Supplementary Material to "Estimation of the Continuous and Discontinuous Leverage Effects"

A Preliminary Technical Results

First we decompose X as $X_t = X'_t + X''_t$, where

$$\begin{aligned} X''_t &= \delta \star \mu_t = \int_0^t \int_{\mathbb{R}} \delta(s,x)\mu(ds,dx), \\ X'_t &= X_t + \int_0^t a'_s ds + \int_0^t \sigma_{s-} dW_s \quad \text{with } a'_s = a_s - \int_{|\delta(t,x)|\leq\kappa} \delta(t,x)\lambda(dx). \end{aligned} \tag{A.1}$$

Observe that X' contains no jump component hence has continuous paths almost surely, while $X''_t = \sum_{s\leq t} \Delta X_s$[13] is a well-defined pure jump process.

Henceforth, we denote by K a positive constant that may change from line to line and we write K_q in case we want to emphasize its dependency on a particular parameter q.

A.1 Localization

As shown in, for example, Jacod and Protter (2011), localization is a simple but very powerful standard procedure to prove limit theorems for discretized processes over a finite time interval. Adopting the localization procedure, it is sufficient to prove our results only under a stronger version of Assumption (H), and the same results will remain valid under the original Assumption (H). In particular, we can strengthen Assumption (H) by replacing the locally boundedness conditions by boundedness, and we only need to prove our results under the boundedness condition. More precisely, we set

[13] This equivalence is a direct result of (2.1.13) and (2.1.14) in Jacod and Protter (2011), with $\delta(s,x) \equiv x$.

Assumption (SH): We have (H) and, for some constant Λ and all (ω, t, x),

$$\left.\begin{array}{lll} |a_t(\omega)| \leq \Lambda, & |\sigma_t(\omega)| \leq \Lambda, & |X_t(\omega)| \leq \Lambda; \\ |\tilde{a}_t(\omega)| \leq \Lambda, & |\tilde{\sigma}_t(\omega)| \leq \Lambda, & |\tilde{b}_t(\omega)| \leq \Lambda; \\ \multicolumn{3}{l}{|\delta(\omega,t,x)| \leq \Lambda(\gamma(x) \wedge 1), \quad |\tilde{\delta}(\omega,t,x)| \leq \Lambda(\tilde{\gamma}(x) \wedge 1);} \\ \multicolumn{3}{l}{\text{the coefficients of } \tilde{\sigma} \text{ are also bounded by } \Lambda.} \end{array}\right\} \tag{A.2}$$

With all the above conditions satisfied, we can choose $\gamma, \tilde{\gamma} < 1$. Additionally, if we further set the truncation parameter $\kappa = 2\Lambda$, then (2.1) and (2.2) can be written in more concise forms, as follows:

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_{s-} dW_s + \delta \star (\mu - \nu)_t, \tag{A.3}$$

$$\sigma_t = \sigma_0^2 + \int_0^t \tilde{a}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{b}_s dB_s + \tilde{\delta} \star (\tilde{\mu} - \tilde{\nu})_t, \tag{A.4}$$

and consequently, by Itô's formula, for any integer $p \geq 2$,

$$\sigma_t^p = \int_0^t \tilde{a}(p)_s ds + \int_0^t p\, \sigma_{s-}^{p-1} \Big(\tilde{\sigma}_s dW_s + \tilde{b}_s dB_s \Big) + g(p) \star (\tilde{\mu} - \tilde{\nu})_t, \tag{A.5}$$

where $\tilde{a}(p)_s$ and $g(p)$ are the coefficients of the drift and jump components, respectively. To save space, we do not display their expressions here. Next, for some t_0, $s > 0$ and integer n, one can get the following result by applying Itô's formula to $Y_s = X'_{t_0+s} - X'_{t_0}$:

$$Y_s^n = \int_0^s \left(n Y_u^{n-1} a'_{t_0+u} + \frac{n(n-1)}{2} Y_u^{n-2} (\sigma_{t_0+u-}^2) \right) du + n \int_0^s Y_u^{n-1} (\sigma_{t_0+u-}) dW_{t_0+u}. \tag{A.6}$$

In what follows, we will frequently use the above two equations.

A.2 Auxiliary Results

Lemma 1. *Denote by $\widehat{\sigma}'^2$ the estimator to σ^2 based on X' instead of X. Let $u_n = n^{b \wedge (1-b)}$, where b satisfies condition (3.2). At any time t_i we have*

$$\sqrt{u_n} \left(\widehat{\sigma}'^2_{i+} - \sigma_{i+}^2, \widehat{\sigma}'^2_{i-} - \sigma_{i-}^2 \right) \xrightarrow{\mathscr{L}_{st}} (V_i^+, V_i^-), \tag{A.7}$$

where (V_i^+, V_i^-) is a vector of normal random variables independent of $\mathcal{F}$. They have zero $\mathcal{F}$-conditional covariance and

$$\mathbb{E}((V_i^{\pm})^2|\mathcal{F}) = \frac{2}{c}\sigma_{i\pm}^4 1_{\{b\in(0,1/2]\}} + \frac{c}{3}\Big(\frac{d\langle\sigma^2,\sigma^2\rangle_t}{dt}\Big|_{t=t_i^{\pm}}\Big)1_{\{b\in[1/2,1)\}}. \tag{A.8}$$

Proof. Applying Itô's formula to $f(x)=x^2$, we get the following equation, after some elementary calculations:

$$\widehat{\sigma}_{i+}^{\prime 2} - \sigma_{i+}^2 = \frac{1}{k_n\Delta_n}\sum_{j\in I_n^+(i)}\Big(2\int_{t_{j-1}^n}^{t_j^n}(X_s' - X_{t_{j-1}^n}')dX_s' + \int_{t_{j-1}^n}^{t_j^n}(\sigma_s^2 - \sigma_{i+}^2)ds\Big). \tag{A.9}$$

Step 1. We are going to show that

$$\sqrt{k_n}\,\xi_{i+}^n(1) := \frac{\sqrt{k_n}}{k_n\Delta_n}\sum_{j\in I_n^+(i)} 2\int_{t_{j-1}^n}^{t_j^n}(X_s' - X_{t_{j-1}^n}')dX_s' \xrightarrow{\mathscr{L}_{st}} \mathcal{N}(0, 2\sigma_{i+}^4).$$

First of all, by Itô's formula, it is easy to verify that

$$\xi_{i+}^{\prime\prime n}(1) := \frac{\sigma_{i+}^2}{k_n\Delta_n}\sum_{j\in I_n^+(i)} 2\int_{t_{j-1}^n}^{t_j^n}\int_{t_{j-1}^n}^{s} dW_u dW_s = \frac{\sigma_{i+}^2}{k_n}\sum_{j\in I_n^+(i)}\left(\left(\frac{\Delta_j^n W}{\sqrt{\Delta_n}}\right)^2 - 1\right),$$

where $\{\frac{\Delta_j^n W}{\sqrt{\Delta_n}}\}$ is an i.i.d. sequence of random variables, with mean and variance being 1 and 2, respectively. Moreover, note that for any martingale M, irrespective of whether $M=W$ or whether M is orthogonal to W, we have that

$$\sqrt{k_n}\,\mathbb{E}\big(\xi_{i+}^{\prime\prime n}(1)(M_{i+k+n} - M_i) \mid \mathcal{F}_i\big) = 0.$$

Consequently, one readily sees that $\sqrt{k_n}\,\xi_{i+}^{\prime\prime n}(1) \xrightarrow{\mathscr{L}_{st}} \mathcal{N}(0, 2\sigma_{i+}^4)$. Next, define

$$\xi_{i+}^{\prime n}(1) := \frac{1}{k_n\Delta_n}\sum_{j\in I_n^+(i)} 2\int_{t_{j-1}^n}^{t_j^n}\int_{t_{j-1}^n}^{s}\sigma_u\sigma_s dW_u dW_s.$$

Then,

$$\begin{aligned}
\big(\mathbb{E}(|\xi'^n_{i+}(1) - \xi''^n_{i+}(1)|)\big)^2 &\le \mathbb{E}(|\xi'^n_{i+}(1) - \xi''^n_{i+}(1)|^2) \\
&= \frac{4}{(k_n\Delta_n)^2}\sum_{j\in I_n^+(i)} \mathbb{E}\left(\int_{t_{j-1}^n}^{t_j^n}\int_{t_{j-1}^n}^{s}(\sigma_u\sigma_s - \sigma_{i+}^2)^2 du ds\right) \le K\Delta_n.
\end{aligned}$$

Therefore,

$$\limsup_n \sqrt{k_n}\,\mathbb{E}(|\xi'^n_{i+}(1) - \xi''^n_{i+}(1)|) \le \limsup_n K\sqrt{k_n\Delta_n} = 0,$$

implying that $\sqrt{k_n}(\xi'^n_{i+}(1) - \xi''^n_{i+}(1)) \xrightarrow{\mathbb{P}} 0$. What remains to be shown is that

$$\sqrt{k_n}(\xi^n_{i+}(1) - \xi'^n_{i+}(1)) \xrightarrow{\mathbb{P}} 0.$$

A sufficient condition is that

$$\limsup_n \sqrt{k_n}\mathbb{E}(|\xi^n_{i+}(1) - \xi'^n_{i+}(1)|) = 0.$$

The difference can be decomposed as follows:

$$\xi^n_{i+}(1) - \xi'^n_{i+}(1) = \frac{1}{k_n\Delta_n}\sum_{j\in I_n^+(i)} 2\int_{t_{j-1}^n}^{t_j^n}\int_{t_{j-1}^n}^{s}\big(a_s a_u du\, ds + a_s\sigma_u dW_u ds + \sigma_s a_u du\, dW_s\big).$$

Note that when the drift coefficient of X' is zero, $\xi^n_{i+}(1) = \xi'^n_{i+}(1)$. So, the above sufficient condition amounts to requiring that the drift term does not affect the asymptotic distribution of $\xi^n_{i+}(1)$. Intuitively, in the high-frequency setting, the magnitude of the drift is much smaller compared to the diffusion, hence is negligible. Formally, we have

$$k_n\mathbb{E}(|\xi^n_{i+}(1) - \xi'^n_{i+}(1)|^2) \le \frac{Kk_n}{(k_n\Delta_n)^2}\left(\sum_{j\in I_n^+(i)}\Delta_n^3 + \sum_{j,l\in I_n^+(i)}\Delta_n^4\right) = K\Delta_n(1 + k_n\Delta_n).$$

Then the desired result follows immediately.

Step 2. Now we turn to the limiting behavior of the second part of (A.9). The result to be proved is

$$\frac{1}{\sqrt{k_n\Delta_n}}\,\xi^n_{i+}(2) := \frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I^+_n(i)}\int_{t^n_{j-1}}^{t^n_j}(\sigma^2_s-\sigma^2_{i+})ds \xrightarrow{\mathscr{L}_{st}} \mathcal{N}(0,1)\cdot\sqrt{\frac{1}{3}\frac{d\langle\sigma^2,\sigma^2\rangle_t}{dt}\Big|_{t=t^\pm_i}}.$$

Different from the previous step, the (volatility) jump component is involved in the integrand above. Recall (A.5) and denote $\theta = g(2)$ for brevity. For any $\epsilon\in(0,1]$, we have $\theta\star(\tilde\mu-\tilde\nu) = A(\epsilon)+B(\epsilon)+C(\epsilon)$, where

$$A(\epsilon) = (\theta 1_{\{|\theta|>\epsilon\}})\star\tilde\mu,\quad B(\epsilon) = (\theta 1_{\{|\theta|\le\epsilon\}})\star(\tilde\mu-\tilde\nu),\quad C(\epsilon) = -(\theta 1_{\{|\theta|>\epsilon\}})\star\tilde\nu.$$

By the localization procedure, we can assume that $|\theta(\omega,t,z)|\le\Gamma(z)$. Let

$$\tilde\gamma_\epsilon = \int_{\Gamma(z)\le\epsilon}\Gamma(z)^v\tilde\lambda(dz),$$

where the constant $v\in[0,2]$ controls the jump activity of the volatility process. We are going to give some estimates of the magnitude of these three processes. First, note that $\Delta^n_j A(\epsilon)$ must be zero if there is no jump larger than ϵ over the interval $((j-1)\Delta_n, j\Delta_n)$. Hence,

$$\begin{aligned}\mathbb{P}^n_{j-1}(\Delta^n_j A(\epsilon)\neq 0) &\le \mathbb{P}^n_{j-1}\big((\Delta^n_j(1_{\{|\Gamma|>\epsilon\}}\star\tilde\mu)>0\big) = \mathbb{E}^n_{j-1}\big(\Delta^n_j(1_{\{|\Gamma|>\epsilon\}}\star\tilde\mu)\big)\\ &= \Delta_n\int_{\Gamma>\epsilon}\tilde\lambda(dz)\le\Delta_n\int_{\Gamma>\epsilon}\frac{\Gamma(z)^v}{\epsilon^v}\tilde\lambda(dz)\le K\Delta_n\epsilon^{-v},\end{aligned}\tag{A.10}$$

where the last inequality results from the assumption that $\int\Gamma(z)^v\tilde\lambda(dz)<\infty$. Second, by the BDG inequality and the property of Γ, we obtain

$$\mathbb{E}^n_{j-1}((\Delta^n_j B(\epsilon))^2)\le\Delta_n\int_{\Gamma\le\epsilon}\Gamma(z)^v\,\Gamma(z)^{2-v}\,\tilde\lambda(dz)\le\Delta_n\tilde\gamma_\epsilon\epsilon^{2-v}.\tag{A.11}$$

Third, if $v\le 1$, i.e., the volatility jump component has finite variation,

$$\mathbb{E}^n_{j-1}(|\Delta^n_j C(\epsilon)|)\le K\Delta_n.$$

Otherwise, we have

$$|\theta| 1_{\{|\theta|>\epsilon\}} = \epsilon \, \frac{|\theta|}{\epsilon} 1_{\{\epsilon\leq|\theta|\leq 1\}} + |\theta| 1_{\{|\theta|>1\}} \leq \epsilon \left(\frac{|\theta|}{\epsilon}\right)^{v} 1_{\{\epsilon\leq|\theta|\leq 1\}} + |\theta| 1_{\{|\theta|>1\}}.$$

Notice that $\theta 1_{\{|\theta|>1\}} \star \tilde{\nu}$ is the compensator of a compound Poisson process, hence it is of integrable variation. Consequently, we get

$$\mathbb{E}^n_{j-1}(|\Delta^n_j C(\epsilon)|) \leq \Delta_n \left(\epsilon^{1-v} \int_{\epsilon\leq|\Gamma|\leq 1} \Gamma(z)^v \tilde{\lambda}(dz) + \int_{\Gamma>1} \Gamma(z)\tilde{\lambda}(dz)\right) \leq K\Delta_n \epsilon^{1-v}.$$

Together, we have

$$\mathbb{E}^n_{j-1}(|\Delta^n_j C(\epsilon)|) \leq K\Delta_n \epsilon^{-(v-1)^+}. \tag{A.12}$$

Now we can rewrite the σ^2 process as

$$\sigma^2_t = \sigma^2_0 + \int_0^t \tilde{a}(2)_s ds + \int_0^t 2\,\sigma_{s-}\Big(\tilde{\sigma}_s dW_s + \tilde{b}_s dB_s\Big) + A(\epsilon)_t + B(\epsilon)_t + C(\epsilon)_t.$$

By a similar argument as in the previous step, we can easily obtain

$$\begin{aligned}\frac{1}{\sqrt{k_n\Delta_n}}\,\xi'^n_{i+}(2) &:= \frac{1}{(k_n\Delta_n)^{3/2}} \sum_{j\in I^+_n(i)} \int_{t^n_{j-1}}^{t^n_j} \int_{t^n_i}^{s} 2\,\sigma_{u-}\Big(\tilde{\sigma}_u dW_u + \tilde{b}_u dB_u\Big) ds \\ &\xrightarrow{\mathscr{L}_{st}} \mathcal{N}(0,1)\cdot \sqrt{\frac{1}{3}\frac{d\langle\sigma^2,\sigma^2\rangle_t}{dt}\Big|_{t=t_i^{\pm}}},\end{aligned}$$

and

$$\frac{1}{(k_n\Delta_n)^{3/2}} \sum_{j\in I^+_n(i)} \int_{t^n_{j-1}}^{t^n_j} \int_{t^n_{j-1}}^{s} b_u du\; ds \xrightarrow{\mathbb{P}} 0.$$

In particular, note that when a martingale M satisfies $M = W$ or $M = B$, we have

$$\frac{1}{\sqrt{k_n\Delta_n}}\,\mathbb{E}\big(\xi'^n_{i+}(2)(M_{i+k_n} - M_i) \mid \mathcal{F}_i\big) = O_p(\sqrt{k_n\Delta_n}) \to 0.$$

When M is orthogonal to W and B, the above conditional expectation is zero. What is left now to be proved is that

$$\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\left(\theta\star(\tilde{\mu}-\tilde{\nu})_s-\theta\star(\tilde{\mu}-\tilde{\nu})_{i+}\right)ds\xrightarrow{\mathbb{P}}0.$$

Recall that by localization, the spot variance is bounded, so is its jump part. Hence, (A.10) yields

$$\begin{aligned}&\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\mathbb{E}|A(\epsilon)_s-A(\epsilon)_{i+}|ds\\ \leq&\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}Kk_n\Delta_n\epsilon^{-v}ds=K\sqrt{k_n\Delta_n}\epsilon^{-v}.\end{aligned}$$

Additionally, (A.11) and (A.12) yield

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\left(B(\epsilon)_{s\wedge u}-B(\epsilon)_{i+}\right)ds\right)^2\\ \leq&\frac{1}{(k_n\Delta_n)^3}\sum_{j\in I_n^+(i)}\sum_{l\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\int_{t_{l-1}^n}^{t_l^n}\mathbb{E}|B(\epsilon)_{s\wedge u}-B(\epsilon)_{i+}|^2ds\,du\\ \leq&\frac{1}{(k_n\Delta_n)^3}\sum_{j\in I_n^+(i)}\sum_{l\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\int_{t_{l-1}^n}^{t_l^n}Kk_n\Delta_n\tilde{\gamma}_\epsilon\epsilon^{2-v}ds=K\tilde{\gamma}_\epsilon\epsilon^{2-v},\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\mathbb{E}|C(\epsilon)_s-C(\epsilon)_{i+}|ds\\ \leq&\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}Kk_n\Delta_n\epsilon^{-(v-1)^+}ds=K\sqrt{k_n\Delta_n}\epsilon^{-(v-1)^+}.\end{aligned}$$

To sum up, we have

$$\begin{aligned}&\lim_{\epsilon\to0}\limsup_n\frac{1}{(k_n\Delta_n)^{3/2}}\sum_{j\in I_n^+(i)}\int_{t_{j-1}^n}^{t_j^n}\mathbb{E}|\theta\star(\tilde{\mu}-\tilde{\nu})_s-\theta\star(\tilde{\mu}-\tilde{\nu})_{i+}|ds\\ \leq&\lim_{\epsilon\to0}\limsup_n K(\sqrt{k_n\Delta_n}\epsilon^{-v}+\sqrt{\gamma_\epsilon\epsilon^{2-v}}+\sqrt{k_n\Delta_n}\epsilon^{-(v-1)^+})=0.\end{aligned}$$

Then the desired result readily follows.

Step 3. One can verify that the previous results hold for $\xi^n_{i-}(1)$ and $\xi^n_{i-}(2)$ too, by similar arguments. Furthermore, the pairwise covariance of $\sqrt{k_n}\xi^n_{i+}(1)$, $\sqrt{k_n}\xi^n_{i-}(1)$, $\xi^n_{i+}(2)/\sqrt{k_n\Delta_n}$ and $\xi^n_{i-}(2)/\sqrt{k_n\Delta_n}$ vanishes asymptotically. Hence, by letting $u_n = n^{b\wedge(1-b)} \propto \sqrt{k_n} \wedge \frac{1}{\sqrt{k_n\Delta_n}}$, one gets the joint convergence stated in the current lemma. $\square$

Lemma 2. *For $i \neq j$ and $|i-j| \leq k_n$, we have*

$$\mathbb{E}_{i\wedge j-k_n}\big((\Delta^n_i X')(\Delta^n_j X')R_iR_j\big) = O_p(\Delta_n^2), \tag{A.13}$$

where R_i is one of $\{\widehat{\sigma}'^2_{i+}, \widehat{\sigma}'^2_{i-}, \Delta^n_i\sigma^2\}$ and R_j is one of $\{\widehat{\sigma}'^2_{j+}, \widehat{\sigma}'^2_{j-}, \Delta^n_j\sigma^2\}$.

Proof. First of all, it is easy to verify the result when both $R_i = \Delta^n_i\sigma^2$ and $R_j = \Delta^n_j\sigma^2$. Next, when $R_\cdot \neq \Delta^n_\cdot\sigma^2$, it amounts to proving that

$$\mathbb{E}_{i\wedge j-k_n}\big((\Delta^n_i X')(\Delta^n_j X')(\Delta^n_u X')^2(\Delta^n_v X')^2\big) = O_p(\Delta_n^4),$$

where $u \in I^\pm_n(i)$ and $v \in I^\pm_n(j)$. The explicit expression on the right-hand side depends on the relative order of i, j, u, v and whether $u = v$. But its order with regard to Δ_n remains the same in all cases. To save space, we just show the calculation in one typical case. Let $i < j < u < v$, and denote $(i-k_n)\Delta_n$ as τ_0. We have

$$\begin{aligned}
&\mathbb{E}_{\tau_0}\big((\Delta^n_i X')(\Delta^n_j X')(\Delta^n_u X')^2(\Delta^n_v X')^2\big)\\
&= \mathbb{E}_{\tau_0}\big((\Delta^n_i X')(\Delta^n_j X')(\Delta^n_u X')^2(\sigma^2_{t^n_{v-1}-})\big)\Delta_n(1+o_p(1))\\
&= \mathbb{E}_{i-k_n}\big((\Delta^n_j X')(\Delta^n_l X')(\sigma^4_{t^n_{u-1}-})\big)\Delta_n^2(1+o_p(1))\\
&= \mathbb{E}_{\tau_0}\big((\Delta^n_i X')(\sigma^4_{t^n_{j-1}-})(a'_{t^n_{j-1}} + 4\tilde{\sigma}_{t^n_{j-1}})\big)\Delta_n^3(1+o_p(1))\\
&= (\sigma^4_{\tau_0-})\Big(\big((a'_{\tau_0})^2 + \Gamma^{(X',a)}_{\tau_0} + 4a'_{\tau_0}\tilde{\sigma}_{\tau_0}\big) + 4\big(a'_{\tau_0}\tilde{\sigma}_{\tau_0} + \Gamma^{(X',\tilde{\sigma})}_{\tau_0} + 4\tilde{\sigma}^2_{\tau_0}\big)\Big)\Delta_n^4(1+o_p(1)),
\end{aligned}$$

where $\Gamma^{(X',Y)}_t := \mathbb{E}_t\big(d\langle X', Y\rangle_t/dt\big)$. Finally, the proof when only one of R_i and R_j equals $\Delta^n_i\sigma^2$ or $\Delta^n_j\sigma^2$ is similar. We omit the details. $\square$

B Proofs of the Main Theorems

B.1 Continuous Leverage Effect

Using the localization procedure, we can and will assume (SH). First of all, we decompose the volatility process into three parts, as follows:

$$\sigma_t^2 = \sigma_t^{2,c} + \sigma_t^{2,j} + \sigma_t^{2,d},$$

where $\sigma_t^{2,c}$ is the continuous part, $\sigma_t^{2,j}$ is the co-jump part and $\sigma_t^{2,d}$ is the disjoint jump part. Next, we decompose the estimation error as follows:

$$\widehat{[X,\sigma^2]}_t^C - [X,\sigma^2]_t^C = T(\alpha_n)_t^n + V_t^n + D(1)_t^n + D(2)_t^n + D(3)_t^n, \tag{B.1}$$

where

$$\begin{aligned}
T(\alpha_n)_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \left(\Delta_i^n X_{\alpha_n} (\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2) - \Delta_i^n X' (\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2) \right), \\
V_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \left(\Delta_i^n X' (\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2) - \Delta_i^n X' \Delta_i^n \sigma^2 \right), \\
D(1)_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \Delta_i^n X' \big(\Delta_i^n \sigma^{2,d} + \Delta_i^n \sigma^{2,j} \big), \\
D(2)_t^n &= -\sum_{i=1}^{k_n} \Delta_i^n X' \Delta_i^n \sigma^2 - \sum_{i=\lfloor t/\Delta_n \rfloor - k_n + 1}^{\lfloor t/\Delta_n \rfloor} \Delta_i^n X' \Delta_i^n \sigma^2, \\
D(3)_t^n &= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_i^n X' \Delta_i^n \sigma^{2,c} - \int_0^t 2\, \sigma_{s-}^2 \tilde{\sigma}_s dt.
\end{aligned}$$

B.1.1 Proof of Theorem 1

In fact, from the subsequent proof of Theorem 3, we have

$$\begin{cases} V_t^n = O_p(1/\sqrt{u_n}), \\ D(j)_t^n = o_p(1/\sqrt{u_n}), \end{cases}$$

implying that they all converge to zero in probability as n goes to infinity. In order to prove $T(\alpha_n)_t^n \xrightarrow{\mathbb{P}} 0$, it is sufficient to prove that

$$\mathbb{E}\big(\big|\Delta_i^n X_{\alpha_n}(\Delta_j^n X_{\alpha_n})^2 - \Delta_i^n X'(\Delta_j^n X')^2\big|\big) = \psi_n \Delta_n^2,$$

instead of $\psi_n \Delta_n^2/\sqrt{u_n}$. Consequently, it is sufficient to have $(2-r)\varpi \geq 1/2$, which amounts to $\varpi \geq \frac{1}{2(2-r)}$.

B.1.2 Proof of Theorem 3

We prove the central limit theory first. In words, we are going to show that the properly scaled truncation error and discretization error converge to zero in probability, while the properly scaled volatility estimation error converges stably in law to the limiting process.

Step 1. To prove $\sqrt{u_n}\, T(\alpha_n)_t^n \xrightarrow{u.c.p.} 0$, it suffices to show that

$$\limsup_{n\to\infty} \sqrt{u_n}\, \mathbb{E}(|T(\alpha_n)_t^n|) = 0.$$

Since

$$\begin{aligned} \sqrt{u_n}\, |T(\alpha_n)_t^n| &= \sqrt{u_n}\Big| \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \Big(\Delta_i^n X_{\alpha_n}(\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2) - \Delta_i^n X'(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2)\Big)\Big| \\ &\leq \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \sum_{j\in I_n^{\pm(i)}} \frac{\sqrt{u_n}}{k_n\Delta_n} \left|\Delta_i^n X_{\alpha_n}(\Delta_j^n X_{\alpha_n})^2 - \Delta_i^n X'(\Delta_j^n X')^2\right|, \end{aligned}$$

it is sufficient to prove that

$$\mathbb{E}\Big(\big|\Delta_i^n X_{\alpha_n}(\Delta_j^n X_{\alpha_n})^2 - \Delta_i^n X'(\Delta_j^n X')^2\big|\Big) = \psi_n \Delta_n^2/\sqrt{u_n},$$

where ψ_n converges to zero as n goes to infinity.

Consider the function $F(x_i, x_j) = x_i(x_j)^2$ and the following vectors:

$$\widetilde{X}_{i,j}^n(1) = \frac{1}{\sqrt{\Delta_n}}\left(\Delta_i^n X, \Delta_j^n X\right),$$
$$\widetilde{X}_{i,j}^n(2) = \frac{1}{\sqrt{\Delta_n}}\left(\Delta_i^n X, \Delta_j^n X'\right),$$
$$\widetilde{X}_{i,j}^n(3) = \frac{1}{\sqrt{\Delta_n}}\left(\Delta_i^n X', \Delta_j^n X'\right).$$

For simplicity, denote $\alpha_n/\sqrt{\Delta_n}$ by v_n and define

$$\phi_{i,j}^n(1) = F_{v_n}\big(\widetilde{X}_{i,j}^n(1)\big) - F_{v_n}\big(\widetilde{X}_{i,j}^n(2)\big),\ \phi_{i,j}^n(2) = F_{v_n}\big(\widetilde{X}_{i,j}^n(2)\big) - F_{v_n}\big(\widetilde{X}_{i,j}^n(3)\big),$$

where $F_v(x_1, x_2) = F(x_1, x_2)\prod_{i=1}^2 1_{\{|x_i|\le v\}}$. Then we have

$$\Delta_i^n X_{\alpha_n}(\Delta_j^n X_{\alpha_n})^2 - \Delta_i^n X'(\Delta_j^n X')^2 = \left(\phi_{i,j}^n(1) + \phi_{i,j}^n(2)\right)\Delta_n^{3/2}.$$

A look at the proof of Lemma 13.2.6 in Jacod and Protter (2011) shows that it does not make an essential difference to use vectors of non-adjacent increments. Therefore, applying that lemma with $r < 1, m = s = 1, s' = p' = 2$, yields, for $l = 1, 2$,

$$\mathbb{E}_{\tau_{i-1}^n}\left(|\phi_{i,j}^n(l)|\right) \le \Big(\Delta_n^{\frac{2-r}{2}} + \Delta_n^{(2-r)\varpi}\Big)\psi_n = \psi_n \Delta_n^{(2-r)\varpi}\left(1 + \Delta_n^{(2-r)(1/2-\varpi)}\right).$$

Thus, it is sufficient to have $(2-r)\varpi \ge 3/4$, which amounts to $\varpi \ge \frac{3}{4(2-r)}$.

Step 2. We are going to show that $\sqrt{u_n}\ D(1)_t^n$ is asymptotically negligible. First, notice that the quadratic covariation between a continuous semimartingale and a pure jump process is identically zero. Therefore, we have

$$D(1)_t^n \xrightarrow{\mathbb{P}} [X', \sigma^{2,d} + \sigma^{2,c}]_t = 0.$$

Next, from Theorem 5.4.2[14] in Jacod and Protter (2011), we know that

$$D(1)^n_t = O_p(\sqrt{\Delta_n}).$$

Then it readily follows that

$$\limsup_n \sqrt{u_n}\, \mathbb{E}\big(|D(1)^n_t|\big) \leq K\sqrt{u_n \Delta_n} \longrightarrow 0.$$

Hence, $\sqrt{u_n}D(1)^n_t$ is asymptotically negligible.

Step 3*.* In this step, we are going to prove the following result for $j = 2, 3$:

$$\sqrt{u_n}\; D(j)^n_t \xrightarrow{u.c.p.} 0.$$

By the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\big(|\Delta^n_i X' \Delta^n_i \sigma^2|\big) \leq \sqrt{\mathbb{E}\big((\Delta^n_i X')^2\big)\; \mathbb{E}\big((\Delta^n_i \sigma^2)^2\big)} \leq K\Delta_n.$$

Then $\sqrt{u_n}\; D(2)^n_t \xrightarrow{u.c.p.} 0$ readily follows from the fact that

$$\limsup_{n\to\infty} \sqrt{u_n}\; \mathbb{E}(|D(2)^n_t|) \leq \limsup_{n\to\infty} K\sqrt{u_n} k_n \Delta_n = 0.$$

Next, Itô's formula yields

$$\begin{aligned}(X'_{t+s} - X'_t)(\sigma^{2,c}_{t+s} - \sigma^{2,c}_t) = &\int_0^s 2\sigma^2_{t+r-}\tilde{\sigma}_{t+r} dr + \int_0^s (X'_{t+r} - X'_t) d\sigma^2_{t+r} \\ &+ \int_0^s (\sigma^{2,c}_{t+r} - \sigma^{2,c}_t) dX'_{t+r}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\mathbb{E}_{i-1}\big(\Delta^n_i X' \Delta^n_i \sigma^{2,c} - \int_{t^n_{i-1}}^{t^n_i} 2\,\sigma^2_{t-}\tilde{\sigma}_t dt\big) &= K\Delta_n^{3/2}, \\ \mathbb{E}_{i-1}\big((\Delta^n_i X' \Delta^n_i \sigma^{2,c} - \int_{t^n_{i-1}}^{t^n_i} 2\,\sigma^2_{t-}\tilde{\sigma}_t dt)^2\big) &\leq K\Delta_n^2.\end{aligned}$$

[14] The assumptions of this theorem are satisfied in the current setting. Also note that this theorem applies to multi-dimensional semimartingales, hence can be used here.

Let $\zeta_i^n = \sqrt{u_n}(\Delta_i^n X' \Delta_i^n \sigma^{2,c} - \int_{t_{i-1}^n}^{t_i^n} 2\,\sigma_{t-}^2 \tilde{\sigma}_t dt)$. Note that ζ_i^n is $\mathcal{F}_{t_i^n}$-measurable. Then the above equations yield

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}_{i-1}\big(\zeta_i^n\big) = K\sqrt{u_n \Delta_n} \longrightarrow 0,$$
$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}_{i-1}\big((\zeta_i^n)^2\big) \leq K\ u_n \Delta_n \longrightarrow 0.$$

Hence, Lemma 4.1 in Jacod (2012) implies that $\sqrt{u_n}\ D(3)_t^n \xrightarrow{u.c.p.} 0$.

Step 4. In this step we analyze $\sqrt{u_n}\ V_t^n$. Define

$$\xi_i^n = \sqrt{u_n}\Big(\Delta_i^n X'(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2) - \Delta_i^n X' \Delta_i^n \sigma^2\Big).$$

The variable ξ_i^n has a vanishing $\mathcal{F}_{(i-1)\Delta_n}$-conditional expectation, but it is not $\mathcal{F}_{i\Delta_n}$-measurable. To induce "some conditional independence" of the successive summands, we split the sum over i into big blocks of size $\tilde{m}k_n$ ($\tilde{m}$ will eventually go to infinity, to ensure that the summation over these big blocks is asymptotically equivalent to the summation over all blocks), separated by small blocks of size $2k_n$; cf. Section 12.2.4 of Jacod and Protter (2011). The condition on $\tilde{m}$ is:

$$\tilde{m} \to \infty \ \text{ and } \ \tilde{m}k_n\Delta_n \to 0.$$

More specifically, define $I(\tilde{m}, n, l) = (l-1)(\tilde{m}+2)k_n + 1$. Then the l-th big block contains ξ_i^n for all i between $I(\tilde{m}, n, l) + k_n + 1$ and $I(\tilde{m}, n, l) + (\tilde{m}+1)k_n$, and the total number of such blocks is $l_n(\tilde{m}, t) = \lfloor \frac{\lfloor t/\Delta_n \rfloor - 1}{(\tilde{m}+2)k_n} \rfloor$. Let

$$\xi(\tilde{m})_i^n = \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \xi_{I(\tilde{m},n,i)+r}, \qquad Z(\tilde{m})_t^n = \sum_{i=1}^{l_n(\tilde{m},t)} \xi(\tilde{m})_i^n,$$
$$\widetilde{\xi}(\tilde{m})_i^n = \sum_{r=-k_n}^{k_n} \xi_{I(\tilde{m},n,i)+r}, \qquad \widetilde{Z}(\tilde{m})_t^n = \sum_{i=2}^{l_n(\tilde{m},t)} \widetilde{\xi}(\tilde{m})_i^n.$$

So $\sqrt{u_n}\, V_t^n = Z(\tilde{m})_t^n + \widetilde{Z}(\tilde{m})_t^n$. We are going to show that $\widetilde{Z}(\tilde{m})_t^n$ is asymptotically negligible first. By successive conditioning, we get

$$\begin{aligned}
\mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\big(\xi_{I(\tilde{m},n,i)+r}\big) &= \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\left(\Delta_i^n X' \sqrt{u_n}\big((\widehat{\sigma}_{i+}'^2 - \sigma_i^2) - (\widehat{\sigma}_{i-}'^2 - \sigma_{i-1}^2)\big)\right) \\
&= \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\left(\Delta_i^n X' O_p(\psi_n) + O_p(\Delta_n)\sqrt{u_n}(\widehat{\sigma}_{i-}'^2 - \sigma_{i-1}^2)\right) \\
&= O_p(\psi_n \Delta_n), \\
\mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\big((\xi_{I(\tilde{m},n,i)+r})^2\big) &= \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\left((\Delta_i^n X')^2 u_n\big((\widehat{\sigma}_{i+}'^2 - \sigma_i^2) - (\widehat{\sigma}_{i-}'^2 - \sigma_{i-1}^2)\big)^2\right) \\
&= \rho_{I(\tilde{m},n,i)-k_n-1}\Delta_n + o_p(\Delta_n).
\end{aligned}$$

As before, $\psi_n \to 0$, and it may change from line to line. Observe that, for any given n and $\tilde{m}$, there is no overlap among the sequence $\widetilde{\xi}(\tilde{m})_i^n$. Then it is easy to verify that

$$\mathbb{E}_{I(\tilde{m},n,i)-k_n-1}(\widetilde{\xi}(\tilde{m})_i^n) = O_p(k_n \Delta_n \psi_n),$$

and

$$\begin{aligned}
\mathbb{E}_{I(\tilde{m},n,i)-k_n-1}(\widetilde{\xi}(\tilde{m})_i^n)^2 = \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}&\left(\sum_{r=-k_n}^{k_n} (\xi_{I(\tilde{m},n,i)+r})^2\right) \\
&+ \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\left(\sum_{r,j=-k_n}^{k_n} 1_{\{j\neq r\}} \xi_{I(\tilde{m},n,i)+r}\xi_{I(\tilde{m},n,i)+j}\right) \\
= O_p(k_n\Delta_n) + O_p(k_n^2\Delta_n^2).&
\end{aligned}$$

The first term on the right-hand side of the expression above is $O_p(k_n\Delta_n)$. For the second term, when $j > r$, by successive conditioning, we get

$$\begin{aligned}
&\mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\big(\xi_{I(\tilde{m},n,i)+r}\xi_{I(\tilde{m},n,i)+j}\big) \\
&= \mathbb{E}_{I(\tilde{m},n,i)-k_n-1}\big(\Delta_r^n X' O_p(1)\, \mathbb{E}_{I(\tilde{m},n,i)+j-1}(\Delta_j^n X' O_p(1))\big),
\end{aligned}$$

where $O_p(1)$ comes from the standardized estimation error of spot volatility. Irrespective of whether $\Delta_j^n X'$ is correlated with its associated $O_p(1)$ or not, the conditional expectation of their product is

$O_p(\Delta_n)$. The same argument applies to $\Delta_r^n X'$. Hence, the above result readily follows. Then, as long as $\tilde{m}$ goes to infinity, Lemma 4.1 in Jacod (2012) yields that $\widetilde{Z}(\tilde{m})_t^n \xrightarrow{u.c.p.} 0$.

Next, notice that the variable $\xi(\tilde{m})_i^n$ has vanishing $\mathcal{F}_{I(\tilde{m},n,i)}$-conditional expectation, and is $\mathcal{F}_{I(\tilde{m},n,i+1)}$-measurable. That is, it behaves like a martingale difference. We are going to prove that

$$\begin{cases} \sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\big(\xi(\tilde{m})_i^n\big)^2 \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} \int_0^t \eta_s^2 ds, \\ \sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\big(\xi(\tilde{m})_i^n\big)^4 \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} 0, \\ \sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\xi(\tilde{m})_i^n \ \Delta_{i,\tilde{m}}^n M \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} 0, \end{cases} \tag{B.2}$$

where $\Delta_{i,\tilde{m}}^n M = M_{(I(\tilde{m},n,i)+(\tilde{m}+1)k_n)\Delta_n} - M_{(I(\tilde{m},n,i)+k_n+1)\Delta_n}$. The remaining difficulty is that the ξ_i^n may have overlaps within the big block. To deal with this, recall that $I_n^-(i) = \{i-k_n, \ldots, i-1\}$ if $i > k_n$ and $I_n^+(i) = \{i+1, \ldots, i+k_n\}$, which define two local windows of length $k_n\Delta_n$ just before and after the time point $i\Delta_n$. Let $I_n^{\pm}(i)$ be the union of them. Furthermore, let

$$J(\tilde{m},n,i,j) = \{I(n,\tilde{m},i)+k_n+1, \ldots, I(\tilde{m},n,i)+(\tilde{m}+1)k_n\} \setminus (I_n^{\pm}(j) \cup j).$$

With these notations, we can decompose the conditional second moment of $\xi(\tilde{m})_i^n$, as follows:

$$\begin{aligned} &\mathbb{E}\Big(\big(\xi(\tilde{m})_i^n\big)^2 \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \\ &= \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j=k_n+1}^{(\tilde{m}+1)k_n} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r} \ \xi_{I(\tilde{m},n,i)+j} | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\ &= \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j=r} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r} \ \xi_{I(\tilde{m},n,i)+j} | \mathcal{F}_{I(\tilde{m},n,i)}\big) + \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\in I_n^{\pm}(r)} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r} \ \xi_{I(\tilde{m},n,i)+j} | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\ &\quad + \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\in J(\tilde{m},n,i,r)} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r} \ \xi_{I(\tilde{m},n,i)+j} | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\ &=: H(\tilde{m},1)_i^n + H(\tilde{m},2)_i^n + H(\tilde{m},3)_i^n. \end{aligned}$$

From Lemmas 1 and 2, we have

$$\sum_{i=1}^{l_n(\tilde{m},t)} H(\tilde{m},1)_i^n = \sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \mathbb{E}\big((\xi_{I(\tilde{m},n,i)+r})^2|\mathcal{F}_{I(\tilde{m},n,i)}\big) \xrightarrow{\mathbb{P}} \int_0^t \eta_s^2 ds,$$

$$\begin{aligned}\sum_{i=1}^{l_n(\tilde{m},t)} H(\tilde{m},2)_i^n &= \sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\in I_n(r)} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r}\ \xi_{I(\tilde{m},n,i)+j}|\mathcal{F}_{I(\tilde{m},n,i)}\big) \\ &\leq \sum_{i=1}^{l_n(\tilde{m},t)} K\tilde{m}k_n^2\Delta_n^2 \to 0.\end{aligned}$$

Next, notice that, when $j \in J(\tilde{m},n,i,r)$, there is no overlap between $\xi_{I(\tilde{m},n,i)+r}$ and $\xi_{I(\tilde{m},n,i)+j}$. Hence, by successive conditioning, we obtain

$$\begin{aligned}\sum_{i=1}^{l_n(\tilde{m},t)} H(\tilde{m},3)_i^n &= \sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\in J(\tilde{m},n,i,r)} \mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r}\ \xi_{I(\tilde{m},n,i)+j}|\mathcal{F}_{I(\tilde{m},n,i)}\big) \\ &\leq \sum_{i=1}^{l_n(\tilde{m},t)} K\tilde{m}^2k_n^2\Delta_n^2\psi_n \to 0.\end{aligned}$$

The calculation of the fourth moments is even more tedious; we present partial results and omit the remainder of the calculations for brevity:

$$
\begin{aligned}
&\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \mathbb{E}\big((\xi_{I(\tilde{m},n,i)+r})^4 | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\
= &\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \mathbb{E}\big((\Delta^n_{I(\tilde{m},n,i)+r} X')^4 O_p(1) | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\
\leq &\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} K\Delta_n^2 = Kt\Delta_n \to 0, \\
&\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\neq r} \mathbb{E}\big((\xi_{I(\tilde{m},n,i)+r})^2 \ (\xi_{I(\tilde{m},n,i)+j})^2 | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\
= &\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \sum_{j\neq r} \mathbb{E}\big((\Delta^n_{I(\tilde{m},n,i)+r} X')^2 (\Delta^n_{I(\tilde{m},n,i)+j} X')^2 O_p(1) | \mathcal{F}_{I(\tilde{m},n,i)}\big) \\
\leq &\sum_{i=1}^{l_n(\tilde{m},t)} K\tilde{m}^2 k_n^2 \Delta_n^2 = Kt\tilde{m}k_n\Delta_n \to 0.
\end{aligned}
$$

As for the last equation in (B.2), we first note that it holds when M is orthogonal to W and B. Besides, when $M = W$ or $M = B$, according to the proof of Lemma 1, one can verify by successive conditioning that

$$
\sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\xi(\tilde{m})_i^n \ \Delta^n_{i,\tilde{m}} M \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \leq O_p(\sqrt{k_n \Delta_n}) \xrightarrow{\mathbb{P}} 0.
$$

In any other case, M can be decomposed into the sum of two components, one driven by W and B and the other orthogonal to B and W. Thus the result readily follows.

Finally, as for the consistency of $\widehat{V}^n_T(X, \alpha_n)$, first note it has been well established that

$$
\frac{4\Delta_n}{15} \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} \Big(\frac{\Delta_i^n X_{\alpha_n}}{\sqrt{\Delta_n}}\Big)^6 \xrightarrow{\mathbb{P}} \frac{1}{c} \int_0^t \sigma^6_{s-}\, ds.
$$

On the other hand, Aït-Sahalia and Jacod (2014) proved that

$$\frac{1}{k_n}\sum_{j=k_n+1}^{\lfloor t/\Delta_n\rfloor-k_n}\left(\frac{3}{2}\big(\widehat{\sigma}^2_{j+}-\widehat{\sigma}^2_{j-}\big)^2-\frac{6}{k_n}\widehat{\sigma}^4_j\right)\xrightarrow{\mathbb{P}}\langle\sigma^2,\sigma^2\rangle_t.$$

Based on this result, one can show that

$$\frac{c}{k_n}\sum_{j=k_n+1}^{\lfloor t/\Delta_n\rfloor-k_n}\big(\Delta^n_j X_{\alpha_n}\big)^2\left(\big(\widehat{\sigma}^2_{j+}-\widehat{\sigma}^2_{j-}\big)^2-\frac{2}{3(k_n\Delta_n)^2}\sum_{l\in I^{\pm}_n(j)}(\Delta^n_l X)^4\right)$$
$$\xrightarrow{\mathbb{P}}\frac{2c}{3}\int_0^t\sigma^2_{s-}\,d\langle\sigma^2,\sigma^2\rangle_s.$$

Therefore, $\widehat{V}^n_t(X,\alpha_n)$ is a consistent estimator of the asymptotic variance $\int_0^t\eta^2_s ds$. This completes the proof.

B.2 Discontinuous Leverage Effect

B.2.1 Proof of Theorem 2

The results of Theorem 2 can be seen to follow as a corollary to Theorem 3.1 in Jacod and Todorov (2010), by verifying that our case (i) satisfies condition (c) and our case (ii) satisfies condition (a) of Theorem 3.1 in Jacod and Todorov (2010), with our particular choice of F.

B.2.2 Proof of Theorem 4

For each integer $m\geq 1$, let $(S(m,q):q\geq 1)$ be the successive jump times of the counting process

$$\mu\left([0,t]\times\left\{x:\frac{1}{m}<\gamma(x)\leq\frac{1}{m-1}\right\}\right).$$

In fact, the two-parameter sequence $(S(m,q) : m, q \geq 1)$ can be relabeled in such a way that it becomes a single sequence $(T_p : p \geq 1)$, which exhausts the jumps of X. Furthermore, we define

$$
\begin{gathered}
\mathcal{T}_m^t = \{p : \exists\, p \geq 1 \text{ and } m' \in \{1, \ldots, m\} \;\; \text{s.t.} \;\; T_p = S(m', q) \leq \lfloor t/\Delta_n \rfloor \Delta_n\}; \\
i(n,p) = \text{the unique integer such that } T_p \in (t_{i-1}^n, t_i^n]; \\
J_{n,m,t} = \{i(n,p) : p \in \mathcal{T}_m^t\}, \quad J'_{n,m,t} = \{1, \ldots, [t/\Delta_n]\} \setminus J_{n,m,t}; \\
T_-(n,p) = t_{i(n,p)-1}^n, \quad T_+(n,p) = t_{i(n,p)}^n; \\
\Omega_{n,m,t} = \bigcap_{p \neq q, p, q \in \mathcal{T}_m^t} \{T_p > \lfloor t/\Delta_n \rfloor \Delta_n \text{ or } |T_p - T_q| > 2\,\Delta_n\}.
\end{gathered}
$$

Note that, for any m, we have

$$
\lim_{n \to \infty} \mathbb{P}(\Omega_{n,m,t}) = 1.
$$

Therefore, it is sufficient to restrict our attention to the set $\Omega_{n,m,t}$.

Let $A_m := \{x : \gamma(x) \leq 1/m\}$ and $\gamma_m := \int_{A_m} \gamma(z)^r \lambda(dz)$. Then we further decompose X'' (recall equation (A.1)) as $X'' = L(m) + J(m)$, where

$$
\begin{cases}
L(m)_t = \displaystyle\int_0^t \int_{A_m} \delta(\omega, u, x) \mu(du, dx), \\
J(m)_t = \displaystyle\int_0^t \int_{(A_m)^c} \delta(\omega, u, x) \mu(du, dx).
\end{cases}
$$

Furthermore, let $X'(m) = X' + L(m)$.

Now the estimation error can be written as

$$
\widehat{[X, \sigma^2]}_t^D - [X, \sigma^2]_t^D = H(1)_t^n + \sum_{j=2}^{7} H(m,j)_t^n,
$$

where

$$H(1)_t^n = \sum_{i=k_n+1}^{\lfloor t/\Delta_n \rfloor - k_n} (\Delta_i^n X^{\alpha_n}) \Big[\big(\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2 \big) - \big(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2 \big) \Big],$$

$$H(m,2)_t^n = \sum_{i \in J_{n,m,t}} \Big[(\Delta_i^n X^{\alpha_n}) - \Delta X_{T_p} \Big] \Big[\big(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2 \big) - \Delta \sigma_{T_p}^2 \Big],$$

$$H(m,3)_t^n = \sum_{p \in \mathcal{T}_m^t} \Delta X_{T_p} \Big[\big(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2 \big) - \Delta \sigma_{T_p}^2 \Big],$$

$$H(m,4)_t^n = \sum_{i \in J'_{n,m,t}} \Big[(\Delta_i^n X^{\alpha_n}) - \Delta_i^n L(m)^{\alpha_n} \Big] \Big[(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2) - \Delta_i^n \sigma^2 \Big],$$

$$H(m,5)_t^n = \sum_{i \in J'_{n,m,t}} (\Delta_i^n L(m)^{\alpha_n}) \Big[\big(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2 \big) - \Delta_i^n \sigma^2 \Big],$$

$$H(m,6)_t^n = \sum_{i \in J'_{n,m,t}} (\Delta_i^n X^{\alpha_n}) \Delta_i^n \sigma^2 - \sum_{p \notin \mathcal{T}_m^t} \Delta X_{T_p} 1_{\{|\Delta X_{T_p}| > \alpha_n\}} \Delta \sigma_{T_p}^2,$$

$$H(m,7)_t^n = \sum_{p \notin \mathcal{T}_m^t} -\Delta X_{T_p} 1_{\{|\Delta X_{T_p}| \le \alpha_n\}} \Delta \sigma_{T_p}^2.$$

For simplicity, denote by $h(m,j)_i^n$ the i-th addend of $H(m,j)_t^n$.

Step 1. We start by considering the first three terms. First of all, note that on the set $\Omega_{n,m,t}$, for any given m, the number of elements of $\mathcal{T}_m^t$ is locally finite. Hence, the following joint convergence result holds

$$\sqrt{u_n} \left(\widehat{\sigma}_{i+}'^2 - \sigma_{T_p+}^2, \widehat{\sigma}_{i-}'^2 - \sigma_{T_p-}^2 \right)_{i \in J_{n,m,t}} \xrightarrow{\mathscr{L}_{st}} (V_i^+, V_i^-)_{i \in J_{n,m,t}}.$$

Moreover, for any $i(n,p) \in J_{n,m,t}$, we have

$$|\Delta_{i(n,p)}^n X^{\alpha_n} - \Delta X_{t_p}| \xrightarrow{\mathbb{P}} 0.$$

Consequently, on the one hand, we obtain that for any t

$$\lim_{m \to 0} \limsup_{n \to 0} \sqrt{u_n}\, \mathbb{E}\big(|H(m,2)_t^n| \big) \le \lim_{m \to 0} \limsup_{n \to 0} K \sum_{i \in J_{n,m,t}} \mathbb{E}\big(|(\Delta_i^n X^{\alpha_n}) - \Delta X_{T_p}| \big) = 0.$$

Thus, Lemma 4.1 in Jacod (2012) yields that $\sqrt{u_n}\,|H(m,2)_t^n| \xrightarrow{u.c.p.} 0$. On the other hand, we have[15]

$$\sqrt{u_n}H(m,3)_t^n \xrightarrow{\mathscr{L}_{st}} \mathcal{D}_t(m) := \sum_{p\in\mathcal{T}_m^p} \Delta X_{T_p}(V_{T_p}^+ - V_{T_p}^-).$$

Observe that

$$\widetilde{\mathbb{E}}\big(|\mathcal{D}_t(m) - \mathcal{D}_t|^2|\mathcal{F}\big) \le K\sum_{s\le t}|\Delta X_s|^{2r}1_{\{|\Delta X_s|\le 1/m\}}.$$

The right-hand side goes to 0 a.s. as $m\to\infty$. Thus, we have $\mathcal{D}_t(m) \xrightarrow{u.c.p.} \mathcal{D}_t$, as m goes to infinity.

Next, note that for any given $\epsilon>0$, we can choose $m = 1/\epsilon$ [16] so that for sufficiently large n, we have

$$\widehat{[X,\sigma^2]}_t^D(\epsilon) - [X,\sigma^2]_t^D(\epsilon) = H(1;\epsilon)_t^n + \sum_{j=2}^{3} H(m,j)_t^n,$$

where

$$H(1;\epsilon)_t^n := \sum_{i=k_n+1}^{\lfloor t/\Delta_n\rfloor - k_n} (\Delta_i^n X^{(\epsilon\vee\alpha_n)})\Big[\big(\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2\big) - \big(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2\big)\Big].$$

When n is sufficiently large, we have $\alpha_n < \epsilon$, yielding $\Delta_i^n X^{(\epsilon\vee\alpha_n)} \equiv \Delta_i^n X^{\epsilon}$. Employing the results of Step 1 of the previous subsection and (A.12) (with v replaced by r), it is straightforward to verify by successive conditioning that

$$\begin{aligned}\sqrt{u_n}\,\mathbb{E}\big(|H(1;\epsilon)_t^n|\big) &\le K\sqrt{u_n}\sum_{i=k_n+1}^{\lfloor t/\Delta_n\rfloor - k_n}\Big(\mathbb{E}\big(|\Delta_i^n X^{\epsilon}|\big)\Delta^{(2-r)\varpi}\psi_n + \Delta_n\epsilon^{-(r-1)^+}\mathbb{E}\big(|\widehat{\sigma}_{i-}^2 - \sigma_{i-}^2|\big)\Big)\\ &\le K\sqrt{u_n}\Delta^{(2-r)\varpi}\psi_n.\end{aligned}$$

[15] A similar argument can be found, for example, in the proof of Lemma 5.4.10 of Jacod and Protter (2011)

[16] The requirement that m is an integer is just for convenience and not crucial. In fact, one can replace $1/m$ by q_m, where the sequence $\{q_m\}_{m=1}^{\infty}$ consists of all rational numbers within $(0,1]$ in descending order. Then, for any $\epsilon\in(0,1]$, one can choose a subsequence $\{q_{m_k}\}$ that converges to ϵ.

Then, in order to obtain

$$\sqrt{u_n}\, H(1;\epsilon)_t^n \xrightarrow{u.c.p.} 0,$$

it is sufficient to have $(2-r)\varpi \geq 1/4$. To guarantee $\varpi < 1/2$, we must require $r < 3/2$. Therefore, we have proved the second statement of the theorem. As for $H(1)_t^n$, adopting a similar argument, we obtain (upon replacing ϵ by α_n)

$$\sqrt{u_n}\,\mathbb{E}\big(|H(1)_t^n|\big) \leq K\sqrt{u_n}\Delta^{[(2-r)-(r-1)^+]\varpi}\psi_n.$$

Thus, it is sufficient to have $r < 5/4$ to make the right-hand side asymptotically negligible, while keeping $\varpi < 1/2$.

Step 2. To analyze $H(m,4)_t^n$, we need additional notation. For some $l > 1$, we denote $q_n = [\alpha_n^{-l}]$ and we suppose that n is sufficiently large so that $1/q_n < \alpha_n < 1/m$. Next, define

$$A'_n = A_m \cap (A_{q_n}^c), \quad N_t^n = \mu([0,t] \times A'_n),$$
$$L'_t = \int_0^t \int_{A'_n} \delta(\omega,u,z)\mu(du,dz), \quad L(q_n) = L(m) - L',$$
$$G(n,i) = \{|\Delta_i^n X'| \leq \alpha_n/4\} \cap \{|\Delta_i^n L(q_n)| \leq \alpha_n/4\} \cap \{\Delta_i^n N^n \leq 1\}.$$

Accordingly, let $X'(q_n) = X' + L(q_n)$.

We then evaluate the probability of $\omega \in G(n,i)$. First, when $r < 1$, it is easy to see that

$$\int_{A_{q_n}} \delta(t,x)\lambda(dx) = \int_{A_{q_n}} \delta^r \delta^{1-r}\lambda(dx) \leq K\Delta_n^{l\varpi(1-r)} \int_{A_{q_n}} \delta^r \lambda(dx) = K\Delta_n^{l\varpi(1-r)}\gamma_{q_n}.$$

And, for any $\varrho \geq r$, we have

$$\mathbb{E}_{i-1}\big(|\Delta_i^n X'|^\varrho\big) \leq K(\Delta_n^{\varrho/2}) \ \text{ and } \ \mathbb{E}_{i-1}\big(|\Delta_i^n L(q_n)|^\varrho\big) \leq \Delta_n^{1+l\varpi(\varrho-r)}\gamma_{q_n}.$$

These results, applied with $\varrho = \frac{4}{1-2\varpi}$ to the first term and with $\varrho = \frac{1+lr\varpi}{\varpi(l-1)}$ to the second one, and Markov's inequality yield

$$\mathbb{P}(|\Delta_i^n X'| > \alpha_n/2) \leq K\Delta_n^2 \ \text{ and } \ \mathbb{P}(|\Delta_i^n L(q_n)| > \alpha_n/2) \leq K\Delta_n^2.$$

Second, N^n is a Poisson process with parameter $\lambda(A'_n) \leq K\gamma_m q_n^r$. Together with $\alpha_n = \alpha\Delta_n^{\varpi}$, we obtain

$$\mathbb{P}(\Delta_i^n N^n = 1) \propto \Delta_n^{1-rl\varpi}\gamma_m \quad \text{and} \quad \mathbb{P}(\Delta_i^n N^n \geq 2) \leq K\Delta_n^{2-2rl\varpi}\gamma_m^2.$$

Let $\Omega(G)_{n,t} = \bigcap_{1\leq i\leq[t/\Delta_n]} G(n,i)$. Then, as long as $l < 1/r$, hence $2\varpi rl < rl < 1$, we have

$$\mathbb{P}(\Omega(G)_{n,t}^c) \leq \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \mathbb{P}(G(n,i)^c) \leq tK\Delta_n^{1-2rl\varpi} \longrightarrow 0.$$

Hence, it is sufficient to prove the desired results on the intersection of $\Omega_{n,m,t}$ and $\Omega(G)_{n,t}$.

Some elementary calculations show that

$$\begin{aligned}
\big||x+y|1_{\{|x+y|>\alpha\}} - |x|1_{\{|x|>\alpha\}}\big| &\leq |y|\cdot 1_{\{|x+y|>\alpha,|x|>\alpha\}} + 0\cdot 1_{\{|x+y|\leq\alpha,|x|\leq\alpha\}} \\
&\quad + |x+y|\cdot 1_{\{|x+y|>\alpha,|x|\leq\alpha\}} + \big|(x+y)-y\big|\cdot 1_{\{|x+y|\leq\alpha,|x|>\alpha\}} \\
&\leq |y|\cdot 1_{\{|x+y|>\alpha\,\cup\,|x|>\alpha\}} + |x|\cdot 1_{\{|x+y|>\alpha,|x|\leq\alpha\}} + |x+y|\cdot 1_{\{|x+y|\leq\alpha,|x|>\alpha\}} \\
&\leq |y|\cdot 1_{\{|x+y|>\alpha\,\cup\,|x|>\alpha\}} + \alpha\cdot 1_{\{|x+y|>\alpha,|x|\leq\alpha\}} + \alpha\cdot 1_{\{|x+y|\leq\alpha,|x|>\alpha\}} \\
&\leq (|y|+\alpha)\cdot 1_{\{|x+y|>\alpha\,\cup\,|x|>\alpha\}}.
\end{aligned}$$

Observe that on the set $G(n,i)$, we have

$$|\Delta_i^n X'(q_n)| \leq |\Delta_i^n X'| + |\Delta_i^n L(q_n)| \leq \alpha_n/2 < \alpha_n.$$

Therefore, for any $i \in J'(n,m,t)$, if $\Delta_i^n N^n = 0$, then

$$|\Delta_i^n L(m)| = |\Delta_i^n L(q_n)| \leq \alpha_n/4,$$
$$|\Delta_i^n X| = |\Delta_i^n X'(m)| = |\Delta_i^n X'(q_n)| \leq \alpha_n/2.$$

Then it is obvious that, conditional on $\Delta_i^n N^n = 0$, $\Delta_i^n X^{\alpha_n} = 0$ and $\Delta_i^n L(m)^{\alpha_n} = 0$. Consequently, taking $x = \Delta_i^n L(m)^{\alpha_n}$, $y = \Delta_i^n X'$ and $\alpha = \alpha_n$, the elementary inequality array above yields

$$\begin{aligned}
\mathbb{E}\big(|(\Delta_i^n X^{\alpha_n}) - (\Delta_i^n L(m)^{\alpha_n})|\big) &\leq \mathbb{E}\big(|\Delta_i^n X'| + \alpha_n\big)\,\mathbb{P}(\Delta_i^n N^n = 1) + 0\cdot\mathbb{P}(\Delta_i^n N^n = 0) \\
&\leq K(\Delta_n^{1/2} + \Delta_n^{\varpi})\Delta_n^{1-rl\varpi}\gamma_m \leq K\Delta_n^{1+(1-rl)\varpi}\gamma_m.
\end{aligned}$$

Together with the fact that $\sqrt{u_n}|\widehat{\sigma}'^2_{i\pm} - \sigma^2_{i\pm}| = O_p(1)$, we obtain by successive conditioning

$$
\begin{aligned}
\sqrt{u_n}\,\mathbb{E}\big(|h(m,4)^n_i|\big) &\le K\,\mathbb{E}\big(|(\Delta^n_i X^{\alpha_n}) - (\Delta^n_i L(m)^{\alpha_n})|\big) + K\,\mathbb{E}\big(\sqrt{u_n}|\widehat{\sigma}'^2_{i-} - \sigma^2_{i-}|\big)\Delta_n^{1+(1-rl)\varpi}\gamma_m \\
&\le K\Delta_n^{1+(1-rl)\varpi}\gamma_m.
\end{aligned}
$$

Thus, it readily follows that

$$
\sqrt{u_n}\,\mathbb{E}\big(|H(m,4)^n_t|\big) \le \sum_{i\in J'_{n,m,t}} \sqrt{u_n}\,\mathbb{E}\big(|h(m,4)^n_t|\big) \le tK\Delta_n^{\varpi(1-rl)}\gamma_m.
$$

So the sufficient condition for $\sqrt{u_n}\,H(m,4)^n_t$ to be asymptotically negligible is $r<1$, which enables us to choose $l\in(1,1/r)$.

As for $H(m,5)^n_t$, as long as $r<1$, we obtain the following inequality by successive conditioning and the boundedness of the respective moments

$$
\sqrt{u_n}\,\mathbb{E}\big(|H(m,5)^n_t|\big) \le K\sum_{i\in J'_{n,m,t}} \Big(\mathbb{E}\big(|\Delta^n_i L(m)^{\alpha_n}|\big) + \Delta_n\gamma_m\mathbb{E}\big(\sqrt{u_n}\,|\widehat{\sigma}'^2_{i-} - \sigma^2_{i-1}|\big)\Big) \le tK\gamma_m.
$$

We first let n and next m go to infinity, and get

$$
\limsup_{n\to\infty}\sqrt{u_n}\,\mathbb{E}\big(|H(m,5)^n_t|\big) \le \limsup_{n\to\infty} tK\gamma_m = 0.
$$

Therefore, we can conclude that $\sqrt{u_n}\,H(m,5)^n_t$ is also asymptotically negligible.

Step 3. Finally, we consider the last two terms, $H(m,6)^n_t$ and $H(m,7)^n_t$. We continue to work on the intersection of $\Omega_{n,m,t}$ and $\Omega(G)_{n,t}$. Recall that by definition, on each $G(n,i)$ with $i\in J'_{n,m,t}$, there is at most one T_p such that $|\Delta X_{T_p}|$ is possibly larger than α_n. Therefore, $H(m,6)^n_t$ can be written as

$$
H(m,6)^n_t = \sum_{p\notin\mathcal{T}^t_m}\Big(\big(\Delta^n_{i(n,p)}X^{\alpha_n} - \Delta X^{\alpha_n}_{T_p}\big)\Delta\sigma^2_{T_p} + \Delta^n_{i(n,p)}X^{\alpha_n}\big(\Delta^n_{i(n,p)}\sigma^2 - \Delta\sigma^2_{T_p}\big)\Big).
$$

By applying the Cauchy-Schwarz inequality and Jensen's inequality, we obtain that

$$
\begin{aligned}
&\sqrt{u_n}\,\mathbb{E}\Big(\sum_{p\notin\mathcal{T}_m^t}\big|\big(\Delta^n_{i(n,p)}X^{\alpha_n}-\Delta X^{\alpha_n}_{T_p}\big)\Delta\sigma^2_{T_p}\big|\Big)\\
&\leq \sqrt{u_n}\,\mathbb{E}\Big(\sum_{p\notin\mathcal{T}_m^t}\big(\Delta^n_{i(n,p)}X^{\alpha_n}-\Delta X^{\alpha_n}_{T_p}\big)^2\sum_{p\notin\mathcal{T}_m^t}\big(\Delta\sigma^2_{T_p}\big)^2\Big)^{1/2}\\
&\leq \sqrt{u_n}\,\Big(\mathbb{E}\sum_{p\notin\mathcal{T}_m^t}\big(\Delta^n_{i(n,p)}X^{\alpha_n}-\Delta X^{\alpha_n}_{T_p}\big)^2\sum_{p\notin\mathcal{T}_m^t}\big(\Delta\sigma^2_{T_p}\big)^2\Big)^{1/2}\\
&\leq K\sqrt{u_n}\,\Big(\mathbb{E}\sum_{p\notin\mathcal{T}_m^t}\big(\Delta^n_{i(n,p)}X^{\alpha_n}-\Delta X^{\alpha_n}_{T_p}\big)^2\Big)^{1/2}.
\end{aligned}
$$

Note that $|\Delta^n_{i(n,p)}X^{\alpha_n}-\Delta X^{\alpha_n}_{T_p}|\leq|\Delta^n_{i(n,p)}X'(\alpha_n)|$. Consequently, the above quantity is smaller than

$$
\begin{aligned}
&K\sqrt{u_n}\,\Big(\mathbb{E}\sum_{p\notin\mathcal{T}_m^t}\big(\Delta^n_{i(n,p)}X'(\alpha_n)\big)^2\Big)^{1/2}\\
&\leq K\sqrt{u_n}\,\Big(Kt\Delta_n+\sum_i\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{A_{\alpha_n}}\delta^{2-r}\delta^r\ F(dx)\Big)^{1/2}\\
&\leq K\sqrt{u_n}\,\Big(t\Delta_n+t\Delta_n^{\varpi(2-r)}\int_{A_m}\delta^r\ F(dx)\Big)^{1/2}\leq K\sqrt{t\,\gamma_m}\,\Delta_n^{\varpi(2-r)/2-1/4}.
\end{aligned}
$$

By a similar procedure, we also obtain

$$
\begin{aligned}
&\sqrt{u_n}\,\mathbb{E}\Big(\sum_{p\notin\mathcal{T}_m^t}\big|\Delta^n_{i(n,p)}X^{\alpha_n}\big(\Delta^n_{i(n,p)}\sigma^2-\Delta\sigma^2_{T_p}\big)\big|\Big)\\
&\leq K\sqrt{u_n}\,\Big(\mathbb{E}\sum_{p\notin\mathcal{T}_m^t}\big(\Delta^n_{i(n,p)}X^{\alpha_n}\big)^2\Big)^{1/2}\leq K\sqrt{t\,\gamma_m}\,\Delta_n^{\varpi(2-r)/2-1/4}.
\end{aligned}
$$

To sum up, we have

$$
\sqrt{u_n}\,\mathbb{E}\big(|H'(m,6)^n_t|\big)\leq K\sqrt{t\,\gamma_m}\Delta_n^{\varpi(2-r)/2-1/4}.
$$

Applying a similar procedure to $H(m,7)$, we also get

$$
\begin{aligned}
&\sqrt{u_n}\,\mathbb{E}\big(|H(m,7)_t^n|\big) \le K\sqrt{u_n}\Big(E\sum_{s\le t}\big(\Delta X_s^{\alpha_n}\big)^2\sum_{s\le t}\big(\Delta\sigma_s^2\big)^2\Big)^{1/2} \\
&\le K\sqrt{u_n}\Big(E\sum_{s\le t}\big(\Delta X_s^{\alpha_n}\big)^2\Big)^{1/2} \le K\sqrt{u_n}\Big(\int_0^t\int_{A_{\alpha_n}}\delta^{2-r}\delta^r\ F(dx)\Big)^{1/2} \\
&\le K\sqrt{t\,\gamma_m}\Delta_n^{\varpi(2-r)/2-1/4}.
\end{aligned}
$$

Now it becomes clear that to ensure that the last two terms are asymptotically negligible, the sufficient condition is

$$\varpi(2-r) \ge \frac{1}{2}.$$

Then $\varpi < 1/2$ implies that we must have $r < 1$. Hence, part (i) has also been proved.

B.3 Total Leverage Effect

B.3.1 Proof of Theorem 5

Analogous to previous proofs, we can make the following decomposition:

$$\widehat{[X,\sigma^2]}_t - [X,\sigma^2]_t = R(1)_t^n + R(2)_t^n + R(3)_t^n,$$

where

$$
\begin{aligned}
R(1)_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n\rfloor-k_n}(\Delta_i^n X)\Big[\big(\widehat{\sigma}_{i+}^2-\widehat{\sigma}_{i-}^2\big)-\big(\widehat{\sigma}_{i+}'^2-\widehat{\sigma}_{i-}'^2\big)\Big], \\
R(2)_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n\rfloor-k_n}(\Delta_i^n X'+\Delta_i^n X'')\Big[\big(\widehat{\sigma}_{i+}'^2-\widehat{\sigma}_{i-}'^2\big)-\Delta_i^n\sigma^2\Big], \\
R(3)_t^n &= \sum_{i=k_n+1}^{\lfloor t/\Delta_n\rfloor-k_n}\Delta_i^n X\Delta_i^n\sigma^2-[X,\sigma^2]_t.
\end{aligned}
$$

As before, we need $r < 3/2$ and $\varpi \geq \frac{1}{4(2-r)}$ to ensure that $\sqrt{u_n}\, R(1)_t^n$ is asymptotically negligible. Moreover, $R(2)_t^n$ can be further decomposed into two parts, which, after scaling by $\sqrt{u_n}$, converge to $\int_0^t \eta_s \, dB_s$ and $\mathcal{D}_t$ (asymptotically independent of each other), respectively. Finally, $R(3)_t^n$ is of order $\sqrt{\Delta_n}$, which implies that $\sqrt{u_n}\, R(3)_t^n$ is also asymptotically negligible. The proof is complete.

B.4 Market Microstructure Noise

B.4.1 Proof of Theorems 6 and 7

For this proof, we decompose $\hat{X}$ into $\bar{X} + \bar{\epsilon}$ (with obvious definitions in view of Assumption 1). Then the estimator becomes:

$$
\begin{aligned}
\widehat{[X, \sigma^2]}_T^C =& \sum_{i=k_nM+1}^{n-k_nM} \Delta_i^n \hat{X}_{\alpha_n} (\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2) \\
=& \sum_{i=k_nM+1}^{n-k_nM} \Delta_i^n \bar{X}_{\alpha_n} \frac{1}{k_n M \Delta_n} \Big(\sum_{j \in J_n^+(i)} (\Delta_j^n \bar{X}_{\alpha_n})^2 - \sum_{j \in J_n^-(i)} (\Delta_j^n \bar{X}_{\alpha_n})^2 \\
& + \sum_{j \in J_n^+(i)} 2(\Delta_j^n \bar{X}_{\alpha_n})(\Delta_j^n \bar{\epsilon}_{\alpha_n}) - \sum_{j \in J_n^-(i)} 2(\Delta_j^n \bar{X}_{\alpha_n})(\Delta_j^n \bar{\epsilon}_{\alpha_n}) \\
& + \sum_{j \in J_n^+(i)} (\Delta_j^n \bar{\epsilon}_{\alpha_n})^2 - \sum_{j \in J_n^-(i)} (\Delta_j^n \bar{\epsilon}_{\alpha_n})^2 \Big) + O_p(k_n M \Delta_n).
\end{aligned}
$$

The $O_p(k_nM\Delta_n)$ terms can be easily verified, thanks to the independence between $\bar{X}$ and $\bar{\epsilon}$, and the fact that $\bar{\epsilon}^2 = O_p(\frac{1}{M})$. In fact, all the terms in the last two lines of the equation above converge to 0 in probability. But since they will contribute to the asymptotic variance, we keep them here. To prove Theorems 6 and 7, we will go through very similar steps as in the proofs of Theorems 1 and 3. So we will only point out the similarities and differences and omit the full derivations for brevity.

Similar to equation (A.9), in the case with microstructure noise,

$$\begin{aligned}\widehat{\sigma}_{i+}^{\prime 2}-\sigma_{i+}^{2}=&\frac{1}{k_nM\Delta_n}\sum_{j\in J_n^+(i)}\frac{3}{2M}\sum_{t_l^n\in(\tau_{j-1}^n,\tau_j^n]}\Big(2\int_{t_l^n}^{t_l^n+M\Delta_n}(X'_s-X'_{t_{j-1}^n})dX'_s\\&+\int_{t_l^n}^{t_l^n+M\Delta_n}(\sigma_s^2-\sigma_{i+}^2)ds\Big)\\&+\frac{1}{k_nM\Delta_n}\sum_{j\in J_n^+(i)}\frac{3}{M}\sum_{\substack{t_l^n\in(\tau_{j-1}^n,\tau_j^n]\\t_k^n\in(\tau_{j-1}^n,\tau_j^n]}}\Big(2\int_{t_k^n}^{t_l^n+M\Delta_n}(X'_s-X'_{t_{j-1}^n})dX'_s\\&+\int_{t_k^n}^{t_l^n+M\Delta_n}(\sigma_s^2-\sigma_{i+}^2)ds\Big).\end{aligned}\tag{B.3}$$

Compared to equation (A.9), equation (B.3) features an extra term. This is because when we replace ΔX_i by $\Delta\bar{X}_i$ to estimate the volatility, we introduce cross product terms. Previously, when we used ΔX_i to estimate the volatility, the Brownian motion part of ΔX_i and that of ΔX_{i+1} had no overlap, since they were taken over disjoint time increments. By contrast, $\Delta\bar{X}_i$ and $\Delta\bar{X}_{i+1}$ are shifted by a small time increment Δt while the difference is taken over a bigger time increment $\Delta\tau$. So the Brownian motion part of $\Delta\bar{X}_i$ and that of $\Delta\bar{X}_{i+1}$ will have overlap, and thus introduce the cross product term. The terms in the first two lines on the right-hand side of equation (B.3), converge to 0 in probability and do not even contribute to the asymptotic variance. Only the terms in the second two lines will contribute to the asymptotic variance. The remainder of the proof of Theorem 6 follows similarly from equation (B.3).

As for Theorem 7, we can still apply the same decomposition as in equation (B.1), except that we replace $\Delta_i^n X_{\alpha_n}$ by $\Delta_i^n\bar{X}_{\alpha_n}$ and $\Delta_i^n X'$ by $\Delta_i^n\bar{X}'$. Let $\Delta'_n=M\Delta_n$. Consider

$$\widehat{[X,\sigma^2]}_t^C-[X,\sigma^2]_t^C=T(\alpha_n)_t^n+V_t^n+D(1)_t^n+D(2)_t^n+D(3)_t^n+D(4)_t^n+D(5)_t^n,$$

where

$$
\begin{aligned}
T(\alpha_n)_t^n &= \sum_{i=k_n+1}^{\lfloor t/(M\Delta_n)\rfloor - k_n} \Big(\Delta_i^n \bar{X}_{\alpha_n}(\widehat{\sigma}_{i+}^2 - \widehat{\sigma}_{i-}^2) - \Delta_i^n \bar{X}'(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2)\Big), \\
V_t^n &= \sum_{i=k_n+1}^{\lfloor t/(M\Delta_n)\rfloor - k_n} \Big(\Delta_i^n \bar{X}'(\widehat{\sigma}_{i+}'^2 - \widehat{\sigma}_{i-}'^2) - \Delta_i^n \bar{X}' \Delta_i^n \sigma^2\Big), \\
D(1)_t^n &= \sum_{i=k_n+1}^{\lfloor t/(M\Delta_n)\rfloor - k_n} \Delta_i^n \bar{X}'\big(\Delta_i^n \sigma^{2,d} + \Delta_i^n \sigma^{2,j}\big), \\
D(2)_t^n &= -\sum_{i=1}^{k_n} \Delta_i^n \bar{X}' \Delta_i^n \sigma^2 - \sum_{i=[t/(M\Delta_n)]-k_n+1}^{\lfloor t/(M\Delta_n)\rfloor} \Delta_i^n \bar{X}' \Delta_i^n \sigma^2, \\
D(3)_t^n &= \sum_{i=1}^{\lfloor t/(M\Delta_n)\rfloor} \Delta_i^n X' \Delta_i^n \sigma^{2,c} - \int_0^t 2\,\sigma_{s-}^2 \tilde{\sigma}_s ds. \\
D(4)_t^n &= \sum_{i=k_nM+1}^{n-k_nM} \Delta_i^n \bar{X}_{\alpha_n} \frac{1}{k_n M \Delta_n}\Big(\sum_{j\in J_n^+(i)} 2(\Delta_j^n \bar{X}_{\alpha_n})(\Delta_j^n \bar{\epsilon}_{\alpha_n}) - \sum_{j\in J_n^-(i)} 2(\Delta_j^n \bar{X}_{\alpha_n})(\Delta_j^n \bar{\epsilon}_{\alpha_n})\Big) \\
D(5)_t^n &= \sum_{i=k_nM+1}^{n-k_nM} \Delta_i^n \bar{X}_{\alpha_n} \frac{1}{k_n M \Delta_n}\Big(\sum_{j\in J_n^+(i)} (\Delta_j^n \bar{\epsilon}_{\alpha_n})^2 - \sum_{j\in J_n^-(i)} (\Delta_j^n \bar{\epsilon}_{\alpha_n})^2\Big).
\end{aligned}
$$

Then we can deploy essentially the same ideas as in the proof of Theorem 3, upon replacing Δ_n by Δ_n' and u_n by $u_n' = n^{\frac{b\wedge(1-b)}{2}}$ in each step. The five terms in the decomposition will still converge to 0.

For the asymptotic variance, if we consider Δ_n' as the unit of time change, then we can keep the same notation as in the proof for the case without microstructure noise. There is only one difference that we need to point out: $\Delta_j^n \bar{X}'$ and $\Delta_{j+1}^n \bar{X}'$ are not conditionally independent while $\Delta_j^n X'$ and $\Delta_{j+1}^n X'$ are. Therefore, except for the conditional second moments, there will be an extra cross product term contributing to the asymptotic variance. In particular,

$$\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n} \mathbb{E}\big((\xi_{I(\tilde{m},n,i)+r})^2 | \mathcal{F}_{I(\tilde{m},n,i)}\big) \xrightarrow{\mathbb{P}} \frac{2}{3}\int_0^t \tilde{\eta}_s^2 ds, \tag{B.4}$$

$$\sum_{i=1}^{l_n(\tilde{m},t)} \sum_{r=k_n+1}^{(\tilde{m}+1)k_n-1} 2\mathbb{E}\big(\xi_{I(\tilde{m},n,i)+r}\ \xi_{I(\tilde{m},n,i)+r+1} | \mathcal{F}_{I(\tilde{m},n,i)}\big) \xrightarrow{\mathbb{P}} \frac{1}{3}\int_0^t \tilde{\eta}_s^2 ds. \tag{B.5}$$

By similar arguments as in the proof of Theorem 3, one can then prove the following equations to establish the CLT:

$$\begin{cases} \displaystyle\sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\big(\xi(\tilde{m})_i^n\big)^2 \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} \int_0^t \tilde{\eta}_s^2 ds, \\ \displaystyle\sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\big(\xi(\tilde{m})_i^n\big)^4 \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} 0, \\ \displaystyle\sum_{i=1}^{l_n(\tilde{m},t)} \mathbb{E}\Big(\xi(\tilde{m})_i^n\ \Delta_{i,\tilde{m}}^n M \Big| \mathcal{F}_{I(\tilde{m},n,i)}\Big) \xrightarrow{\mathbb{P}} 0. \end{cases}$$

The first equation is proven by the summation of (B.4) and (B.5). The remaining two equations can be proven through tedious calculations, analogous to the ones in the proof of Theorem 3. As a result, we complete the proof of Theorem 7.