Worst VaR Scenarios with Given Marginals

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1. Introduction [1]

In many instances in insurance and finance one has to do with multiple sources of risk and it is of vital importance to model the dependence between these risks.

Examples:

- risk taxonomy for financial conglomerates (market, credit, insurance, operational, liquidity, concentration)
- asset classes (stocks, bonds, real estate, ...)
- credit risk (between sectors, intrasectorial)
- etc.

Multivariate probabilistic and statistical modeling is much more complicated than univariate modeling because the number of degrees of freedom ramifies rapidly with dimension.

Introduction [2]

Quoting the CEO of Van Lanschot while explaining the main causes of the current credit crunch in The Dutch House of Representatives:

"Over the past few years, we have invested tremendously in improving our risk management systems, but our models appeared to be unable to appropriately capture the interdependences between risks." (NRC Handelsblad, November 27, 2008)

An edited volume

A recent collection of papers on the topic can be found in the edited volume:

Genest, Christian, Hans U. Gerber, Marc J. Goovaerts & Roger J.A. Laeven (Eds.) (2009). *Modeling and Measurement of Multivariate Risk in Insurance and Finance*, Elsevier.

This talk is based on

1. Kaas, Rob, Roger J.A. Laeven & Roger B. Nelsen (2009). "Worst VaR scenarios with given marginals and measures of association," *Insurance: Mathematics and Economics* 44 (2), 146-158.
2. Laeven, Roger J.A. (2009). "Worst VaR scenarios: A remark," *Insurance: Mathematics and Economics* 44 (2), 159-163.

Our main message

Copula-based modeling, as better alternative to correlation-based modeling, may fail DRAMATICALLY when applied “rücksichtslos”.

- This is true in particular in the context of capital allocation, when modeling extreme adverse shocks.

The formal problem statement

We study generalized versions of the following problem, which was attributed to A.N. Kolmogorov by Makarov (1981):

- Let X and Y be two random variables with given distribution functions F_1 and F_2, respectively.
- Let G_ψ denote the distribution function of a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ (e.g., the sum) of X and Y.

Find $\underline{G}_\psi(x) = \inf\{G_\psi(x)\}$, where the infimum is taken over the *Fréchet-Hoeffding class* $\mathbf{F}(F_1, F_2)$ consisting of all joint distribution functions with marginals F_1 and F_2.

We will be concerned with generalized versions allowing for partial information on the dependence structure between X and Y, in which case the infimum is taken over well-defined subclasses of $\mathbf{F}(F_1, F_2)$.

The practical relevance of the problem

Besides being theoretically challenging, the problem is highly relevant in risk management: it is equivalent to finding worst case scenarios for the Value-at-Risk for a function of two risks when the marginal distributions of the risks are known but the dependence structure between the risks is unknown, or only partially known.

- One of our aims is to investigate and illustrate the effectiveness (or in some cases rather, lack thereof) of different types of information on the dependence structure in bounding the Value-at-Risk.
- Educative warning: contrary to what is quite often thought in the insurance and financial industry, the Value-at-Risk may vary widely even when the marginals and a nonparametric dependence measure, such as the value of a measure of association, are fixed.
- Readers should thus beware of adopting in this context the commonly used multivariate inference techniques that implicitly assume otherwise.

Outline

2.1 Modeling dependence with copulas [1]

Copulas are of interest for two main reasons:

1. As a way of studying scale-free measures of dependence;
2. As a starting point for constructing families of multivariate distributions.

Modeling dependence with copulas [2]

A bivariate copula is a function $C : [0,1]^2 \rightarrow [0,1]$ that satisfies the boundary conditions

$$C(u,0) = C(0,v) = 0, \; C(u,1) = u, \; C(1,v) = v,$$

for every $u, v \in [0,1]$, and is 2-increasing on $[0,1]^2$, i.e.,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \tag{1}$$

for all $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$. Equivalently, a bivariate copula is a bivariate d.f. with domain $[0,1]^2$ and uniform $(0,1)$ marginals.

Every bivariate copula C satisfies the *Fréchet-Hoeffding inequality*

$$W(u,v) = \max\{0; \; u+v-1\} \leq C(u,v) \leq \min\{u; \; v\} = M(u,v),$$

for every $(u,v) \in [0,1]^2$. Here, the Fréchet-Hoeffding bounds W and M are themselves bivariate copulas. A third copula that plays an important role is the *product copula* $\Pi(u,v) = uv$.

Modeling dependence with copulas [3]

It is straightforward to verify that for a given bivariate copula C and given marginals F_1 and F_2, the function F defined by

$$F(x,y) = C(F_1(x), F_2(y)), \qquad (x,y) \in \mathbb{R}^2, \tag{2}$$

is a bivariate d.f. with marginals F_1 and F_2. Sklar (1959) proved that also the converse is true: for a given bivariate d.f. F with marginals F_1 and F_2 there exists a bivariate copula C satisfying (2). Whenever F_1 and F_2 are continuous, C is unique.

Dual of a copula

For later reference, we also define the dual of a copula: the dual of a copula C is the function $\widetilde{C}$ defined by $\widetilde{C}(u, v) = u + v - C(u, v)$. We note that $\widetilde{C}$ is not a copula. However, when C is the copula of a pair of r.v.'s X and Y (in the sense of Sklar),

$$\mathbb{P}[X \leq x \text{ or } Y \leq y] = \widetilde{C}(F_1(x), F_2(y)).$$

2.2 VaR of a sum of random variables [1]

- Recall: $\mathrm{VaR}_\alpha[X]$ is (essentially) the α-quantile of the random variable X:
  Definition The Value-at-Risk (VaR) at probability level $p \in [0,1]$ for a r.v. X with d.f. F is defined as

$$\mathrm{VaR}_\alpha[X] = \inf \{x \in \mathbb{R} \mid F(x) \geq \alpha\}, \tag{3}$$

  where $\inf\{\emptyset\} = +\infty$ by convention.
- Notice that VaR_α is a left-continuous function of α. It is the generalized inverse of F, meaning that upon inversion bounds on F transform into equivalent bounds on VaR_α.
- Value-at-Risk is a risk measure that is (the most) frequently used in practice.
- In many cases one wants to draw inferences and conclusions about the VaR of a sum of risks from the VaRs of the individual risks.

VaR of a sum of random variables [2]

Stylized facts:

- For comonotonic variables X and Y, we have $\mathrm{VaR}_\alpha[X+Y] = \mathrm{VaR}_\alpha[X] + \mathrm{VaR}_\alpha[Y]$.
- If X and Y follow elliptic distributions of the same type, then $\mathrm{VaR}_\alpha[X+Y] \le \mathrm{VaR}_\alpha[X] + \mathrm{VaR}_\alpha[Y]$.
- In general it may happen, that $\mathrm{VaR}_\alpha[X+Y] > \mathrm{VaR}_\alpha[X] + \mathrm{VaR}_\alpha[Y]$.
- Specifically, this may happen i) in the case of skewed distributions, ii) in the case of variables with peculiar dependencies, and iii) in the case of variables with heavy tails.

2.3 Measures of association [1]

In what follows, we consider three measures of association: Kendall's tau, Spearman's rho and Blomqvist's beta.

1. Let (X_1, Y_1) and (X_2, Y_2) be two independent and identically distributed random vectors, with joint d.f. F. Then the population version of Kendall's tau is defined by

$$\tau = \tau_{X,Y} = \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0];$$

   it is the probability of concordance minus the probability of discordance.
2. The population version of Spearman's rho is simply the linear (or Pearson's product-moment) correlation coefficient of $F_1(X)$ and $F_2(Y)$.
3. Finally, Blomqvist's beta is defined by

$$\beta = \beta_{X,Y} = \mathbb{P}[(X - \tilde{x})(Y - \tilde{y}) > 0] - \mathbb{P}[(X - \tilde{x})(Y - \tilde{y}) < 0],$$

   where $\tilde{x}$ and $\tilde{y}$ denote medians of X and Y, respectively.

Measures of association [2]

When X and Y are continuous r.v.'s with copula C, Kendall's tau, Spearman's rho and Blomqvist's beta for X and Y are functions of C, and are given by

$$\tau_{X,Y} = \tau(C) = 4 \int\int_{[0,1]^2} C(u,v) dC(u,v) - 1; \tag{4}$$

$$\rho_{X,Y} = \rho(C) = 12 \int\int_{[0,1]^2} C(u,v) du dv - 3; \tag{5}$$

$$\beta_{X,Y} = \beta(C) = 4C(1/2, 1/2) - 1. \tag{6}$$

2.4 PQD [1]

The r.v.'s X and Y are said to be positively quadrant dependent (PQD) if for all $(x, y) \in \mathbb{R}^2$

$$\mathbb{P}[X \leq x, Y \leq y] \geq \mathbb{P}[X \leq x]\mathbb{P}[Y \leq y],$$

or equivalently,

$$\mathbb{P}[Y \leq y \mid X \leq x] \geq \mathbb{P}[Y \leq y].$$

When X and Y are continuous r.v.'s with copula C they are PQD if and only if

$$C(u, v) \geq \Pi(u, v),$$

for all $(u, v) \in [0, 1]^2$. As is well-known, $C \geq \Pi$ implies that $\tau(C), \rho(C) \geq 0$. However, the converse is not true.

PQD [2]

Beware that $C \geq \Pi$ is a rather strong assumption and assuming it while "not fully" appropriate may lead to a significant undervaluation of the worst VaR scenario.

This is a consequence of the fact that the notion of PQD establishes a partial rather than complete order in the class of bivariate copulas. Assuming $C \geq \Pi$ excludes all copulas that are not comparable to Π.

2.5 Improved copula bounds [1]

Knowing the value of a measure of association may allow one to improve the Fréchet-Hoeffding bounds.

In particular, if $\underline{C}_\alpha(u,v) = \inf\{C(u,v) \mid C \text{ is a copula, } \alpha(C) = \alpha\}$,

$$\underline{C}_\tau(u,v) = \max\left\{0;\ u+v-1;\ \frac{1}{2}\left(u+v-\sqrt{(u-v)^2+1-\tau}\right)\right\}; \tag{7}$$

$$\underline{C}_\rho(u,v) = \max\left\{0;\ u+v-1;\ \frac{1}{2}\left(u+v-\phi(u,v,\rho)\right)\right\}; \tag{8}$$

$$\underline{C}_\beta(u,v) = \max\left\{0;\ u+v-1;\ \frac{\beta+1}{4}-\left(\frac{1}{2}-u\right)_+-\left(\frac{1}{2}-v\right)_+\right\}; \tag{9}$$

with

$$\phi(u,v,\rho) = \frac{1}{3}\left(\left(9(1-\rho)+3\sqrt{9(1-\rho)^2-3(u-v)^6}\right)^{\frac{1}{3}} + \left(9(1-\rho)-3\sqrt{9(1-\rho)^2-3(u-v)^6}\right)^{\frac{1}{3}}\right),$$

and $x_+ = \max\{0;\ x\}$.

Improved copula bounds [2]

Some remarks:

- Notice that whenever the value of Kendall's tau is positive, $\underline{C}_\tau$ is an improvement over W.
- Furthermore, whenever Spearman's rho is larger than $-\frac{1}{2}$, $\underline{C}_\rho$ is an improvement over W.
- Finally, for any $\beta > -1$, $\underline{C}_\beta$ is an improvement over W.

The bounds are best-possible.

2.6 Using the improved copula bounds [1]

- When a lower bound on the copula is available, following Williamson & Downs (1990), pointwise best-possible upper and lower bounds on the d.f. of a function of the r.v.'s X and Y can be derived.
- Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function that is non-decreasing in both coordinates and left-continuous in the second coordinate.
- Furthermore, let the copula $\underline{C}$ be such that

$$F(x,y) \geq \underline{C}(F_1(x), F_2(y)), \tag{10}$$

  for all $(x,y) \in \mathbb{R}^2$.
- We denote by F_i^- the left-continuous version of the d.f. F_i, $i = 1, 2$.

Using the improved copula bounds [2]

Formally, for any given number $s \in \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \left\{ \underline{C}(F_1(x), F_2^-(\psi_{\hat{x}}(s))) \right\} \leq \mathbb{P}[\psi(X,Y) < s] \tag{11}$$

$$\leq \mathbb{P}[\psi(X,Y) \leq s] \leq \inf_{x\in\mathbb{R}} \left\{ \underline{\widetilde{C}}(F_1(x), F_2(\psi_{\tilde{x}}(s))) \right\}.$$

Here

$$\psi_{\hat{x}}(s) = \sup\{y \in \mathbb{R} \mid \psi(x,y) < s\}, \quad \text{and} \quad \psi_{\tilde{x}}(s) = \sup\{y \in \mathbb{R} \mid \psi(x,y) \leq s\},$$

for fixed $x \in \mathbb{R}$.

Moreover, these bounds are best-possible.

Using the improved copula bounds [3]

Recall: The VaR is the generalized inverse of the d.f., meaning that upon inversion bounds on the d.f. (c.f. (11)) transform into equivalent bounds on the VaR.

Example

- Let $\psi(x, y) = x + y$ and let F_1 and F_2 be uniform (0,1) laws.
- Furthermore, let $\underline{C}$ be as in (7) for a given number τ.

In this case, $\psi_{\hat{x}}(s) = s - x$ and the LHS of (11) becomes

$$\sup_{u \in (0,1)} \left\{ \max \left\{ 0;\; s-1;\; \frac{1}{2}\left(s - \sqrt{(2u-s)^2 + 1 - \tau} \right) \right\} \right\}.$$

Hence, we find that

$$\inf \{ \mathbb{P}[U + V < s] \mid \tau_{U,V} \} = \max \left\{ 0;\; s-1;\; \frac{1}{2}\left(s - \sqrt{1 - \tau_{U,V}} \right) \right\},$$

with U and V two uniform (0,1) r.v.'s having a Kendall's tau of $\tau_{U,V}$.

3. Some new copula bounds

- When the risks are PQD and τ or ρ is given;
- When both τ or ρ and β are given;
- When the risks are PQD and β is given.

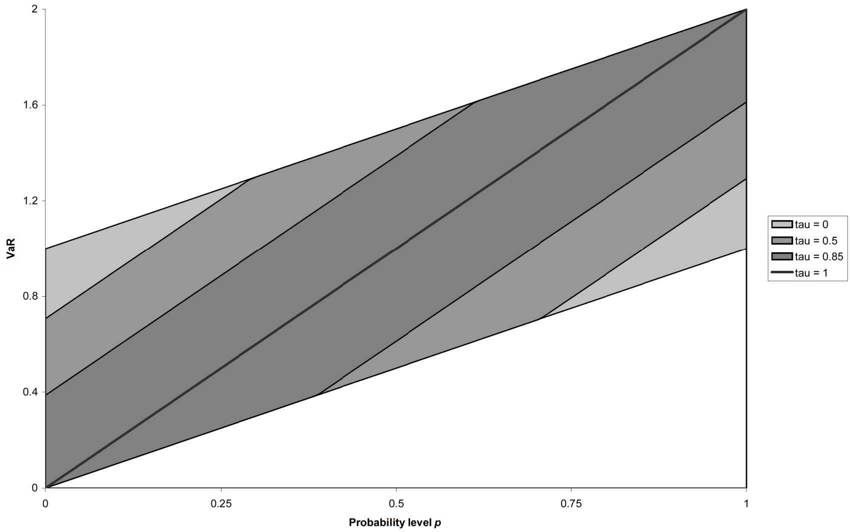
4. Worst VaR Scenarios at work

Based on these copula bounds we derive explicit upper and lower bounds on the VaR under various sets of assumptions on the marginals, the measures of association and other dependence properties.

Summary of main findings:

When τ, ρ or β is given [1]

- For probability levels relevant in risk pricing and risk management ($p \geq 0.75$, say) the upper bound on the VaR is improved only for large values of τ or ρ.
- Furthermore, information on β appears to be useless for this purpose.
- On the contrary, the lower bound on the VaR when the probability level tends to 1 is improved for all $\tau > 0$, $\rho > -0.5$ and $\beta > -1$.


2
1.6
1.2
0.8
0.4
0
VaR
0
0.25
0.5
0.75
1
Probability level p
tau = 0
tau = 0.5
tau = 0.85
tau = 1

When τ, ρ or β is given [2]

It is perhaps slightly counterintuitive that the upper bound on the VaR decreases when τ, ρ or β increases. This can be explained by the fact that the larger τ, ρ or β the larger is the "distance" between the (unimproved) lower bound W and $\underline{C}_\tau$, $\underline{C}_\rho$ and $\underline{C}_\beta$, respectively.

When the risks are PQD and τ, ρ or β is given

- The information that the risks are PQD is rather effective in bounding the VaR, both from above and from below.
- For probability levels relevant in risk pricing and risk management, additional information on τ or ρ leads to improved upper bounds only when τ or ρ are large.
- Information on β seems to be useless for this purpose.
- The lower bound on the VaR when the probability level tends to 1 is improved for most values of $\tau > 3/4$, $\rho > 13/16$ and $\beta > 0$.

When both τ and β are given

- Depending on the particular values of the measures of association τ and β, the upper and lower bound on the VaR may be well improved for probability levels relevant in risk pricing and risk management.
- This means that knowing both τ and β could conceivably be significantly more informative in this context than knowing only τ or β.

5. VaR and comonotonic scenarios

Recall: In general the comonotonic dependence structure does not lead to the worst VaR scenario.

- At first sight, this may be surprising: comonotonicity is generally perceived to be the strongest dependence notion.
- The worst VaR scenario depends on the VaR probability level.
- This is highly inconvenient in practice!

Though comonotonicity does not lead to the worst VaR scenario for a given probability level, the (conditionally) comonotonic dependence structure (see Kaas, Dhaene & Goovaerts (2000) and Dhaene *et al.* (2002)) gives rise to the VaR scenario that is most adverse on average.

- This result supports the use of (conditionally) comonotonic scenarios also in VaR-based risk management.
- All concave distortion risk measures (which include, most noticeably, the Tail-Value-at-Risk) assume their worst-case scenario when the risks are comonotonic. Therefore, comonotonic scenarios play a clear-cut role in risk management based on concave distortion risk measures.

6. Conclusion [1]

- Our analysis has demonstrated that even when the marginal distributions are given and the values of some commonly used nonparametric dependence measures are known, the Value-at-Risk for a function of two random variables may still vary widely.
- It warns the reader against using multivariate inference techniques that implicitly suppose that this is not so.
- Commonly adopted methods may fail dramatically.

Conclusion [2]

- From our analysis it becomes explicit that the dependence measures Kendall's tau, Spearman's rho and Blomqvist's beta, which effectively measure dependence in the central part of the bivariate distribution, contain little probabilistic information for the purpose of bounding the Value-at-Risk for probability levels relevant in risk pricing and risk management.
- It motivates the analysis and use of appropriate tail dependence measures.
- The information that the risks are positively quadrant dependent is rather effective in bounding the Value-at-Risk, both from above and from below.
- Joint information on both Kendall's tau and Blomqvist's beta may well improve the upper and lower bounds on the Value-at-Risk when compared with the situation in which only one of the two measures of association is given.

Conclusion [3]

(Conditionally) comonotonic scenarios are valuable since (conditional) comonotonicity is

- the most adverse dependence structure in stop-loss and supermodular order and hence in Tail-VaR-based risk management;
- the on-average-most-adverse dependence structure in VaR-based risk management.

Worst-case scenarios are not only very instructive under incomplete information, they are also crucial in stress-testing procedures.

The recent credit crunch vividly illustrates the major importance of adequate stress tests for insurance and financial institutions.