathbb{1}_{A_t^c} = \mathrm{F}_{jt}(A^c)[h_{jt}]^\sharp_{A_t^c} \mathbb{1}_{A_t^c}, \quad j \in \{0, 1\},$$

where the unknown densities $h_{jt} = \frac{\mathrm{dH}_{jt}}{\mathrm{d}\mu}$ can be seen as infinite-dimensional nuisance parameters. Explicating the implied assumption on A_t^c in the hypotheses and phrasing them in terms of a statement about the whole sample distribution leads to the following equivalent hypotheses:

$$\mathbb{H}_j : p(\mathbf{y}) = f_j(\mathbf{y}) := \prod_{t=0}^{T-1} \left(f_{jt}(y_{t+1}) \mathbb{1}_{A_t}(y_{t+1}) + \mathrm{F}_{jt}(A^c)[h_{jt}]^{\sharp}_{A_t^c}(y_{t+1}) \mathbb{1}_{A_t^c}(y_{t+1}) \right), \quad j \in \{0, 1\}.$$

Since the densities f_{jt} are fixed, and the densities h_{jt} are unrestricted under both hypotheses, the class of densities satisfying hypothesis $\mathbb{H}_j$ can alternatively be written as

$$\mathbb{F}_j = \left\{ \prod_{t=0}^{T-1} \left(f_j(y_{t+1}) \mathbb{1}_{A_t}(y_{t+1}) + \mathrm{F}_{jt}(A^c)[h_{jt}]^{\sharp}_{A_t^c}(y_{t+1}) \mathbb{1}_{A_t^c}(y_{t+1}) \right), h_j \in \mathcal{H} \right\}, \quad j \in \{0, 1\},$$

in which $\mathcal{H}$ denotes the space of all densities on $\mathcal{Y}^T$. In terms of the index set of all observations $\mathcal{I} = \{1, \ldots, T\}$, this space can also be denoted as $\mathcal{Y}(\mathcal{I}) = \prod_{t \in \mathcal{I}} \mathcal{Y}_t$.

Fixing an $\alpha \in (0, 1)$, the aim is to find a uniformly most powerful (UMP) test ϕ^* of size α for testing problem (8), i.e., a solution to the maximization problem

$$\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi, \qquad \Phi(\alpha) = \{\phi : \sup_{f_0 \in \mathbb{F}_0} \mathbb{E}_{f_0} \phi \leq \alpha\}. \tag{A.4}$$

Now fix an $h_1 \in \mathcal{H}$ so that the distribution under the alternative is completely known. Given the fact that the hypotheses are, in the end, silent about the shape of the densities on A_t^c, we conjecture that a UMP test neglects the information about the shape of the densities on A_t^c, $\forall t$. If $T = 2$, for example, and we consider the optimal test on $A_1 \times A_2^c$, our intuition is that an optimal test is not concerned about the shape of $[h_2]^{\sharp}_{{A_2}^c}$, that is, the specific values $[h_2]^{\sharp}_{{A_2}^c}(y_2)$ for all $y_2 \in A_2^c$, but only about the total probability of an outcome falling into A_2^c. In other words, we expect that a solution to problem (A.4) has integrated out the dependence on the nuisance densities.

Although it is obvious that marginalizing out the (still assumed to be fixed) density $h_1 \in \mathcal{H}$ is harmless in terms of power, it is non-trivial that this is an affordable strategy in terms of size for all $h_0 \in \mathcal{H}$. Lemma A4 and its proof show that the subclass of tests disregarding information about the shape of h_1 is guaranteed to be size correct. In our search for the UMP test, Corollary A5 then formalizes the idea that we can restrict our attention to tests of the conjectured kind.

Lemma A4. *Consider testing problem* (8) *and suppose that the outcomes* $(y_t)_{t\in\mathcal{I}_A}$ *are in* A_t, *and the remaining* $n-k$, *with* $k = |\mathcal{I}_A|$, *observations* $(y_t)_{t\in\mathcal{I}_{A^c}}$ *in* A_t^c. *For an arbitrary but fixed density* $h_1 \in \mathcal{H}$, *the test*

$$\psi_{h_1} : \mathcal{Y}^T \to [0,1], \quad \psi_{h_1} = \int_{\mathcal{Y}(\mathcal{I}_{A^c})} \phi_{h_1}^* \prod_{t\in\mathcal{I}_{A^c}} [h_{1t}]^\sharp_{A_t^c} \mathbb{1}_{A_t^c} \mathrm{d}\mu^{\otimes|\mathcal{I}_{A^c}|},$$

where $\phi_{h_1}^*$ *denotes a solution to problem* (A.4), *is such that* $\psi_{h_1} \in \Phi(\alpha)$.

Proof. Due to the integral over $\mathcal{Y}(\mathcal{I}_{A^c})$, any test ψ_{h_1} is constant in arguments varying in $\mathcal{Y}(\mathcal{I}_{A^c})$. We can use this observation to simplify the size of a test ψ_{h_1}. In particular,

$\forall h_1 \in \mathcal{H}$, we have that

$$
\begin{aligned}
\sup_{f_0 \in \mathbb{F}_0} \mathbb{E}_{f_0} \psi_{h_1} &= \left(\prod_{t \in \mathcal{I}_{A^c}} \mathrm{F}_0(A_t^c) \right) \sup_{h_0 \in \mathcal{H}} \int_{\mathcal{Y}^T} \psi_{h_1} \prod_{t \in \mathcal{I}_A} f_{0t} \mathbb{1}_{A_t} \prod_{t \in \mathcal{I}_{A^c}} [h_{0t}]^{\sharp}_{A_t^c} \mathbb{1}_{A_t^c} \mathrm{d}\mu^{\otimes T} \\
&= \left(\prod_{t \in \mathcal{I}_{A^c}} \mathrm{F}_0(A_t^c) \right) \int_{\mathcal{Y}(\mathcal{I}_A)} \psi_{h_1} \prod_{t \in \mathcal{I}_A} f_{0t} \mathbb{1}_{A_t} \mathrm{d}\mu^{\otimes |\mathcal{I}_A|} \\
&= \left(\prod_{t \in \mathcal{I}_{A^c}} \mathrm{F}_0(A_t^c) \right) \int_{\mathcal{Y}^T} \phi^*_{h_1} \prod_{t \in \mathcal{I}_{A^c}} [h_{1t}]^{\sharp}_{A_t^c} \mathbb{1}_{A_t^c} \mathrm{d}\mu^{\otimes |\mathcal{I}_{A^c}|} \prod_{t \in \mathcal{I}_A} f_{0t} \mathbb{1}_{A_t} \mathrm{d}\mu^{\otimes |\mathcal{I}_A|} \\
&\leq \left(\prod_{t \in \mathcal{I}_{A^c}} \mathrm{F}_0(A_t^c) \right) \sup_{h_0 \in \mathcal{H}} \int_{\mathcal{Y}^T} \phi^*_{h_1} \prod_{t \in \mathcal{I}_{A^c}} [h_{0t}]^{\sharp}_{A_t^c} \mathbb{1}_{A_t^c} \mathrm{d}\mu^{\otimes |\mathcal{I}_{A^c}|} \prod_{t \in \mathcal{I}_A} f_{0t} \mathbb{1}_{A_t} \mathrm{d}\mu^{\otimes |\mathcal{I}_A|} \\
&= \sup_{f_0 \in \mathbb{F}_0} \mathbb{E}_{f_0} \phi^*_{h_1} \\
&\leq \alpha,
\end{aligned}
$$

since $\phi^*_{h_1} \in \Phi(\alpha)$. Hence, $\psi_{h_1} \in \Phi(\alpha)$. $\square$

Corollary A5. *Consider testing problem* (8) *and assume that outcomes* $y_t \in A_t, \forall t \in \mathcal{I}_A$, *and* $y_t \in A_t^c, \forall t \in \mathcal{I}_{A^c}$. *Let* $\Psi(\alpha) \subseteq \Phi(\alpha)$ *denote the class of size* α *tests on* $\mathcal{Y}^T$ *that are constant in arguments varying in* $\mathcal{Y}(\mathcal{I}_{A^c})$. *Then,*

$$
\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi = \max_{\psi \in \Psi(\alpha)} \mathbb{E}_{f_1} \psi, \qquad \forall h_1 \in \mathcal{H}.
$$

Proof. Fix an arbitrary $h_1 \in \mathcal{H}$. Since $\Psi(\alpha) \subseteq \Phi(\alpha)$, we trivially have that $\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi \geq \max_{\psi \in \Psi(\alpha)} \mathbb{E}_{f_1} \psi$. Now suppose that $\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi < \max_{\psi \in \Psi(\alpha)} \mathbb{E}_{f_1} \psi$. Then, we can always define the test $\tilde{\psi} = \int_{\mathcal{Y}(\mathcal{I}_{A^c})} \phi^* \prod_{t \in \mathcal{I}_{A^c}} [h_{1t}]^{\sharp}_{A_t^c} \mathbb{1}_{A_t^c} \mathrm{d}\mu^{\otimes |\mathcal{I}_{A^c}|}$, with $\phi^* \in \arg\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi$, satisfying $\mathbb{E}_{f_1} \phi^* = \mathbb{E}_{f_1} \tilde{\psi}$. But, by Lemma A4, $\tilde{\psi} \in \Psi(\alpha)$, in which case $\max_{\phi \in \Phi(\alpha)} \mathbb{E}_{f_1} \phi = \max_{\psi \in \Psi(\alpha)} \mathbb{E}_{f_1} \tilde{\psi}$, contradicting the assumed strict inequality. $\square$

For any fixed $h_1 \in \mathcal{H}$, the reduced optimization problem resulting from Corollary A5 simplifies to a simple versus simple hypothesis in terms of the censored measures $\mathrm{d}[\mathrm{F}_{jt}]^{\flat}_{A_t} = \mathbb{1}_{A_t}\mathrm{dF}_{jt} + \mathrm{F}_{jt}(A_t^c)\mathrm{d}\delta_*$, allowing us to apply Neyman and Pearson (1933).

Specifically, for any fixed $h_1 \in \mathcal{H}$, the most powerful test of size α is a solution to the following restricted maximization problem:

$$
\begin{aligned}
&\max_{\phi\in\Phi(\alpha)} \mathbb{E}_{f_1}\phi \\
&= \max_{\boldsymbol{\alpha}\in\Delta_{\bar{T}}(\alpha)} \sum_{k=0}^{T}\sum_{s=1}^{\binom{T}{k}} \max_{\phi_{k,s}\in\Phi(\alpha_{k,s})} \mathbb{E}_{f_1}\left(\phi_{k,s}|y_t\in A_t, \forall i\in\mathcal{I}_A(k,s) \wedge y_t\in A_t^c, \forall i\in\mathcal{I}_{A^c}(k,s)\right) \\
&= \max_{\boldsymbol{\alpha}\in\Delta_{\bar{T}}(\alpha)} \sum_{k=0}^{T}\sum_{s=1}^{\binom{T}{k}} \max_{\phi_{k,s}\in\Psi(\alpha_{k,s})} \mathbb{E}_{f_1}\left(\phi_{k,s}|y_t\in A_t, \forall i\in\mathcal{I}_A(k,s) \wedge y_t\in A_t^c, \forall i\in\mathcal{I}_{A^c}(k,s)\right) \\
&= \max_{\boldsymbol{\alpha}\in\Delta_{\bar{T}}(\alpha)} \sum_{k=0}^{T}\sum_{s=1}^{\binom{T}{k}} \max_{\phi_{k,s}\in\Psi(\alpha_{k,s})} \left(\prod_{t\in\mathcal{I}_{A^c}} \mathrm{F}_1(A_t^c)\right)\int_{\mathcal{Y}(\mathcal{I}_A)} \phi_{k,s}\prod_{t\in\mathcal{I}_A} f_{1t}\mathbb{1}_{A_t}\mathrm{d}\mu^{\otimes T} \\
&= \max_{\boldsymbol{\alpha}\in\Delta_{\bar{T}}(\alpha)} \sum_{k=0}^{T}\sum_{s=1}^{\binom{T}{k}} \max_{\phi_{k,s}\in\Psi(\alpha_{k,s})} \int_{\mathcal{Y}^T} \phi_{k,s}\prod_{t=0}^{T-1}\mathrm{d}[\mathrm{F}_t]^{\flat}_{A_t} \\
&= \max_{\boldsymbol{\alpha}\in\Delta_{\bar{T}}(\alpha)} \sum_{k=0}^{T}\sum_{s=1}^{\binom{T}{k}} \max_{\phi_{k,s}\in\Phi(\alpha_{k,s})} \int_{\mathcal{Y}^T} \phi_{k,s}\prod_{t=0}^{T-1}\mathrm{d}[\mathrm{F}_t]^{\flat}_{A_t},
\end{aligned}
$$

where $\bar{T} = \sum_{k=0}^{T}\binom{T}{k}$ and $\Delta_{\bar{T}}(\alpha) = \{\boldsymbol{\alpha}\in[0,\alpha]^{\bar{T}} : \boldsymbol{\iota}_{\bar{T}}'\boldsymbol{\alpha} = \alpha\}$, with $\boldsymbol{\iota}_{\bar{T}}$ denoting column vector of ones of length $\bar{T}$. The first equality exploits the fact that the test function can be decomposed into test functions operating on a single part of the partitioning of the outcome space $\mathcal{Y}^T$, in which case the maximization problem can be split into finding an optimal test on each of the partitioned parts conditional on the amount of size spent on each part and the optimal distribution of size over the partition of the outcome space.

The second equality holds by Corollary A5, the third equality uses that the optimal test is constant in arguments varying in $A^c := \prod_{t=0}^{T-1} A_t^c$, the fourth equality holds by definition of the censored measure and the fifth equality uses that all tests that are non-constant in arguments varying in A^c map under the censored measure onto tests that are constant in arguments varying in A^c.

Finally, the result follows by observing that the final maximization problem is equivalent to finding the optimal test $\phi_A^\flat$ for the testing problem $\mathbb{H}_j : p = \prod_{t=0}^{T-1} [f_j]_{A_t}^\flat$, $j \in \{0, 1\}$, for which $\phi_A^\flat$ is the UMP test by the Fundamental Lemma of Neyman and Pearson (1933). By the equivalence, $\phi_A^\flat$ is, for any $h_1 \in \mathcal{H}$, also the most powerful test for testing problem (8). But, since the test $\phi^\flat$ is independent of h_1, it is the UMP test for testing problem (8). $\square$

B Additional Proofs

B.1 Proof of the censored density in Equation (3)

Since $(\mu + \delta_*)(E) = 0$ implies that both $\mu(E) = 0$ and $\delta_*(E) = 0$, $\forall E \in \mathcal{G}$, we have that both $\mu \ll \mu + \delta_*$ and $\delta_* \ll \mu + \delta_*$. As a consequence,

$$f_w^\flat := \frac{\mathrm{dF}_w^\flat}{\mathrm{d}(\mu + \delta_*)} = w \frac{\mathrm{dF}}{\mathrm{d}(\mu + \delta_*)} + \bar{F}_w \frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu + \delta_*)}$$

is the censored $(\mu + \delta_*)$-density of $\mathrm{F}_w^\flat$.

We can simplify this density as follows. Understanding that

$$\frac{\mathrm{dF}}{\mathrm{d}(\mu + \delta_*)} = \frac{\mathrm{dF}}{\mathrm{d}\mu} \frac{\mathrm{d}\mu}{\mathrm{d}(\mu + \delta_*)},$$

we recall from the Radon-Nikodym theorem that $\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\delta_*)}$ is the solution of

$$\int_{\mathcal{Y}} \mathbb{1}_E \mathrm{d}\mu = \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\mu}{\mathrm{d}(\mu + \delta_*)} \mathrm{d}(\mu + \delta_*) = \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\mu}{\mathrm{d}(\mu + \delta_*)} \mathrm{d}\mu + \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\mu}{\mathrm{d}(\mu + \delta_*)} \mathrm{d}\delta_*.$$

By the same theorem, the solution of this equation is guaranteed to exist uniquely.

A glance at this equation reveals that a reasonable candidate is 1 μ-a.e. and 0 δ_*-a.s. We conclude that $\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\delta_*)} = \mathbb{1}_{\mathcal{Y}\setminus\{*\}}$ is the unique solution for the Radon-Nikodym derivative. By the same token, we conclude from

$$\int_{\mathcal{Y}} \mathbb{1}_E \mathrm{d}\delta_* = \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu+\delta_*)} \mathrm{d}(\mu+\delta_*) = \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu+\delta_*)} \mathrm{d}\mu + \int_{\mathcal{Y}} \mathbb{1}_E \frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu+\delta_*)} \mathrm{d}\delta_*,$$

that a reasonable candidate for $\frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu+\delta_*)}$ is 0 μ-a.e. and 1 δ_*-a.s. More specifically, we deduce that $\frac{\mathrm{d}\delta_*}{\mathrm{d}(\mu+\delta_*)} = \mathbb{1}_*$ is the unique solution for the Radon-Nikodym derivative.

Put together, we arrive at

$$f_w^\flat(y) = w(y)\frac{\mathrm{dF}}{\mathrm{d}\mu}(y)\mathbb{1}_{\mathcal{Y}\setminus\{*\}}(y) + \bar{F}_w \mathbb{1}_*(y) = w(y)f(y)\mathbb{1}_{y\neq *} + \bar{F}_w \mathbb{1}_{y=*}, \quad y \in \mathcal{Y},$$

where f denotes the μ-density of F. $\square$

B.2 Proof of Corollary 1

The test based on $\tilde{\lambda}(\mathbf{y})$ is equivalent to the UMP test in Theorem 3, since

$$\begin{aligned}
\tilde{\lambda}(\mathbf{y}) &= \sum_{t=0}^{T-1} \left(\mathrm{LogS}_{A_t}^\flat(f_{1t}, y_{t+1}) - \mathrm{LogS}_{A_t}^\flat(f_{0t}, y_{t+1})\right) \\
&= \sum_{t=0}^{T-1} \Bigg(\log\left([f_{1t}]_{A_t}^\flat(y_{t+1})\right) - \log\left([f_{0t}]_{A_t}^\flat(y_{t+1})\right) \Bigg) \\
&= \log \lambda(\mathbf{y}),
\end{aligned}$$

and hence $\lambda(\mathbf{y}) \underset{<}{\overset{\geq}{=}} c \iff \tilde{\lambda}(\mathbf{y}) \underset{<}{\overset{\geq}{=}} \tilde{c}$, with $\tilde{c} = \log c$.

B.3 Proof of Corollary 2

We show that $\phi_A^\sharp$ is not UMP by a specific counterexample in which the power of $\phi_A^\sharp$ is strictly smaller than the power of $\phi_A^\flat$. In particular, suppose that $T = 1$ and consider two densities f_0 and f_1 that are different on $A = [r, \infty)$, for some constant $r > 0$. Furthermore, assume that

$$\int_{\{y:\lambda(y)>r\}}^{\infty} \mathrm{F}_0(\mathrm{d}y) > \alpha, \qquad \lambda(y) = \frac{f_1(y)}{f_0(y)}. \tag{B.1}$$

For $T = 1$, the likelihood ratios of the conditional and censored test simplify to

$$\begin{aligned}
\lambda_A^\sharp(y) &= \frac{[f_1]_A^\sharp(y)}{[f_0]_A^\sharp(y)} = \frac{\frac{f_1(y)}{\mathrm{F}_1(A)}}{\frac{f_0(y)}{\mathrm{F}_0(A)}} \mathbb{1}_A(y) = \frac{\mathrm{F}_0(A^c)}{\mathrm{F}_1(A^c)} \frac{f_1(y)}{f_0(y)} \mathbb{1}_A(y) \\
\lambda_A^\flat(y) &= \frac{[f_1]_A^\flat(y)}{[f_0]_A^\flat(y)} = \frac{f_1(y)}{f_0(y)} \mathbb{1}_A(y) + \frac{\mathrm{F}_1(A^c)}{\mathrm{F}_0(A^c)} \mathbb{1}_{A^c}(y).
\end{aligned}$$

Due to restriction (B.1), the corresponding critical regions $C^\sharp = \left[c^\sharp, \infty\right)$ and $C^\flat = \left[c^\flat, \infty\right)$ are both contained in A. Hence, an example in which $\sharp$ has higher power than $\flat$, would not only be a counterexample to Theorem 3 but also to the fundamental lemma of Neyman and Pearson (1933).

There exist many examples for which the power of the censored test is strictly larger than the power of the conditional test. For instance, suppose that $y \sim \mathrm{Exp}(\theta_j)$, $j \in \{0, 1\}$, with $\theta_0 > \theta_1$. Then, the critical regions follow from the equation

$$\alpha = \int_{\{y:\lambda(y)>c^*\}}^{\infty} \theta_0 \mathrm{e}^{-\theta_0 y} \mathrm{d}y = \int_{\{y:a^*\left(\frac{\theta_1}{\theta_0}\right)\mathrm{e}^{-(\theta_1-\theta_0)y}>c^*\}}^{\infty} \theta_0 \mathrm{e}^{-\theta_0 y} \mathrm{d}y = 1 - F_0\left(\frac{\log(\theta_0/\theta_1)}{\theta_0-\theta_1} \frac{c^*}{a^*}\right),$$

where $a^\sharp = \frac{1-F_0(r)}{1-F_1(r)} = \mathrm{e}^{-(\theta_0-\theta_1)r}$ and $a^\flat = 1$. Isolating c^*, gives

$$c^* = ba^*, \qquad b = \frac{\theta_0}{\theta_1}\mathrm{e}^{(\theta_0-\theta_1)F_0^{-1}(1-\alpha)} > 0.$$

Now, the power of the conditional test is only weakly larger than the power of the censored test, if

$$\int_{\{y:\lambda(y)>c^\sharp\}}^{\infty} \theta_1\mathrm{e}^{-\theta_1 y}\mathrm{d}y \geq \int_{\{y:\lambda(y)>c^\flat\}}^{\infty} \theta_1\mathrm{e}^{-\theta_1 y}\mathrm{d}y \iff c^\sharp \geq c^\flat \iff (\theta_0-\theta_1)r \leq 0.$$

But then, as $\theta_0 > \theta_1$ and $r > 0$, it follows that the power of the conditional test is always strictly smaller than the power of the censored test. Consequently, the conditional test $\phi_A^\sharp$ is not UMP. $\square$

C Derivations for Table 1

In the following subsections, we explicitly derive the results summarized in Table 1. The censored density $f_w^\flat(y)$ is given by Equation (3) and for the conditional density we revisit Equation (1) to deduce $f_w^\sharp(y) = w(y)f(y)/(1-\bar{F}_w)$. The assumption on the nuisance density h in the caption of Table 1, that is, its support is a subset of $\{w = 0\} \subseteq \mathcal{Y}$, implies that $w(y)h(y) = 0$, $\forall y \in \mathcal{Y}$. Additionally observing that $f_w(y) = 0, \forall y \in \{w = 0\}$, this facilitates the simplification of the expressions below. Since the results for the focused scoring rules hold by means of having the same expected score differences, a.s.-equivalent scoring rules and candidate distribution independent additive terms can be neglected, denoted by '$\overset{\text{a.s.}}{=}$' and '$\overset{\text{eqv.}}{=}$', respectively. To save space, some obvious results are omitted.

C.1 LogS

Following the order in which the assertions pertaining to the Logarithmic scoring rule $\mathrm{LogS}(f,y)=\log f(y)$ appear in Table 1, they can be easily verified as follows:

$$
\begin{aligned}
\mathrm{LogS}(\tilde f,\tilde y) &= \log \tilde f(\tilde y) = \log f(y) - \log|b| \overset{\text{eqv.}}{=} \log f(y),\\
\mathrm{LogS}^{\sharp}_{w}(f,y) &= w(y)\log\left(\frac{w(y)f(y)}{1-\bar F_w}\right)\\
&\overset{\text{eqv.}}{=} w(y)\log\left(\frac{f(y)}{1-\bar F_w}\right)\\
&= S^{\mathrm{CL}}_{w}(f,y),\\
\mathrm{LogS}^{\flat}_{w}(f,y) &= w(y)\log\big(w(y)f(y)\mathbb{1}_{y\neq *} + \bar F_w\mathbb{1}_{y=*}\big) + \big(1-w(y)\big)\log\bar F_w\\
&\overset{\text{a.s.}}{=} w(y)\log\big(w(y)f(y)\big) + \big(1-w(y)\big)\log\bar F_w\\
&\overset{\text{eqv.}}{=} w(y)\log\big(f(y)\big) + \big(1-w(y)\big)\log\bar F_w\\
&= S^{\mathrm{CSL}}_{w}(f,y),\\
\mathrm{LogS}^{\flat}_{w,h}(f,y) &= w(y)\log f^{\flat}_{w,h}(y) + \big(1-w(y)\big)\int_{\mathcal Y}\log f^{\flat}_{w,h}(q)h(q)\mu(\mathrm{d}q)\\
&= w(y)\Big(\log\big(f_w(y)\big)\mathbb{1}_{w>0} + \log\big(\bar F_w h(y)\big)\mathbb{1}_{w=0}\Big)\\
&\qquad + \big(1-w(y)\big)\int_{\{w=0\}}\Big(\log\big(f_w(q)\big)\mathbb{1}_{w>0} + \log\big(\bar F_w h(q)\big)\mathbb{1}_{w=0}\Big)h(q)\mu(\mathrm{d}q)\\
&= w(y)\log f_w(y) + \big(1-w(y)\big)\int_{\{w=0\}}\log\big(\bar F_w h(q)\big)\,h(q)\mu(\mathrm{d}q)\\
&\overset{\text{eqv.}}{=} w(y)\log f(y) + \big(1-w(y)\big)\log\bar F_w,\\
&= S^{\mathrm{CSL}}_{w}(f,y).
\end{aligned}
$$

C.2 PsSphS_α

As for LogS, we follow the order of the table for the derivations concerning the Pseudo-Spherical family $\mathrm{PsSphS}_\alpha(f,y) = \frac{f(y)^{\alpha-1}}{\|f\|_\alpha^{\alpha-1}}$, where $\alpha > 1$. In particular,

$$\mathrm{PsSphS}_\alpha(\tilde{f},\tilde{y}) = \frac{\tilde{f}(\tilde{y})}{\|\tilde{f}\|_\alpha^{\alpha-1}} = \frac{\left(\frac{1}{|b|}\right)^{\alpha-1} f(y)^{\alpha-1}}{\left(\frac{1}{|b|}\right)^{\frac{(\alpha-1)^2}{\alpha}} \|f\|_\alpha^{\alpha-1}} = \left(\frac{1}{|b|}\right)^{\frac{\alpha-1}{\alpha}} \mathrm{PsSphS}_\alpha(f,y).$$

Next, we show the limit. Rescaling the PsSphS_α family by a factor $\frac{1}{\alpha-1}$, we obtain

$$\begin{aligned}
&\lim_{\alpha\downarrow 1} \frac{1}{\alpha-1}\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} \\
&= \lim_{\alpha\downarrow 1} \frac{(\alpha-1)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1}}{(\alpha-1)^2} \\
&= \lim_{\alpha\downarrow 1} \frac{\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} + (\alpha-1)\left(\log\left(\frac{f(y)}{\|f\|_\alpha}\right) + (\alpha-1)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{-1}\frac{\partial}{\partial\alpha}\frac{f(y)}{\|f\|_\alpha}\right)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1}}{2(\alpha-1)} \qquad \text{(C.1)} \\
&= \frac{1}{2}\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} + \frac{1}{2}\lim_{\alpha\downarrow 1}\log\left(\frac{f(y)}{\|f\|_\alpha}\right)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} \\
&\quad + \frac{1}{2}\lim_{\alpha\downarrow 1}(\alpha-1)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-2}\frac{\partial}{\partial\alpha}\frac{f(y)}{\|f\|_\alpha},
\end{aligned}$$

and hence

$$\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} = \log f(y), \qquad \text{(C.2)}$$

since $\|f\|_1 = 1$. It might be helpful to note that Equation (C.1) follows from l'Hôpital's rule combined with the following derivative:

$$\frac{\partial}{\partial\alpha}\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1} = \log\left(\left(\frac{f(y)}{\|f\|_\alpha}\right) + (\alpha-1)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{-1}\frac{\partial}{\partial\alpha}\frac{f(y)}{\|f\|_\alpha}\right)\left(\frac{f(y)}{\|f\|_\alpha}\right)^{\alpha-1}.$$

For the conditional PsSphS_α family, we find

$$\begin{aligned}
\mathrm{PsSphS}^\sharp_{\alpha,w}(f,y) &= w(y)\frac{\left(\frac{f_w(y)}{1-\bar{F}_w}\right)^{\alpha-1}}{\left(\int_{\mathcal{Y}}\left(\frac{f_w}{1-\bar{F}_w}\right)^{\alpha}\mathrm{d}\mu\right)^{\frac{\alpha-1}{\alpha}}} \\
&= w(y)\frac{f_w(y)^{\alpha-1}}{\|f_w\|_\alpha^{\alpha-1}} \\
&= w(y)\left(\frac{f_w(y)^{\alpha}}{\|f_w\|_\alpha^{\alpha}}\right)^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

By the close similarity with Equation (C.2), it is straightforward to obtain the following limit:

$$\begin{aligned}
\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PsSphS}^\sharp_{\alpha,w}(f,y) &= w(y)\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\left(\frac{f_w(y)}{\|f_w\|_\alpha}\right)^{\alpha-1} \\
&= w(y)\log\left(\frac{f_w(y)}{\|f_w\|_1}\right) \\
&= w(y)\log f^\sharp_w(y) \\
&= \mathrm{LogS}^\sharp_w(f,y),
\end{aligned}$$

since $\|f_w\|_1 = \int_{\mathcal{Y}} wf\mathrm{d}\mu = 1-\bar{F}_w$. Clearly, this result also follows directly from the linearity of limits, as

$$\begin{aligned}
\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PsSphS}^\sharp_{\alpha}(f,y) &= w(y)\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PsSphS}_{\alpha}(f^\sharp_w,y) \\
&= w(y)\log f^\sharp_w(y) = \mathrm{LogS}^\sharp_w(f,y). \qquad \text{(C.3)}
\end{aligned}$$

Moreover, for the censored PsSphS_α family, it follows that

$$\begin{aligned}
\mathrm{PsSphS}_w^\flat(f,y) &= \frac{w(y)\left(f_w(y)\mathbb{1}_{y\neq *} + \bar{F}_w\mathbb{1}_{y=*}\right)^{\alpha-1} + \big(1-w(y)\big)\bar{F}_w^{\alpha-1}}{\left(\int_{\mathcal{Y}}\left(f_w(y)\mathbb{1}_{y\neq *} + \bar{F}_w\mathbb{1}_{y=*}\right)^{\alpha}(\mu+\delta_*)(\mathrm{d}y)\right)^{\frac{\alpha-1}{\alpha}}} \\
&= \frac{w(y)\left(f_w(y)^{\alpha-1}\mathbb{1}_{y\neq *} + \bar{F}_w^{\alpha-1}\mathbb{1}_{y=*}\right) + \big(1-w(y)\big)\bar{F}_w^{\alpha-1}}{\left(\int_{\mathcal{Y}}\left(f_w(y)\right)^{\alpha}\mathrm{d}y + \bar{F}_w^{\alpha}\right)^{\frac{\alpha-1}{\alpha}}} \\
&\overset{\text{a.s.}}{=} \frac{w(y)f_w(y)^{\alpha-1} + \big(1-w(y)\big)\bar{F}_w^{\alpha-1}}{\left(\|f_w(y)\|_\alpha^\alpha + \bar{F}_w^\alpha\right)^{\frac{\alpha-1}{\alpha}}}.
\end{aligned}$$

For the limit as $\alpha \downarrow 1$, we cannot directly apply Equation (C.2) as we did for the conditional case. Nevertheless, we obtain a similarly satisfying result, namely

$$\begin{aligned}
&\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PsSphS}_w^\flat(f,y) \\
&= w(y)\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\left(\frac{f_w(y)}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-1} \\
&\qquad + \big(1-w(y)\big)\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\left(\frac{\bar{F}_w}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-1} \\
&= w(y)\left(\lim_{\alpha\downarrow 1}\log\left(\frac{f_w(y)}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)\left(\frac{f_w(y)}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-1}\right. \\
&\qquad \left. + \lim_{\alpha\downarrow 1}(\alpha-1)\left(\frac{f_w(y)}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-2}\frac{\partial}{\partial\alpha}\frac{f_w(y)}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right) \\
&\qquad + \big(1-w(y)\big)\left(\lim_{\alpha\downarrow 1}\log\left(\frac{\bar{F}_w}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)\left(\frac{\bar{F}_w}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-1}\right. \\
&\qquad \left. + \lim_{\alpha\downarrow 1}(\alpha-1)\left(\frac{\bar{F}_w}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\alpha-2}\frac{\partial}{\partial\alpha}\frac{\bar{F}_w}{\left(\|f_w\|_\alpha^\alpha+\bar{F}_w^\alpha\right)^{\frac{1}{\alpha}}}\right) \\
&= w(y)\log f_w(y) + \big(1-w(y)\big)\log\bar{F}_w \\
&= \mathrm{LogS}_w^\flat(f,y),
\end{aligned}$$

where we have used that $\|f_w\|_1 + \bar{F}_w = 1 - \bar{F}_w + \bar{F}_w = 1$.

C.3 $\mathbf{PowS}_\alpha$

For results related to the PowS_α family $\mathrm{PowS}_\alpha(f, y) = \alpha f(y)^{\alpha-1} - (\alpha - 1)\|f\|_\alpha^\alpha$, where $\alpha > 1$, we start by verifying that

$$\begin{aligned}
\mathrm{PowS}_\alpha(\tilde{f}, \tilde{y}) &= \alpha\big(\tilde{f}(\tilde{y})\big)^{\alpha-1} - (\alpha-1)\|\tilde{f}\|_\alpha^\alpha \\
&= \alpha\left(\frac{1}{|b|}\right)^{\alpha-1} f(y) - (\alpha-1)\left(\frac{1}{|b|}\right)^{\alpha-1}\|f\|_\alpha^\alpha \\
&= \left(\frac{1}{|b|}\right)^{\alpha-1} \mathrm{PowS}_\alpha(f, y),
\end{aligned}$$

for which we rely on the result expressed in

$$\begin{aligned}
\|\tilde{f}\|_\alpha^\alpha &= \int_{\tilde{\mathcal{Y}}} \tilde{f}(\tilde{y})^\alpha \mu(\mathrm{d}\tilde{y}) \\
&= \left(\frac{1}{|b|}\right)^{\alpha-1} \int_{\tilde{\mathcal{Y}}} \left(f\left(\frac{\tilde{y}-a}{b}\right)\right)^\alpha \frac{1}{|b|}\mu(\mathrm{d}\tilde{y}) \\
&= \left(\frac{1}{|b|}\right)^{\alpha-1} \int_{\mathcal{Y}} (f\,(y))^\alpha \mu(\mathrm{d}y) \\
&= \left(\frac{1}{|b|}\right)^{\alpha-1} \|f\|_\alpha^\alpha.
\end{aligned}$$

Next, we verify the limit for the non-focused family. Specifically,

$$\begin{aligned}
\lim_{\alpha\downarrow 1} \frac{1}{\alpha-1}\mathrm{PowS}_\alpha &= \lim_{\alpha\downarrow 1} \frac{1}{\alpha-1}\left(\alpha f(y)^{\alpha-1} - (\alpha-1)\|f\|_\alpha^\alpha\right) \\
&= \lim_{\alpha\downarrow 1} \frac{(\alpha-1)\alpha f(y)^{\alpha-1}}{(\alpha-1)^2} - 1 \\
&= \lim_{\alpha\downarrow 1} \frac{\alpha f(y)^{\alpha-1} + (\alpha-1) f(y)^{\alpha-1}\big(1 + \alpha \log f(y)\big)}{2(\alpha-1)} - 1 \\
&= \frac{1}{2}\left(\lim_{\alpha\downarrow 1} \frac{1}{\alpha-1}\alpha f(y)^{\alpha-1} - 1\right) + \frac{1}{2}\left(\lim_{\alpha\downarrow 1} f(y)^{\alpha-1}\big(1 + \alpha\log f(y)\big) - 1\right)
\end{aligned}$$

and hence

$$\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PowS}_\alpha(f,y)=\log f(y).$$

Furthermore, the conditional version of the PowS_α family displayed in Table 1 is nothing but a direct application of the conditioning procedure. For the limit of the $\mathrm{PowS}^\sharp_{\alpha,w}$ family, we recall Equation (C.3) and immediately conclude that $\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PowS}^\sharp_{\alpha,w}(f,y)=\mathrm{LogS}^\sharp_w(f,y)$.

Turning to the censored focusing method, we recall from the analysis in Appendix C.2 that $\|f^\flat_w\|^\alpha_\alpha=\|f_w(y)\|^\alpha_\alpha+\bar{F}^\alpha_w$. Using this result, we obtain

$$\begin{aligned}\mathrm{PowS}^\flat_{\alpha,w}(f,y)&=w(y)\alpha\left(f_w(y)\mathbb{1}_{y\neq *}+\bar{F}_w\mathbb{1}_{y=*}\right)^{\alpha-1}+\big(1-w(y)\big)\alpha\bar{F}_w^{\alpha-1}-(\alpha-1)\|f^\flat_w\|^\alpha_\alpha\\&\overset{\text{a.s.}}{=}w(y)\alpha f_w(y)^{\alpha-1}+\big(1-w(y)\big)\alpha\bar{F}_w^{\alpha-1}-(\alpha-1)\left(\|f_w\|^\alpha_\alpha+\bar{F}^\alpha_w\right),\end{aligned}$$

which bears the following limit

$$\begin{aligned}&\lim_{\alpha\downarrow 1}\frac{1}{\alpha-1}\mathrm{PowS}^\flat_{\alpha,w}(f,y)\\&=w(y)\lim_{\alpha\downarrow 1}\frac{(\alpha-1)\alpha f_w(y)^{\alpha-1}}{(\alpha-1)^2}+\big(1-w(y)\big)\lim_{\alpha\downarrow 1}\frac{(\alpha-1)\alpha\bar{F}_w^{\alpha-1}}{(\alpha-1)^2}-1\\&=\frac{1}{2}\lim_{\alpha\downarrow 1}\left(w(y)\left(\frac{1}{\alpha-1}\alpha f_w(y)^{\alpha-1}-1\right)+\big(1-w(y)\big)\left(\frac{1}{\alpha-1}\alpha\bar{F}_w^{\alpha-1}-1\right)\right)\\&\quad+\frac{1}{2}\lim_{\alpha\downarrow 1}\Bigg(w(y)\left(f_w(y)^{\alpha-1}\big(1+\alpha\log f_w(y)\big)-1\right)\\&\quad+\big(1-w(y)\big)\left(\bar{F}_w^{\alpha-1}\big(1+\alpha\log\bar{F}_w\big)-1\right)\Bigg).\end{aligned}$$

Therefore,

$$\lim_{\alpha \downarrow 1} \frac{1}{\alpha - 1} \mathrm{PowS}^{\flat}_{\alpha,w}(f,y) = w(y) \log f_w(y) + \big(1 - w(y)\big) \log \bar{F}_w = \mathrm{LogS}^{\flat}_w(f,y).$$

D Monte Carlo Simulation

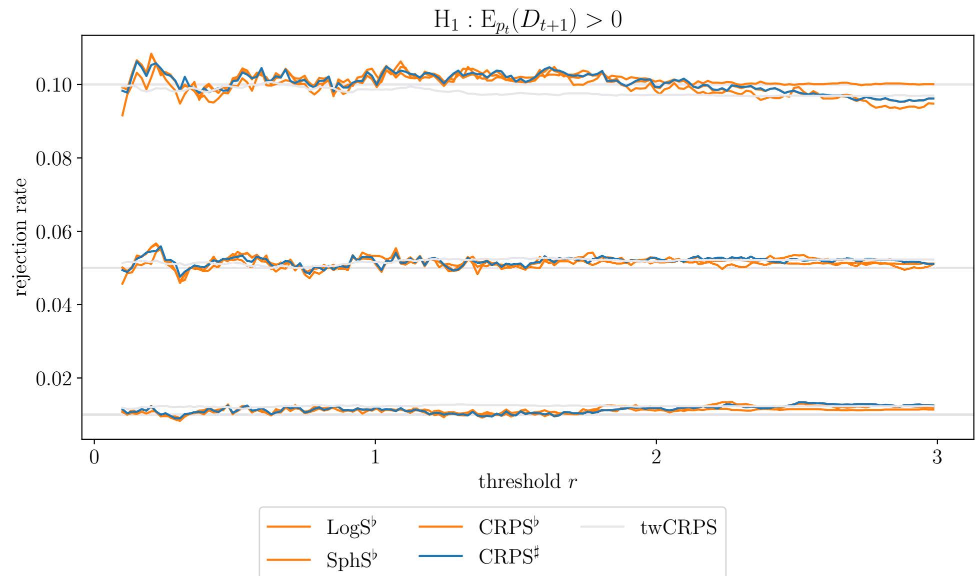
In this section, we compare the size and power properties of the conditional and censored scoring rules of a selection of regular scoring rules based on the Giacomini and White (2006) (GW) test, for the null hypothesis of equal expected scores of two candidates $\hat{f}_t$ and $\hat{g}_t$, by means of the Diebold and Mariano (2002) (DM) type test statistic; see Section 3.5.

D.1 Size

As explained by Diks et al. (2011), the null hypothesis of the GW test forces a particularly symmetric design. We adopt the design of Diks et al. (2011), using a center-indicator weight function $w(y) = \mathbb{1}_{[-r,r]}(y)$ combined with an i.i.d. standard normal DGP and normal candidates with unit variance and means $\mu_f = -0.2$ and $\mu_g = 0.2$. Due to the symmetry, the norms and $\bar{F}_w$-probabilities of the candidates are equivalent, leading to coinciding DM statistics based on QS and SphS scoring rules. Additionally, the equal norms and discrete probabilities also imply the censoring and conditioning rules to be equivalent within a semi-local scoring rule family since observations outside the region of interest obtain the same scores under both candidates in this case.

Figure D.1.a displays the rejection rates for rejection the null of equal predictive ability against the one-sided alternative that candidate f is statistically closer to p than g. The rejection rates are given at nominal significance levels 0.01, 0.05 and 0.10, for focused versions of the LogS, SphS and CRPS scoring rules, based on 10,000 simulations. Given the

Figure D.1.a: Size properties of the GW test

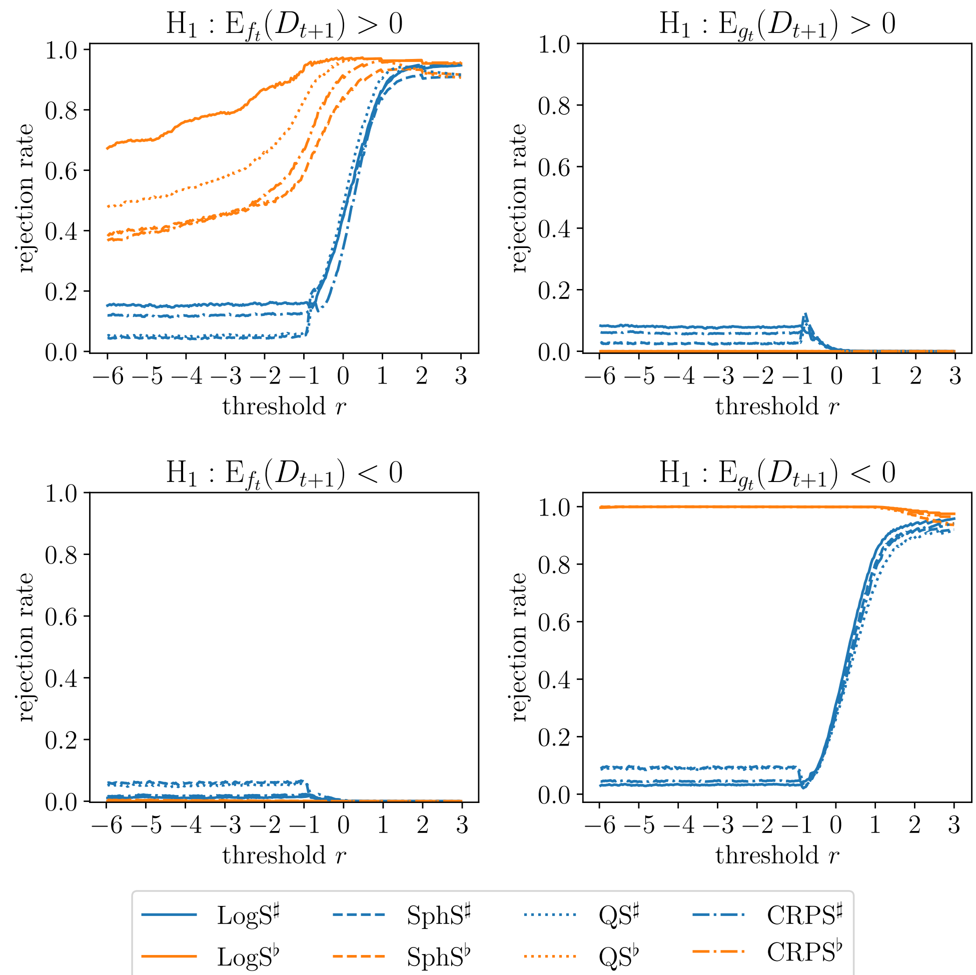


discussion above, this gives a complete picture of the selection $\{\mathrm{LogS}, \mathrm{SphS}, \mathrm{QS}, \mathrm{CRPS}\} \times \{\sharp, \flat\}$. The twCRPS is added since it will also be included as a benchmark in the power studies based on weight functions for which the censored CRPS variants do not reduce to the twCRPS. None of the displayed rejection rates give reason to doubt the size correctness of the tests.

D.2 Power

Laplace tails Our first simulation experiment studies the consequences of the lack of the conditional rule to disentangle two proportional tails when using the left tail indicator function $w(y) = \mathbb{1}_{(-\infty,r)}(y)$. In particular, we analyze two Laplace candidates with different location $\mu_f = -1$ and $\mu_g = 1$ but equivalent scale $\theta_f = \theta_g = 1$. Interestingly, even if $\mu_p \to \mu_f$, the conditional scoring rule does not have any power against the null of the candidates being statistically equally far away from p, that is, for thresholds $r < \mu_f$, for which the conditional distributions on $(-\infty, r)$ coincide. Since movements of p in terms of μ_p are invisible through the lens of a conditional score divergence, this is essentially not a lack of power against $\mathbb{H}_0$, which is based on the conditional scoring rule. Yet, it is a lack of power against the distributions being statistically equally far away from the actual density on $\{w > 0\}$ through the lens of the regular score divergence and, therefore, still a lack of local discriminative ability. More fundamentally, the GW test degenerates in this case, as the score differences are exactly zero.

Figure D.2.b: Laplace experiment ($c = 20$)



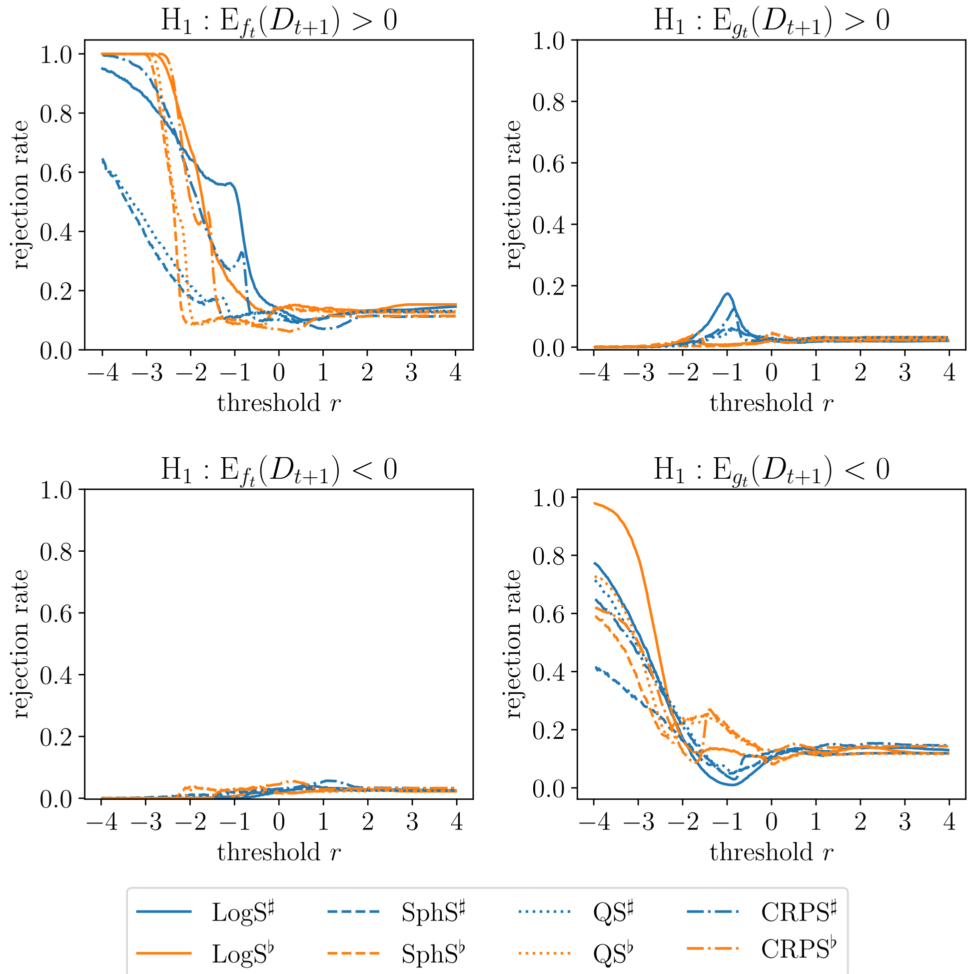
One-sided rejection rates of the GW-test of equal predictive ability of the candidates f_t (Laplace$(-1, 1)$ and g_t (Laplace$(1, 1.1)$) at a nominal significance level of 0.05 based on 10,000 simulations. The DGP is either f_t (left-hand side) or g_t (right-hand side). Moreover, rejections in the top panels are in favor of f_t, while rejections in the bottom panels are in favor of g_t. The incorporated weight function is $w(y) = 1_{(-\infty,r)}(y)$ and the number of expected observations in the region of interest is kept constant at $c = 20$.

Leaving this extreme case, we analyze what happens if the scale parameters are not exactly the same, but close. Specifically, we let $\theta_f = 1$ and $\theta_g = 1.1$. Figure D.2.b shows the rejection rates of the GW test if the DGP is f (left-hand side) or g (right-hand side) in favor of f (top) or g (bottom). Since both candidates are now also different through the lens of the conditional rule, the subfigures on the diagonal display actual power, while the off-diagonal ones show spurious power. Concerning the selection of scoring rules, it is good to remember that the censored CRPS coincides with the twCRPS for the selected weight function.

Three observations are clearly apparent. First, the increase in power from the conditional operator to the censoring operator is immense for all four scoring rules and thresholds $r < \mu_f$. The difference decreases over the interval $r \in (\mu_f, \mu_g)$, after which both conditioning and censoring have close to unit power. This observation is in line with the lack of discriminative ability of proportional and apparently close to proportional tails. Second, there is a clear difference in spurious power between the focusing operators: The censoring operator does seemingly not suffer from spurious power at all, whereas the conditional rules have spurious power up to 0.10 for thresholds smaller than $\mu_f = -1$. Third, we note that the censored likelihood score dominates the other scoring rules in terms of power.

Normal versus Student-t: Left-tail. Figure D.2.c shows the rejection rates of the GW test, where f is standard normal and g Student-t_5. Again, we consider the left-tail indicator function $w(y) = 1_{(-\infty,r)}(y)$ for varying values of r. The combination of the selected candidates and the left-tail region of interest make the current setting particularly interesting for financial risk management applications. As revealed by the figure, the rejection rate plots are now less monotonic, intersecting the graph of the competing focusing operator rejection rates. The latter occurs by construction since the densities of the candidates intersect as well, see Diks et al. (2011) for a discussion. Starting with the clearest differences, we note

Figure D.2.c: $\mathcal{N}(0,1)$ versus Student-t_5: Left-tail ($c = 20$)

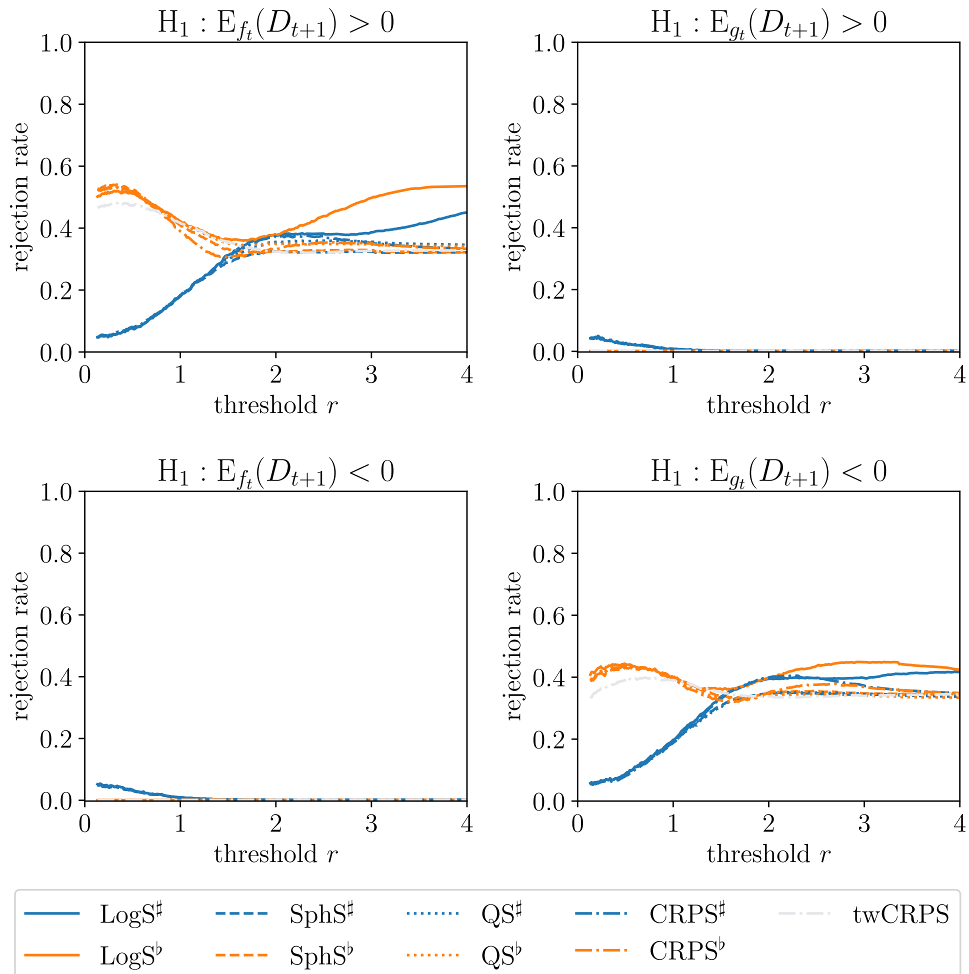


One-sided rejection rates of the GW-test of equal predictive ability of the candidates f_t (standard normal) and g_t (Student-t_5) at a nominal significance level of 0.05 based on 10,000 simulations. The DGP is either f_t (left-hand side) or g_t (right-hand side). Moreover, rejections in the top panels are in favor of f_t, while rejections in the bottom panels are in favor of g_t. The incorporated weight function is $w(y) = 1_{(-\infty,r)}(y)$ and the number of expected observations in the region of interest is kept constant at $c = 20$.

the spurious power humps of the conditional rules if the Student-t_5 distribution is the DGP. By contrast, the censored scoring rules have almost no spurious power. The rejection rates in the bottom right panel of Figure D.2.c reveal a clear preference for the censoring operator. Indeed, the exceptions of higher conditional power are rather weak, while the difference between the rejection rates (far) into the left-tail is particularly large for the Logarithmic and Spherical scoring rules. On the other hand, if the standard normal distribution is the

DGP, then there is hardly (a difference in) spurious power. The differences between the rejection rates representing power are more extreme when the data is generated from the standard Normal distribution, yet so is their drop between $r = -2$ and $r = -1$, clouding a clear preference for either of the focusing operators for these intermediate tail values of r.

Figure D.2.d: $\mathcal{N}(0,1)$ versus Student-t_5: Center ($c = 200$)



One-sided rejection rates of the GW-test of equal predictive ability of the candidates f_t (standard normal) and g_t (Student-t_5) at a nominal significance level of 0.05 based on 10,000 simulations. The DGP is either f_t (left-hand side) or g_t (right-hand side). Moreover, rejections in the top panels are in favor of f_t, while rejections in the bottom panels are in favor of g_t. The incorporated weight function is $w(y) = 1_{[-r,r]}(y)$ and the number of expected observations in the region of interest is kept constant at $c = 200$.

Normal versus Student-t: Center. In our third Monte Carlo experiment, we focus on the center of the candidate distributions by implementing the weight function $w(y) = \mathbb{1}_{[-r,r]}(y)$.

Figure D.2.d displays the rejection rates for the same selection of regular scoring rules as in the previous experiments. Based on Figure D.2.d, the added value of censoring relative to conditioning is overwhelming; censoring leads to higher power and lower spurious power, in particular for values smaller than $r = 1$, which are of particular interest in applications.

The $\mathrm{CRPS}_w^\flat$ displayed in the Figure D.2.d is the generalized censored scoring rule based on the generalized censored measure in Equation (7). Due to the symmetry of the setup, there is visually no difference between using the suggested value $\gamma = \frac{1}{2}$ (included in Figure D.2.d) or the estimated proportion $\hat{\gamma}$. We have also calculated the $\mathrm{CRPS}^\dagger(\mathrm{F}, y)$ introduced in Section 3.3, which visually coincides with the twCRPS in this case.

E Additional Tables

E.1 Risk management

Table E.1.a: MCS p-values for risk management application.

			LogS		QS		SphS		CRPS	
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$
0.01	1	RGARCH-t	1.00	0.60	0.45	0.95	0.65	0.88	0.73	0.97
		TGARCH-t	0.99	1.00	0.63	1.00	0.88	1.00	0.81	1.00
		GARCH-t	0.53	0.69	0.34	0.84	0.65	0.88	0.81	0.91
		RGARCH-$\mathcal{N}$	**0.09**	**0.19**	1.00	0.95	1.00	0.66	0.81	0.97
		TGARCH-$\mathcal{N}$	**0.03**	**0.09**	0.63	0.95	0.88	0.38	1.00	0.97
		GARCH-$\mathcal{N}$	**0.01**	**0.09**	0.45	0.84	0.65	0.45	0.81	0.64
	5	RGARCH-t	0.37	0.87	**0.12**	1.00	**0.24**	1.00	0.57	1.00
		TGARCH-t	0.83	1.00	0.86	0.45	1.00	0.40	0.65	0.63
		GARCH-t	1.00	0.96	**0.17**	0.38	0.47	0.40	0.65	0.44
		RGARCH-$\mathcal{N}$	**0.01**	**0.05**	**0.12**	0.81	**0.18**	0.40	0.57	**0.18**
		TGARCH-$\mathcal{N}$	**0.01**	**0.05**	1.00	0.75	1.00	**0.23**	1.00	**0.16**
		GARCH-$\mathcal{N}$	**0.01**	**0.04**	**0.17**	0.75	0.41	0.27	0.65	**0.09**
0.05	1	RGARCH-t	1.00	0.79	**0.05**	1.00	**0.01**	1.00	0.40	0.79
		TGARCH-t	**0.11**	1.00	**0.02**	0.83	**0.01**	0.74	0.40	1.00
		GARCH-t	**0.01**	0.79	**0.00**	0.83	**0.00**	0.74	**0.14**	0.79
		RGARCH-$\mathcal{N}$	**0.09**	**0.06**	1.00	1.00	1.00	0.60	1.00	0.79
		TGARCH-$\mathcal{N}$	**0.00**	**0.03**	**0.05**	0.83	**0.01**	**0.24**	0.50	0.65
		GARCH-$\mathcal{N}$	**0.00**	**0.01**	**0.00**	0.83	**0.00**	**0.25**	0.40	0.37
	5	RGARCH-t	0.75	**0.23**	0.31	0.26	**0.25**	0.29	0.49	0.55
		TGARCH-t	0.98	0.75	0.98	1.00	1.00	1.00	1.00	1.00
		GARCH-t	1.00	1.00	0.31	0.36	0.41	0.57	0.56	0.55
		RGARCH-$\mathcal{N}$	**0.01**	**0.01**	0.98	**0.14**	0.92	**0.01**	0.49	**0.03**
		TGARCH-$\mathcal{N}$	**0.01**	**0.01**	1.00	0.36	0.81	**0.02**	0.74	**0.06**
		GARCH-$\mathcal{N}$	**0.01**	**0.01**	0.31	0.26	**0.25**	**0.01**	0.49	**0.06**
0.1	1	RGARCH-t	1.00	0.73	1.00	0.70	0.35	0.88	**0.12**	0.74
		TGARCH-t	**0.05**	1.00	**0.16**	1.00	**0.02**	1.00	**0.05**	1.00
		GARCH-t	**0.00**	0.40	**0.01**	0.43	**0.00**	0.37	**0.01**	**0.20**
		RGARCH-$\mathcal{N}$	**0.05**	**0.03**	0.49	0.70	1.00	**0.21**	1.00	0.74
		TGARCH-$\mathcal{N}$	**0.00**	**0.01**	**0.06**	0.39	**0.00**	**0.03**	**0.12**	**0.04**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.06**	**0.00**	**0.01**	**0.03**	**0.01**
	5	RGARCH-t	0.46	**0.15**	0.35	**0.16**	0.55	**0.11**	0.47	0.43
		TGARCH-t	1.00	0.36	1.00	1.00	1.00	0.41	1.00	1.00
		GARCH-t	0.56	1.00	0.35	0.70	0.55	1.00	0.60	0.43
		RGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.01**	**0.16**	0.26	**0.03**	0.53	**0.01**
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.01**	**0.16**	**0.21**	**0.00**	0.60	**0.00**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.01**	**0.16**	**0.20**	**0.00**	0.46	**0.00**

Table E.1.a (Continued): MCS p-values for risk management application.

0.15	1	RGARCH-t	1.00	1.00	1.00	1.00	1.00	1.00	**0.08**	0.84
		TGARCH-t	**0.03**	0.79	**0.07**	0.64	**0.01**	**0.19**	**0.01**	0.88
		GARCH-t	**0.00**	**0.18**	**0.00**	**0.19**	**0.00**	**0.10**	**0.00**	**0.11**
		RGARCH-$\mathcal{N}$	**0.03**	**0.02**	**0.07**	0.64	0.52	**0.10**	1.00	1.00
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.38	**0.00**	**0.01**	**0.03**	**0.11**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.04**	**0.00**	**0.00**	**0.01**	**0.00**
	5	RGARCH-t	1.00	1.00	1.00	1.00	1.00	1.00	**0.08**	0.84
		TGARCH-t	**0.03**	0.79	**0.07**	0.64	**0.01**	**0.19**	**0.01**	0.88
		GARCH-t	**0.00**	**0.18**	**0.00**	**0.19**	**0.00**	**0.10**	**0.00**	**0.11**
		RGARCH-$\mathcal{N}$	**0.03**	**0.02**	**0.07**	0.64	0.52	**0.10**	1.00	1.00
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.38	**0.00**	**0.01**	**0.03**	**0.11**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.04**	**0.00**	**0.00**	**0.01**	**0.00**
0.2	1	GARCH-$\mathcal{N}$	1.00	1.00	1.00	0.40	1.00	0.26	**0.10**	0.42
		GARCH-t	**0.02**	**0.23**	**0.10**	**0.06**	**0.02**	**0.02**	**0.01**	**0.14**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.06**	**0.06**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**
		QGARCH-I-t	**0.01**	**0.04**	**0.00**	1.00	**0.02**	1.00	1.00	1.00
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.00**	**0.01**	**0.09**
		QGARCH-II-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**
	5	RGARCH-t	**0.15**	0.36	**0.00**	0.89	**0.02**	0.97	0.37	0.67
		TGARCH-t	1.00	0.36	0.77	1.00	1.00	0.97	1.00	1.00
		GARCH-t	0.78	1.00	1.00	0.89	0.50	0.97	0.37	0.67
		RGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.89	**0.00**	1.00	0.32	0.67
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.79	**0.00**	0.90	0.37	**0.02**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.79	**0.00**	0.76	**0.23**	**0.01**
0.25	1	RGARCH-t	1.00	1.00	1.00	**0.04**	1.00	**0.02**	**0.17**	**0.11**
		TGARCH-t	**0.03**	0.28	0.74	**0.08**	**0.03**	**0.02**	**0.01**	**0.10**
		GARCH-t	**0.00**	**0.04**	0.74	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**
		RGARCH-$\mathcal{N}$	**0.01**	**0.04**	**0.00**	1.00	**0.00**	1.00	1.00	1.00
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.08**	**0.00**	**0.02**	**0.01**	**0.11**
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**
	5	RGARCH-t	**0.12**	0.33	**0.00**	0.76	**0.00**	0.96	0.28	0.49
		TGARCH-t	1.00	0.49	**0.16**	1.00	0.84	0.96	1.00	1.00
		GARCH-t	0.99	1.00	1.00	0.76	1.00	0.93	0.28	0.52
		RGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.76	**0.00**	0.96	**0.12**	0.50
		TGARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.76	**0.00**	1.00	**0.13**	0.44
		GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	0.66	**0.00**	0.71	**0.12**	**0.08**

NOTE: This table presents the MCS p-values implied by censored ($\flat$) and conditional ($\sharp$) scoring rules, based on h-step ahead density forecasts. The computation of these p-values is conducted using the R package `MCS`, developed by Bernardi and Catania (2018). The emphasis is on the left tail, incorporated by the weight function $w_t(y_t) = \mathbb{1}_{(-\infty,\hat{r}_t^q)}(y_t)$, where $\hat{r}_t^q$ is the empirical q-th quantile based on the estimation window. Bold (and underlined) p-values signify a forecast method's elimination from $\text{MCS}_{0.75}$ (and $\text{MCS}_{0.90}$). TR is the selected statistic, using $B = 10,000$ simulations and block length $k = 5$.

Table E.1.b: Overview and robustness of results risk management application.

		$MCS_{0.90}$							$MCS_{0.75}$						
		|MCS|			VaR		ES		**|MCS|**			VaR		ES	
T_{est}	**Stat.**	$\leq$	$<$	$\sharp/\flat$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\leq$	$<$	$\sharp/\flat$	$\flat$	$\sharp$	$\flat$	$\sharp$
								$h=1$							
1000	TR_{20}	96%	71%	2.28	83%	83%	83%	92%	92%	63%	2.04	83%	79%	75%	88%
	TR_{100}	96%	71%	2.30	83%	83%	83%	92%	92%	63%	2.04	83%	83%	83%	100%
	$Tmax_{20}$	88%	46%	1.58	100%	88%	96%	96%	92%	71%	2.07	83%	88%	83%	96%
	$Tmax_{100}$	88%	46%	1.48	100%	88%	96%	96%	88%	67%	2.02	83%	88%	83%	96%
750	TR_{20}	88%	54%	1.81	88%	75%	88%	88%	88%	50%	1.80	88%	71%	71%	83%
	$Tmax_{20}$	71%	33%	1.19	100%	88%	96%	96%	83%	62%	2.00	96%	83%	79%	92%
1250	TR_{100}	92%	63%	1.22	83%	83%	83%	100%	96%	50%	1.99	83%	83%	79%	83%
	$Tmax_{20}$	83%	58%	1.74	92%	92%	88%	100%	83%	50%	1.83	92%	88%	79%	83%
								$h=5$							
1000	TR_{20}	75%	38%	1.69	83%	100%	96%	100%	58%	50%	1.72	83%	79%	96%	100%
	TR_{100}	75%	38%	1.69	83%	100%	96%	100%	58%	50%	1.66	83%	79%	96%	100%
	$Tmax_{20}$	75%	42%	1.43	100%	100%	100%	100%	58%	50%	1.59	100%	83%	100%	100%
	$Tmax_{100}$	83%	38%	1.44	100%	100%	100%	100%	63%	46%	1.59	100%	83%	100%	100%
750	TR_{100}	54%	33%	1.44	100%	92%	100%	100%	50%	33%	1.54	100%	75%	100%	100%
	$Tmax_{20}$	75%	29%	1.53	100%	96%	100%	100%	63%	42%	1.53	100%	83%	100%	100%
1250	TR_{20}	63%	33%	1.61	96%	96%	100%	100%	67%	50%	1.61	92%	75%	100%	100%
	$Tmax_{20}$	71%	42%	1.46	100%	96%	100%	100%	63%	42%	1.46	96%	83%	100%	100%

NOTE: The table summarizes MCS and backtesting results using varying values for the estimation window length T_{est}, equivalence test statistics TR_k and $Tmax_k$, block length k, across forecast horizons $h=1$ and $h=5$, based on $B=10,000$ bootstrap replications. Columns labeled $\leq$ ($<$) display the percentage of cases where $MCS^{\flat}$ contains (strictly) fewer forecast methods than $MCS^{\sharp}$ and the column labeled $\sharp/\flat$ reports the factor $|MCS^{\sharp}|/|MCS^{\flat}|$. Each of the results represents and average over a set of quantiles $q \in \{0.01, 0.05, 0.10, 0.15, 0.20, 0.25\}$ and scoring rules $S \in \{\text{LogS}, \text{QS}, \text{SphS}, \text{CRPS}\}$. The VaR (ES) column shows the percentage of cases where $MCS^{\sharp}$ and $MCS^{\flat}$ contain one of the top three models based on VaR (ES) backtesting results.

E.2 Inflation

Table E.2.a: MSE and MAE for inflation application.

	MSE			MAE		
Method	$h = 6$	$h = 12$	$h = 24$	$h = 6$	$h = 12$	$h = 24$
Random Walk	8.74	**3.75**	**1.60**	1.97	**1.48**	**1.00**
AR	7.91	5.52	**3.67**	1.89	1.71	**1.46**
Bagging	5.74	**3.58**	4.67	1.83	1.56	1.69
CSR	**5.70**	5.18	8.11	**1.60**	1.76	2.14
LASSO	**5.66**	4.32	5.21	**1.66**	**1.55**	1.74
Random Forest	**5.40**	**2.76**	**1.93**	**1.54**	**1.18**	**1.11**

NOTE: MSE and MAE of forecast methods used in the inflation examples, for the incorporated horizons $h \in \{6, 12, 24\}$. Bold face numbers indicate the three best models per performance measure.

Table E.2.b: MCS p-values for inflation application.

			LogS		QS		SphS		CRPS		
q	h	Method	♭	♯	♭	♯	♭	♯	♭	♯	tw
						Center					
1	6	Random Walk	**0.10**	0.54	**0.04**	0.71	**0.01**	0.68	**0.07**	0.68	**0.18**
		AR	**0.09**	0.95	**0.02**	0.71	**0.00**	0.84	**0.04**	0.82	0.27
		Bagging	**0.00**	**0.00**	**0.00**	**0.05**	**0.00**	**0.01**	**0.00**	**0.00**	**0.15**
		CSR	0.37	0.99	**0.04**	0.71	**0.02**	0.84	**0.09**	0.82	0.94
		LASSO	**0.10**	0.26	**0.04**	0.67	**0.01**	0.45	**0.09**	0.37	1.00
		Random Forest	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.94
	12	Random Walk	1.00	1.00	0.33	1.00	**0.11**	1.00	0.43	1.00	0.71
		AR	**0.10**	**0.22**	0.33	**0.19**	**0.11**	**0.11**	0.43	**0.25**	**0.18**
		Bagging	**0.00**	**0.16**	0.33	0.27	**0.02**	**0.07**	0.43	**0.25**	0.71
		CSR	**0.02**	0.31	0.33	0.37	**0.07**	0.29	0.43	0.40	**0.14**
		LASSO	**0.10**	0.31	0.33	0.48	**0.11**	0.29	0.43	0.40	0.79
		Random Forest	0.46	0.31	1.00	0.48	1.00	0.29	1.00	0.40	1.00
	24	Random Walk	1.00	0.64	**0.06**	0.96	**0.01**	0.99	**0.06**	0.84	0.92
		AR	**0.23**	1.00	**0.06**	1.00	**0.01**	1.00	**0.06**	1.00	**0.06**
		Bagging	**0.16**	0.32	**0.13**	0.77	**0.01**	**0.05**	**0.23**	**0.17**	0.92
		CSR	**0.19**	0.51	**0.06**	0.89	**0.01**	0.61	**0.06**	0.75	**0.03**
		LASSO	**0.23**	0.48	**0.06**	0.96	**0.01**	0.68	**0.06**	0.76	0.92
		Random Forest	0.84	0.67	1.00	0.96	1.00	0.99	1.00	0.96	1.00
1.5	6	Random Walk	0.26	0.67	**0.06**	0.30	**0.04**	0.60	**0.12**	0.62	**0.13**
		AR	**0.20**	0.97	**0.04**	0.26	**0.04**	0.60	**0.04**	0.62	**0.18**
		Bagging	**0.00**	**0.01**	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**	**0.01**	**0.13**
		CSR	0.67	1.00	**0.17**	0.44	**0.08**	0.60	**0.12**	0.62	0.99
		LASSO	**0.20**	0.56	**0.06**	0.44	**0.01**	0.60	**0.09**	0.60	0.99
		Random Forest	1.00	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	12	Random Walk	1.00	1.00	**0.07**	1.00	**0.13**	1.00	0.42	1.00	0.54
		AR	**0.06**	0.37	**0.07**	0.53	**0.13**	0.48	0.33	0.39	**0.08**
		Bagging	**0.00**	**0.02**	**0.03**	**0.01**	**0.01**	**0.01**	**0.10**	**0.05**	0.54
		CSR	**0.00**	0.37	**0.07**	0.41	**0.10**	0.29	0.33	0.39	**0.08**
		LASSO	**0.02**	0.37	**0.07**	0.56	**0.10**	0.48	0.42	0.39	0.54
		Random Forest	0.60	0.37	1.00	0.80	1.00	0.48	1.00	0.39	1.00
	24	Random Walk	1.00	0.91	**0.00**	0.98	0.38	1.00	**0.02**	0.92	0.63
		AR	**0.15**	0.91	**0.00**	0.98	0.26	0.88	**0.00**	0.92	**0.04**
		Bagging	**0.00**	**0.07**	**0.19**	**0.06**	0.26	**0.00**	0.35	**0.08**	0.63
		CSR	**0.00**	0.91	**0.00**	0.72	**0.03**	0.77	**0.02**	0.92	**0.04**
		LASSO	**0.02**	0.71	**0.00**	0.98	**0.13**	0.77	**0.03**	0.92	0.63
		Random Forest	0.38	1.00	1.00	1.00	1.00	0.77	1.00	1.00	1.00

Table E.2.b (Continued): MCS p-values for inflation application.

			LogS		QS		SphS		CRPS		
q	h	Method	♭	♯	♭	♯	♭	♯	♭	♯	tw
2	6	Random Walk	0.36	0.66	**0.03**	0.53	**0.05**	0.48	**0.19**	0.69	**0.07**
		AR	**0.12**	0.82	**0.04**	0.54	**0.05**	0.69	**0.01**	0.89	**0.12**
		Bagging	**0.00**	**0.01**	**0.00**	**0.03**	**0.00**	**0.00**	**0.00**	**0.02**	**0.07**
		CSR	1.00	1.00	**0.21**	0.69	0.29	0.81	0.32	1.00	0.95
		LASSO	0.36	0.61	**0.06**	0.69	**0.01**	0.65	0.32	0.77	0.95
		Random Forest	0.68	0.66	1.00	1.00	1.00	1.00	1.00	0.92	1.00
	12	Random Walk	1.00	1.00	**0.01**	0.64	**0.05**	1.00	0.26	1.00	0.39
		AR	**0.16**	0.44	**0.01**	0.64	**0.05**	0.44	**0.06**	0.67	**0.03**
		Bagging	**0.00**	**0.02**	**0.00**	**0.04**	**0.00**	**0.00**	**0.00**	**0.09**	0.39
		CSR	**0.10**	**0.24**	**0.00**	**0.22**	**0.04**	**0.13**	**0.12**	0.37	**0.03**
		LASSO	**0.16**	0.44	**0.00**	0.64	**0.04**	0.44	0.26	0.67	0.39
		Random Forest	0.76	0.44	1.00	1.00	1.00	0.79	1.00	0.87	1.00
	24	Random Walk	1.00	0.74	**0.00**	1.00	0.59	1.00	0.39	1.00	1.00
		AR	0.38	1.00	**0.00**	0.70	**0.14**	0.29	**0.00**	0.92	**0.09**
		Bagging	**0.07**	**0.00**	**0.07**	**0.09**	**0.14**	**0.01**	0.35	**0.01**	**0.09**
		CSR	**0.20**	0.57	**0.00**	0.70	**0.02**	**0.22**	**0.02**	0.90	**0.09**
		LASSO	0.38	0.42	**0.00**	0.70	**0.14**	**0.22**	0.29	0.90	**0.09**
		Random Forest	0.49	0.57	1.00	0.70	1.00	**0.22**	1.00	0.90	0.91
						Tails					
1	6	Random Walk	**0.24**	0.46	**0.02**	0.26	**0.03**	0.31	0.33	**0.18**	**0.06**
		AR	**0.00**	**0.13**	**0.00**	0.31	**0.01**	0.44	1.00	**0.21**	**0.03**
		Bagging	**0.00**	**0.07**	**0.00**	0.26	**0.00**	**0.08**	0.30	0.84	**0.03**
		CSR	1.00	1.00	**0.13**	1.00	**0.11**	1.00	**0.09**	0.84	0.29
		LASSO	0.53	0.78	**0.03**	0.94	**0.03**	0.81	0.29	1.00	**0.16**
		Random Forest	0.53	0.46	1.00	0.94	1.00	0.81	**0.05**	0.84	1.00
	12	Random Walk	1.00	0.50	**0.18**	**0.07**	**0.12**	0.28	**0.08**	**0.00**	**0.06**
		AR	0.27	0.46	**0.05**	**0.07**	**0.08**	0.28	1.00	**0.00**	**0.06**
		Bagging	**0.03**	**0.07**	**0.00**	**0.07**	**0.00**	**0.03**	0.48	0.88	**0.06**
		CSR	**0.14**	0.46	**0.02**	**0.04**	**0.04**	**0.16**	0.47	**0.00**	**0.06**
		LASSO	0.27	1.00	**0.04**	**0.07**	**0.05**	0.28	0.45	0.88	**0.06**
		Random Forest	0.56	0.53	1.00	1.00	1.00	1.00	**0.08**	1.00	1.00
	24	Random Walk	1.00	0.58	1.00	0.74	0.80	0.69	0.43	0.31	1.00
		AR	**0.11**	0.58	**0.00**	0.74	**0.00**	0.69	0.94	0.31	**0.00**
		Bagging	**0.00**	**0.15**	**0.00**	0.69	**0.00**	0.69	1.00	**0.19**	**0.00**
		CSR	**0.02**	0.44	**0.00**	0.69	**0.00**	0.58	0.94	**0.09**	**0.00**
		LASSO	**0.11**	1.00	**0.00**	1.00	**0.00**	1.00	0.94	1.00	**0.00**
		Random Forest	**0.20**	0.31	**0.07**	0.66	1.00	0.50	0.28	0.31	0.31

Table E.2.b (Continued): MCS p-values for inflation application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
1.5	6	Random Walk	**0.15**	0.25	**0.02**	0.43	**0.01**	0.55	0.39	**0.24**	**0.14**
		AR	**0.00**	0.39	**0.02**	0.61	**0.00**	0.66	1.00	0.46	**0.06**
		Bagging	**0.01**	0.25	**0.00**	0.38	**0.00**	**0.02**	0.27	0.63	**0.03**
		CSR	1.00	0.68	**0.21**	0.80	**0.16**	0.82	**0.16**	0.63	**0.21**
		LASSO	0.52	1.00	**0.02**	1.00	**0.01**	1.00	0.27	1.00	**0.15**
		Random Forest	0.49	0.39	1.00	0.80	1.00	0.82	**0.08**	0.63	1.00
	12	Random Walk	1.00	0.40	0.33	**0.01**	**0.06**	0.43	**0.25**	**0.02**	**0.11**
		AR	**0.20**	0.40	**0.07**	**0.02**	**0.06**	0.43	1.00	**0.14**	**0.06**
		Bagging	**0.06**	0.40	**0.01**	**0.17**	**0.01**	0.26	0.31	1.00	**0.06**
		CSR	**0.12**	0.40	**0.03**	**0.01**	**0.04**	0.38	0.31	**0.14**	**0.06**
		LASSO	**0.20**	1.00	**0.05**	**0.17**	**0.04**	0.43	0.28	0.27	**0.06**
		Random Forest	0.52	0.40	1.00	1.00	1.00	1.00	**0.16**	0.27	1.00
	24	Random Walk	1.00	0.32	1.00	0.53	0.89	0.49	0.49	0.38	1.00
		AR	**0.06**	0.32	**0.00**	0.58	**0.01**	0.49	0.75	0.26	**0.00**
		Bagging	**0.06**	0.32	**0.00**	0.91	**0.02**	0.99	1.00	0.38	**0.01**
		CSR	**0.06**	0.32	**0.00**	0.41	**0.00**	0.44	0.75	0.26	**0.00**
		LASSO	**0.06**	1.00	**0.00**	1.00	**0.02**	1.00	0.75	1.00	**0.01**
		Random Forest	**0.08**	0.35	**0.10**	0.91	1.00	0.89	0.49	0.38	0.31
2	6	Random Walk	**0.16**	**0.24**	**0.10**	0.67	**0.02**	0.64	0.31	0.32	**0.18**
		AR	**0.01**	**0.23**	**0.05**	0.67	**0.01**	0.64	1.00	0.32	**0.10**
		Bagging	**0.04**	**0.24**	**0.01**	0.92	**0.00**	0.58	0.31	0.76	**0.04**
		CSR	1.00	**0.24**	0.68	0.92	0.34	1.00	0.28	0.68	**0.22**
		LASSO	0.67	1.00	**0.06**	0.92	**0.01**	0.96	0.31	1.00	**0.19**
		Random Forest	0.60	**0.24**	1.00	1.00	1.00	0.96	**0.21**	0.68	1.00
	12	Random Walk	1.00	0.26	0.86	0.28	**0.24**	0.62	**0.22**	**0.17**	0.31
		AR	**0.22**	**0.22**	**0.03**	0.34	**0.06**	0.62	1.00	**0.13**	**0.11**
		Bagging	**0.22**	0.66	**0.03**	1.00	**0.01**	0.94	0.31	1.00	**0.08**
		CSR	**0.22**	0.27	**0.03**	0.34	**0.06**	0.62	0.31	**0.17**	**0.10**
		LASSO	0.34	1.00	**0.07**	0.58	**0.06**	1.00	0.31	0.29	**0.11**
		Random Forest	0.46	0.46	1.00	0.58	1.00	0.94	0.27	0.29	1.00
	24	Random Walk	1.00	0.26	1.00	**0.24**	1.00	**0.24**	0.42	0.36	1.00
		AR	**0.07**	0.26	**0.00**	**0.24**	**0.00**	**0.24**	1.00	0.36	**0.01**
		Bagging	**0.03**	0.35	**0.00**	**0.24**	**0.00**	0.31	0.34	0.42	**0.01**
		CSR	**0.02**	0.26	**0.00**	**0.24**	**0.00**	**0.24**	0.34	0.36	**0.00**
		LASSO	**0.07**	1.00	**0.00**	1.00	**0.00**	1.00	0.34	1.00	**0.03**
		Random Forest	**0.18**	0.26	**0.11**	**0.24**	0.84	**0.22**	0.42	0.36	0.52

NOTE: This table mimics the setup of Table E.1.a, albeit with one additional column for the twCRPS, as the twCRPS and $\text{CRPS}^\flat_w$ no longer coincide by construction. The emphasis is on the center or tails, incorporated by the weight function $w(y_t) = \mathbb{1}_{[2-q,2+q]}(y_t)$ and its complement. All other settings are consistent with Table E.1.a.

E.3 Climate

Table E.3.a: MSE and MAE for climate application.

	MSE			MAE		
Method	$h=1$	$h=2$	$h=3$	$h=1$	$h=2$	$h=3$
GARCH-$\mathcal{N}$	5.13	10.14	13.92	1.80	2.54	2.99
GARCH-t	**5.10**	**10.06**	**13.74**	**1.79**	**2.52**	**2.97**
QGARCH-I-$\mathcal{N}$	5.17	10.30	14.27	1.81	2.56	3.04
QGARCH-I-t	5.15	10.26	14.18	1.80	2.55	3.02
QGARCH-II-$\mathcal{N}$	**4.82**	**8.26**	**9.93**	**1.73**	**2.26**	**2.49**
QGARCH-II-t	**4.82**	**8.26**	**9.94**	**1.73**	**2.26**	**2.49**

NOTE: MSE and MAE of forecast methods used in the climate examples, for the incorporated horizons $h \in \{1, 2, 3\}$. Bold face numbers indicate the three best models per performance measure.

Table E.3.b: MCS p-values for climate application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
		Center									
1	1	GARCH-$\mathcal{N}$	**0.00**	0.27	**0.00**	0.48	**0.00**	0.43	**0.00**	0.44	**0.00**
		GARCH-t	**0.00**	0.62	**0.00**	0.48	**0.00**	0.43	**0.00**	0.46	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.04**	**0.00**	**0.02**	**0.00**	**0.02**	**0.00**	**0.03**	**0.00**
		QGARCH-I-t	**0.00**	**0.06**	**0.00**	**0.02**	**0.00**	**0.02**	**0.00**	**0.04**	**0.00**
		QGARCH-II-$\mathcal{N}$	0.76	**0.22**	1.00	0.51	1.00	0.43	1.00	0.44	1.00
		QGARCH-II-t	1.00	1.00	0.48	1.00	0.35	1.00	**0.08**	1.00	**0.12**
	2	GARCH-$\mathcal{N}$	**0.01**	**0.03**	**0.00**	**0.02**	**0.00**	**0.01**	**0.00**	**0.04**	**0.00**
		GARCH-t	**0.01**	1.00	**0.00**	0.42	**0.00**	0.69	**0.00**	0.74	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.04**	**0.02**	1.00	**0.08**	1.00	**0.02**	1.00	**0.04**	1.00
		QGARCH-II-t	1.00	0.98	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.04**
	3	GARCH-$\mathcal{N}$	**0.01**	**0.05**	**0.00**	**0.01**	**0.00**	**0.01**	**0.00**	**0.03**	**0.00**
		GARCH-t	**0.01**	0.31	**0.00**	**0.06**	**0.00**	**0.14**	**0.00**	**0.19**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.01**	**0.05**	1.00	**0.02**	1.00	**0.01**	1.00	**0.03**	1.00
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.01**
2	1	GARCH-$\mathcal{N}$	**0.00**	**0.11**	**0.00**	**0.03**	**0.00**	**0.09**	**0.00**	**0.05**	**0.00**
		GARCH-t	**0.00**	**0.10**	**0.00**	**0.01**	**0.00**	**0.06**	**0.00**	**0.05**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.02**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**
		QGARCH-I-t	**0.00**	**0.02**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**
		QGARCH-II-$\mathcal{N}$	0.36	0.82	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		QGARCH-II-t	1.00	1.00	0.77	**0.20**	0.64	0.66	0.90	0.48	**0.18**
	2	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.02**	**0.00**	**0.00**	**0.00**
		GARCH-t	**0.00**	**0.05**	**0.00**	**0.01**	**0.00**	**0.04**	**0.00**	**0.01**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.02**	**0.02**	1.00	0.79	1.00	**0.04**	1.00	**0.21**	1.00
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**
	3	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		GARCH-t	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.01**	1.00	0.58	1.00	**0.00**	1.00	**0.17**	1.00
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.01**

Table E.3.b (Continued): MCS p-values for climate application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
4	1	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.02**	**0.00**	**0.00**	**0.00**
		GARCH-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.02**	**0.00**	**0.00**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.01**	**0.02**	1.00	0.55	1.00	**0.02**	**0.25**	0.66	1.00
		QGARCH-II-t	1.00	1.00	0.57	1.00	0.34	1.00	1.00	1.00	0.28
	2	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**
		GARCH-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	0.66	1.00	**0.00**	1.00	0.86	1.00
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**
	3	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		GARCH-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	**0.24**	1.00	**0.00**	1.00	0.66	1.00
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**
		Tails									
0.99	1	GARCH-$\mathcal{N}$	0.70	1.00	**0.06**	1.00	0.27	0.70	0.40	0.56	
		GARCH-t	0.75	0.32	**0.06**	0.85	0.27	1.00	0.40	1.00	
		QGARCH-I-$\mathcal{N}$	0.48	0.33	**0.06**	0.67	**0.01**	0.64	**0.04**	**0.05**	
		QGARCH-I-t	0.47	0.32	**0.06**	0.67	**0.03**	0.64	**0.05**	**0.05**	
		QGARCH-II-$\mathcal{N}$	0.39	**0.07**	1.00	**0.06**	1.00	**0.07**	0.48	**0.04**	
		QGARCH-II-t	1.00	**0.07**	0.81	**0.06**	0.80	**0.07**	1.00	**0.04**	
	2	GARCH-$\mathcal{N}$	**0.01**	**0.11**	**0.00**	0.68	**0.00**	0.25	**0.00**	**0.19**	
		GARCH-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.01**	**0.04**	**0.00**	**0.04**	**0.00**	**0.03**	**0.00**	**0.19**	
		QGARCH-I-t	**0.01**	**0.04**	**0.00**	**0.04**	**0.00**	**0.03**	**0.00**	**0.19**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.02**	0.44	**0.04**	**0.10**	**0.03**	0.25	**0.19**	
		QGARCH-II-t	0.62	**0.03**	1.00	**0.04**	1.00	**0.03**	1.00	**0.19**	
	3	GARCH-$\mathcal{N}$	**0.00**	**0.04**	**0.00**	**0.24**	**0.00**	**0.03**	**0.00**	**0.10**	
		GARCH-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.03**	**0.00**	**0.04**	**0.00**	**0.01**	**0.00**	**0.10**	
		QGARCH-I-t	**0.00**	**0.03**	**0.00**	**0.04**	**0.00**	**0.01**	**0.00**	**0.10**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.02**	1.00	**0.03**	1.00	**0.01**	0.94	**0.07**	
		QGARCH-II-t	0.92	**0.03**	0.27	**0.24**	0.99	**0.01**	1.00	**0.07**	

Table E.3.b (Continued): MCS p-values for climate application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
0.95	1	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.03}}$	0.80	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	**0.19**	
		GARCH-t	$\underline{\mathbf{0.07}}$	1.00	$\underline{\mathbf{0.00}}$	**0.11**	$\underline{\mathbf{0.00}}$	**0.09**	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.05}}$	$\underline{\mathbf{0.00}}$	**0.11**	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.09}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.06}}$	
		QGARCH-I-t	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.06}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.03}}$	**0.14**	1.00	**0.23**	0.36	$\underline{\mathbf{0.09}}$	0.48	**0.19**	
		QGARCH-II-t	1.00	**0.15**	0.49	**0.23**	1.00	$\underline{\mathbf{0.09}}$	1.00	**0.19**	
	2	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	0.59	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		GARCH-t	$\underline{\mathbf{0.05}}$	1.00	$\underline{\mathbf{0.00}}$	0.72	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.02}}$	1.00	$\underline{\mathbf{0.01}}$	1.00	$\underline{\mathbf{0.05}}$	
		QGARCH-II-t	1.00	$\underline{\mathbf{0.01}}$	0.30	0.72	0.56	$\underline{\mathbf{0.01}}$	0.36	$\underline{\mathbf{0.05}}$	
	3	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	0.73	$\underline{\mathbf{0.00}}$	0.37	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.08}}$	
		GARCH-t	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	0.73	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.08}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.08}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.01}}$	1.00	$\underline{\mathbf{0.01}}$	1.00	$\underline{\mathbf{0.08}}$	
		QGARCH-II-t	1.00	$\underline{\mathbf{0.01}}$	0.97	1.00	0.85	$\underline{\mathbf{0.03}}$	0.53	$\underline{\mathbf{0.08}}$	
0.9	1	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.01}}$	0.41	$\underline{\mathbf{0.00}}$	0.78	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		GARCH-t	$\underline{\mathbf{0.01}}$	1.00	$\underline{\mathbf{0.00}}$	0.78	$\underline{\mathbf{0.00}}$	0.92	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.00}}$	**0.13**	$\underline{\mathbf{0.00}}$	0.52	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.00}}$	**0.13**	$\underline{\mathbf{0.00}}$	0.51	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.07}}$	1.00	0.78	1.00	0.52	0.31	$\underline{\mathbf{0.05}}$	
		QGARCH-II-t	1.00	0.73	$\underline{\mathbf{0.03}}$	1.00	0.57	0.92	1.00	$\underline{\mathbf{0.05}}$	
	2	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.14**	$\underline{\mathbf{0.00}}$	0.48	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		GARCH-t	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	**0.14**	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	**0.17**	1.00	$\underline{\mathbf{0.06}}$	1.00	$\underline{\mathbf{0.05}}$	
		QGARCH-II-t	1.00	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.01}}$	1.00	**0.16**	0.48	0.30	$\underline{\mathbf{0.05}}$	
	3	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	**0.15**	$\underline{\mathbf{0.00}}$	0.45	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		GARCH-t	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	**0.15**	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	0.31	1.00	$\underline{\mathbf{0.05}}$	1.00	$\underline{\mathbf{0.07}}$	
		QGARCH-II-t	1.00	$\underline{\mathbf{0.02}}$	$\underline{\mathbf{0.02}}$	1.00	$\underline{\mathbf{0.04}}$	0.45	0.27	$\underline{\mathbf{0.07}}$	

Table E.3.b (Continued): MCS p-values for climate application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
0.85	1	GARCH-$\mathcal{N}$	**0.00**	**0.19**	**0.00**	**0.08**	**0.00**	**0.25**	**0.00**	0.56	
		GARCH-t	**0.00**	0.32	**0.00**	**0.08**	**0.00**	0.74	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	**0.05**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.05**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.03**	1.00	**0.21**	1.00	**0.05**	0.56	**0.05**	
		QGARCH-II-t	1.00	1.00	**0.02**	1.00	**0.13**	1.00	1.00	**0.05**	
	2	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.10**	**0.00**	**0.05**	
		GARCH-t	**0.00**	1.00	**0.00**	**0.00**	**0.00**	1.00	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.05**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.05**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	**0.03**	1.00	**0.00**	1.00	**0.05**	
		QGARCH-II-t	1.00	0.43	**0.00**	1.00	**0.00**	0.98	**0.16**	**0.05**	
	3	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.11**	**0.00**	**0.10**	
		GARCH-t	**0.00**	1.00	**0.00**	**0.00**	**0.00**	1.00	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.10**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.10**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	**0.01**	1.00	**0.00**	1.00	**0.10**	
		QGARCH-II-t	1.00	0.58	**0.00**	1.00	**0.01**	0.95	**0.09**	**0.06**	
0.8	1	GARCH-$\mathcal{N}$	**0.00**	**0.07**	**0.00**	**0.02**	**0.00**	0.66	**0.00**	**0.21**	
		GARCH-t	**0.00**	**0.07**	**0.00**	**0.02**	**0.00**	0.66	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.01**	**0.00**	0.31	**0.00**	**0.04**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.11**	**0.00**	**0.04**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.03**	1.00	1.00	1.00	0.86	1.00	**0.05**	
		QGARCH-II-t	1.00	1.00	**0.05**	0.51	**0.15**	1.00	0.95	**0.05**	
	2	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0.31	**0.00**	**0.04**	
		GARCH-t	**0.00**	0.61	**0.00**	**0.00**	**0.00**	0.57	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.05**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.05**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	0.45	1.00	**0.04**	1.00	**0.04**	
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.02**	**0.04**	
	3	GARCH-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0.40	**0.00**	**0.09**	
		GARCH-t	**0.00**	0.30	**0.00**	**0.00**	**0.00**	0.40	**0.00**	1.00	
		QGARCH-I-$\mathcal{N}$	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.09**	
		QGARCH-I-t	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.09**	
		QGARCH-II-$\mathcal{N}$	**0.00**	**0.00**	1.00	0.27	1.00	**0.03**	1.00	**0.07**	
		QGARCH-II-t	1.00	1.00	**0.00**	1.00	**0.00**	1.00	**0.03**	**0.06**	

Table E.3.b (Continued): MCS p-values for climate application.

			LogS		QS		SphS		CRPS		
q	h	Method	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	$\flat$	$\sharp$	tw
0.75	1	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.12**	$\underline{\mathbf{0.00}}$	0.26	
		GARCH-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.02}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.10**	$\underline{\mathbf{0.00}}$	0.79	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.02}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.10}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.02}}$	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.02}}$	1.00	1.00	1.00	0.72	1.00	**0.19**	
		QGARCH-II-t	1.00	1.00	0.34	0.61	**0.10**	1.00	0.62	**0.19**	
	2	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.15**	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.03}}$	
		GARCH-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.05}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.15**	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.07}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	**0.12**	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	1.00	1.00	**0.15**	1.00	**0.12**	
		QGARCH-II-t	1.00	1.00	$\underline{\mathbf{0.00}}$	0.60	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.09}}$	
	3	GARCH-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.08}}$	
		GARCH-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.03}}$	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-I-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.06}}$	
		QGARCH-I-t	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	
		QGARCH-II-$\mathcal{N}$	$\underline{\mathbf{0.00}}$	$\underline{\mathbf{0.00}}$	1.00	0.73	1.00	$\underline{\mathbf{0.03}}$	1.00	**0.11**	
		QGARCH-II-t	1.00	1.00	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.00}}$	1.00	$\underline{\mathbf{0.01}}$	$\underline{\mathbf{0.09}}$	

NOTE: This table mimics the setup of Table E.2.b. The emphasis is on the center or right tail, incorporated by the weight functions $w(y_t) = \mathbb{1}_{[18-q,18+q]}(y_t)$ and $w_t(y_t) = \mathbb{1}_{(\hat{r}_t^q,\infty)}(y_t)$, respectively. For the latter weight function, the twCRPS is equivalent to the $\mathrm{CRPS}_w^\flat$ by construction, and is therefore excluded from the table. All other settings are consistent with those outlined in the caption of Table E.2.b, except for the block length $k = 200$.

References

Bernardi, M. and L. Catania (2018), "The Model Confidence Set Package for R", *International Journal of Computational Economics and Econometrics*, *8*(2), 144–158.

Diebold, F. X. and R. S. Mariano (2002), "Comparing Predictive Accuracy", *Journal of Business & Economic Statistics*, *20*(1), 134–144.

Diks, C., V. Panchenko, and D. Van Dijk (2011), "Likelihood-based Scoring Rules for Comparing Density Forecasts in Tails", *Journal of Econometrics*, *163*(2), 215–230.

Giacomini, R. and H. White (2006), "Tests of Conditional Predictive Ability", *Econometrica*, *74*(6), 1545–1578.

Neyman, J. and E. Pearson (1933), "IX. On the Problem of the Most Efficient Tests of Statistical Hypotheses", *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, *231*(694-706), 289–337.