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Abstract

We give weak conditions for a continuous time optimal control problem with one-dimensional state space to have monotone optimal trajectories. These conditions include a concavity condition on the Lagrange function. If that condition is changed to strict concavity, then all optimal trajectories are monotone.

Time-autonomous infinite horizon optimal control problems with one-dimensional state spaces have monotone optimal trajectories. As far as we know, there is only a single full paper discussing this fundamental observation: Hartl (1987) proved it under the assumption that the problem has a unique solution among trajectories tending to a limiting value. The proof is elementary, and we give a paraphrase below, but the uniqueness requirement severely restricts its applicability: general optimal control problems in one dimension may have 'indifference' or 'Skiba' states (Skiba, 1978), that are initial states to several optimal solutions.

Monotonicity of optimal trajectories is proved as an auxiliary result in a number of papers, usually with methods that require differentiability assumptions or that are tailored to the specific application. Ngo Van Long et al. (1997) and Askenazy and Cuong Le Van (1999) base their arguments on the costate equation and the implied continuity of the choice variable. Wagener (2003) also uses the costate equation, but rather relies on the Poincaré–Bendixson theorem. Recently Akao et al. (2025), for a Ramsey growth model, gave a proof based on Jensen's inequality.

In our work on the structure of the set of all feedback Nash equilibria of differential games with one-dimensional state spaces (Jaakkola & Wagener, 2024), our theoretical arguments require a general version of this result without assumptions on uniqueness and limit behaviour, and under weak regularity assumptions. The first theorem proved in this article provides such a result, showing that under a natural concavity hypothesis, time-autonomous infinite horizon optimal control problems have monotone trajectories. The second theorem shows that if concavity is replaced by strict concavity, all optimal trajectories are monotone. Thus our results highlight the difference between concavity and strict concavity, an aspect that is absent from the earlier literature. The proof of the first theorem is elementary; for the second theorem, we need some basic measure-theoretical notions.

The results are formulated in terms of assumptions on the set of trajectories over which the objective is optimised, rather than on the restrictions which generate this set. We show how to

verify the assumptions for a standard capital accumulation problem. Our results generalise all previous results we have referenced above.

1 Context

We consider the problem to optimise an objective

$$\int_0^\infty L(x(t), \dot{x}(t)) \exp(-rt)\, \mathrm{d}t, \tag{1}$$

where $r > 0$, over a set $\mathcal{X}$ of absolutely continuous functions $x : [0, \infty) \to \mathbb{R}$, or *trajectories*.

Assumption 1. *The set $\mathcal{X}$ of trajectories and the function $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfy the following conditions.*

(i) The trajectories are equi-Lipschitz: there is $C > 0$ such that $|x(t_1) - x(t_2)| \leq C|t_1 - t_2|$ for all $x \in \mathcal{X}$ and all $t_1, t_2 \geq 0$.

(ii) If a sequence $\{x_k\}$ of trajectories converges to a function y, uniformly on compact sets, then y is also a trajectory.

(iii) For each trajectory $x \in \mathcal{X}$, the function $L(x(t), \dot{x}(t)) \exp(-rt)$ is integrable on $[0, \infty)$.

(iv) The function $L(x, v)$ is upper semi-continuous in (x, v) and concave in v for fixed x.

(v) If x is a trajectory such that $x(t_1) = x(t_2)$ for some $0 \leq t_1 \leq t_2$, then the functions

$$y(t) = \begin{cases} x(t) & \text{if } 0 \leq t < t_1, \\ x(t + t_2 - t_1) & \text{if } t \geq t_1, \end{cases}$$

and

$$z(t) = \begin{cases} x(t) & \text{if } 0 \leq t < t_2, \\ x(t - t_2 + t_1) & \text{if } t \geq t_2, \end{cases}$$

are also trajectories.

The first four assumptions are standard for proving the existence of maximisers (Vinter, 2000). As we take the existence of maximising trajectories for granted, in our situation the first four assumptions imply that a bounded sequence of maximising trajectories has a convergent subsequence whose limit is again a maximising trajectory.

The last assumption says that if a trajectory contains a cycle, then removing this cycle, or adding it a second time, generates another trajectory. In the following, we use this property to construct from a given maximising trajectory a sequence of maximising trajectories that converges to a monotone trajectory.

In economic terms, the first assumption is significant: it rules out, for instance, capital models where disinvestment can happen at an arbitrarily high rate. This is usually only a technical convenience: in practice, setting C to a large value will have a similar effect.

The assumptions typically hold for time-autonomous optimal control problems with bounded

action spaces. We give as example the classical capital accumulation problem, to maximise

$$\int_0^\infty u(c)\exp(-rt)\,\mathrm{d}t,$$

with $r > 0$ and $u(c)$ continuous, increasing and concave, subject to

$$\dot{x} = f(x) - c - \delta x, \qquad x(0) = x_0,$$

where $f(x) \geq 0$ is output of production, $0 \leq c \leq f(x)$ the rate of consumption, $\delta > 0$ the rate of capital depreciation, and x_0 the initial state. By performing the substitutions $c = f(x) - \delta x - \dot{x}$ and $L(x, \dot{x}) = u(f(x) - \delta x - \dot{x})$, the objective is brought into the form (1). The set $\mathcal{X}$ of trajectories is given as

$$\mathcal{X} = \{x : x(0) = x_0 \text{ and } \dot{x}(t) \in [-\delta x(t), f(x(t)) - \delta x(t)] \text{ a.e.}\}.$$

We assume that for $x \geq 0$ the function f is continuous, bounded, and non-negative. This implies that for all $x \in \mathcal{X}$, the trajectories $x(t)$ and the derivatives $\dot{x}(t)$ are uniformly bounded, with the bound depending on the initial state x_0. The equi-Lipschitz property follows. The closedness of the set of trajectories under uniform convergence on compact sets is a consequence of the continuity of f (Filippov, 1988). As f is bounded, so is c and hence also $L(x, \dot{x}) = u(c)$, implying integrability of $L(x(t), \dot{x}(t))\exp(-rt)$. Continuity of f and continuity and concavity of u take care of the fourth assumption. The last assumption is a direct consequence of the fact that the problem is time-autonomous.

Our main result is the following theorem, proved in Section 3.

Theorem 1. *Let Assumption 1 hold. If a trajectory x maximises the objective* (1)*, then there is a monotone trajectory y that also maximises the objective.*

The theorem implies Hartl's result: if there is a unique maximiser, then the trajectories x and y coincide, and x is monotone.

The conclusion of Theorem 1 is optimal in the sense that there are optimal control problems satisfying the assumptions for which non-monotone optimal trajectories exist: take for instance $L(x, v) \equiv 0$. The theorem can be strengthened if concavity is replaced by strict concavity.

Assumption 2. *The function $L(x, v)$ is strictly concave in v for fixed x.*

Theorem 2. *Let Assumptions 1 and 2 hold. If x is a trajectory that maximises the objective, then it is monotone.*

2 Hartl's lemma

We first reformulate Hartl's argument.

A *section* s of a trajectory x is the restriction of x to an interval $[a, b]$ translated to $[0, b - a]$; that is

$$s(t) = x(t - a).$$

We call a and b, respectively, the *start time* and the *end time* of the section. Their difference $\theta = b - a$ is the *duration* of the section. The points $s(0) = x(a)$ and $s(\theta) = x(b)$ are respectively

the *initial point* and the *terminal point* of the section.

The *value* of a section s with duration θ is

$$J(s) = \int_0^\theta L(s(t), \dot{s}(t)) \exp(-rt)\,\mathrm{d}t.$$

Two sections s_1 and s_2, with durations θ_1 and θ_2, can be *concatenated* if the duration θ_1 of the first section is finite and if the terminal point of the first section coincides with the initial point of the second. The result is a new section $s_1 \cdot s_2$ given as

$$(s_1 \cdot s_2)(t) = \begin{cases} s_1(t) & 0 \le t < \theta_1, \\ s_2(t - \theta_1) & \theta_1 \le t. \end{cases}$$

A finite concatenation $s_1 \cdot \ldots \cdot s_n$ is a trajectory if s_n is of infinite duration.

The value of a concatenation satisfies

$$J(s_1 \cdot s_2) = J(s_1) + \exp(-r\theta_1) J(s_2).$$

We have that

$$J(s_0 \cdot s_1) < J(s_0 \cdot s_2) \quad \text{if and only if} \quad J(s_1) < J(s_2).$$

A *cycle* is a section whose initial and terminal points coincide.

Lemma 1 (Hartl's lemma)**.** *Let x be an optimal non-monotone trajectory. Then all trajectories obtained by adding or removing cycles from x are also optimal.*

Proof. As x is non-monotone, there are $0 \le a < b$ such that $x(a) = x(b)$ and x restricted to $[a, b]$ is not constant. Hence x has a non-constant cycle c, and it can be written as

$$x = s_1 \cdot c \cdot s_2,$$

where s_2 has infinite duration.

By Assumption 1(v), the section $s_1 \cdot s_2$, obtained from x by removing the cycle, is also a trajectory. Assume this trajectory is not optimal. Then

$$J(s_1 \cdot s_2) < J(s_1 \cdot c \cdot s_2)$$

and hence

$$J(s_2) < J(c \cdot s_2). \tag{2}$$

Consider now $s_1 \cdot c \cdot c \cdot s_2$, with the cycle added to x: by Assumption 1(v) this is again a trajectory. Equation (2) implies

$$J(c \cdot s_2) < J(c \cdot c \cdot s_2)$$

and hence

$$J(x) = J(s_1 \cdot c \cdot s_2) < J(s_1 \cdot c \cdot c \cdot s_2).$$

But this contradicts the optimality of x. We conclude that $s_1 \cdot s_2$ is optimal.

The proof for the addition of cycles is similar. □

3 Concave optimisation problems have monotone maximisers

Proof of Theorem 1. We introduce the functions

$$x^+(t) = \max\{x(s) \,:\, 0 \le s \le t\}, \qquad x^-(t) = \min\{x(s) \,:\, 0 \le s \le t\}.$$

The function $x^+(t)$ is increasing, as it is a maximum taken over increasingly large sets; similarly $x^-(t)$ is decreasing. If, for instance, $x(t)$ is increasing, then $x(t) = x^+(t)$ for all t. We set

$$\sigma = \sup\left\{T \ge 0 \,:\, x(t) = x^-(t) \text{ for all } 0 \le t \le T\right\},$$
$$\tau = \sup\left\{T \ge 0 \,:\, x(t) = x^+(t) \text{ for all } 0 \le t \le T\right\}.$$

That is, $[0, \sigma]$ is the largest interval containing 0 for which x is decreasing, and $[0, \tau]$ is the largest interval containing 0 for which x is increasing.

If either $\sigma = \infty$ or $\tau = \infty$, the trajectory x is monotone, and there is nothing to prove. If both are finite, assume that $\tau \ge \sigma$, the other case being similar.

Lemma 2. *There is a sequence of intervals $[\rho_k, \tau_k]$ such that $\rho_k < \tau_k$ for all k, $\rho_k, \tau_k \to \tau$ as $k \to \infty$, and x restricted to each of these intervals is a cycle.*

Proof. While x restricted to $[0, \tau]$ is monotone, for every $k > 0$, the function x restricted to $[0, \tau + 1/k]$ is not monotone. This implies that there are $0 \le t_1^{(k)} < \sigma_k < t_2^{(k)} \le \tau + 1/k$ such that either $x(\sigma_k) < \min\{x(t_1^{(k)}), x(t_2^{(k)})\}$ or $x(\sigma_k) > \max\{x(t_1^{(k)}), x(t_2^{(k)})\}$.

Assume that we are in the first situation, the proof for the second situation being similar. By the intermediate value theorem, there are numbers ρ_k, τ_k such that $t_1^{(k)} \le \rho_k < \sigma_k < \tau_k \le t_2^{(k)}$ and $x(\sigma_k) < x(\rho_k) = x(\tau_k) = \min\{x(t_1^{(k)}), x(t_2^{(k)})\}$. Then x restricted to $[\rho_k, \tau_k]$ is a cycle. Note that, because of non-monotonicity, we have $\tau < \tau_k \le \tau + 1/k$.

If $\rho_k \to \tau$ as $k \to \infty$, the lemma is proved. If not, then a subsequence, which can be taken to the sequence itself, converges to a limit $\rho < \tau$, and $x(\rho_k) = x(\tau_k) \to x(\rho) = x(\tau)$. Hence $x(t) = x(\tau)$ for all $\rho \le t \le \tau$ as x is monotone on this interval. We then redefine $\rho_k = \max\{\rho, \tau - 1/k\}$ to prove the lemma also in this situation. □

As a consequence of the lemma we write

$$x = s_1^{(k)} \cdot c_k \cdot s_2^{(k)},$$

with $s_1^{(k)}$ the restriction of x to $[0, \rho_k]$ and $s_2^{(k)}$ its restriction to $[\tau_k, \infty)$. We construct a new trajectory by repeating the cycle c_k infinitely often:

$$x_k = s_1^{(k)} \cdot c_k \cdot c_k \cdot c_k \cdot \ldots$$

By Assumption 1(ii), x_k is a trajectory, being the limit of the sequence of trajectories $x_k^{(1)} = s_1^{(k)} \cdot c_k \cdot s_2^{(k)}$, $x_k^{(2)} = s_1^{(k)} \cdot c_k \cdot c_k \cdot s_2^{(k)}$, … which converges uniformly on compact intervals.

Hartl's lemma implies that all $x_k^{(j)}$ are optimal. Therefore all values $J(x_k^{(j)})$ are the same. For $j \to \infty$, we have that

$$|J(x_k^{(j)}) - J(x_k)| = \exp(-r(\rho_k + j\theta_k))|J(s_2^{(k)}) - J(c_k \cdot c_k \cdot \ldots)| = C_k \exp(-jr\theta_k) \to 0,$$

where C_k is independent of j, and hence $J(x_k)$, as the limit of a constant sequence, equals $J(x_k^{(j)})$ for all j. We conclude that x_k is optimal.

We set $\xi = x(\tau)$. We moreover let ℓ be the infinite duration segment given as $\ell(t) = \xi$ for all $t \geq 0$, and we introduce s_1 as the restriction of x to $[0, \tau]$. Using these sections, we construct

$$y = s_1 \cdot \ell.$$

Lemma 3. *The trajectories x_k converge uniformly to the trajectory y as $k \to \infty$.*

Proof. Introduce $\psi_k = \min\{\rho_k, \tau\}$, and note that $x_k(t) = x(t)$ for all $t \in [0, \tau_k]$. Hence $x_k(t) = y(t)$ for $0 \leq t \leq \psi_k$.

Take $\varepsilon > 0$. By continuity of x, there is $\delta > 0$ such that $|x(t) - \xi| < \varepsilon$ whenever $|t - \tau| < \delta$. As $\rho_k, \tau_k \to \tau$, there is $K > 0$ such that $|\rho_k - \tau| < \delta$ and $|\tau_k - \tau| < \delta$ for $k > K$ and consequently

$$|x_k(t) - \xi| = |x(t) - \xi| < \varepsilon \tag{3}$$

for all $\psi_k \leq t \leq \tau_k$. In particular $|c_k(t) - \xi| < \varepsilon$ if $0 \leq t < \theta_k = \tau_k - \rho_k$. It follows that $|x_k(t) - y(t)| < \varepsilon$ if $t \geq \psi_k$.

We conclude that for every $\varepsilon > 0$ there is $K > 0$ such that if $k > K$, then $|x_k(t) - y(t)| < \varepsilon$ for all $t \geq 0$. This shows uniform convergence of x_k to y. By Assumption 1(ii) it follows that y is also a trajectory. □

To show that the limit trajectory y is optimal, introduce the section $x_1^{(k)} = s_1^{(k)} \cdot c_k$ defined on $[0, \tau_k]$, and $y_1^{(k)}$ and ℓ_k as the restriction of y to, respectively, $[0, \tau_k]$ and $[\tau_k, \tau_k + \theta_k]$. Then $y_1^{(k)}(t) = x_1^{(k)}(t)$ for $0 \leq t \leq \psi_k$ and

$$x_k = x_1^{(k)} \cdot c_k \cdot c_k \cdot \ldots \quad \text{and} \quad y = y_1^{(k)} \cdot \ell_k \cdot \ell_k \cdot \ldots \quad .$$

Setting $\alpha_k = \exp(-r\tau_k)$ and $\beta_k = \exp(-r\theta_k)$, we have

$$J(y) - J(x_k) = J(y_1^{(k)}) - J(x_1^{(k)}) + \alpha_k \sum_{n=0}^{\infty} \beta_k^n (J(\ell_k) - J(c_k)).$$

We estimate the two differences on the right hand side. First

$$J(y_1^{(k)}) - J(x_1^{(k)}) = \int_{\psi_k}^{\tau_k} \Big(L(y_1^{(k)}(t), \dot{y}_1^{(k)}(t)) - L(x_1^{(k)}(t), \dot{x}_1^{(k)}(t)) \Big) \exp(-rt)\, \mathrm{d}t.$$

As the integrand is an integrable function and the length of the domain of integration tends to 0, for given $\varepsilon > 0$ there is $K_1 > 0$ such that for $k > K_1$ we have that

$$|J(y_1^{(k)}) - J(x_1^{(k)})| < \varepsilon/3.$$

To estimate $J(\ell_k) - J(c_k)$, we introduce the excess function

$$E(v) = L(\xi, 0) + pv - L(\xi, v),$$

where we choose p, using the concavity of $L(\xi, v)$ in v, such that the graph of $L(\xi, 0) + pv$ is a support line of the hypograph of $L(\xi, v)$, implying that $E(v) \geq 0$ for all v.

Using the excess function, we obtain the equality

$$L(\xi, 0) - L(x(t), \dot{x}(t)) = E(\dot{x}(t)) - p\dot{x}(t) + L(\xi, \dot{x}(t)) - L(x(t), \dot{x}(t)). \tag{4}$$

Set $M = |x(0)| + C\tau_1$. The values of the trajectories x_k as well as their limit y are all contained in the compact set $x([0, \tau_1]) \subset [-M, M]$, and the values of the derivatives $\dot{x}_k$ are bounded by C. The function L is uniformly upper semicontinuous on the compact set $[-M, M] \times [-C, C]$. Lemma 3 and the upper semi-continuity of L then imply that for every $\varepsilon > 0$ there is $K_2 > 0$ such that if $k > K_2$, then

$$L(\xi, \dot{x}_k(t)) - L(x_k(t), \dot{x}_k(t)) \geq -\varepsilon/3. \tag{5}$$

We use the cycle condition $c_k(0) = c_k(\theta_k)$ as well as partial integration to calculate

$$\begin{aligned}
\int_0^{\theta_k} \dot{c}_k(t) \exp(-rt)\, \mathrm{d}t &= c_k(t) \exp(-rt) \Big|_0^{\theta_k} + r \int_0^{\theta_k} c_k(t) \exp(-rt)\, \mathrm{d}t \\
&= c_k(0) \exp(-rt) \Big|_0^{\theta_k} + r \int_0^{\theta_k} c_k(t) \exp(-rt)\, \mathrm{d}t \\
&= r \int_0^{\theta_k} (c_k(t) - c_k(0)) \exp(-rt)\, \mathrm{d}t.
\end{aligned}$$

The Lipschitz condition for c_k, which is the same as that for x, then implies

$$\left| \int_0^{\theta_k} \dot{c}_k(t) \exp(-rt)\, \mathrm{d}t \right| \leq r \int_0^{\theta_k} |c_k(t) - c_k(0)|\, \mathrm{d}t \leq Cr \int_0^{\theta_k} t\, \mathrm{d}t = Cr\theta_k^2. \tag{6}$$

Using (4), (5), and (6), as well as the nonnegativity of E, we obtain for $k > K_2$ that

$$\begin{aligned}
J(\ell_k) - J(c_k) &= \int_0^{\theta_k} [L(\xi, 0) - L(c_k(t), \dot{c}_k(t))] \exp(-rt)\, \mathrm{d}t \\
&\geq -|p| Cr\theta_k^2 - (1 - \exp(-r\theta_k))\varepsilon/3 \geq -|p| Cr\theta_k^2 - (1 - \beta_k)\varepsilon/3.
\end{aligned}$$

Using the inequality $x/(1 - \exp(-x)) \leq \exp(x)$ with $x = r\theta_k$, we obtain

$$\sum_{n=0}^{\infty} \beta_k^n (J(\ell_k) - J(c_k)) \geq -|p| C \exp(r\theta_k)\theta_k - \varepsilon/3.$$

Choose $K_3 > 0$ such that for $k > K_3$ the first term satisfies $|p| C \exp(r\theta_k)\theta_k < \varepsilon/3$. Then we infer for $k \geq \max_i K_i$ that

$$J(y) - J(x_k) \geq -\alpha_k \varepsilon \geq -\varepsilon.$$

Since all $J(x_k)$ are optimal, they have a common value $J_{\max}$. We have therefore found that $J(y) \geq J_{\max} - \varepsilon$ for every $\varepsilon > 0$. As $\varepsilon > 0$ was arbitrary, we conclude that $J(y) \geq J_{\max}$, and that

y is optimal as well. □

4 All maximisers of strictly concave optimisation problems are monotone

Proof of Theorem 2. The idea of the proof is, assuming the existence of a non-monotone maximiser, to construct a sequence x_k with an infinitely often repeated cycle for which the excess function $E(\dot{x}_k)$ is bounded away from 0 on sets of sufficiently large measure. Here, as everywhere in the following, measure is understood to mean Lebesgue measure.

Assume that $x \in \mathcal{X}$ is a non-monotone maximiser and let c be a cycle of x with start time τ and duration θ. We write $x = s_0 \cdot c \cdot s_\infty$.

Lemma 4. *There are times $t_1, t_2 \in [0, \theta]$ and a constant $\mu > 0$ such that $c(t_1) = c(t_2)$ and $\dot{c}(t_1) > \mu$ and $\dot{c}(t_2) < -\mu$.*

Proof. By the Weierstrass theorem, there are $\tau_1, \tau_2 \in [0, \theta]$ such that $\xi_1 = c(\tau_1) \leq c(t) \leq c(\tau_2) = \xi_2$ for all $t \in [0, \theta]$. For $\mu > 0$ introduce the sets $S^-, S^+ \subset [0, \theta]$ of times t such that c is moving respectively slowly downward or upward, that is

$$S^- \;:\; -\mu \leq \dot{c}(t) \leq 0, \qquad S^+ \;:\; 0 \leq \dot{c}(t) \leq \mu.$$

Likewise, let $F^-, F^+ \subset [0, \theta]$ be the sets of times that c is respectively moving fast downward or upward

$$F^- \;:\; \dot{c}(t) < -\mu, \qquad F^+ \;:\; \mu < \dot{c}(t).$$

Interpreting all integrals as Lebesgue integrals, and denoting by $\lambda(S)$ the measure of a measurable set S, we have for $i \in \{-, +\}$ that

$$\lambda(c(S^i)) = \int_{c(S^i)} \mathrm{d}x = \int_{S^i} |\dot{c}(t)| \; \mathrm{d}t \leq \mu \lambda(S^i) \leq \mu\theta.$$

Choosing $\mu = \frac{1}{3\theta}(\xi_2 - \xi_1)$ we find that

$$\lambda(c(F^i)) = \lambda([\xi_1, \xi_2] - c(S^i)) \geq \tfrac{2}{3}(\xi_2 - \xi_1).$$

The sets $c(F^-)$ and $c(F^+)$ are both subsets of the interval $[\xi_1, \xi_2]$ and the sum of their measures is strictly larger than the measure of the interval: hence the sets have a nonempty intersection. This shows the lemma. □

We use the lemma to construct a sequence $\{x_k\}$ of optimal trajectories whose limit y has a strictly higher value, thereby reaching a contradiction. We first give the construction of the sequence.

If $A \subset \mathbb{R}$ is a measurable set with positive measure, a point $t_0 \in A$ is a *Lebesgue density point* of A if $\lambda([t_0 - h, t_0 + k] \cap A)/(k + h) \to 1$ as $h, k \to 0$. Almost all points in a measurable set are Lebesgue density points.

Let $\tilde{F}^i$ denote the set of Lebesgue density points of F^i: then $Z^i = F^i - \tilde{F}^i$ is a measure zero set. Moreover, as $|\dot{c}(t)| \leq C$, the map $c : F \to c(F)$ maps measure zero sets to measure zero sets.

Hence $c(Z^i) = 0$, the sets $c(F^i) = c(\tilde{F}^i) \cup c(Z^i)$ and $c(\tilde{F}^i)$ have the same measure, and we may choose t_1 and t_2 such that they are Lebesgue density points of F^+ and F^- respectively.

Lemma 5. *There are numbers h_k^-, h_k^+ and a constant $K > 0$, such that for all $k > K$ we have*

$$(1/C)k^{-1} < h_k^-, h_k^+ < (2/\mu)k^{-1}$$

and $c(t_1 + h_k^+) = c(t_2 - h_k^-) = c(t_1) + 1/k$.

Proof. We introduce

$$\eta = (C + \tfrac{1}{2}\mu)/(C + \mu),$$

with C the Lipschitz constant from Assumption 1(i), and we note that $0 < \eta < 1$. We moreover introduce the sets

$$F_h^+ = F^+ \cap [t_1, t_1 + h] \quad \text{and} \quad F_h^- = F^- \cap [t_2 - h, t_2].$$

writing $(F_h^+)^c$ and $(F_h^-)^c$ for the complement of these sets in the intervals $[t_1, t_1 + h]$ and $[t_2 - h, t_2]$ respectively.

As t_1 is a Lebesgue density point of F^+, we can find $h_0 > 0$ such that

$$\lambda(F_h^+)/h = \lambda([t_1, t_1 + h] \cap F^+)/h \geq \eta$$

if $0 < h < h_0$. We have the inequality $\dot{c}(t) > \mu$, which holds if $t \in F^+$, and $\dot{c}(t) \geq -C$, which holds everywhere. They imply

$$\begin{aligned} c(t_1 + h) - c(t_1) = \int_{t_1}^{t_1+h} \dot{c}(t)\,\mathrm{d}t &= \int_{F_h^+} \dot{c}(t)\ \mathrm{d}t + \int_{(F_h^+)^c} \dot{c}(t)\ \mathrm{d}t \\ &\geq \mu \int_{F_h^+} \mathrm{d}t - C \int_{(F_h^+)^c} \mathrm{d}t \geq \mu\eta h - C(1-\eta)h = \frac{\mu}{2}h. \end{aligned}$$

Set $K = 2/(\mu h_0)$. Then $c(t_1 + h_0) \geq c(t_1) + 1/K$ and for all $k > K$ the equation

$$c(t_1 + h_k^+) = c(t_1) + 1/k.$$

has a solution $1/(Ck) < h_k^+ < 2/(\mu k)$. Similarly we find h_k^- such that $1/(Ck) < h_k^- < 2/(\mu k)$, which solves the equation

$$c(t_2 - h_k^-) = c(t_2) + 1/k = c(t_1) + 1/k = c(t_1 + h_k^+). \tag{7}$$

This shows the lemma. □

Consider first the situation $t_1 < t_2$. If k is sufficiently large, then $t_1 + h_k^+ < t_2 - h_k^-$. We write

$$c = s_1 \cdot s_2^{(k)} \cdot s_3^{(k)} \cdot s_4^{(k)} \cdot s_5,$$

where s_1 is the restriction of c to $[0, t_1]$, $s_2^{(k)}$ that to $[t_1, t_1 + h_k^+]$, $s_3^{(k)}$ to $[t_1 + h_k^+, t_2 - h_k^-]$, $s_4^{(k)}$ to $[t_2 - h_k^-, t_2]$, and s_5 the restriction to $[t_2, \theta]$. Equation (7) shows that $s_3^{(k)}$ is a cycle. Removing it

results in a new trajectory

$$s_0 \cdot s_1 \cdot s_2^{(k)} \cdot s_4^{(k)} \cdot s_5 \cdot s_\infty$$

which is optimal by the Hartl lemma. The concatenation $c_k = s_2^{(k)} \cdot s_4^{(k)}$ is also a cycle, as $c(t_1) = c(t_2)$, of duration $\theta_k = h_k^+ + h_k^-$. We set

$$x_k = s_0 \cdot s_1 \cdot c_k \cdot c_k \cdot c_k \cdot \ldots,$$

which is again optimal by the Hartl lemma.

The trajectories x_k are equi-Lipschitz, as they are concatenations of segments of x. Let ℓ be the constant segment $\ell(t) = \xi \equiv x(t_1) = c_k(0)$ of infinite duration, and set

$$y = s_0 \cdot s_1 \cdot \ell.$$

For $t \in [0, \theta_k]$, we have

$$|c_k(t) - \xi| = |c_k(t) - c_k(0)| \le Ct \le C\theta_k.$$

As $\theta_k \to 0$ as $k \to \infty$, it follows that the trajectories x_k converge uniformly to y.

We estimate the difference

$$\begin{aligned} J(y) - J(x_k) &= J(s_0 \cdot s_1 \cdot \ell) - J(s_0 \cdot s_1 \cdot c_k \cdot c_k \cdot \ldots) = \beta_0\big(J(\ell) - J(c_k \cdot c_k \cdot \ldots)\big) \\ &= \beta_0 \sum_{n=0}^{\infty} \beta_k^n (J(\ell_k) - J(c_k)), \end{aligned} \tag{8}$$

where $\beta_0 = \exp(-r(\tau + t_1))$, $\beta_k = \exp(-r\theta_k)$, and ℓ_k is the restriction of ℓ to the interval $[0, \theta_k]$. Using equation (4), we obtain

$$\begin{aligned} J(\ell_k) - J(c_k) &= \int_0^{\theta_k} (L(\xi, 0) - L(c_k(t), \dot{c}_k(t))) \exp(-rt)\, \mathrm{d}t \\ &= \int_0^{\theta_k} \big(E(\dot{c}_k(t)) - p\dot{c}_k(t) + L(\xi, \dot{c}_k(t)) - L(c_k(t), \dot{c}_k(t))\big) \exp(-rt)\, \mathrm{d}t \end{aligned}$$

As $L(x, v)$ is strictly concave, there is $p \in \mathbb{R}$ such that the excess function

$$E(v) = L(\xi, 0) + pv - L(\xi, v)$$

is positive definite as well as strictly concave: in particular, $E(v) > 0$ for all $v \neq 0$ and $E(v)$ is decreasing for $v < 0$ and increasing for $v > 0$. Introducing the indicator function $\chi_A(t)$ of A, we have for $t \in [0, \theta_k]$ that

$$E(\dot{c}_k(t)) \ge E(\mu)\chi_{F^+_{h_k^+}}(t) + E(-\mu)\chi_{F^-_{h_k^-}}(t).$$

Let $m = \min\{E(\mu), E(-\mu)\} > 0$. By integrating over a cycle c_k we obtain

$$\int_0^{\theta_k} E(\dot{c}_k(t)) \exp(-rt)\, \mathrm{d}t \ge \beta_k(E(\mu)\lambda(F^+_{h_k^+}) + E(-\mu)\lambda(F^-_{h_k^-})) \ge \beta_k m(\eta h_k^+ + \eta h_k^-) = \beta_k m \eta \theta_k.$$

We obtain that the excess function contribution to the difference (8) is bounded away from 0 by estimating

$$\beta_0 \sum_{n=0}^{\infty} \beta_k^n \int_0^{\theta_k} E(\dot{c}_k(t)) \exp(-rt)\, \mathrm{d}t \geq \beta_0 m\eta/r.$$

As in the proof of Theorem 1, the other contributions to (8) can be bounded from below by $-\varepsilon$ if k is sufficiently large. On choosing $\varepsilon > 0$ sufficiently small and k sufficiently large, we therefore obtain that $J(y) > J(x_k)$, contradicting the optimality of x_k. We conclude that a non-monotone maximiser cannot exist.

The proof for the situation that $t_2 < t_1$ is similar. □

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References

Akao, K.-I., Kamihigashi, T., & Nishimura, K. (2025). Critical capital stock in a continuous-time growth model with a convex-concave production function. *Journal of Mathematical Economics*, *119*, 103146.

Askenazy, P., & Cuong Le Van. (1999). A model of optimal growth strategy. *Journal of Economic Theory*, *85*, 24–51.

Filippov, A. (1988). *Differential equations with discontinuous righthand sides*. Springer.

Hartl, R. (1987). A simple proof of the monotonicity of the state trajectories in autonomous control problems. *Journal of Economic Theory*, *41*, 211–215.

Jaakkola, N., & Wagener, F. (2024, April). *Differential games of public investment* [Working paper].

Ngo Van Long, Nishimura, K., & Shimomura, K. (1997). Endogeneous growth, trade, and specialization under variable returns to scale: The case of a small, open economy. In B. Jensen & K. Wong (Eds.), *Dynamics, economic growth, and international trade* (pp. 127–150). University of Michigan Press.

Skiba, A. (1978). Optimal growth with a convex–concave production function. *Econometrica*, *46*, 527–539.

Vinter, R. (2000). *Optimal control*. Birkhäuser.

Wagener, F. (2003). Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics and Control*, *27*, 1533–1561.